Equivariant Tamagawa Numbers and Non-Commutative Fitting Invariants

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Declaration

This thesis is a presentation of my original research work. Wherever contributions of others are involved, every effort is made to indicate this clearly, with due reference to the literature.

Signature:.. Date...................................

Abstract

Motivated by the Equivariant Tamagawa Number Conjecture (ETNC) of Burns and Flach we develop an abstract framework of 'equivariant leading term conjectures'. We also define a natural Fitting invariant for modules over group rings of arbitrary finite groups and use this to derive explicit consequences of our equivariant leading term conjectures. We then describe applications of our approach in several concrete arithmetic settings. In particular, by these means we derive conjectural non-commutative generalizations of Brumer's Conjecture, of the Coates-Sinnott Conjecture and of the 'Strong Main Conjecture' of Mazur and Tate.

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Introduction

Let L/K be an abelian CM extension of number fields with Galois group G and T a finite set of primes of K containing all primes which ramify in L/K. Let $\mathcal{O}_L$ be the ring of integers of L and μ_L the roots of unity of L. Then the Brumer-Stickelberger element is defined to be

$$\Theta(T, L/K) := \sum_{\sigma \in G} \zeta_{K,T}(\sigma, 0)\sigma^{-1} \in \mathbb{C}G$$

where the $\zeta_{K,T}(\sigma, s)$ is the partial zeta function given by

$$\zeta_{K,T}(\sigma, s) := \sum_{\substack{(\mathfrak{a},T)=1 \\ (\mathfrak{a},L/K)=\sigma}} N\mathfrak{a}^{-s}$$

for all complex numbers s with sufficiently large real part. The set
$\mathrm{Ann}_{\mathbb{Z}G}(\mu_L)\Theta(T, L/K)$ is known to be an ideal of $\mathbb{Z}G$ and so the following conjecture is plausible.

Conjecture(Brumer) $\mathrm{Ann}_{\mathbb{Z}G}(\mu_L)\Theta(T, L/K) \subset \mathrm{Ann}_{\mathbb{Z}G}(\mathrm{Pic}(\mathcal{O}_L))$.

It is reasonable to ask whether this conjecture can be generalized to non-abelian extensions. In this thesis our approach to this question is motivated by the formulation of the *Equivariant Tamagawa Number Conjecture* (ETNC) by Burns and Flach in [9]. The ETNC is a leading term conjecture which can be applied to pairs of the form $(M, \mathfrak{A})$, where M is a motive defined over a number field and $\mathfrak{A}$ is an 'order' in the coefficient algebra of M. We denote the ETNC for a pair $(M, \mathfrak{A})$ by $\mathrm{ETNC}(M, \mathfrak{A})$. Both Brumer's conjecture and $\mathrm{ETNC}(h^0(\mathrm{Spec}\, L), \mathbb{Z}G)$ are conjectures concerning $\Theta(T, L/K)$ and so one can ask whether $\mathrm{ETNC}(h^0(\mathrm{Spec}\, L), \mathbb{Z}G)$ recovers Brumer's Conjecture. Moreover, since the ETNC makes no restriction on the extension L/K being abelian one can ask whether this case of the ETNC could also be used to provide a non-abelian generalization to Brumer's Conjecture.

We now let L/K denote an arbitrary Galois extension of number fields with Galois group G and S a finite G-stable set of places of L containing the set of archimedean places of L and the set of places which ramify in L/K. For the pair $(h^0(\operatorname{Spec} L), \mathbb{Z}G)$ one can show that the ETNC is equivalent to an equality in a relative algebraic K-group between the Euler characteristic of an 'S-Tate Sequence' of L/K and the leading term $L_S^\times(0)$ of the S-truncated equivariant L-function of L/K. To be more explicit, an 'S-Tate sequence' is an exact sequence representing a canonical element T_S of $\operatorname{Ext}^2_{\mathbb{Z}G}(X_{L,S}, \mathcal{O}^\times_{L,S})$, where $\mathcal{O}^\times_{L,S}$ is the group of S-units of L and $X_{L,S}$ is a $\mathbb{Z}G$-lattice of known structure. The leading term $L_S^\times(0)$ is an element of the multiplicative group of the center $\zeta(\mathbb{R}G)^\times$ of the group algebra $\mathbb{R}G$, and one can view this as an element of an algebraic K-group $K_0(\mathbb{Z}G, \mathbb{R}G)$ via the extended boundary homomorphism $\widehat{\delta} : \zeta(\mathbb{R}G)^\times \to K_0(\mathbb{Z}G, \mathbb{R}G)$. The assertion of ETNC$(h^0(\operatorname{Spec} L), \mathbb{Z}G)$ is then that

$$\widehat{\delta}(L_S^\times(0)) = -\chi(T_S) \in K_0(\mathbb{Z}G, \mathbb{R}G),$$

where $\chi(T_S)$ is a natural 'Euler characteristic' of T_S calculated with respect to a 'trivialization' induced by the Dirichlet regulator map Reg_S.

Motivated by this reformulation of the ETNC we shall develop a framework of 'abstract leading term conjectures' and study some of their properties. In order to do this we define an *augmented trivialized extension* (or a.t.e. for short), which is a triple $\tau = (\epsilon_\tau, \lambda_\tau, \mathcal{L}^\times_\tau)$ comprising the following data. A 'perfect extension' $\epsilon_\tau \in \operatorname{Ext}^2_{\mathbb{Z}G}(H^1_\tau, H^0_\tau)$ where the H^i_τ are specified finitely generated G-modules, an isomorphism of $\mathbb{R}G$-modules $\lambda_\tau : \mathbb{R} \otimes H^0_\tau \xrightarrow{\sim} \mathbb{R} \otimes H^1_\tau$ and an element $\mathcal{L}^\times_\tau$ of $\zeta(\mathbb{R}G)^\times$. The extension ϵ_τ can be thought of as an 'abstract Tate sequence', λ_τ as an 'abstract regulator map' and $\mathcal{L}^\times_\tau$ as an 'abstract leading term'. We then show that the vanishing of the natural Euler characteristic that is associated to τ implies relations between $\mathcal{L}^\times_\tau$ and structural invariants of the modules H^i_τ.

The ETNC for the pair $(h^0(\operatorname{Spec} L), \mathbb{Z}G)$ naturally gives rise to an a.t.e. $\tau_0 = (T_S, \operatorname{Reg}_S, L_S^\times(0))$. Furthermore, our abstract framework is constructed in such a way that the validity of $\mathrm{ETNC}(h^0(\operatorname{Spec} L), \mathbb{Z}G)$ is equivalent to the vanishing of the Euler characteristic of τ_0. When G is abelian, applying our general framework under the assumption that $\mathrm{ETNC}(h^0(\operatorname{Spec} L), \mathbb{Z}G)$ is valid will give a relation between the Fitting ideal of $\mathrm{Pic}(\mathcal{O}_{L,S})$ and the leading term of the L-function of $h^0(\operatorname{Spec} L)$, which is precisely in the style of Brumer's Conjecture. When G is non-abelian the Fitting ideal of a G-module is no longer defined. In this situation we define a natural notion of Fitting invariant for G-modules. Applying our general framework under the assumption $\mathrm{ETNC}(h^0(\operatorname{Spec} L), \mathbb{Z}G)$ is valid will give a relation between the Fitting invariant of $\mathrm{Pic}(\mathcal{O}_{L,S})$ and the leading term of the L-function of $h^0(\operatorname{Spec} L)$. This thereby gives a natural non-commutative generalization of Brumer's Conjecture.

The benefit of our abstract approach is that it applies not only to $\mathrm{ETNC}(h^0(\operatorname{Spec} L), \mathbb{Z}G)$ but also to many other cases of the ETNC (see Chapter 6). Examples of these cases include $\mathrm{ETNC}(h^0(\operatorname{Spec} L)(r), \mathbb{Z}G)$ for $r < 0$ and, under certain hypotheses, $\mathrm{ETNC}(h^0(\operatorname{Spec} L) \otimes_{h^0(\operatorname{Spec} K)} h^1(E)(1), \mathbb{Z}G)$ where E is an elliptic curve defined over K. In the first case our abstract framework will recover the Coates-Sinnott Conjecture for L/K and indeed give a natural generalization of the conjecture to the case that L/K is non-abelian. In the second case this will give a 'Strong Main Conjecture' of the kind that Mazur and Tate ask for in [29] and, again, give a natural generalization when L/K is non-abelian.

Let S now be any finite G-stable set of places of L containing all archimedean places. The S-Tate sequence of L/K was originally defined in [42] for large S, where 'large' means that S contains all the places which ramify in L/K and $\mathrm{Pic}(\mathcal{O}_{L,S})$ is trivial, and has been used in the study of the Galois module structure of $\mathcal{O}_{L,S}^{\times}$ by Chinburg in [16]. In [38] Ritter and Weiss remove the restriction that S is large. They construct an element of $\mathrm{Ext}^2_{\mathbb{Z}G}(\nabla_{L,S}, \mathcal{O}_{L,S}^{\times})$ where $\nabla_{L,S}$ is a module which fits into an exact sequence of G-modules

$$0 \longrightarrow \mathrm{Pic}(\mathcal{O}_{L,S}) \longrightarrow \nabla_{L,S} \longrightarrow X_{L,S} \longrightarrow 0\,.$$

Their methods are fairly involved and it is unclear how their construction, when S is not large, relates to the ETNC. Furthermore, the module $\nabla_{L,S}$ is non-canonical in the sense that it is only defined up to isomorphisms which are the identity on its torsion submodule and its torsion free quotient. We resolve these issues by showing that the Tate sequence constructed by Ritter and Weiss can be interpreted canonically in terms of certain étale cohomology complexes (when it makes sense to use étale cohomology). This will give a concise and natural construction of the Tate Sequence of [38]. Furthermore, this approach will give a canonical description of the module $\nabla_{L,S}$ in loc. cit. and prove natural functorial properties of the Tate Sequence of loc. cit.. It will also show how the Tate sequence of Ritter and Weiss is related to the general framework of the ETNC.

It should be noted that the more general Tate Sequence of Ritter and Weiss has already been used, for example, by Greither in [23] and Popescu in [36].

The structure of this thesis is as follows. In Chapter 2 we recall some basics, including the relevant details on algebraic K-theory and Fitting ideals. In Chapter 3 we define a Fitting invariant for modules over arbitrary finite groups and investigate some of its basic properties. In Chapter 4 we develop a general framework of abstract leading term conjectures and interpret the vanishing of the Euler characteristic of an arbitrary a.t.e. in terms of equalities between Fitting invariants. This chapter is purely algebraic in nature and requires no prior knowledge of the ETNC. In Chapter 5 we quickly review the ETNC. In Chapter 6 we describe some explicit arithmetic applications of the approach developed in Chapter 4. Finally, Chapter 7 will be devoted to the study of the Tate sequence of Ritter and Weiss in [38] and can be read independently from the preceding chapters.

Preliminaries

2.1 General Notation

In this thesis all rings will be both associative and unital. Let R be a ring. We write $\zeta(R)$ for the center of R. By an idempotent of R we shall always mean an idempotent of $\zeta(R)$. Unless stated otherwise, by an R-module we shall mean a left R-module. If M is an R-module, then we write $\mathrm{pd}_R(M)$ for the projective dimension of M over R. We define $\mathrm{PMod}(R)$ to be the category of finitely generated projective R-modules. Given a right R-module M and an R-module N we write $M \otimes_R N$ for the tensor product over R. When the tensor product $M \otimes N$ has no subscript we shall always mean the tensor product over $\mathbb{Z}$. When $R = \mathbb{Z}G$ for a finite group G we write $M \otimes_G N$ for the tensor product over $\mathbb{Z}G$. Given G-modules M and N we can view M as a right G-module via the contragredient G-action, i.e. $mg := g^{-1}m$ for all $g \in G$ and $m \in M$. In this way we can define $M \otimes_G N$ for any G-modules M and N.

Given a ring homomorphism $f : R \to S$ we write $S \otimes_R M$ in place of $S \otimes_f M$, which will cause no ambiguity as the homomorphism f will always be clear from context. We will also frequently abbreviate $S \otimes_R M$ to M_S. Given a homomorphism of R-modules $\phi : M \to N$ we write $S \otimes \phi$ for the induced homomorphism of S-modules $M_S \to N_S$.

For any $\mathbb{Z}$-module M we write M_{tor} for the torsion submodule of M and $\overline{M}$ for the quotient module M/M_{tor}. Given a prime p we write $\mathbb{Z}_{(p)}$ for the localization of $\mathbb{Z}$ at p and $\mathbb{Z}_p$ for the completion of $\mathbb{Z}$ at p. Given any $\mathbb{Z}$-module M we set $M_{(p)} := \mathbb{Z}_{(p)} \otimes M$ and $M_p := \mathbb{Z}_p \otimes M$. When M is finite $M_{(p)}$ and M_p both identify with the p-power torsion submodule of M.

If G is a finite group and K is a field, then we write $\mathrm{Irr}_K(G)$ for the set of irreducible K-valued characters of G.

2.2 Algebraic K-Theory

In this section we recall the definition and basic properties of the relevant algebraic K-groups. For a detailed discussion we refer the reader to, for example, [19, Ch. 5] or [40].

2.2.1 $K_1(R)$

The Whitehead Group $K_1(R)$ of a ring R is defined to be the abelian group generated by symbols $[P, f]$, where P is a finitely generated projective R-module and f is an automorphism of P, subject to the following relations:-

i) For each exact commutative diagram of finitely generated projective R-modules

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P' & \longrightarrow & P & \longrightarrow & P'' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f''} & & \\ 0 & \longrightarrow & P' & \longrightarrow & P & \longrightarrow & P'' & \longrightarrow & 0 \end{array}$$

we have $[P, f] = [P', f'] + [P'', f'']$, where f, f' and f'' are R-automorphisms;

ii) If P is a finitely generated projective R-module and f and g are automorphisms of P, then $[P, g \circ f] = [P, g] + [P, f]$.

Given a ring homomorphism $R \to S$ one has an induced homomorphism of groups $\mu_{R,S} : K_1(R) \to K_1(S)$ given on generators by $[P, f] \mapsto [S \otimes_R P, S \otimes f]$. We thus see that $K_1(R)$ is functorial in R.

We give an alternative description of $K_1(R)$ which will be useful later. Let n be a natural number, $i, j \in \{1, \ldots, n\}$ with $i \neq j$ and $\lambda \in R$. We define an $n \times n$ matrix $E_{ij}(\lambda)$ by setting

$$(E_{ij}(\lambda))_{kl} := \begin{cases} 1 & k = l \\ \lambda & k = i, l = j \\ 0 & \text{otherwise.} \end{cases}$$

We define the group of $n \times n$ *elementary matrices* $E_n(R)$ to be the group generated by the $E_{ij}(\lambda)$. For any natural number n we have an injection $GL_n(R) \to GL_{n+1}(R)$ given by $A \mapsto \left(\begin{array}{c|c} A & 0 \\ \hline 0 & 1 \end{array}\right)$. The groups $GL_n(R)$ form a directed system under these maps and we define the infinite general linear group $GL(R)$ of R to be the corresponding direct limit $\varinjlim GL_n(R)$. Similarly we can define $E(R) := \varinjlim E_n(R)$, since the above directed system maps elementary matrices to elementary matrices. One then has a canonical isomorphism $K_1(R) \cong GL(R)/E(R)$ given as follows. Let $[P, f] \in K_1(R)$ and let P' be a finitely generated R-module such that $P \oplus P'$ is free of rank n say. From the relations in $K_1(R)$ it is clear that $[P', \mathrm{id}] = 0$ and so $[P, f] = [P \oplus P', f \oplus \mathrm{id}]$, hence we can assume that P is free. By choosing a basis of P we can represent f by an element A_f of $GL_n(R)$. The required map is given by mapping $[P, f]$ to the class of A_f in $GL(R)/E(R)$. For a proof that this is an isomorphism we refer the reader to [40].

