Explicit units and a refinement of the Rubin-Stark conjecture

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1. Abelian Stark conjecture

Let L/K be an abelian extension of global fields with group G. Let S be a finite nonempty set of primes of K containing all archimedean and ramified primes. Let $U_{K,S} = S$-units of K, $U_{L,S}$ similarly for L. Let w_L, w_K be the numbers of roots of unity.

Suppose S contains one place v which splits completely in L/K. Let w be a place of L above v.

Abelian Stark Conjecture (Tate's form)
There is $\varepsilon \in U_{L,S}$ such that
1) $w_L \cdot \zeta'_{L/K,S}(0, \sigma) = -\ln \mid \varepsilon^\sigma \mid_w \quad \forall \sigma \in G$;
2) $L(\varepsilon^{1/w_L})$ is abelian over K.

Example $K = \mathbb{Q}, L = \mathbb{Q}(\zeta_m)^+, S = \mathrm{Supp}(m) \cup \{\infty\}, \varepsilon = (-\zeta_m^{-1})(1-\zeta_m)^2$.

Rubin reformulated Stark's conjecture: let T be a finite set of places of K disjoint from S such that $U_{L,S,T}$ (=S-units of L congruent to 1 modulo every place of L above a place in T) is torsion free. I.e. if $w_{L,T}$ = number of roots of unity in $U_{L,S,T}$ then $w_{L,T} = 1$. Set $\delta_T = \prod_{v \in T}(1 - N_v \sigma_v^{-1})$.

Abelian Stark Conjecture (Rubin's reformulation)
For (almost) all $T = \{v\}$, there is $u \in U_{L,S,T}$ such that
$\zeta'_{L/K,S}(0,\sigma) - N_v \zeta'_{L/K,S}(0, \sigma_v^{-1}\sigma) = -\ln \mid u^\sigma \mid_w.$

Connection: $u = (\varepsilon^{1/w_L})^{\delta_T}$.
Note: also formulated for any number of splitting primes.

Advantage of Rubin's form: no w_L. Fix K, S, T, let L vary. For number fields, if L/K unramified above p and $p-1 > [K:\mathbb{Q}]$ then $p \nmid w_L$ and, for any v with residue characteristic p, $w_{L,v} = 1$. So such a $T = \{v\}$ works for all L simultaneously.

Conjecture determines u uniquely. So get system of special units over K satisfying Euler system distribution relation. Get norm-compatible systems $u_S \in \varprojlim_L U_{L,S,T}$, distribution relation under norm operator

$$\varprojlim_{L} U_{L,S\cup\{v'\},T} \to \varprojlim_{L} U_{L,S,T} \cdots$$

2. Explicit units

Let K be a global field with ≤ 1 archimedean place. Let ∞ be the archimedean place (or any place for K a function field). Let $\mathcal{O}$ = the ∞-integers of K.

For any ideal $\mathfrak{m}$ of $\mathcal{O}$, have ray class field mod $\mathfrak{m}$ (totally split at ∞) $H(\mathfrak{m})$. This corresponds to the idèle subgroup $K^\times U(\mathfrak{m})$, where idèles in $U(\mathfrak{m})$ are local units away from ∞ and congruent to 1 $\pmod{\mathfrak{m}}$.

Fix $\mathfrak{m} = \mathfrak{p}_1^{e_1}\mathfrak{p}_2^{e_2}\cdots\mathfrak{p}_n^{e_n}$. For a subset $I \subseteq \{1, \ldots, n\}$ define $\mathfrak{m}_I = \prod_{i \in I} \mathfrak{p}_i^{e_i}$. Abbreviate $w_I := w_{K,\mathfrak{m}_I}$.

Fix an embedding $\infty : H(\mathfrak{m}) \hookrightarrow \overline{\widehat{\widehat{K}}}$.

Proposition There exists a system $\widehat{\lambda}_I \in K^{ab}$ for $I \subseteq \{1, \ldots, n\}$ such that for $\mathfrak{t}$ coprime to $3\mathfrak{m}w_{H(\mathfrak{m})}$, $\lambda_I := \widehat{\lambda}_I^{\delta_\mathfrak{t}}$ satisfies

1) $\lambda_I \in U_{H(\mathfrak{m}_I), S_I, \mathfrak{t}}$;

2a) $N_{H(\mathfrak{m}_J)/H(\mathfrak{m}_I)}(\lambda_J^{w_I/w_J}) = \lambda_I^{\prod_{J-I}(1-\sigma_{p_i}^{-1})}$;

2b) $N_{H(\mathfrak{m}_i)/H}(\lambda_i^{w_K/w_i})\mathcal{O}_H = \mathfrak{p}_i^{1-N_\mathfrak{t}}\mathcal{O}_H$
($\mathcal{O}_H$ = ring of integers of Hilbert class field);

3) $w_I\big(\zeta'_{H(\mathfrak{m}_I)/K,S_I}(0,\sigma) - N_v\zeta'_{H(\mathfrak{m}_I)/K,S_I}(0,\sigma_\mathfrak{t}^{-1}\sigma)\big)$

$$= -\ln \mid \lambda_I^\sigma \mid_\infty .$$

Remark: 1) (3) $\implies$ (2a). Does it imply (2b)?!