2.2.2 $K_0(R, S)$

Let $R \to S$ be a ring homomorphism. The *relative algebraic K-group* $K_0(R, S)$ is defined to be the abelian group generated by symbols $[P, \varphi, Q]$, where P and Q are finitely generated projective R-modules and $\varphi : P_S \to Q_S$ is an isomorphism of S-modules, subject to the following relations:-

i) Let $0 \longrightarrow P' \longrightarrow P \longrightarrow P'' \longrightarrow 0$ and $0 \longrightarrow Q' \longrightarrow Q \longrightarrow Q'' \longrightarrow 0$ be exact sequences of finitely generated projective R-modules. If there exist S-isomorphisms φ, φ' and φ'' fitting into an exact commutative diagram of S-modules

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P'_S & \longrightarrow & P_S & \longrightarrow & P''_S & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \varphi'} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi''} & & \\ 0 & \longrightarrow & Q'_S & \longrightarrow & Q_S & \longrightarrow & Q''_S & \longrightarrow & 0, \end{array}$$

then $[P, \varphi, Q] = [P', \varphi', Q'] + [P'', \varphi'', Q'']$;

ii) If $[P, \varphi, Q], [Q, \phi, L] \in K_0(R, S)$, then $[P, \phi \circ \varphi, L] = [P, \varphi, Q] + [Q, \phi, L]$.

We recall that $K_0(R,S)$ is functorial in the following sense. Given a commutative diagram of ring homomorphisms

$$\begin{array}{ccc} R & \longrightarrow & S \\ \downarrow & & \downarrow \\ R' & \longrightarrow & S', \end{array}$$

there is an induced homomorphism of groups $K_0(R,S) \to K_0(R',S')$ given on generators by $[P,\varphi,Q] \mapsto [R' \otimes_R P, S' \otimes \varphi, R' \otimes_R Q]$.

2.2.3 The Localization Sequence

Let $R \to S$ be a ring homomorphism. There is an exact sequence of abelian groups

$$K_1(R) \xrightarrow{\mu_{R,S}} K_1(S) \xrightarrow{\delta_{R,S}} K_0(R,S)$$

where the right hand map is defined as follows. As in §2.2.1 we can assume that every element of $K_1(S)$ is of the form $[S^n, f]$. One defines $\delta_{R,S}([S^n, f]) := [R^n, f, R^n]$. For a proof of the exactness we refer the reader to [40, Th. 15.5]. When $R \to S$ is clear from context we will drop the subscripts on $\mu_{R,S}$.

The maps in this *localization sequence* are natural in the following sense. Given a commutative diagram of ring homomorphisms

$$\begin{array}{ccc} R & \longrightarrow & S \\ \downarrow & & \downarrow \\ R' & \longrightarrow & S' \end{array}$$

the following diagram commutes

$$\begin{array}{ccccc} K_1(R) & \xrightarrow{\mu_{R,S}} & K_1(S) & \xrightarrow{\delta_{R,S}} & K_0(R,S) \\ \downarrow & & \downarrow & & \downarrow \\ K_1(R') & \xrightarrow{\mu_{R',S'}} & K_1(S') & \xrightarrow{\delta_{R',S'}} & K_0(R',S'), \end{array}$$

where the vertical maps arise from the functoriality of both $K_1(-)$ and $K_0(-,-)$.

2.2.4 The Reduced Norm Map

Let L be either $\mathbb{R}$, $\mathbb{C}$, a number field or a finite extension of $\mathbb{Q}_p$ for some prime p. For any semi-simple L-algebra A there exists a *reduced norm* map

$$\mathrm{Nrd}_A : A \longrightarrow \zeta(A)$$

which is defined componentwise on each Wedderburn component (cf. [37, §9]). The reduced norm map induces an injective homomorphism of abelian groups

$$\mathrm{Nrd}_A : K_1(A) \longrightarrow \zeta(A)^\times \qquad (2.1)$$

which we also denote by Nrd_A. We write Nrd for Nrd_A if the algebra A is clear from context.

We denote the image of the reduced norm map (2.1) by $\zeta(A)^{\times+}$. A precise description of $\zeta(A)^{\times+}$ is given by the Hasse-Schilling-Maass Norm Theorem [37, Ch. 8, Th. 33.15].

A case of particular interest will be when $A = LG$ for some finite group G. If L is either $\mathbb{C}$ or a finite extension of $\mathbb{Q}_p$, then $\zeta(LG)^{\times+} = \zeta(LG)^\times$. If G has no non-trivial irreducible symplectic character, then $\zeta(\mathbb{R}G)^{\times+} = \zeta(\mathbb{R}G)^\times$. We note that if G is abelian, then it satisfies this hypothesis. If G has a non-trivial irreducible symplectic character, then $\zeta(\mathbb{R}G)^{\times+} \subsetneq \zeta(\mathbb{R}G)^\times$.

2.2.5 The Extended Boundary Homomorphism

One can define a canonical *extended boundary homomorphism*

$$\widehat{\delta} : \zeta(\mathbb{R}G)^{\times} \longrightarrow K_0(\mathbb{Z}G, \mathbb{R}G)$$

as follows. Given any prime number p one has a canonical map $K_0(\mathbb{Z}G, \mathbb{Q}G) \to K_0(\mathbb{Z}_pG, \mathbb{Q}_pG)$ arising from the functoriality of $K_0(-,-)$. If $x \in K_0(\mathbb{Z}G, \mathbb{Q}G)$, then we denote by x_p the image of x under this map. One then has a canonical isomorphism of groups

$$K_0(\mathbb{Z}G, \mathbb{Q}G) \xrightarrow{\sim} \bigoplus_p K_0(\mathbb{Z}_pG, \mathbb{Q}_pG) \tag{2.2}$$

given by $x \mapsto (x_p)_p$, where the sum ranges over all prime numbers p (see the discussion following [19, (49.12)]). If $x \in \zeta(\mathbb{R}G)^{\times}$, then one can choose, using the weak approximation theorem (see [15, Ch. II, §6]) and the explicit description of $\zeta(\mathbb{R}G)^{\times +}$, an element λ of $\zeta(\mathbb{Q}G)^{\times}$ such that $\lambda x \in \zeta(\mathbb{R}G)^{\times +}$. One then defines

$$\widehat{\delta}(x) := \delta_{\mathbb{Z}G,\mathbb{R}G} \circ \mathrm{Nrd}_{\mathbb{R}G}^{-1}(\lambda x) - \sum_p \delta_{\mathbb{Z}_pG,\mathbb{Q}_pG} \circ \mathrm{Nrd}_{\mathbb{Q}_pG}^{-1}(\lambda)$$

where the sum is considered as an element of $K_0(\mathbb{Z}G, \mathbb{R}G)$ via (2.2). It can be shown that this map is well defined. Furthermore, it is clear from the definition that if $x \in \zeta(\mathbb{R}G)^{\times +}$, then $\widehat{\delta}(x) = \delta_{\mathbb{Z}G,\mathbb{R}G} \circ \mathrm{Nrd}_{\mathbb{R}G}^{-1}(x)$. For a more conceptual description of $\widehat{\delta}$ we refer the reader to [5, Lemma 2.2].

Given an idempotent e of $\mathbb{Q}G$ one can define a homomorphism $\widehat{\delta}_e : \zeta(\mathbb{R}Ge)^{\times} \to K_0(\mathbb{Z}Ge, \mathbb{R}Ge)$ in the same way as $\widehat{\delta}$. Furthermore, given two idempotents e, e' of $\mathbb{Q}G$ such that $ee' = e'$ it is straightforward to check that the maps $\widehat{\delta}_e$ and $\widehat{\delta}_{e'}$ commute with maps induced on the localization sequence by the commutative square

$$\begin{array}{ccc} \mathbb{Z}Ge & \longrightarrow & \mathbb{R}Ge \\ \downarrow & & \downarrow \\ \mathbb{Z}Ge' & \longrightarrow & \mathbb{R}Ge'. \end{array}$$

Finally, given a commutative diagram of ring homomorphisms

$$\begin{array}{ccc} \mathbb{Z}Ge & \longrightarrow & \mathbb{R}Ge \\ \downarrow & & \| \\ R & \longrightarrow & \mathbb{R}Ge \end{array}$$

we have a composite map $\widehat{\delta}_{e,R}$ given by

$$\zeta(\mathbb{R}Ge)^\times \xrightarrow{\widehat{\delta}_e} K_0(\mathbb{Z}Ge, \mathbb{R}Ge) \longrightarrow K_0(R, \mathbb{R}Ge).$$

In our applications the ring R will always be clear from context and so we will drop the subscript R on $\widehat{\delta}_{e,R}$.

2.3 Homological algebra

2.3.1 Complexes

Let R be a ring. We write $\mathrm{Kom}(R)$ for the category of complexes of R-modules and $\mathcal{D}(R)$ for its derived category. Unless stated otherwise, given a complex A of R-modules we will denote the i^{th} differential of A by d_A^i. Given a morphism of complexes $f : A \to B$ we write $\mathrm{Cone}(f)$ for the 'mapping cone' complex whose objects are $\mathrm{Cone}(f)^i := A^{i+1} \oplus B^i$ and whose differentials are $d^i_{\mathrm{Cone}(f)}(a_{i+1}, b_i) := (d_A^{i+1}(a_{i+1}), -f^{i+1}(a_{i+1}) - d_B^i(b_i))$. We regard the derived category as triangulated with respect to this choice of mapping cone. An exact triangle in $\mathcal{D}(R)$ will always be written in the form

$$A \longrightarrow B \longrightarrow C \longrightarrow \cdot$$

For any complex of R-modules C we write $C[n]$ for the shifted complex whose objects are $C[n]^i := C^{i+n}$ and whose differentials are $d^i_{C[n]} := (-1)^n d_C^{i+n}$. Given an R-module M we write $M[k]$ for the complex consisting of M concentrated in degree $-k$ and hence have an equality $M[k][n] = M[k+n]$.

We write $\mathcal{C}^b(\mathrm{PMod}(R))$ for the category of bounded complexes of finitely generated projective R-modules. We say a complex of R-modules is *perfect* if it is isomorphic in $\mathcal{D}(R)$ to an object of $\mathcal{C}^b(\mathrm{PMod}(R))$. We write $\mathcal{D}^{\mathrm{perf}}(R)$ for the full triangulated subcategory of $\mathcal{D}(R)$ consisting of perfect complexes.

2.3.2 Derived Functors

We briefly discuss how to compute certain derived functors. A full discussion of this material can be found in [22, III.6].

Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories and $F : \mathcal{A} \to \mathcal{B}$ a right exact functor. We say that a class of objects $\mathcal{T}$ of $\mathcal{A}$ is *adapted to* F if the following hypotheses hold:-

- Every object of $\mathcal{A}$ is a quotient of an object of $\mathcal{T}$;
- $\mathcal{T}$ is closed under direct sums;
- If $0 \to A' \to A \to A'' \to 0$ is an exact sequence in $\mathcal{T}$, then $0 \to F(A') \to F(A) \to F(A'') \to 0$ is exact in $\mathcal{B}$.

Let $\mathcal{D}(\mathcal{A})$, resp. $\mathcal{D}(\mathcal{B})$, denote the derived category of $\mathcal{A}$, resp. $\mathcal{B}$, and let $\mathcal{D}^b(\mathcal{A})$ denote the subcategory of $\mathcal{D}(\mathcal{A})$ of bounded complexes. Given a class $\mathcal{T}$ adapted to F one can compute the left derived functor of F,

$$L(F)(-) : \mathcal{D}^b(\mathcal{A}) \longrightarrow \mathcal{D}(\mathcal{B}),$$

as follows. Let C be an object of $\mathcal{D}^b(\mathcal{A})$ and choose a bounded above complex T all of whose objects are in $\mathcal{T}$ and such that T is isomorphic to C in $\mathcal{D}(\mathcal{A})$. One defines $L(F)(C) := F(T)$. It can be shown that $L(F)(C)$ is independent of the choice of T, i.e. given another choice T', one has a canonical isomorphism $F(T) \cong F(T')$ in $\mathcal{D}(\mathcal{B})$.

2.3.3 Tate Cohomology

Let G be a finite group. Given a G-module M and an integer i we write $\widehat{H}^i(G,M)$ for the Tate cohomology group defined in [15, Ch. IV]. We remark that, by definition, $\widehat{H}^i(G,M) = \mathrm{Ext}^i_{\mathbb{Z}G}(\mathbb{Z},M)$ for all $i \geq 1$. We say that M is *cohomologically trivial* if the groups $\widehat{H}^i(J,M)$ vanish for all subgroups J of G and for all integers i. We write M^G for the largest submodule of M upon which G acts trivially and M_G for the largest quotient module of M upon which G acts trivially. For any G-modules M and N we have a natural identification of abelian groups $M \otimes_G N \cong (M \otimes N)_G$.

Lemma 2.3.1. *A G-module M is cohomologically trivial if and only if $pd_{\mathbb{Z}G}(M) \leq 1$.*

Proof See [15, Ch. IV, Th. 9].□

Lemma 2.3.2. *Let M be a torsion free G-module and C a cohomologically trivial G-module. Then $M \otimes C$ is a cohomologically trivial G-module with respect to the diagonal G-action. Furthermore, there is a canonical isomorphism $M \otimes_G C \cong (M \otimes C)^G$.*

Proof By Lemma 2.3.1 there exists an exact sequence of G-modules $0 \to P_1 \to P_0 \to C \to 0$ with P_0 and P_1 projective. Since M is torsion free, applying the functor $M \otimes -$ to this exact sequence gives an exact sequence of G-modules

$$0 \longrightarrow M \otimes P_1 \longrightarrow M \otimes P_0 \longrightarrow M \otimes C \longrightarrow 0. \tag{2.3}$$

Now if P is a projective G-module, then $M \otimes P$ is a cohomologically trivial G-module (cf. [15, Ch. IV, §9]). Hence (2.3) implies that $M \otimes C$ is a cohomologically trivial G-module.

By definition of the Tate cohomology groups there is a tautological exact sequence of abelian groups

$$0 \longrightarrow \widehat{H}^{-1}(G, M \otimes C) \longrightarrow (M \otimes C)_G \xrightarrow{N_G} (M \otimes C)^G \longrightarrow \widehat{H}^0(G, M \otimes C) \longrightarrow 0,$$

where N_G denotes multiplication by the element $\sum_{g \in G} g$ of $\mathbb{Z}G$. As $M \otimes C$ is cohomologically trivial we therefore have a canonical isomorphism

$$N_G : (M \otimes C)_G \xrightarrow{\sim} (M \otimes C)^G$$

and since $(M \otimes C)_G$ identifies with $M \otimes_G C$ we are done.□

2.4 Fitting Ideals

In this section R is a commutative noetherian ring.

Definition 2.4.1. *Let M be a finitely generated R-module. Choose an exact sequence of R-modules of the form*

$$R^r \xrightarrow{\theta} R^g \longrightarrow M \longrightarrow 0.$$

With respect to the standard basis we can identify θ with a $g \times r$ matrix over R. The Fitting Ideal of M *is defined to be*

$$\mathrm{Fitt}_R(M) := \begin{cases} \textit{The ideal of } R \textit{ generated by all } g \times g \textit{ minors of } \theta & \textit{if } g \leq r \\ 0 & \text{otherwise.} \end{cases}$$

It is not difficult to show that this ideal is independent of the choice of presentation of M (see [35, Th. 1]).

Non-Commutative Fitting Invariants

We now generalize the definition of the Fitting ideal given in §2.4 to group rings $\mathbb{Z}G$, where G is an arbitrary finite group. For an alternative definition of Fitting invariants over non-commutative rings we refer the reader to [25].

3.1 The Definition

Let G be a finite group and M a finitely generated G-module. Let K be a splitting field for $\mathbb{Q}G$ and set $M_K := K \otimes M$. We define

$$\Upsilon(M) := \{\chi \in \mathrm{Irr}_K(G) | e_\chi M_K = 0\}$$

where e_χ is the primitive (central) idempotent of KG corresponding to χ. We set

$$e_M := \sum_{\chi \in \Upsilon(M)} e_\chi.$$

There is a natural action of $\mathrm{Aut}_{\mathbb{Q}}(K)$ on the set of irreducible idempotents of KG. Given a character χ in $\Upsilon(M)$, all idempotents in the $\mathrm{Aut}_{\mathbb{Q}}(K)$-orbit of e_χ annihilate M_K (since we can pick a basis of $K \otimes M$ which belongs to $\mathbb{Q} \otimes M$). Thus e_M is a sum over $\mathrm{Aut}_{\mathbb{Q}}(K)$-orbits of idempotents and so belongs to $\mathbb{Q}G$. The definition of e_M implies that the module $\mathbb{Z}Ge_M \otimes_G M$ is finite, which can be seen by applying the functor $\mathbb{Q} \otimes -$.

Let Λ be either $\mathbb{Z}_{(p)}$ or $\mathbb{Z}_p$ for some prime p, L the quotient field of Λ and e an idempotent of $\mathbb{Q}G$.

Hypothesis Hyp(M, Λ, e) *M is a G-module for which there exists an exact sequence* *of ΛGe-modules of the form*

$$0 \longrightarrow (\Lambda Ge)^n \xrightarrow{\theta} (\Lambda Ge)^n \longrightarrow \Lambda Ge \otimes_G M \longrightarrow 0\ . \tag{3.1}$$

If $e = e_M$, then we write $\mathrm{Hyp}(M, \Lambda)$ *in place of* $\mathrm{Hyp}(M, \Lambda, e)$.

Remark 3.1.1. From the definition of e_M it is easy to see that $\mathrm{Hyp}(M, \Lambda, e)$ holds only if e is an idempotent such that $e.e_M = e$.

Definition 3.1.2. *If* $\mathrm{Hyp}(M, \Lambda, e)$ *holds, then for any presentation of $\Lambda Ge \otimes_G M$ of the form* (3.1) *we define*

$$\mathrm{Fitt}_{\Lambda G}(M, e) := \mathrm{Nrd}_{LGe}(L \otimes_\Lambda \theta) \in \zeta(LGe)^\times \ \textit{mod}\ Nrd_{LGe}(\mu_{\Lambda Ge, LGe}(K_1(\Lambda Ge))).$$

Furthermore, if $e = e_M$ we set $\mathrm{Fitt}_{\Lambda G}(M) := \mathrm{Fitt}_{\Lambda G}(M, e_M)$ *and call this the* Fitting invariant of M.

In the next section we shall show that $\mathrm{Fitt}_{\Lambda G}(M, e)$ is independent of the choice of the presentation θ in (3.1), but first we describe criteria that are sufficient to ensure that $\mathrm{Hyp}(M, \Lambda, e)$ is satisfied.

Proposition 3.1.3. *Let M be a G-module which fits into an exact sequence of G-modules of the form*

$$0 \longrightarrow K \longrightarrow C \longrightarrow F \longrightarrow M \longrightarrow 0, \tag{3.2}$$

where F is finitely generated and free, C is finitely generated and cohomologically trivial and $\mathbb{Q} \otimes C$ is isomorphic to $\mathbb{Q} \otimes F$ as $\mathbb{Q}G$-modules.

If p is any prime for which $(K_{tor})_p$ is a cohomologically trivial G-module, then $\mathrm{Hyp}(M, \mathbb{Z}_{(p)})$, $\mathrm{Hyp}((K_{tor})_p, \mathbb{Z}_{(p)})$, $\mathrm{Hyp}(M, \mathbb{Z}_p)$ *and* $\mathrm{Hyp}((K_{tor})_p, \mathbb{Z}_p)$ *all hold.*

Proof We first need the following auxiliary lemma.

Lemma 3.1.4. *Let* Λ, G *and* e *be as above. Let*

$$0 \longrightarrow H^0 \longrightarrow C^0 \longrightarrow C^1 \longrightarrow H^1 \longrightarrow 0 \tag{3.3}$$

be an exact sequence of G*-modules with* C^0 *and* C^1 *cohomologically trivial. The induced sequence of* ΛGe*-modules*

$$0 \longrightarrow (\Lambda Ge \otimes H^0)^G \longrightarrow \Lambda Ge \otimes_G C^0 \longrightarrow \Lambda Ge \otimes_G C^1 \longrightarrow \Lambda Ge \otimes_G H^1 \longrightarrow 0$$

is exact, where ΛGe *acts on the first factor of the tensors in the natural way.*

Proof Given a G-module M we make $\Lambda Ge \otimes M$ into a $\Lambda Ge \times \mathbb{Z}G$-module via the action $(g_1 e, g_2)(x \otimes m) := g_1 x g_2^{-1} \otimes g_2 m$. Since ΛGe is torsion free, applying $\Lambda Ge \otimes -$ to (3.3) gives an exact sequence of $\Lambda Ge \times \mathbb{Z}G$-modules

$$0 \longrightarrow \Lambda Ge \otimes H^0 \longrightarrow \Lambda Ge \otimes C^0 \longrightarrow \Lambda Ge \otimes C^1 \longrightarrow \Lambda Ge \otimes H^1 \longrightarrow 0. \tag{3.4}$$

Applying the functor $(-)^G$, resp. $(-)_G$, to (3.4) gives an exact sequence of ΛGe-modules

$$0 \longrightarrow (\Lambda Ge \otimes H^0)^G \longrightarrow (\Lambda Ge \otimes C^0)^G \longrightarrow (\Lambda Ge \otimes C^1)^G,$$

resp.

$$\Lambda Ge \otimes_G C^0 \longrightarrow \Lambda Ge \otimes_G C^1 \longrightarrow \Lambda Ge \otimes_G H^1 \longrightarrow 0.$$

Lemma 2.3.2 implies that we have a canonical isomorphism $(\Lambda Ge \otimes C^i)^G \cong \Lambda Ge \otimes_G C^i$ for both $i = 0$ and 1. Combining this with the previous two exact sequences gives the exact sequence of the lemma.□

Applying Lemma 3.1.4 to (3.2) gives an exact sequence of ΛGe-modules

$$0 \longrightarrow (\Lambda Ge \otimes K)^G \longrightarrow \Lambda Ge \otimes_G C \longrightarrow \Lambda Ge \otimes_G F \longrightarrow \Lambda Ge \otimes_G M \longrightarrow 0\,. \tag{3.5}$$

We set $e = e_M$ and write T for K_{tor}. By definition of e we see that $(\Lambda Ge \otimes K)^G$ is finite and thus equal to $(\Lambda Ge \otimes T)^G$. Let p be a prime such that T_p is a cohomologically trivial G-module and fix $\Lambda = \mathbb{Z}_{(p)}$. Lemma 2.3.2 implies that we have an isomorphism of $\mathbb{Z}_{(p)}Ge$-modules $\mathbb{Z}_{(p)}Ge \otimes_G T_p \cong (\mathbb{Z}_{(p)}Ge \otimes T_p)^G$. Furthermore, $\mathbb{Z}_{(p)}Ge \otimes_G T_p$ identifies with $\mathbb{Z}_{(p)}Ge \otimes_G T$ and so (3.5) gives an exact sequence of $\mathbb{Z}_{(p)}Ge$-modules

$$0 \longrightarrow \mathbb{Z}_{(p)}Ge \otimes_G T_p \longrightarrow \mathbb{Z}_{(p)}Ge \otimes_G C \longrightarrow \mathbb{Z}_{(p)}Ge \otimes_G F \longrightarrow \mathbb{Z}_{(p)}Ge \otimes_G M \longrightarrow 0\,. \tag{3.6}$$

We need the following lemma, which is a p-local version of [15, Ch. IV, Th. 8].

Lemma 3.1.5. *Let M be a G-module, $T := M_{tor}$ and p a prime. If M and T_p are cohomologically trivial G-modules then $\overline{M}_{(p)}$ is a projective $\mathbb{Z}_{(p)}G$-module.*

Proof The proof will be similar to [15, Ch. IV, Th. 8]. Choose an exact sequence of G-modules of the form

$$0 \longrightarrow Q \longrightarrow F \longrightarrow M/T_{(p)} \longrightarrow 0$$

with F free. We have a natural identification $\overline{M}_{(p)} = \mathbb{Z}_{(p)} \otimes M/T_p$ and hence an exact sequence of $\mathbb{Z}_{(p)}G$-modules

$$0 \longrightarrow Q_{(p)} \longrightarrow F_{(p)} \longrightarrow \overline{M}_{(p)} \longrightarrow 0. \tag{3.7}$$

As $\overline{M}_{(p)}$ is torsion free the functor $\mathrm{Hom}_{\mathbb{Z}_{(p)}}(\overline{M}_{(p)}, -)$ is exact, thus applying this functor to (3.7) gives a short exact sequence of $\mathbb{Z}_{(p)}G$-modules

$$0 \longrightarrow \mathrm{Hom}_{\mathbb{Z}_{(p)}}(\overline{M}_{(p)}, Q_{(p)}) \longrightarrow \mathrm{Hom}_{\mathbb{Z}_{(p)}}(\overline{M}_{(p)}, F_{(p)}) \longrightarrow \mathrm{Hom}_{\mathbb{Z}_{(p)}}(\overline{M}_{(p)}, \overline{M}_{(p)}) \longrightarrow 0,$$

where G acts on the Hom-groups via the contragredient action. Applying the functor $(-)^G$ to this exact sequence gives a sequence of $\mathbb{Z}_{(p)}$-modules

$$0 \longrightarrow \mathrm{Hom}_{\mathbb{Z}_{(p)}G}(\overline{M}_{(p)}, Q_{(p)}) \longrightarrow \mathrm{Hom}_{\mathbb{Z}_{(p)}G}(\overline{M}_{(p)}, F_{(p)}) \longrightarrow \mathrm{Hom}_{\mathbb{Z}_{(p)}G}(\overline{M}_{(p)}, \overline{M}_{(p)}) \longrightarrow 0 \tag{3.8}$$

which will be exact if $\mathrm{Ext}^1_{\mathbb{Z}_{(p)}G}(\mathbb{Z}_{(p)}, \mathrm{Hom}_{\mathbb{Z}_{(p)}}(\overline{M}_{(p)}, Q_{(p)})) = 0$. If this is the case, then any pre-image of the identity map $\mathrm{id}_{\overline{M}_{(p)}}$ in (3.8) will split (3.7), showing $\overline{M}_{(p)}$ is projective as required.

Let G_p be a Sylow p-subgroup of G. We recall that the Tate cohomology groups $\widehat{H}^i(G, A)$ of a G-module A are annihilated by $|G|$ ([15, Ch. IV, Cor. 1]). Hence if T is a finite G-module such that $(|G|, |T|) = 1$, then T is cohomologically trivial. We thus see that T/T_p is a cohomologically trivial G_p-module. Furthermore, since both M and T_p are cohomologically trivial G_p-modules, then so is M/T_p. Thus applying the long exact sequence of Tate cohomology to the tautological exact sequence of G_p-modules

$$0 \longrightarrow T/T_p \longrightarrow M/T_p \longrightarrow \overline{M} \longrightarrow 0$$

shows that $\overline{M}$ is a cohomologically trivial G_p-module.

For all G-modules N and integers $i \geq 1$ we have a commutative diagram of groups

$$\begin{array}{ccc} \mathrm{Ext}^i_{\mathbb{Z}G}(\mathbb{Z}, N)_{(p)} & \xrightarrow{\sim} & \mathrm{Ext}^i_{\mathbb{Z}_{(p)}G}(\mathbb{Z}_{(p)}, N_{(p)}) \\ \downarrow & & \downarrow \\ \mathrm{Ext}^i_{\mathbb{Z}G_p}(\mathbb{Z}, N) & \xrightarrow{\sim} & \mathrm{Ext}^i_{\mathbb{Z}_{(p)}G_p}(\mathbb{Z}_{(p)}, N_{(p)}), \end{array} \tag{3.9}$$

where the horizontal arrows are the natural maps and the vertical arrows are the restriction maps. The first vertical map in (3.9) is injective since the restriction map $\mathrm{res}^G_{G_p} : \widehat{H}^i(G, A) \longrightarrow \widehat{H}^i(G_p, A)$ is injective on the p-part of $\widehat{H}^i(G, A)$ ([15, Ch. IV, Cor. 3]). Now since $\overline{M}$ is a cohomologically trivial G_p-module [loc. cit., Ch. IV, Th. 7] implies that $\mathrm{Hom}(\overline{M}, Q)$ is a cohomologically trivial G_p-module, i.e. $\mathrm{Ext}^i_{\mathbb{Z}G_p}(\mathbb{Z}, \mathrm{Hom}(\overline{M}, Q)) = 0$ for all $i \geq 1$. Furthermore, $\mathrm{Hom}(\overline{M}, Q)_{(p)}$ is isomorphic to $\mathrm{Hom}_{\mathbb{Z}_{(p)}}(\overline{M}_{(p)}, Q_{(p)})$ and so setting $N = \mathrm{Hom}(\overline{M}, Q)$ in diagram (3.9) shows that $\mathrm{Ext}^i_{\mathbb{Z}_{(p)}G}(\mathbb{Z}_{(p)}, \mathrm{Hom}(\overline{M}, Q)_{(p)}) = 0$ as required.$\square$

Applying Lemma 3.1.5 to the G-module C shows that $\overline{C}_{(p)}$ is a projective $\mathbb{Z}_{(p)}G$-module. Furthermore, $\overline{C}_{(p)}$ identifies canonically with $\overline{\mathbb{Z}_{(p)}G \otimes_G C}$. There is a natural surjective $\mathbb{Z}_{(p)}Ge$-homomorphism $\mathbb{Z}_{(p)}Ge \otimes_{\mathbb{Z}_{(p)}G} \overline{\mathbb{Z}_{(p)}G \otimes_G C} \to \overline{\mathbb{Z}_{(p)}Ge \otimes_G C}$. As

$\overline{\mathbb{Z}_{(p)}G \otimes_G C}$ is projective, $\mathbb{Z}_{(p)}Ge \otimes_{\mathbb{Z}_{(p)}G} \overline{\mathbb{Z}_{(p)}G \otimes_G C}$ is torsion free and so this homomorphism is also injective. We thus see that $\overline{\mathbb{Z}_{(p)}Ge \otimes_G C}$ is a projective $\mathbb{Z}_{(p)}Ge$-module. One also sees, using the previous isomorphism, that $\mathbb{Z}_{(p)}Ge \otimes_G T_p = (\mathbb{Z}_{(p)}Ge \otimes_G C)_{tor}$ and so (3.6) induces an exact sequence of $\mathbb{Z}_{(p)}Ge$-modules

$$0 \longrightarrow \overline{\mathbb{Z}_{(p)}Ge \otimes_G C} \longrightarrow \mathbb{Z}_{(p)}Ge \otimes_G F \longrightarrow \mathbb{Z}_{(p)}Ge \otimes_G M \longrightarrow 0\,.$$

Recalling that $e := e_M$, we see that $\mathbb{Z}_{(p)}e \otimes_G M$ is finite. Hence applying the functor $\mathbb{Q} \otimes_{\mathbb{Z}_{(p)}} -$ to the previous exact sequence shows that $\mathbb{Q} \otimes_{\mathbb{Z}_{(p)}} \overline{\mathbb{Z}_{(p)}Ge \otimes_G C}$ is isomorphic to $\mathbb{Q} \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_{(p)}Ge \otimes_G F$ as a $\mathbb{Q}G$-module. The following well known result now shows that $\overline{\mathbb{Z}_{(p)}Ge \otimes_G C}$ is a free $\mathbb{Z}_{(p)}G$-module and hence that $\mathrm{Hyp}(M, \mathbb{Z}_{(p)})$ holds.

Lemma 3.1.6. *Let R be a discrete valuation ring with quotient field K and G a finite group. Let M and N be projective RG-modules such that $K \otimes_R M \cong K \otimes_R N$ as KG-modules. Then M is isomorphic to N as RG-modules.*

Proof See [19, Ch. 4, Th. 32.1].□

Next, as T_p is a cohomologically trivial G-module Lemma 2.3.1 implies T_p has a length one projective resolution over $\mathbb{Z}G$. Applying $\mathbb{Z}_{(p)} \otimes -$ to such a resolution gives an exact sequence of $\mathbb{Z}_{(p)}G$-modules

$$0 \longrightarrow P \longrightarrow P' \longrightarrow T_p \longrightarrow 0 \tag{3.10}$$

where P is a projective and, without loss of generality, P' is both finitely generated and free. Applying Lemma 3.1.6 to (3.10) shows $\mathrm{Hyp}(T_p, \mathbb{Z}_{(p)})$ holds. Finally, the hypotheses $\mathrm{Hyp}(M, \mathbb{Z}_p)$ and $\mathrm{Hyp}(T_p, \mathbb{Z}_p)$ now clearly hold as the functor $\mathbb{Z}_p \otimes_{\mathbb{Z}_{(p)}} -$ is exact.□

3.2 Fitting Invariants are Well Defined

We now return to the situation that Λ is either $\mathbb{Z}_{(p)}$ or $\mathbb{Z}_p$ for some prime p. We fix a G-module M and an idempotent e of $\mathbb{Q}G$ such that $\mathrm{Hyp}(M, \Lambda, e)$ holds.