2) This verifies abelian Rubin-Stark conjecture in these contexts via embedding L into a $H(\mathfrak{m})$ for sufficiently large $\mathfrak{m}$ with $w_{K,\mathfrak{m}} = 1$.

Remarks on the proof

For $K = \mathbb{Q}$, we take $\widehat{\lambda}_I = (1 - \zeta_{m_I})$, so that $\lambda_I = (1 - \zeta_{m_I})^{\delta_\mathfrak{t}} = \frac{(1-\zeta)}{(1-\zeta^{t^{-1}})^t}$.

If K is a function field, we use sign-normalised rank-one Drinfeld modules $\rho : \mathcal{O} \to \mathbb{C}_\infty[T]_\circ$. The $\mathfrak{m}_I$-torsion $\rho[\mathfrak{m}_I] \subseteq \mathbb{C}_\infty$ is naturally an $\mathcal{O}$-module isomorphic to $\mathcal{O}/\mathfrak{m}_I$. We choose $\widehat{\lambda}_I$ to be an $\mathcal{O}$-generator of this module. Then $\widehat{\lambda}_I$ generates a "narrow" ray class field and λ_I satisfies all the properties of the Proposition.

For K an imaginary quadratic field, we consider the function defined relative to an inclusion of lattices $L \subseteq L'$:

$$F(z, L, L') = \sqrt[12]{\frac{\Delta(L)^{N_{\mathfrak{t}}}}{\Delta(L')}} \prod_u \Big(\wp_L(z) - \wp_L(u) \Big)^{-1}$$
$$= \frac{\phi(z, \omega)^{N_{\mathfrak{t}}}}{\sqrt[12]{1} \cdot \phi(z, \omega')}$$

where u runs through $(L'/L)/\{\pm 1\} - \{0\}$, $\omega = (w_1, w_2)$ with $L = \mathbb{Z}w_1 + \mathbb{Z}w_2$, the twelfth roots are carefully chosen to ensure certain distribution relations, and ϕ is the function in Kronecker's second limit formula. We let $L = \mathfrak{m}_I$ and $L' = \mathfrak{t}^{-1}\mathfrak{m}_I$. Then most of the properties are known to hold for $\lambda_I = F(1, \mathfrak{m}_I, \mathfrak{t}^{-1}\mathfrak{m}_I)$ (the last is the limit formula). To see the congruence mod $\mathfrak{t}$, note that $\phi(1, \omega) \in H(12 f_{\mathfrak{m}_I}^2)$. Can show that $\phi(1, \omega)^{\sigma_{\mathfrak{t}}} = \sqrt[12]{1} \cdot \phi(1, \omega')$. This shows that $F(1, \mathfrak{m}_I, \mathfrak{t}^{-1}\mathfrak{m}_I) = \phi(1, \omega)^{N_{\mathfrak{t}} - \sigma_{\mathfrak{t}}}$.

3. A refinement of the conjecture

Back to setting of the abelian Stark conjecture (L/K abelian, S containing one place which splits completely).

Let $\phi \in \mathrm{Hom}_{\mathbb{Z}[G]}(U_{L,S,T}, \mathbb{Z}[G])$. What is the image $\phi(u) \in \mathbb{Z}[G]$? When there are no splitting places, Gross's conjecture postulates a congruence for

$$\Theta_{L/K,S,T}(0) := \delta_T \sum_{\sigma \in G} \zeta_{L/K,S}(0,\sigma)\sigma^{-1} \in \mathbb{Z}[G];$$

it should be $\equiv h_{K,S,T}\mathrm{Reg} \pmod{I^{\#S}}$, where $I =$ augmentation ideal and Reg is an "algebraic regulator". Burns has refined Rubin's conjecture to accommodate this idea. The formulation originates in explicit-unit interpretations of the equivariant Tamagawa number conjecture.

In the one-place-splitting case, Burns's conjecture says: (set $n := \#S - 1$)

Conjecture $\phi(u) \equiv h_{K,S,T}\mathrm{Reg}^\phi \pmod{I^n}$ where

$$\mathrm{Reg}^\phi = \begin{vmatrix} \phi^1(u_1) & \cdots & \phi^1(u_n) \\ f_{v_1}(u_1)-1 & \cdots & f_{v_1}(u_n)-1 \\ \vdots & \ddots & \vdots \\ f_{v_{n-1}}(u_1)-1 & \cdots & f_{v_{n-1}}(u_n)-1 \end{vmatrix}$$

$$= \sum_{i=1}^n (-1)^{i+1}\phi^1(u_i) \underbrace{\begin{vmatrix} f_{v_1}(u_1)-1 & \cdots & f_{v_1}(u_n)-1 \\ \vdots & \ddots & \vdots \\ f_{v_{n-1}}(u_1)-1 & \cdots & f_{v_{n-1}}(u_n)-1 \end{vmatrix}}_{\text{excluding } u_i}$$

- $\phi^1(x) \in \mathbb{Z}$ is the coefficient of $1 \in G$ in $\phi(x)$;
- u_j is a basis of $U_{K,S,T}$;
- v_i are the nonsplit places;
- $f_v : K_v^\times \to G$ is the local reciprocity map for L/K localised at v.