Lemma 3.2.1. *Given two exact sequences of ΛGe-modules of the form*

$$0 \longrightarrow (\Lambda Ge)^n \xrightarrow{\theta} (\Lambda Ge)^n \xrightarrow{\pi} \Lambda Ge \otimes_G M \longrightarrow 0$$

$$0 \longrightarrow (\Lambda Ge)^m \xrightarrow{\theta'} (\Lambda Ge)^m \xrightarrow{\pi'} \Lambda Ge \otimes_G M \longrightarrow 0,$$

one has $Nrd(\theta) = Nrd(\theta')Nrd(u)$ for some $u \in \mu(K_1(\Lambda Ge))$. In particular $\mathrm{Fitt}_{\Lambda G}(M, e)$ *is independent of the choice of presentation (3.1).*

Proof To ease notation we set $\Gamma := \Lambda Ge$ and $M_e := \Lambda Ge \otimes_G M$.

Consider the exact sequence of Γ-modules

$$0 \longrightarrow \Gamma^n \xrightarrow{\theta} \Gamma^n \xrightarrow{\pi} M_e \longrightarrow 0\ .$$

Let $\{\epsilon_i\}_{i=1}^n$ denote the standard basis of Γ^n. Clearly $\{m_i := \pi(\epsilon_i)\}_{i=1}^n$ is a set of generators of M_e. Let m be an element of M_e and $\{\epsilon_i'\}_{i=1}^{n+1}$ denote the standard basis of Γ^{n+1}. We define a map $\pi' : \Gamma^{n+1} \to M_e$ by setting

$$\pi'(\epsilon_i') = \begin{cases} m_i & 1 \le i \le n \\ m & i = n+1. \end{cases}$$

We now have an exact commutative diagram of Γ-modules

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Gamma^n & \xrightarrow{\theta} & \Gamma^n & \xrightarrow{\pi} & M_e & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & K & \xhookrightarrow{\iota} & \Gamma^{n+1} & \xrightarrow{\pi'} & M_e & \longrightarrow & 0 \end{array}$$

where the middle vertical arrow is the natural injection of Γ^n into the first n factors of Γ^{n+1}, $K := \ker(\pi')$ and the first vertical arrow is the unique map making the diagram commute. Completing the diagram via the Snake Lemma we obtain the following exact commutative diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Gamma^n & \xrightarrow{\theta} & \Gamma^n & \xrightarrow{\pi} & M_e & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & K & \xrightarrow{\iota} & \Gamma^{n+1} & \xrightarrow{\pi'} & M_e & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle\sigma} & & \downarrow & & \\ 0 & \longrightarrow & C & \xrightarrow{\sim} & \Gamma & \longrightarrow & 0 & & \\ & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0 & & & & \end{array} \tag{3.11}$$

where σ is the projection of the $(n+1)^{\text{th}}$ component of Γ^{n+1} onto Γ. We extract the following split exact sequence from the diagram

$$0 \longrightarrow \Gamma^n \longrightarrow K \xrightarrow{\sigma\circ\iota} \Gamma \longrightarrow 0.$$

Let $(\lambda_1, \ldots, \lambda_n, e)$ be a pre-image of e under $\sigma \circ \iota$. This pre-image defines an isomorphism of Γ-modules $K \cong \Gamma^{n+1}$. Together with (3.11) this gives an exact commutative diagram of Γ-modules

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Gamma^n & \xrightarrow{\theta} & \Gamma^n & \xrightarrow{\pi} & M_e & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & \Gamma^{n+1} & \xrightarrow{\phi} & \Gamma^{n+1} & \xrightarrow{\pi'} & M_e & \longrightarrow & 0 \end{array}$$

where, with respect to the standard basis, ϕ is represented by the matrix

$$\left(\begin{array}{c|c} \theta & 0 \\ \hline \lambda_1, \ldots, \lambda_n & e \end{array}\right).$$

Let $0 \longrightarrow \Gamma^n \xrightarrow{\theta} \Gamma^n \xrightarrow{\pi} M_e \longrightarrow 0$ and $0 \longrightarrow \Gamma^m \xrightarrow{\theta'} \Gamma^m \xrightarrow{\pi'} M_e \longrightarrow 0$ be exact sequences of Γ-modules and $\{m_i\}_{i=1}^n$, resp. $\{m_i'\}_{i=1}^m$, the images of the standard basis vectors of Γ^n, resp. Γ^m, under π, resp. π'. Using the above argument inductively we obtain two exact sequences

$$0 \longrightarrow \Gamma^{n+m} \xrightarrow{\phi} \Gamma^{n+m} \xrightarrow{\Pi} M_e \longrightarrow 0$$

$$0 \longrightarrow \Gamma^{n+m} \xrightarrow{\phi'} \Gamma^{n+m} \xrightarrow{\Pi'} M_e \longrightarrow 0$$

where, if $\{\epsilon_i\}_{i=1}^{n+m}$ is the standard basis of Γ^{n+m}, then Π and Π' are defined by

$$\Pi(\epsilon_i) = \begin{cases} m_i & i = 1, \ldots, n \\ m'_i & i = n+1, \ldots, n+m \end{cases} \qquad \Pi'(\epsilon_i) = \begin{cases} m'_i & i = 1, \ldots, m \\ m_i & i = m+1, \ldots, n+m. \end{cases}$$

Furthermore, the maps ϕ and ϕ' are represented by matrices of the form

$$\phi = \left(\begin{array}{c|ccc} \theta & 0 & \ldots & 0 \\ \hline * & e & 0 & 0 \\ * & * & \ddots & 0 \\ * & * & * & e \end{array}\right) \qquad \phi' = \left(\begin{array}{c|ccc} \theta' & 0 & \ldots & 0 \\ \hline * & e & 0 & 0 \\ * & * & \ddots & 0 \\ * & * & * & e \end{array}\right). \tag{3.12}$$

We can construct an exact commutative diagram of Γ-modules

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Gamma^{n+m} & \xrightarrow{\phi} & \Gamma^{n+m} & \xrightarrow{\Pi''} & M_e & \longrightarrow & 0 \\ & & \downarrow V & & \downarrow U & & \| & & \\ 0 & \longrightarrow & \Gamma^{n+m} & \xrightarrow{\phi'} & \Gamma^{n+m} & \xrightarrow{\Pi'} & M_e & \longrightarrow & 0 \end{array}$$

where U is the obvious change of basis isomorphism making the right hand square commute and V is the unique map completing the diagram. The Snake Lemma implies that V is an isomorphism thus we can represent U and V by elements of $GL_{n+m}(\Gamma)$. The equality $\phi = U^{-1}\phi' V$, together with (3.12) and the definition of Nrd, implies that $\mathrm{Nrd}(\theta) = \mathrm{Nrd}(\phi) = \mathrm{Nrd}(U^{-1})\mathrm{Nrd}(\phi')\mathrm{Nrd}(V) = \mathrm{Nrd}(U^{-1})\mathrm{Nrd}(\theta')\mathrm{Nrd}(V)$. Since $\mathrm{Nrd}(U^{-1})$ and $\mathrm{Nrd}(V)$ are clearly in $\mathrm{Nrd}(\mu(K_1(\Gamma)))$ we are done.$\square$

3.3 Basic Properties

3.3.1 Abelian Groups

For this section G will be an abelian group. Let Λ be as in §3.2, L the quotient field of Λ and $\overline{L}$ an algebraic closure of L. Let e be an idempotent of $\mathbb{Q}G$ and M a G-module for which $\mathrm{Hyp}(M,\Lambda,e)$ holds. We set $C(e) := \{\chi \in \mathrm{Irr}_{\overline{L}}(G) | e_\chi e \neq 0\}$, where e_χ is the irreducible idempotent of $\overline{L}G$ corresponding to χ. There is a unique isomorphism of $\overline{L}$-algebras

$$\rho : \overline{L}G \xrightarrow{\sim} \prod_{\chi \in \mathrm{Irr}_{\overline{L}}(G)} \overline{L}.$$

For any natural number m we denote by ρ_e the isomorphism of $\overline{L}$-algebras

$$\rho_e : M_m(\overline{L}Ge) \xrightarrow{\sim} \prod_{\chi \in C(e)} M_m(\overline{L})$$

induced by ρ. We then have a commutative diagram

$$\begin{array}{ccc} M_m(\Lambda Ge) & \xrightarrow{\rho_e} & \prod_{\chi \in C(e)} M_m(\overline{L}) \\ \downarrow{\scriptstyle \det} & & \downarrow{\scriptstyle (\det)_\chi} \\ \Lambda Ge & \xrightarrow{\rho_e|_{\Lambda Ge}} & \prod_{\chi \in C(e)} \overline{L}\ . \end{array} \tag{3.13}$$

Since $\mathrm{Hyp}(M,\Lambda,e)$ holds we can choose an exact sequence of ΛGe-modules of the form

$$0 \longrightarrow (\Lambda Ge)^n \xrightarrow{\theta} (\Lambda Ge)^n \longrightarrow \Lambda Ge \otimes_G M \longrightarrow 0.$$

We then have

$$\mathrm{Fitt}_{\Lambda Ge}(\Lambda Ge \otimes_G M) = \det(\theta)\Lambda Ge$$

and

$$\mathrm{Fitt}_{\Lambda G}(M,e)=\mathrm{Nrd}_{LGe}(L\otimes\theta)\ \mathrm{mod}\ \mathrm{Nrd}_{LGe}(\mu(K_1(\Lambda Ge))).$$

As G is abelian we have $\mathrm{Nrd}_{LGe}(\mu(K_1(\Lambda Ge)))=\rho_e((\Lambda Ge)^{\times})$. By definition of the reduced norm and the commutativity of diagram (3.13) we see that $\mathrm{Nrd}_{LGe}(L\otimes\theta)=\rho_e(\det(\theta))$. We have thus proved the following result.

Lemma 3.3.1. *Let G be an abelian group and M a G-module for which $Hyp(M,\Lambda,e)$ holds. Then $Fitt_{\Lambda Ge}(\Lambda Ge\otimes_G M)$ is a principal ideal of ΛGe. Furthermore, if x is any generator of this ideal then*

$$\mathrm{Fitt}_{\Lambda G}(M,e)=\rho_e(x)\ \mathit{mod}\ \rho_e((\Lambda Ge)^{\times}).\square$$

In view of this lemma there should be no confusion in using the notation $\mathrm{Fitt}_{\Lambda G}(-)$ to denote both Fitting invariants and Fitting ideals.

Lemma 3.3.2. *Let Λ, G and e be as above and M a G-module. Then:-*

i) $\mathrm{Fitt}_{\Lambda Ge}(\Lambda Ge\otimes_G M)=\mathrm{Fitt}_{\Lambda G}(M)e$*;*

ii) If $e=e_M$, then $\mathrm{Fitt}_{\Lambda G}(M)e=\mathrm{Fitt}_{\Lambda G}(M)$*.*

Proof i) Choose an exact sequence of ΛG-modules

$$(\Lambda G)^r\xrightarrow{\theta}(\Lambda G)^s\longrightarrow M\longrightarrow 0$$

and identify θ with an $s\times r$ matrix over ΛG in the usual way. Applying the functor $\Lambda Ge\otimes_{\Lambda G}-$ to this exact sequence gives an exact sequence of ΛGe-modules

$$(\Lambda Ge)^r\xrightarrow{\Lambda Ge\otimes\theta}(\Lambda Ge)^s\longrightarrow\Lambda Ge\otimes_{\Lambda G}M\longrightarrow 0\,.$$

Clearly $\Lambda Ge\otimes\theta$ is represented by the matrix obtained from θ by multiplying each of its entries by e, hence $\det(\theta)e=\det(\Lambda Ge\otimes\theta)$. Claim i) now follows from Definition 2.4.1.

ii) By definition of e_M we have $1 - e_M = \sum_{\chi \in \mathrm{Irr}_{\overline{L}}(G) \setminus \Upsilon(M)} e_\chi$, where e_χ is the primitive idempotent of $\overline{L}G$ corresponding to χ. For any $x \in \mathrm{Fitt}_{\Lambda G}(M)$ one has $x = xe_M + x(1 - e_M)$, thus it suffices to show that $\mathrm{Fitt}_{\Lambda G}(M)e_\chi = 0$ for all $\chi \in \mathrm{Irr}_{\overline{L}}(G) \setminus \Upsilon(M)$. Given such a character χ we let Λ_χ be the ring extension of Λ generated by the values of χ and set $\Gamma_\chi := \Lambda_\chi G e_\chi$. Choose an exact sequence of ΛG-modules of the form

$$(\Lambda G)^r \xrightarrow{\theta} (\Lambda G)^s \longrightarrow M \longrightarrow 0\,.$$

Applying $\Gamma_\chi \otimes_{\Lambda G} -$ gives an exact sequence of Γ_χ-modules

$$\Gamma_\chi^r \xrightarrow{\theta_\chi} \Gamma_\chi^s \longrightarrow M_{\Gamma_\chi} \longrightarrow 0$$

where $\theta_\chi := \Gamma_\chi \otimes \theta$. Definition 2.4.1 implies that $\mathrm{Fitt}_{\Lambda G}(M)\Gamma_\chi = \mathrm{Fitt}_{\Gamma_\chi}(M_{\Gamma_\chi})$ and furthermore that $\mathrm{im}(\bigwedge^s \theta_\chi) = \mathrm{Fitt}_{\Gamma_\chi}(M_{\Gamma_\chi})$ in $\bigwedge^s_{\Gamma_\chi} \Gamma_\chi^s \cong \Gamma_\chi$. However, since $\chi \notin \Upsilon(M)$ we know that M_{Γ_χ} has rank at least one over Γ_χ and thus $\mathrm{im}(\theta_\chi)$ has rank at most $s-1$. This in turn implies that $\mathrm{im}(\bigwedge^s \theta_\chi)$ is a torsion submodule of Γ_χ and is thus zero as required.□

3.3.2 Multiplicativity

For commutative rings Cornacchia and Greither have shown that, under certain conditions, Fitting ideals are 'multiplicative' on short exact sequences (cf. [18, Lemma 3]). We have the following analogous result in the non-commutative setting.

Proposition 3.3.3. *Let Λ be as in §3.3.1, G a finite group and*

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

a short exact sequence of ΛG-modules such that both Hyp(M, Λ, e) and Hyp(M'', Λ, e) hold and Tor$^1_{\Lambda G}(\Lambda Ge, M'') = 0$. Then Hyp$(M', \Lambda, e)$ holds and

$$\mathrm{Fitt}_{\Lambda G}(M, e) = \mathrm{Fitt}_{\Lambda G}(M', e)\,\mathrm{Fitt}_{\Lambda G}(M'', e).$$

Proof Let $\Gamma = \Lambda Ge$. Under the assumption that $\mathrm{Tor}^1_{\Lambda G}(\Gamma, M'') = 0$, applying $\Gamma \otimes_{\Lambda G} -$ to the exact sequence in the statement of the proposition gives a short exact sequence of Γ-modules

$$0 \longrightarrow \Gamma \otimes_{\Lambda G} M' \longrightarrow \Gamma \otimes_{\Lambda G} M \longrightarrow \Gamma \otimes_{\Lambda G} M'' \longrightarrow 0\,. \tag{3.14}$$

Since $\mathrm{Hyp}(M, \Lambda, e)$ and $\mathrm{Hyp}(M'', \Lambda, e)$ both hold we can choose short exact sequences of Γ-modules of the form

$$\begin{gathered} 0 \longrightarrow \Gamma^n \xrightarrow{\theta} \Gamma^n \longrightarrow \Gamma \otimes_{\Lambda G} M \longrightarrow 0 \\ 0 \longrightarrow \Gamma^{n''} \xrightarrow{\theta''} \Gamma^{n''} \longrightarrow \Gamma \otimes_{\Lambda G} M'' \longrightarrow 0. \end{gathered} \tag{3.15}$$

We will show that this implies the existence of a short exact sequence

$$0 \longrightarrow \Gamma^{n'} \xrightarrow{\theta'} \Gamma^{n'} \longrightarrow \Gamma \otimes_{\Lambda G} M' \longrightarrow 0$$

and that $\mathrm{Nrd}(\theta) = \mathrm{Nrd}(\theta')\mathrm{Nrd}(\theta'')$ in $\zeta(LGe)^\times$.