Remarks

- Formulated for any number of places splitting.
- Proved for extensions of degree two.
- Provides interpretation for "minus-unit" conjectures of Darmon and Gross.
- Generalised version deduced from ETNC in many cases (recent work of Burns).

Question: how much can be proved using the explicit units of §2 in the case where K has ≤ 1 archimedean place, and ∞ splits completely in L?

4. Congruence for explicit units

Let K be a global field with at most one infinite place, $\mathfrak{m}$ an ideal of $\mathcal{O}$, $S = \mathrm{Supp}(\mathfrak{m}) \cup \{\infty\}$, $T = \{\mathfrak{t}\}$.

Theorem Let b be an integer such that $w_K \mid b$ and there is a section to the restriction map $\mathrm{Gal}(H(\mathfrak{m})/K) \to \mathrm{Gal}(H/K)$ whose b-th power is multiplicative. Then Burns's conjecture is true after multiplication by b on both sides.

Sketch-proof: Step 1: using the distribution relations, produce an intermediate congruence involving $\sigma_{\mathfrak{p}_i}$.

Step 2: Relate the global symbols to the local ones using product formula.

Step 1 involves some interesting combinatorics. The proof is by induction on $\#S$, using a combinatorial identity concerning minors of matrices with zero row sum – cf. work of Greither and Kučera.

Fix a_{ij} in some ring for $1 \leq i \neq j \leq n$. For $I \subseteq \{1, \ldots, n\}$ let A^I be the $\#I \times \#I$ matrix indexed by I with off-diagonal entries a_{ij} and zero row-sum. For $J \subseteq \{1, \ldots, n\}$, let A^I_J be the determinant of the minor matrix given by deleting the rows and columns indexed by $I \cap J$ in A^I. The *Matrix Tree Theorem* shows how to express A^I_J in terms of graph-theoretic "arborescences".

Matrix Tree Theorem (S. Chaiken and D. J. Kleitman)

$A_J^I = (-1)^{\#I-\#J} \sum_T A(T)$, where T runs through all arborescences on I with root J.

Example

$$A_{\{3,4\}}^{\{1,2,3,4\}} = \det \begin{pmatrix} -a_{12} - a_{13} - a_{14} & a_{12} \\ a_{21} & -a_{21} - a_{23} - a_{24} \end{pmatrix}$$

$= a_{13}a_{21} + a_{12}a_{23} + a_{13}a_{23} + a_{13}a_{24} + a_{14}a_{23} + a_{14}a_{24} + a_{12}a_{24} + a_{14}a_{21},$

corresponding to the arborescences

$$\begin{array}{cccccccccccccccc}
\multicolumn{2}{c}{\underbrace{a_{13}a_{21}}} & \multicolumn{2}{c}{\underbrace{a_{12}a_{23}}} & \multicolumn{2}{c}{\underbrace{a_{13}a_{23}}} & \multicolumn{2}{c}{\underbrace{a_{13}a_{24}}} & \multicolumn{2}{c}{\underbrace{a_{14}a_{23}}} & \multicolumn{2}{c}{\underbrace{a_{14}a_{24}}} & \multicolumn{2}{c}{\underbrace{a_{12}a_{24}}} & \multicolumn{2}{c}{\underbrace{a_{14}a_{21}}} \\
2 & & 1 & & & & & & & & & & & 1 & & 2 \\
\downarrow & & \downarrow & & & & & & & & & & & \downarrow & & \downarrow \\
1 & & 2 & & 1 & 2 & 1 & 2 & 2 & 1 & 1 & 2 & & 2 & & 1 \\
\downarrow & & \downarrow & & \downarrow & \swarrow & \downarrow & \downarrow & \downarrow & \downarrow & \searrow & \downarrow & & \downarrow & & \downarrow \\
3 & 4 & 3 & 4 & 3 & 4 & 3 & 4 & 3 & 4 & 3 & 4 & 3 & 4 & 3 & 4
\end{array}$$

The identity we need is, for fixed $J \subseteq \{1, \ldots, n\}$:

$$\sum_I (-1)^{\# I' \cap J} F(\# I) A_J^I \prod_{i \in I' \text{-} J} \Big(\sum_{j \in I} a_{ij}\Big) \;=\; 0,$$

where $F(X) = (1 - X)^{\# J - 1}$.

By interpreting this as counting "leaves" on "trees", reduce this expression to

$$\pm \sum_T A(T) \sum_{r=0}^{m} (-1)^r \binom{m}{r} F(n - r)$$

where $m =$ number of leaves on arborescence T (vertices $\{1, \ldots, n\}$, root J). This is then zero by

Theorem (Euler) If $0 \leq t < m$,

$$\sum_{r=0}^{m} (-1)^r \binom{m}{r} r^t = 0.$$