For all Γ-modules N the long exact sequence corresponding to (3.14) and the functors $\mathrm{Ext}^i_\Gamma(-, N)$, together with (3.15), shows that $\mathrm{Ext}^2_\Gamma(\Gamma \otimes_{\Lambda G} M', N) = 0$. This implies that $\mathrm{pd}_\Gamma(\Gamma \otimes_{\Lambda G} M') \leq 1$. Hence we can pick a projective resolution of $\Gamma \otimes_{\Lambda G} M'$ of the form

$$0 \longrightarrow P' \xrightarrow{\theta'} \Gamma^{n'} \longrightarrow \Gamma \otimes_{\Lambda G} M' \longrightarrow 0\,.$$

We obtain a commutative diagram of short exact sequences in which the solid lines have already been given and the dotted lines are constructed using the Horseshoe

Lemma

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P' & \xrightarrow{\theta'} & \Gamma^{n'} & \longrightarrow & \Gamma \otimes_{\Lambda G} M' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \dashrightarrow & P' \oplus \Gamma^{n''} & \overset{\widetilde{\theta}}{\dashrightarrow} & \Gamma^{n'} \oplus \Gamma^{n''} & \dashrightarrow & \Gamma \otimes_{\Lambda G} M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Gamma^{n''} & \xrightarrow{\theta''} & \Gamma^{n''} & \longrightarrow & \Gamma \otimes_{\Lambda G} M'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & & .
\end{array} \tag{3.16}
$$

By applying Schanuel's Lemma ([19, (38.37)]) to the middle row of this diagram and the free resolution of $\Gamma \otimes_{\Lambda G} M$ in (3.15), we can assume that all the objects in the top left square of (3.16) are free. Furthermore, the map $\widetilde{\theta}$ can be represented by a matrix of the form

$$\widetilde{\theta} \quad = \quad \left(\begin{array}{c|c} \theta' & 0 \\ \hline * & \theta'' \end{array} \right)$$

Our claim now follows from the equality $\mathrm{Nrd}(\widetilde{\theta}) = \mathrm{Nrd}(\theta')\mathrm{Nrd}(\theta'')$.$\Box$

Remark 3.3.4. It is natural to ask if the methods of this section can be adapted to associate to each finitely generated G-module M a more general non-commutative invariant. For example, one could try to construct an ideal in a maximal order $\mathfrak{M}$ containing $\mathbb{Z}G$ as follows. Mimicking Definition 2.4.1 one could take a presentation matrix A for M and look at the ideal $\mathfrak{A}_g(A)$ of $\mathfrak{M}$ generated by the reduced norms of all $g \times g$ submatrices of A. However in order to show such an ideal is independent of the choice of resolution of M one would need a proof similar to [35, Ch. 3, Lemma 2] and this idea fails for the following reason. Given a $g \times g$ elementary matrix E it is an easy exercise using the reduced norm to see that in general $\mathfrak{A}_g(EA) \neq \mathfrak{A}_g(A)$. This appears to be the obstacle in making other similar definitions in the non-commutative setting.

Abstract Leading Term Conjectures

4.1 Euler Characteristics

Let R be a ring. For any graded R-module C we define the following R-modules

$$C^{\mathrm{ev}} := \bigoplus_{i \text{ even}} C^i \quad C^{\mathrm{odd}} := \bigoplus_{i \text{ odd}} C^i,$$

the sums being taken over integers i.

Let R now be a Dedekind domain of characteristic zero, F its field of fractions and A a finite dimensional semi-simple F-algebra. We let Γ be an R-algebra which is a full R-lattice in A. Such an R-algebra is said to be an *R-order* in A.

Definition 4.1.1. *Let C be an object of $\mathcal{D}^{perf}(\Gamma)$ and $t : H^{ev}(C_F) \xrightarrow{\sim} H^{odd}(C_F)$ an isomorphism of A-modules. We say that t is a* trivialization *of C and refer to the pair (C,t) as a* trivialized complex.

To any trivialized complex (C,t) one can associate a canonical (refined) Euler characteristic $\chi_{\Gamma,A}(C,t)$ in $K_0(\Gamma, A)$. We will recall the basic ingredients of this construction but for full details and properties the reader is referred to [6].

Let C be an object of $\mathcal{D}^{\mathrm{perf}}(\Gamma)$ and t a trivialization of C. Choose a complex $P \in \mathcal{C}^b(\mathrm{PMod}(R))$ which is quasi-isomorphic to C and splittings of the tautological exact sequences

$$\begin{gathered} 0 \longrightarrow B^i(P_F) \longrightarrow Z^i(P_F) \longrightarrow H^i(P_F) \longrightarrow 0 \\ 0 \longrightarrow Z^i(P_F) \longrightarrow P_F^i \longrightarrow B^{i+1}(P_F) \longrightarrow 0, \end{gathered} \tag{4.1}$$

where $B^i(P_F)$, resp. $Z^i(P_F)$, denotes the degree i coboundaries, resp. degree i cocycles, of P_F. We obtain an isomorphism

$$\begin{aligned} P_F^{\mathrm{odd}} &\cong \bigoplus_{i\in\mathbb{Z}} B^i(P_F) \oplus H^{\mathrm{odd}}(P_F) \\ &\cong \bigoplus_{i\in\mathbb{Z}} B^i(P_F) \oplus H^{\mathrm{ev}}(P_F) \\ &\cong P_F^{\mathrm{ev}} \end{aligned}$$

where the first and third maps are obtained from the chosen splittings of the exact sequences in (4.1) and the second map is induced by t^{-1}. We write $\phi(t^{-1})$ for this isomorphism, which is clearly dependent upon the chosen splittings.

One defines

$$\chi_{\Gamma,A}(C,t) := [P^{\mathrm{odd}}, \phi(t^{-1}), P^{\mathrm{ev}}] \in K_0(\Gamma, A)$$

which can be shown to be independent of all choices made in the construction. In particular it is independent of the choice of splittings in (4.1).

We recall the following useful result.

Lemma 4.1.2. *Let* $\Delta := [T[0] \longrightarrow Q \longrightarrow R \longrightarrow]$ *be an exact triangle in* $\mathcal{D}^{perf}(\Gamma)$ *where* T *is a finite* Γ*-module. The complex* $T[0]$ *has a unique trivialization which we denote by* 0*. The triangle* Δ *induces isomorphisms of* A*-modules* $H^{ev}(Q_F) \cong H^{ev}(R_F)$ *and* $H^{odd}(Q_F) \cong H^{odd}(R_F)$*. Thus any trivialization* $t : H^{ev}(Q_F) \xrightarrow{\sim} H^{odd}(Q_F)$ *of* Q *induces a trivialization of* R*, which we also denote by* t*. One has*

$$\chi_{\Gamma,A}(Q,t) = \chi_{\Gamma,A}(T[0],0) + \chi_{\Gamma,A}(R,t).$$

Proof See [6, Th. 2.8].□

4.2 Trivialized Extensions and Abstract L-values

Let Λ be a finitely generated subring of $\mathbb{Q}$ and G a finite group. Let H^0 and H^1 be finitely generated ΛG-modules and fix an element ϵ of $\mathrm{Ext}^2_{\Lambda G}(H^1, H^0)$. Let

$$0 \longrightarrow H^0 \longrightarrow \Phi^0 \longrightarrow \Phi^1 \longrightarrow H^1 \longrightarrow 0$$

be an exact sequence of ΛG-modules representing ϵ. One obtains a complex Φ_ϵ of ΛG-modules $\Phi^0 \longrightarrow \Phi^1$ where the first term is placed in degree zero and $H^i(\Phi) = H^i$ for $i = 0, 1$ (whilst the complex Φ_ϵ is dependent upon the choice of Yoneda extension representing ϵ, we shall only make constructions which are independent of this choice). Conversely, given any complex Φ which is concentrated in degrees zero and one, one obtains an element of $\mathrm{Ext}^2_{\Lambda G}(H^1(\Phi), H^0(\Phi))$ in the obvious way.

Definition 4.2.1. *An element ϵ of $\mathrm{Ext}^2_{\Lambda G}(H^1, H^0)$ is said to be a* perfect extension *if the complex Φ_ϵ is perfect. An* augmented trivialized extension (a.t.e.) *(with respect to ΛG) is a triple $\tau = (\epsilon_\tau, \lambda_\tau, \mathcal{L}^\times_\tau)$ comprising a perfect extension ϵ_τ in $\mathrm{Ext}^2_{\Lambda G}(H^1_\tau, H^0_\tau)$, an isomorphism of $\mathbb{R}G$-modules*

$$\lambda_\tau : \mathbb{R} \otimes_\Lambda H^0_\tau \xrightarrow{\sim} \mathbb{R} \otimes_\Lambda H^1_\tau$$

and an element $\mathcal{L}^\times_\tau$ of $\zeta(\mathbb{R}G)^\times$. We shall call $\mathcal{L}^\times_\tau$ the leading term of τ.

The existence of the isomorphism λ_τ implies that $\Upsilon(H^0_\tau) = \Upsilon(H^1_\tau)$ and so we write $\Upsilon(\tau)$ for this set and e_τ for the idempotent associated to $\Upsilon(\tau)$. With respect to the natural multiplicative action of $\zeta(\mathbb{R}G)^\times$ on $\zeta(\mathbb{R}Ge_\tau)^\times$ we set $\mathcal{L}_\tau := \mathcal{L}^\times_\tau e_\tau \in \zeta(\mathbb{R}G)$.

We will omit τ from the notation when the context is clear.

Definition 4.2.2. *Given an a.t.e. $\tau = (\epsilon, \lambda, \mathcal{L}^\times)$ the Euler characteristic of τ is defined to be*

$$\chi(\tau) := \chi_{\Lambda G, \mathbb{R}G}(\Phi_\epsilon, \lambda) + \widehat{\delta}(\mathcal{L}^\times) \in K_0(\Lambda G, \mathbb{R}G).$$

4.3 Euler Characteristics and Fitting Invariants

In this section we state and prove the main algebraic result of this thesis (Theorem 4.3.1). This shows that the vanishing of the Euler characteristic $\chi(\tau)$ of an a.t.e. τ gives explicit congruences between the leading term of τ and structural invariants of the modules associated to τ. In Chapter 6 we will show that a.t.e.'s arise naturally in the context of the ETNC in such a way that the validity of the ETNC is equivalent to the vanishing of the Euler characteristic of the given a.t.e.. In this way one can therefore view the conjectural equality $\chi(\tau) = 0$ as an 'abstract leading term conjecture'.

Let G be a finite group, p a prime, Λ either $\mathbb{Z}_{(p)}$, $\mathbb{Z}_p$ or a finitely generated subring of $\mathbb{Q}$ and L the quotient field of Λ. For any ΛG-module M we write M^*, resp. $M^\vee$, for the ΛG-module $\mathrm{Hom}_\Lambda(M, \Lambda)$, resp. $\mathrm{Hom}_\Lambda(M, L/\Lambda)$, endowed with the contragredient G-action. Given a Λ-order $\mathfrak{A}$ in LG we say that $\mathfrak{A}$ is *relatively Gorenstein* if $\mathfrak{A}^*$ is isomorphic to $\mathfrak{A}$ as an ΛG-module.

Let Λ now be a finitely generated subring of $\mathbb{Q}$. Let $\tau := (\epsilon, \lambda, \mathcal{L}^\times)$ be an a.t.e. (with respect to ΛG). We set $T := (H^0)_{tor}$ and write Υ, resp. e, for $\Upsilon(\tau)$, resp. e_τ.

Theorem 4.3.1. *If $\chi(\tau)$ vanishes, then each of the following claims is valid.*

i) $\mathcal{L} \in \mathbb{Q}G$.

ii) If p is any prime such that T_p is a cohomologically trivial G-module, then $\mathrm{Fitt}_{\mathbb{Z}_pG}(T_p^\vee)$ and $Fitt_{\mathbb{Z}_pG}(H_p^1)$ are both defined and

$$Fitt_{\mathbb{Z}_pG}(T_p^\vee)\mathcal{L} = Fitt_{\mathbb{Z}_pG}(H_p^1).$$

iii) If G is abelian and p is a prime such that either T_p is a cohomologically trivial G-module or $\mathbb{Z}_{(p)}Ge$ is relatively Gorenstein, then

$$Fitt_{\mathbb{Z}_{(p)}G}(T_p^\vee)\mathcal{L} = Fitt_{\mathbb{Z}_{(p)}G}(H_{(p)}^1).$$

Proof

Claim i): We have an isomorphism of $\mathbb{R}G$-modules $\lambda : \mathbb{R} \otimes H^0 \xrightarrow{\sim} \mathbb{R} \otimes H^1$ which implies the existence of an isomorphism of $\mathbb{Q}G$-modules $\varphi : \mathbb{Q} \otimes H^0 \xrightarrow{\sim} \mathbb{Q} \otimes H^1$ (see, for example, the arguments in [39, Ch. 12]). We fix such an isomorphism φ and given $\psi \in \mathrm{Irr}_{\mathbb{C}}(G)$ we define

$$A_\varphi(\psi) := \mathcal{L}_\psi^\times . \det_{\mathbb{C}}((\mathbb{C} \otimes \varphi^{-1}) \circ (\mathbb{C} \otimes \lambda) \mid_{\mathrm{Hom}_{\mathbb{C}G}(V_\psi, \mathbb{C} \otimes_\Lambda H^0)}) \in \mathbb{C}^\times$$

where V_ψ is a $\mathbb{C}$-vector space realizing ψ.

Remark 4.3.2. Our definition of $A_\varphi(\psi)$ is motivated by Tate's formulation of the Stark conjectures [41, Ch. 4].

Lemma 4.3.3. *$\chi(\tau)$ is an element of $K_0(\Lambda G, \mathbb{Q}G)$ if and only if $A_\varphi(\psi^\omega) = A_\varphi(\psi)^\omega$ for all $\psi \in Irr_{\mathbb{C}}(\ G)$ and $\omega \in Aut(\mathbb{C})$.*

Proof As in §4.2 we let Φ_ϵ be a complex of ΛG-modules concentrated in degrees zero and one and such that the extension class it defines is ϵ. Recall that $\chi(\tau) := \chi_{\Lambda G, \mathbb{R}G}(\Phi_\epsilon, \lambda) + \widehat{\delta}(\mathcal{L}^\times)$. Let $\alpha \in \zeta(\mathbb{Q}G)^\times$ such that $\alpha\mathcal{L}^\times \in \zeta(\mathbb{R}G)^{\times +}$ (such an element exists after the discussion in §2.2.5). Then

$$\begin{aligned} \widehat{\delta}(\mathcal{L}^\times) &= \widehat{\delta}(\alpha\mathcal{L}^\times) - \widehat{\delta}(\alpha) \\ &= \delta_{\Lambda G, \mathbb{R}G} \circ \mathrm{Nrd}^{-1}(\alpha\mathcal{L}^\times) - \widehat{\delta}(\alpha) \end{aligned}$$

and $\widehat{\delta}(\alpha) \in K_0(\Lambda G, \mathbb{Q}G)$. It will thus suffice to prove the result in the case that $\mathcal{L}^\times \in \zeta(\mathbb{R}G)^{\times +}$.

We have two trivializations of Φ_ϵ, one with respect to $\mathbb{Q}G$, namely φ, and one with respect to $\mathbb{R}G$, namely λ. Using [6, Th. 2.1 (2)] one has

$$\chi_{\Lambda G, \mathbb{R}G}(\Phi_\epsilon, \lambda) - \chi_{\Lambda G, \mathbb{Q}G}(\Phi_\epsilon, \varphi) = \delta_{\Lambda G, \mathbb{R}G}([\mathbb{R} \otimes H^0, (\mathbb{R} \otimes \varphi^{-1}) \circ \lambda])$$

and so $\chi_{\Lambda G,\mathbb{R}G}(\Phi_\epsilon,\lambda) \equiv \delta_{\Lambda G,\mathbb{R}G}([\mathbb{R}\otimes H^0,(\mathbb{R}\otimes\varphi^{-1})\circ\lambda]) \bmod K_0(\Lambda G,\mathbb{Q}G)$. Hence

$$\begin{aligned}
& \chi(\tau) \in K_0(\Lambda G,\mathbb{Q}G) \\
\iff\ & \chi_{\Lambda G,\mathbb{R}G}(\Phi_\epsilon,\lambda) + \delta_{\Lambda G,\mathbb{R}G}\circ \mathrm{Nrd}^{-1}(\mathcal{L}^\times) \in K_0(\Lambda G,\mathbb{Q}G) \\
\iff\ & \delta_{\Lambda G,\mathbb{R}G}([\mathbb{R}\otimes H^0,(\mathbb{R}\otimes\varphi^{-1})\circ\lambda]) + \delta_{\Lambda G,\mathbb{R}G}\circ \mathrm{Nrd}^{-1}(\mathcal{L}^\times) \in K_0(\Lambda G,\mathbb{Q}G) \\
\iff\ & \mathrm{Nrd}((\mathbb{R}\otimes\varphi^{-1})\circ\lambda)\mathcal{L}^\times \in \zeta(\mathbb{Q}G)^\times
\end{aligned} \tag{4.2}$$

Furthermore, by definition of Nrd we have $\mathrm{Nrd}((\mathbb{R}\otimes\varphi^{-1})\circ\lambda)\mathcal{L}^\times = (A_\varphi(\psi))_\psi$ in $\zeta(\mathbb{C}G)^\times$. Hence the condition that $A_\varphi(\psi^\omega) = A_\varphi(\psi)^\omega$ for all ψ and ω is equivalent to the bottom statement in (4.2) as $\zeta(\mathbb{Q}G)^\times = H^0(\mathrm{Aut}(\mathbb{C}),\zeta(\mathbb{C}G)^\times)$.□

Recall the definition of $\Upsilon(\tau)$ from 4.2.1 and that we defined $\Upsilon := \Upsilon(\tau)$. For any $\psi\in\Upsilon$ we have $(H^0\otimes\mathbb{C})e_\psi = 0$ and so $A_\varphi(\psi) = \mathcal{L}_\psi^\times$. Our assumption that $\chi(\tau) = 0$ trivially implies that $\chi(\tau)\in K_0(\Lambda G,\mathbb{Q}G)$. By Definition 4.2.1, $\mathcal{L}$ is the element of $\zeta(\mathbb{R}G)$ given by

$$\mathcal{L}_\psi = \begin{cases} \mathcal{L}_\psi^\times & \text{if } \psi\in\Upsilon \\ 0 & \text{otherwise.}\end{cases}$$

Since Υ is stable under $\mathrm{Aut}(\mathbb{C})$, Lemma 4.3.3 implies that $\mathcal{L}^\times_{\psi^\omega} = (\mathcal{L}^\times_\psi)^\omega$ for all $\omega \in\mathrm{Aut}(\mathbb{C})$ and $\psi\in\Upsilon$. Thus $\mathcal{L}\in\zeta(\mathbb{Q}G)$ which proves claim i) of Theorem 4.3.1.

Since claims ii) and iii) are local statements, for the rest of the proof we will assume $\Lambda = \mathbb{Z}$.

Claim ii): Let

$$0 \longrightarrow H^0 \longrightarrow \Phi^0 \stackrel{d}{\longrightarrow} \mathbb{Z}G^s \longrightarrow H^1 \longrightarrow 0 \tag{4.3}$$

be an exact sequence of G-modules representing ϵ and Φ_ϵ the complex $[\Phi^0\to\mathbb{Z}G^s]$ with Φ^0 placed in degree zero. The assumption that ϵ is a perfect extension is equivalent to Φ^0 being a cohomologically trivial G-module. We can thus apply Proposition 3.1.3 to deduce that the Fitting invariants in claim ii) of the theorem are both defined.

We now fix a prime p for which T_p is a cohomologically trivial G-module. The functor $\mathbb{Z}_{(p)}\otimes -$ is exact, thus applying it to (4.3) gives an exact sequence of $\mathbb{Z}_{(p)}G$-modules $0\longrightarrow H^0_{(p)}\longrightarrow \Phi^0_{(p)}\longrightarrow \mathbb{Z}_{(p)}G^s\longrightarrow H^1_{(p)}\longrightarrow 0$. Let $\Phi_{(p)}$ denote the complex $[\Phi^0_{(p)} \to \mathbb{Z}_{(p)}G^s]$, where the first term is placed in degree zero. Since ϵ is perfect, the exactness of $\mathbb{Z}_{(p)} \otimes -$ implies that $\Phi_{(p)} \in \mathcal{D}^{perf}(\mathbb{Z}_{(p)}G)$. The trivialization λ of Φ_ϵ induces a trivialization of $\Phi_{(p)}$ and, furthermore, under the natural map $K_0(\mathbb{Z}G, \mathbb{R}G) \to K_0(\mathbb{Z}_{(p)}G, \mathbb{R}G)$ one has $\chi_{\mathbb{Z}G,\mathbb{R}G}(\Phi_\epsilon, \lambda) \mapsto \chi_{\mathbb{Z}_{(p)}G,\mathbb{R}G}(\Phi_{(p)}, \lambda)$.

Let $\overline{\Phi}_{(p)}$ be the complex $[\overline{\Phi^0}_{(p)} \to \mathbb{Z}_{(p)}G^s]$ with the first term placed in degree zero. We have a tautological exact triangle in $\mathcal{D}(\mathbb{Z}G)$

$$T_p[0] \longrightarrow \Phi_{(p)} \longrightarrow \overline{\Phi}_{(p)} \longrightarrow \cdot \tag{4.4}$$

Now, Lemma 3.1.5 implies that $\overline{\Phi^0}_{(p)}$ is a projective $\mathbb{Z}_{(p)}G$-module and so $\overline{\Phi}_{(p)} \in \mathcal{C}^b(\mathrm{PMod}(\mathbb{Z}_{(p)}G))$. Furthermore, since $\mathrm{Hyp}(T_p, \mathbb{Z}_{(p)}G)$ holds $T_p[0] \in \mathcal{D}^{perf}(\mathbb{Z}_{(p)}G)$. Thus (4.4) is an exact triangle in $\mathcal{D}^{perf}(\mathbb{Z}_{(p)}G)$ and so we can apply Lemma 4.1.2 to obtain

$$\chi_{\mathbb{Z}_{(p)}G,\mathbb{R}G}(\Phi_{(p)}, \lambda) = \chi_{\mathbb{Z}_{(p)}G,\mathbb{R}G}(T_p[0], 0) + \chi_{\mathbb{Z}_{(p)}G,\mathbb{R}G}(\overline{\Phi}_{(p)}, \lambda). \tag{4.5}$$

We now compute this expression. By definition of $\chi_{\mathbb{Z}_{(p)}G,\mathbb{R}G}(-,-)$ we have

$$\chi_{\mathbb{Z}_{(p)}G,\mathbb{R}G}(\overline{\Phi}_{(p)}, \lambda) = [\mathbb{Z}_{(p)}G^s, \phi(\lambda^{-1}), \overline{\Phi^0}_{(p)}],$$

where $\phi(\lambda^{-1})$ is defined in §4.1. Since $\mathrm{Hyp}(T_p, \mathbb{Z}_{(p)})$ holds we can choose an exact sequence of $\mathbb{Z}_{(p)}G$-modules of the form

$$0 \longrightarrow \mathbb{Z}_{(p)}G^r \xrightarrow{\theta} \mathbb{Z}_{(p)}G^r \longrightarrow T_p \longrightarrow 0.$$

By definition of $\chi_{\mathbb{Z}_{(p)}G,\mathbb{R}G}(-,-)$ we have $\chi_{\mathbb{Z}_{(p)}G,\mathbb{R}G}(T_p[0], 0) = [\mathbb{Z}_{(p)}G^r, \mathbb{R} \otimes \theta, \mathbb{Z}_{(p)}G^r]$. Hence the equality $\chi(\tau) = 0$, together with (4.5), implies that

$$-\widehat{\delta}(\mathcal{L}^\times) = [\mathbb{Z}_{(p)}G^r, \mathbb{R} \otimes \theta, \mathbb{Z}_{(p)}G^r] + [\mathbb{Z}_{(p)}G^s, \phi(\lambda^{-1}), \overline{\Phi^0}_{(p)}] \in K_0(\mathbb{Z}_{(p)}G, \mathbb{R}G). \tag{4.6}$$

We write Γ for the ring $\mathbb{Z}_{(p)}Ge$ and, in keeping with our conventions, Γ_p for the ring

$\mathbb{Z}_pGe$. Consider the following natural commutative diagram of groups

$$\begin{array}{ccc} \zeta(\mathbb{R}G)^\times & \xrightarrow{\widehat{\delta}} & K_0(\mathbb{Z}_{(p)}G, \mathbb{R}G) \\ \downarrow & & \downarrow \\ \zeta(\mathbb{R}Ge)^\times & \xrightarrow{\widehat{\delta}_e} & K_0(\Gamma, \mathbb{R}Ge). \end{array} \tag{4.7}$$

We denote the right hand vertical map in (4.7) by π. Given a homomorphism of $\mathbb{R}G$-modules $f : M \to N$ we write fe for the induced map $fe : Me \to Ne$. Diagram (4.7), together with (4.6), implies that

$$-\widehat{\delta}_e(\mathcal{L}) = [\Gamma^r, (\mathbb{R} \otimes \theta)e, \Gamma^r] + [\Gamma^s, \phi(\lambda^{-1})e, \Gamma \otimes_G \overline{\Phi^0}] \in K_0(\Gamma, \mathbb{R}Ge). \tag{4.8}$$

The argument directly following the proof of Theorem 3.1.5 implies that $\Gamma \otimes_G \overline{\Phi^0}$ is a free Γ-module. Furthermore, by definition of $K_0(-,-)$, the rank of $\Gamma \otimes_G \overline{\Phi^0}$ is s. We choose a basis of $\Gamma \otimes_G \overline{\Phi^0}$ and henceforth make an identification of $\Gamma \otimes_G \overline{\Phi^0}$ with Γ^s. Recall that $e := e_\tau$ and for both $i = 0$ and $i = 1$ we have $e_\tau = e_{H^i}$. By definition of e_{H^i} we have $(\mathbb{R} \otimes H^i)e = 0$ for $i = 0, 1$. Recall the map d from (4.3). By definition of $\phi(\lambda^{-1})$ we see that the isomorphism $\phi(\lambda^{-1})e$ is equal to the isomorphism

$$\mathbb{R} \otimes_{\mathbb{Q}} (\mathbb{Q} \otimes \mathbb{Z}G^s)e \xrightarrow{\mathbb{R}\otimes_{\mathbb{Q}}((\mathbb{Q}\otimes d)e)^{-1}} \mathbb{R} \otimes_{\mathbb{Q}} (\mathbb{Q} \otimes \Phi^0)e.$$

Thus

$$[\Gamma^s, \phi(\lambda^{-1})e, \Gamma^s] = [\Gamma^s, ((\mathbb{Q} \otimes d)e)^{-1}, \Gamma^s] \in K_0(\Gamma, \mathbb{Q}Ge).$$

Likewise we have

$$[\Gamma^r, (\mathbb{R} \otimes \theta)e, \Gamma^r] = [\Gamma^r, (\mathbb{Q} \otimes \theta)e, \Gamma^r] \in K_0(\Gamma, \mathbb{Q}Ge).$$

Hence (4.8) now implies

$$\begin{aligned} -\widehat{\delta}_e(\mathcal{L}) &= [\Gamma^r, (\mathbb{Q} \otimes \theta)e, \Gamma^r] + [\Gamma^s, ((\mathbb{Q} \otimes d)e)^{-1}, \Gamma^s] \\ &= [\Gamma^r, (\mathbb{Q} \otimes \theta)e, \Gamma^r] - [\Gamma^s, (\mathbb{Q} \otimes d)e, \Gamma^s] \\ &= \delta_{\Gamma,\mathbb{R}Ge}([(\mathbb{Q}Ge)^r, (\mathbb{Q} \otimes \theta)e] - [(\mathbb{Q}Ge)^s, (\mathbb{Q} \otimes d)e]), \end{aligned} \tag{4.9}$$

where the second equality follows from the relations in $K_0(\Gamma, \mathbb{Q}Ge)$.

Remark 4.3.4. Let $\Gamma \otimes_G \Phi$ be the complex $[\Gamma \otimes_G \Phi^0 \xrightarrow{\Gamma\otimes d} \Gamma \otimes_G \mathbb{Z}G^s]$, with the first term placed in degree zero, obtained by applying $\Gamma \otimes_G -$ to the middle two terms of (4.3). The cohomology of $\Gamma \otimes_G \Phi$ is finite and so we have a trivialized complex $(\Gamma \otimes_G \Phi, 0)$. It can be shown that the right hand side of (4.9) is equal to $\chi_{\Gamma,\mathbb{R}Ge}(\Gamma \otimes_G \Phi, 0)$. We leave the details for the interested reader.

In the case that G has no non-trivial irreducible symplectic characters we know $\mathrm{Nrd}_{\mathbb{Q}G}$ is bijective, hence $\widehat{\delta}_e = \delta_{\Gamma,\mathbb{R}Ge} \circ \mathrm{Nrd}^{-1}_{\mathbb{R}Ge}$. Thus (4.9) implies we have the following congruence in $\zeta(\mathbb{Q}Ge)^\times$

$$\mathcal{L}^{-1} \equiv \mathrm{Nrd}_{\mathbb{Q}Ge}((\mathbb{Q} \otimes \theta)e)\,(\mathrm{Nrd}_{\mathbb{Q}Ge}(\mathbb{Q} \otimes d)e)^{-1} \mod \mathrm{Nrd}_{\mathbb{Q}Ge}(\mu(K_1(\Gamma))).$$

By definition of the Fitting invariant of a $\mathbb{Z}_{(p)}G$-module this is equivalent to the statement

$$\mathrm{Fitt}_{\mathbb{Z}_{(p)}G}(T_p)\mathcal{L} = \mathrm{Fitt}_{\mathbb{Z}_{(p)}G}(H^1_{(p)}). \tag{4.10}$$

Now, it is easy to see that the argument in the proof of [12, Lemma 6] generalizes to show that $\mathrm{Fitt}_{\mathbb{Z}_{(p)}G}(T_p^\vee) = \mathrm{Fitt}_{\mathbb{Z}_{(p)}G}(T_p)$. Thus (4.10) implies that

$$\mathrm{Fitt}_{\mathbb{Z}_{(p)}G}(T_p^\vee)\mathcal{L} = \mathrm{Fitt}_{\mathbb{Z}_{(p)}G}(H^1_{(p)}), \tag{4.11}$$

which is the analogous p-local statement of Theorem 4.3.1 ii).

In the general case $\mathrm{Nrd}_{\mathbb{Q}G}$ can fail to be surjective and so we must 'pass to p-completions'. Since $\mathrm{Nrd}_{\mathbb{Q}_pGe}$ is bijective we have the following commutative diagram

$$\begin{array}{ccc} \zeta(\mathbb{Q}Ge)^\times & \xrightarrow{\widehat{\delta}_e} & K_0(\Gamma, \mathbb{Q}Ge) \\ \downarrow & & \downarrow \\ \zeta(\mathbb{Q}_pGe)^\times & \xrightarrow{\delta_{\Gamma_p,\mathbb{Q}_pGe}\circ \mathrm{Nrd}^{-1}_{\mathbb{Q}_pGe}} & K_0(\Gamma_p, \mathbb{Q}_pGe). \end{array} \tag{4.12}$$

Together with (4.9) this implies that

$$\mathcal{L}^{-1} \equiv \mathrm{Nrd}_{\mathbb{Q}_pGe}((\mathbb{Q}_p \otimes \theta)e)\left(\mathrm{Nrd}_{\mathbb{Q}_pGe}((\mathbb{Q}_p \otimes d)e)\right)^{-1} \mod \mathrm{Nrd}_{\mathbb{Q}_pGe}(\mu(K_1(\Gamma_p)))$$

which, by definition of the Fitting invariant of a $\mathbb{Z}_pG$-module, is equivalent to claim ii) of Theorem 4.3.1.

Claim iii): We recall that an abelian group has no non-trivial irreducible symplectic characters. The statement of claim iii), under the hypothesis that p is a prime for which T_p is a cohomologically trivial G-module, then follows from (4.11). Thus it remains to prove that the same statement holds under the hypothesis that p is a prime for which Γ is relatively Gorenstein.

Using Lemma 3.1.4 we see that applying $\Gamma \otimes_G -$ to the middle two terms of (4.3) induces an exact sequence of Γ-modules

$$0 \longrightarrow (\Gamma \otimes T)^G \longrightarrow \Gamma \otimes_G \Phi^0 \longrightarrow \Gamma^s \longrightarrow \Gamma \otimes_G H^1 \longrightarrow 0\,.$$

Let P be an object of $\mathcal{C}^b(\mathrm{PMod}(\Gamma))$ such that P is isomorphic to Φ in $\mathcal{D}(\mathbb{Z}G)$. The left derived functor $L(\Gamma \otimes_G -)$ can be computed by applying $\Gamma \otimes_G -$ to P. However, given an exact sequence of cohomologically trivial G-modules

$$0 \longrightarrow C' \longrightarrow C \longrightarrow C'' \longrightarrow 0\,,$$

Lemma 3.1.4 implies that the sequence

$$0 \longrightarrow \Gamma \otimes_G C' \longrightarrow \Gamma \otimes_G C \longrightarrow \Gamma \otimes_G C'' \longrightarrow 0$$

is also exact. This implies that cohomologically trivial G-modules are adapted to the functor $\Gamma \otimes_G -$. The discussion on derived functors in §2.3.2 therefore implies that we can compute $L(\Gamma \otimes_G -)$ using resolutions of cohomologically trivial G-modules. Hence we have a canonical isomorphism $\Gamma \otimes_G P \cong \Gamma \otimes_G \Phi$ in $\mathcal{D}(\Gamma)$. By definition of the Euler characteristic of a trivialized complex one thus sees that, under the functorial map

$K_0(\mathbb{Z}G, \mathbb{R}G) \longrightarrow K_0(\Gamma, \mathbb{R}Ge)$, one has $\chi_{\mathbb{Z}G,\mathbb{R}G}(\Phi, \lambda) \mapsto \chi_{\Gamma,\mathbb{R}Ge}(\Gamma \otimes_G \Phi, 0)$. Thus the equality $\chi(\tau) = 0$ implies that

$$\chi_{\Gamma,\mathbb{R}Ge}(\Gamma \otimes_G \Phi, 0) + \delta_{\Gamma,\mathbb{R}Ge} \circ \mathrm{Nrd}^{-1}_{\mathbb{R}Ge}(\mathcal{L}) = 0.$$

As in the proof of claim ii) we see that this expression belongs to $K_0(\Gamma, \mathbb{Q}Ge)$ and so we can 'pass to p-completions' to obtain

$$\chi_{\Gamma_p,\mathbb{Q}_pGe}(\Gamma_p \otimes_G \Phi, 0) + \delta_{\Gamma_p,\mathbb{Q}_pGe} \circ \mathrm{Nrd}^{-1}_{\mathbb{Q}_pGe}(\mathcal{L}) = 0. \tag{4.13}$$

We need the following auxiliary result.

Lemma 4.3.5. *Let C be a complex in $\mathcal{C}^b(PMod(\mathbb{Z}_pGe))$ concentrated in degrees zero and one and such that $H^i := H^i(C)$ is finite for all i. There exists a complex of $\mathbb{Z}_pGe$-modules $A := [A^0 \to A^1]$, with the first term placed in degree zero, and an isomorphism $\nu : C \to A$ in $\mathcal{D}(\mathbb{Z}_pGe)$ such that:-*

i) A^0 and A^1 are finite;

ii) $pd_{\mathbb{Z}_pGe}(A^0) = pd_{\mathbb{Z}_pGe}(A^1) = 1$;

iii) $H^i(\nu)$ is the identity map for $i = 0$, 1.

Proof To ease notation we assume that H^1 is a cyclic module. The argument in the general case is entirely similar. We also continue to denote $\mathbb{Z}_pGe$ by Γ_p.

If n is a sufficiently large integer, then we can chose an exact sequence of Γ_p-modules of the form

$$0 \longrightarrow K \longrightarrow \Gamma_p/p^n\Gamma_p \longrightarrow H^1 \longrightarrow 0. \tag{4.14}$$

This induces an exact sequence of Ext-groups

$$\cdots \longrightarrow \mathrm{Ext}^1_{\Gamma_p}(K, H^0) \xrightarrow{\delta} \mathrm{Ext}^2_{\Gamma_p}(H^1, H^0) \longrightarrow \mathrm{Ext}^2_{\Gamma_p}(\Gamma_p/p^n\Gamma_p, H^0) \longrightarrow \cdots. \tag{4.15}$$

Since $\mathrm{pd}_{\Gamma_p}(\Gamma_p/p^n\Gamma_p) = 1$ the group $\mathrm{Ext}^2_{\Gamma_p}(\Gamma_p/p^n\Gamma_p, H^0)$ is trivial and so the map δ of (4.15) is surjective. Let $[C]$ denote the class defined by C in $\mathrm{Ext}^2_{\Gamma_p}(H^1, H^0)$. Chose a pre-image of $[C]$ under δ and let

$$0 \longrightarrow H^0 \longrightarrow T \longrightarrow K \longrightarrow 0 \tag{4.16}$$

be an exact sequence of Γ_p-modules representing the chosen pre-image. The definition of the map δ in terms of Yoneda extensions shows that 'splicing' (4.16) with (4.14) gives an exact sequence of Γ_p-modules

$$0 \longrightarrow H^0 \longrightarrow T \longrightarrow \Gamma_p/p^n\Gamma_p \longrightarrow H^1 \longrightarrow 0$$

which is Yoneda equivalent to the tautological exact sequence

$$0 \longrightarrow H^0 \longrightarrow C^0 \longrightarrow C^1 \longrightarrow H^1 \longrightarrow 0$$

induced by C. If we let A be the complex $[T \to \Gamma_p/p^n\Gamma_p]$, with T placed in degree zero, then we have shown that there exists an isomorphism $\nu : C \to A$ in $\mathcal{D}(\Gamma_p)$ satisfying claim iii) of the lemma. Since the module K in (4.14) is clearly finite, so is T. Thus A satisfies claim i) of the lemma. Furthermore, $\mathrm{pd}_{\Gamma_p}(\Gamma_p/p^n\Gamma_p) = 1$ and so to complete the proof it remains to show that $\mathrm{pd}_{\Gamma_p}(T) = 1$. Since C is a perfect complex, so is A. The tautological exact triangle

$$\Gamma_p/p^n\Gamma_p[-1] \longrightarrow A \longrightarrow T[0] \longrightarrow$$

thus implies that $T[0]$ is perfect, which is equivalent to $\mathrm{pd}_{\Gamma_p}(T) < \infty$. As $\mathbb{Z}_pG$ is a product of local rings of Krull dimension one, so is Γ_p. Thus using [1, Th. 1.9] we see that $\mathrm{pd}_{\Gamma_p} = 1$, which completes the proof of the lemma.$\square$

Let A be the complex given by Lemma 4.3.5 when $C = \Gamma_p \otimes_G \Phi$. Applying Lemma 4.1.2 to the tautological exact triangle $A^1[-1] \longrightarrow A \longrightarrow A^0[0] \longrightarrow$ one sees that

$$\begin{aligned} \chi_{\Gamma_p,\mathbb{Q}_pGe}(\Gamma_p \otimes_G \Phi, 0) &= \chi_{\Gamma_p,\mathbb{Q}_pGe}(A, 0) \\ &= \chi_{\Gamma_p,\mathbb{Q}_pGe}(A^0[0], 0) + \chi_{\Gamma_p,\mathbb{Q}_pGe}(A^1[-1], 0). \end{aligned} \tag{4.17}$$

The argument of [3, Lemma 2.6] shows that $K_0(\Gamma_p, \mathbb{Q}_pGe)$ can be identified with the multiplicative group of invertible Γ_p-lattices in $\mathbb{Q}_pGe$. Moreover, under this identification the right hand side of (4.17) is $\mathrm{Fitt}_{\Gamma_p}(A^0)\,\mathrm{Fitt}_{\Gamma_p}(A^1)^{-1}$ (since A^0 and A^1 have projective dimension one) and $\delta_{\Gamma_p,\mathbb{Q}_pGe} \circ \mathrm{Nrd}^{-1}_{\mathbb{Q}_pGe}(\mathcal{L}) = \mathcal{L}\Gamma_p$. Thus (4.13) implies that

$$\mathcal{L}\Gamma_p = \mathrm{Fitt}_{\Gamma_p}(A^0)^{-1}\,\mathrm{Fitt}_{\Gamma_p}(A^1). \tag{4.18}$$

Applying [12, Lemma 5] to the tautological exact sequence of Γ_p-modules

$$0 \longrightarrow (\Gamma_p \otimes T)^G \longrightarrow A^0 \longrightarrow A^1 \longrightarrow \Gamma_p \otimes_G H^1 \longrightarrow 0$$

implies that

$$\mathrm{Fitt}_{\Gamma_p}(((\Gamma_p \otimes T)^G)^\vee)^{-1}\,\mathrm{Fitt}_{\Gamma_p}(\Gamma_p \otimes_G H^1) = \mathrm{Fitt}_{\Gamma_p}(A^0)^{-1}\,\mathrm{Fitt}_{\Gamma_p}(A^1),$$

which combines with (4.18) to give

$$\mathrm{Fitt}_{\Gamma_p}(((\Gamma_p \otimes T)^G)^\vee)\mathcal{L} = \mathrm{Fitt}_{\Gamma_p}(\Gamma_p \otimes_G H^1). \tag{4.19}$$

Lemma 3.3.2 implies that $\mathrm{Fitt}_{\Gamma_p}(\Gamma_p \otimes_G H^1) = \mathrm{Fitt}_{\mathbb{Z}_pG}(H^1)$, thus (4.19) implies that, after replacing completions by localizations,

$$\mathrm{Fitt}_{\Gamma}(((\Gamma \otimes T)^G)^\vee)\mathcal{L} = \mathrm{Fitt}_{\mathbb{Z}_{(p)}G}(H^1). \tag{4.20}$$

Lemma 4.3.6. *Let G be a finite group, e an idempotent of $\mathbb{Q}G$, Γ a $\mathbb{Z}_{(p)}$-order in $\mathbb{Q}Ge$ and T a finite Γ-module. One has an isomorphism of Γ-modules*

$$(\Gamma \otimes T)^\vee \cong \Gamma^* \otimes T^\vee,$$

where Γ^* and $\Gamma^\vee$ have the contragredient G-action.

Proof We have an isomorphism of Γ-modules

$$(\Gamma \otimes T)^\vee \cong \mathrm{Hom}_{\mathbb{Z}_{(p)}}(\Gamma, T^\vee)$$

given by the 'adjointness of Hom and ⊗'. Let $\{v_i\}_{i=1}^n$ be a $\mathbb{Z}_{(p)}$-basis of Γ and let $\{\delta_i\}_{i=1}^n$ be the $\mathbb{Z}_{(p)}$-basis of $\mathrm{Hom}_{\mathbb{Z}_{(p)}}(\Gamma, \mathbb{Z}_{(p)})$ dual to $\{v_i\}_{i=1}^n$. We define a homomorphism of Γ-modules from $\mathrm{Hom}_{\mathbb{Z}_{(p)}}(\Gamma, T^\vee)$ to $\Gamma^* \otimes T^\vee$ by

$$f \longmapsto \sum_{i=1}^n \delta_i \otimes f(v_i).$$

We leave it to the reader to check that this is an isomorphism.□

For any G-module M it is well known that $(M^G)^\vee$ is canonically isomorphic to $(M^\vee)_G$. We thus have

$$\begin{aligned} ((\Gamma \otimes T)^G)^\vee &\cong ((\Gamma \otimes T)^\vee)_G \\ &\cong (\Gamma^* \otimes T^\vee)_G \\ &\cong (\Gamma \otimes T^\vee)_G \\ &\cong \Gamma \otimes_G T^\vee. \end{aligned}$$

where the second isomorphism follows from Lemma 4.3.6 and the third from the assumption that Γ is relatively Gorenstein. This isomorphism, together with (4.20) and Lemma 3.3.2, implies that

$$\mathrm{Fitt}_{\mathbb{Z}_{(p)}G}(T^\vee)\mathcal{L} = \mathrm{Fitt}_{\mathbb{Z}_{(p)}G}(H^1),$$

which completes the proof of Theorem 4.3.1.□

4.4 A remark on Euler Characteristics

The following technical remark is not needed for the rest of the thesis.

In [4] the authors give an alternative definition of the Euler characteristic of a trivialized complex. This in turn leads to an alternative definition of the Euler characteristic of an a.t.e. (used for example by Burns in [7]). We chose not to use this definition in order to keep this thesis as simple as possible, since the alternative definition involves determinant functors and virtual objects. For any interested reader we now compare the two definitions and show that the difference between the two does not affect the results in this thesis.

Let F, Γ and A be as in §4.1 and recall the definitions made in §4.2.1. For this section only we write $\chi^{\mathrm{old}}_{\Gamma,A}(-,-)$ for the Euler characteristic of Definition 4.2.2 and $\chi_{\Gamma,A}(-,-)$ for the Euler characteristic defined in [4, Def. 5.5]. In [4, Th. 6.2] the authors show that for any trivialized complex (C,t) one has

$$-\chi_{\Gamma,A}(C,t) = \chi^{old}_{\Gamma,A}(C,t^{-1}) + \delta_{\Gamma,A}([B^{odd}(C_F), -\mathrm{id}]). \tag{4.21}$$

Given an a.t.e. $\tau = (\epsilon, \lambda, \mathcal{L}^{\times})$ we write $\chi^{old}(\tau)$ for the Euler characteristic of τ given in Definition 4.2.2. As in §4.2 we let Φ_ϵ be a complex of ΛG-modules concentrated in degrees zero and one, such that the extension class it defines is ϵ. Furthermore, we assume Φ_ϵ is of the form $[\Phi^0 \xrightarrow{d} F]$, where F is finitely generated and free. Using the Euler characteristic of [4] an alternative definition of the Euler characteristic of an a.t.e. is

$$\chi(\tau) := \chi_{\Lambda G, \mathbb{R}G}(\Phi_\epsilon, \lambda) - \widehat{\delta}(\mathcal{L}^{\times}).$$

Using (4.21) we have

$$\begin{aligned} \chi(\tau) &= -\chi^{old}_{\Lambda G,\mathbb{R}G}(\Phi_\epsilon, t^{-1}) - \delta_{\Lambda G,\mathbb{R}G}([(\mathrm{im}\, d)_{\mathbb{R}}, -\mathrm{id}]) - \widehat{\delta}(\mathcal{L}^{\times}) \\ &= -\chi^{old}(\tau) - \delta_{\Lambda G,\mathbb{R}G}([(\mathrm{im}\, d)_{\mathbb{R}}, -\mathrm{id}]) \end{aligned} \tag{4.22}$$

and thus in general the vanishing of $\chi(\tau)$ is not equivalent to the vanishing of $\chi^{old}(\tau)$.

However, by definition of e_τ the complex $[(\Phi^0_{\mathbb{R}})e_\tau \xrightarrow{(\mathbb{R}\otimes d)e_\tau} (F_{\mathbb{R}})e_\tau]$ is acyclic and so $(\Phi^0_{\mathbb{R}})e_\tau$ is a free $\mathbb{R}Ge_\tau$-module. Thus under the natural map $\pi : K_0(\Lambda G, \mathbb{R}G) \to K_0(\Lambda G e_\tau, \mathbb{R}G e_\tau)$ the element $\delta_{\Lambda G,\mathbb{R}G}([(\mathrm{im}\, d)_{\mathbb{R}}, -\mathrm{id}])$ maps to zero. Hence (4.22) implies that $\pi(\chi(\tau)) = -\pi(\chi^{old}(\tau))$. Since the proof of Theorem 4.3.1 only requires $\pi(\chi^{old}(\tau))$ to vanish, one can replace the hypothesis $\chi^{old}(\tau) = 0$ of the theorem by

$\chi(\tau) = 0$ without affecting the validity of claims i), ii) and iii).

The Equivariant Tamagawa Number Conjecture

Throughout this chapter we fix the following notation. Let L/K be a Galois extension of number fields with Galois group G. We write $\overline{K}$ for a fixed algebraic closure of K containing L and G_K for $\mathrm{Gal}(\overline{K}/K)$. Let S be a finite G-stable set of places of L containing the set S_∞ of archimedean places. We write $S(K)$, resp. $S_\infty(K)$, for the set of places of K below those in S, resp. S_∞. Given $v \in S_\infty(K)$ we write σ_v for a fixed embedding $\overline{K} \hookrightarrow \mathbb{C}$ which restricts to v. Given a place v of K we write K_v for the completion of K at v, $\overline{K}_v$ for a fixed algebraic closure of K_v and G_v for $\mathrm{Gal}(\overline{K}_v/K_v)$. Finally we let $X \to \mathrm{Spec}(K)$ be a smooth projective variety over K.

5.1 Determinants and Virtual Objects

Let R be a ring and $\mathrm{PMod}_{is}(R)$ the subcategory of $\mathrm{PMod}(R)$ consisting of the same objects as $\mathrm{PMod}(R)$ and whose morphism sets consist of all isomorphisms in $\mathrm{PMod}(R)$.

In [20, §4] Deligne constructs a category $V(R)$ of *virtual objects* and a universal determinant functor

$$[-] : \mathrm{PMod}(R)_{is} \longrightarrow V(R) .$$

By definition $V(R)$ is a Picard category (cf. [9, §2.1] for basics on Picard categories), whose tensor bi-functor we will denote by $\boxtimes$ and $[-]$ is a functor satisfying certain natural properties (see [loc. cit., §2.3]).

We fix a unit object $1_{V(R)}$ of $V(R)$ and write $\pi_0(V(R))$ for the group of isomorphism classes of objects in $V(R)$ and $\pi_1(V(R))$ for $\mathrm{Aut}_{V(R)}(1_{V(R)})$. By the construction of $V(R)$ one has natural isomorphisms $\pi_i(V(R)) \cong K_i(R)$ for $i = 0, 1$. Furthermore,

$V(R)$ is functorial in R, i.e. given a ring homomorphism $R \to S$ we get an induced functor $V(R) \to V(S)$, which we denote by $S \otimes_R -$.

Let $\mathcal{D}^{\mathrm{perf}}_{is}(R)$ be the subcategory of $\mathcal{D}^{\mathrm{perf}}(R)$ consisting of the same objects as $\mathcal{D}^{\mathrm{perf}}(R)$ and whose morphism sets consist of all isomorphisms in $\mathcal{D}^{\mathrm{perf}}(R)$. We have a full embedding of categories $\mathrm{PMod}(R) \to \mathcal{D}^{\mathrm{perf}}(R)$ given by $P \mapsto P[0]$ and one can show that $[-]$ extends to a determinant functor

$$[-] : \mathcal{D}^{\mathrm{perf}}_{is}(R) \to V(R)$$

(see [9, Prop. 2.1]).

We specialize the above construction to the following setting. Let A be a finite dimensional semi-simple $\mathbb{Q}$-algebra, R a finitely generated subring of $\mathbb{Q}$ and $\mathfrak{A}$ an R-order in A. We set $\widehat{\mathfrak{A}} := \widehat{\mathbb{Z}} \otimes \mathfrak{A}$ and $\widehat{A} := \widehat{\mathbb{Z}} \otimes A$ where $\widehat{\mathbb{Z}}$ is the profinite completion of $\mathbb{Z}$. Using the fibre product of categories ([loc. cit., §2.2]) one defines

$$\mathbb{V}(\mathfrak{A}) := V(\widehat{\mathfrak{A}}) \times_{V(\widehat{A})} V(A) \qquad \text{and} \qquad \mathbb{V}(\mathfrak{A}, \mathbb{R}) := \mathcal{P}_0 \times_{V(A_{\mathbb{R}})} \mathbb{V}(\mathfrak{A})$$

where $\mathcal{P}_0$ is the Picard category with unique object $1_{\mathcal{P}_0}$ and $\mathrm{Aut}(1_{\mathcal{P}_0}) = 0$. Both $\mathbb{V}(\mathfrak{A})$ and $\mathbb{V}(\mathfrak{A}, \mathbb{R})$ are naturally Picard categories. Furthermore, the isomorphisms $\pi_i(V(\mathfrak{A})) \cong K_i(\mathfrak{A})$ and $\pi_i(V(A_{\mathbb{R}})) \cong K_i(A_{\mathbb{R}})$ induce a natural isomorphism $\pi_0(\mathbb{V}(\mathfrak{A}, \mathbb{R})) \cong K_0(\mathfrak{A}, A_{\mathbb{R}})$.

5.2 Motives

We view a motive as defined in terms of its realizations as follows. The motive $M := h^n(X)(r)$ is the following collection of data:-

i) For each prime ℓ, a finite dimensional $\mathbb{Q}_\ell$-vector space $H_\ell(M) := H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)(r)$ with a continuous action of G_K;

ii) For each archimedean place v of K, a $\mathbb{Q}$-Hodge structure $H_v(M) := H^n(\sigma_v X(\mathbb{C}), (2\pi$

$i)^r\mathbb{Q})$ over K_v;

iii) A filtered K-space $H_{dR}(M) := H^n_{dR}(X/K)$ with its natural decreasing filtration $\{F^iH^n_{dR}(X/K)\}_{i\in\mathbb{Z}}$ shifted by r.

To say that M has coefficients in a finite dimensional $\mathbb{Q}$-algebra A will mean that A acts on each of the above realizations of M in such a way that the actions are compatible under the various comparison isomorphisms.

Examples

i) If $M = h^0(\operatorname{Spec} L)(r)$ for some $r \in \mathbb{Z}$, then $H_\ell(M) = H^0_{\text{ét}}(\operatorname{Spec}(L \otimes_K \overline{K}), \mathbb{Q}_\ell(r))$, $H_v(M) = (2\pi i)^r\mathbb{Q}G$ for each archimedean place v of K and $H_{dR}(M) = L$, with $F^i(L) = L$ if $i \leq -r$ and $F^i(L) = 0$ otherwise. The coefficients of M lie in $\mathbb{Q}G$ where G acts on both $H_\ell(M)$ and $H_{dR}(M)$ via L and on $H_v(M)$ in the obvious way.

ii) Let E be an elliptic curve defined over K. If $M = h^1(E)_L(1)$ where $h^1(E)_L := h^0(\operatorname{Spec} L) \otimes_{h^0(\operatorname{Spec} K)} h^1(E)$, then $H_\ell(M) = \mathbb{Q}_\ell[G] \otimes_{\mathbb{Z}_\ell} T_\ell(E)$, where $T_\ell(E)$ is the ℓ-adic Tate module of E, $H_v(M) = H^1(\sigma_v E(\mathbb{C}), (2\pi i)\mathbb{Q}G)$ and $H_{dR}(M)/F^0H_{dR}(M)$ is isomorphic to $L \otimes_K H^1(E, \mathcal{O}_E)$. The coefficients for M lie in $\mathbb{Q}G$, where G acts on $H_\ell(M)$ on the left of G, on $H_v(M)$ in the obvious way and on $H_{dR}(M)/F^0H_{dR}(M)$ via L.

For the rest of this chapter M will always denote a motive with the action of some semi-simple $\mathbb{Q}$-algebra A.

5.3 The Fundamental Exact Sequence

For $v \in S_\infty(K)$ one defines the Deligne cohomology of $H_v(M)$ to be the cohomology of the complex

$$R\Gamma_D(K_v, H_v(M) \otimes_\mathbb{Q} \mathbb{R}) := ((H_v(M) \otimes_\mathbb{Q} \mathbb{R})^{G_v} \xrightarrow{\alpha_v} (H_v(M) \otimes_\mathbb{R} \overline{K}_v)^{G_v}/F^0)$$

where α_v is induced by the natural inclusion $H_v(M) \otimes_{\mathbb{Q}} \mathbb{R} \to H_v(M) \otimes \overline{K}_v$. The canonical comparison isomorphism

$$H_v(M) \otimes_{\mathbb{Q}} \overline{K}_v \xrightarrow{\sim} H_{dR}(M) \otimes_{K,v} \overline{K}_v$$

is G_v equivariant and so we can write

$$\begin{aligned} R\Gamma_D(K,M) &:= \bigoplus_{v \in S_\infty} R\Gamma_D(K_v, H_v(M) \otimes_{\mathbb{Q}} \mathbb{R}) \\ &= \left[\bigoplus_{v \in S_\infty} (H_v(M) \otimes_{\mathbb{Q}} \mathbb{R})^{G_v} \xrightarrow{\alpha_M} \bigoplus_{v \in S_\infty} H_{dR}(M) \otimes_{K,v} K_v/F^0 \right] \end{aligned}$$

where α_M is defined in the natural way. For $i = 0, 1$ one can also define the motivic cohomology $H^i(K, M)$ of M and its finite parts $H^i_f(K, M)$. They are A-modules with $H^0_f(K, M)$ finite dimensional over $\mathbb{Q}$ and $H^1_f(K, M)$ conjecturally so.

Conjecture 5.3.1. *There exists a canonical exact sequence of finitely generated $A_{\mathbb{R}}$-modules*

$$\begin{aligned} 0 \longrightarrow H^0(K,M) \otimes_{\mathbb{Q}} \mathbb{R} \xrightarrow{\epsilon} \ker \alpha_M \xrightarrow{r_B^*} (H^1_f(K, M^*(1)) \otimes_{\mathbb{Q}} \mathbb{R})^* \\ \xrightarrow{\delta} H^1_f(K,M) \otimes_{\mathbb{Q}} \mathbb{R} \xrightarrow{r_B} \operatorname{coker} \alpha_M \xrightarrow{\epsilon^*} (H^0(K, M^*(1)) \otimes_{\mathbb{Q}} \mathbb{R})^* \longrightarrow 0 \end{aligned}$$

where ϵ is the cycle class map into the singular cohomology, r_B is the Beilinson regulator map and δ is the height pairing. In the case of modules the superscript $*$ denotes the linear dual and in the case of motives it denotes the dual motive.

We define an element of $V(A)$ by setting

$$\begin{aligned} \Xi(M) := \; & [H^0_f(K,M)] \boxtimes [H^1_f(K,M)]^{-1} \boxtimes [H^1_f(K, M^*(1))^*] \\ & \boxtimes [H^0_f(K, M^*(1))^*]^{-1} \boxtimes \boxtimes_{v \in S_\infty} [H_v(M)^{G_v}]^{-1} \boxtimes [H_{dR}(M)/F^0] \end{aligned}$$

where $[-] : D^{perf}(A) \to V(A)$ is as in §5.1. Thus Conjecture 5.3.1 induces a canonical isomorphism

$$\vartheta_\infty : A_{\mathbb{R}} \otimes_A \Xi(M) \cong 1_{V_{A_{\mathbb{R}}}}$$

in $V(A_{\mathbb{R}})$.

Examples

i) $M = h^0(\mathrm{Spec}\, L)(r)$.

One has $M^*(1) = h^0(\mathrm{Spec}\, L)(1-r)$,

$$H^0_f(K,M) = \begin{cases} \mathbb{Q} & \text{if } r = 0 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad H^1_f(K,M) = \begin{cases} \mathbb{Q} \otimes K_{2r-1}(\mathcal{O}_L) & \text{if } r \geq 1 \\ 0 & \text{otherwise,} \end{cases}$$

where $K_i(\mathcal{O}_L)$ is the i^{th} algebraic K-group of $\mathcal{O}_L$ as defined by Quillen.

$\underline{r = 0}$

Let $Y_{L,S}$ be the free abelian group generated by S. There exists a short exact sequence of G-modules

$$0 \longrightarrow X_{L,S} \longrightarrow Y_{L,S} \longrightarrow \mathbb{Z} \longrightarrow 0 \tag{5.1}$$

where the third map sends every element of S to 1 and $X_{L,S}$ is defined to be the kernel of this map. The Dirichlet S-regulator is the $\mathbb{R}G$-equivariant isomorphism $\mathcal{O}^\times_{L,S} \otimes \mathbb{R} \xrightarrow{\sim} X_{L,S} \otimes \mathbb{R}$ which is defined by

$$\mathrm{Reg}_S(u) := -\sum_{w \in S} \log |u|_w . w$$

for all $u \in \mathcal{O}^\times_{L,S}$ and where $|u|_w$ is the absolute value corresponding to w normalized in the usual way. Using (5.1) one has $\Xi(M) = [\mathbb{Q} \otimes \mathcal{O}^\times_{L,S_\infty}] \boxtimes [(\mathbb{Q} \otimes X_{L,S_\infty})^*]^{-1}$ and $r_B = \mathrm{Reg}_{S_\infty}$.

$\underline{r < 0}$

The r^{th}-Beilinson regulator is a $\mathbb{R}G$-equivariant isomorphism

$$\text{Beil}_r : \mathbb{R} \otimes K_{1-2r}(\mathcal{O}_L) \xrightarrow{\sim} \left(\bigoplus_{L \hookrightarrow \mathbb{C}} (2\pi i)^{-r}\mathbb{R} \right)^+$$

where $\text{Gal}(\mathbb{C}/\mathbb{R})$ acts on $\bigoplus_{L \hookrightarrow \mathbb{C}} (2\pi i)^{-r}\mathbb{R}$ diagonally and $(-)^+ := H^0(\text{Gal}(\mathbb{C}/\mathbb{R}), -)$. One has $\Xi(M) = [(\mathbb{Q} \otimes K_{1-2r}(\mathcal{O}_L))^*] \boxtimes [(\bigoplus_{L \hookrightarrow \mathbb{C}} (2\pi i)^{-r}\mathbb{Q})^+]^{-1}$ and $\alpha_M = \text{Beil}_r$.

ii) $M = h^1(E)_L(1)$.

One has $M^*(1) = M$, $H^0_f(K, M) = 0$, $H^1_f(K, M) = E(L)$, $\Xi(M) = [\mathbb{Q} \otimes E(L)]^{-1} \boxtimes [(\mathbb{Q} \otimes E(L))^*] \boxtimes \boxtimes_{v \in S_\infty(K)} [H^1(\sigma_v E(\mathbb{C}), (2\pi i)\mathbb{Q})] \boxtimes [H^1(E, \mathcal{O}_E)]$, $\ker(\alpha_M) = 0$, $\text{coker}(\alpha_M) = 0$ and the map δ of Conjecture 5.3.1 is induced by the Néron-Tate height pairing.

5.4 Motivic L-functions

Let v be a non-archimedean place of K and ℓ a prime not equal to the residue characteristic p of v. Let $\text{Frob}_v \in G_v$ be a choice of Frobenius of v and I_v denote the inertia group of v in G_v. We define

$$L_v(M, T) := \text{Nrd}_{\mathbb{Q}_\ell \otimes A}(1 - \text{Frob}_v^{-1} T | H_\ell(M)^{I_v}) \in \zeta(\mathbb{Q}_\ell \otimes A)[T].$$

where the reduced norm extends to polynomial rings over semi-simple algebras in the obvious way.

A compatibility conjecture of Tate implies that $L_v(M, T)$ belongs to $\zeta(A)[T]$ and is independent of the choice of ℓ; this is known to be true if X has good reduction at p.

One defines the S-truncated (equivariant motivic) L-function of M to be

$$L_S(M, s) := \prod_{v \notin S(K)} L_v(M, Nv^{-s})^{-1} \in \zeta(A_\mathbb{C})^\times,$$

which is defined for s with sufficiently large real part. If $S = S_\infty$ then we drop the subscript S. We now assume that $L_S(M, s)$ has analytic continuation to $s = 0$. We regard $\mathrm{ord}_{s=0}L(M, s)$ as a locally constant function on $\mathrm{Spec}(\zeta(A_\mathbb{C}))$ in the obvious way and then define the (equivariant) leading term of $L_S(M, s)$ to be

$$L_S^\times(M, s) := \lim_{s \to 0} \quad s^{-\mathrm{ord}_{s=0}L(M,s)} L(M, s) \in \zeta(A_\mathbb{C})^\times.$$

When M is clear from context we will write $L_S^\times(s)$ for $L_S^\times(M, s)$.

Examples

i) Let $M = h^0(\mathrm{Spec}\, L)(r)$ with $r \leq 0$, and $A = \mathbb{Q}G$. Then for each $s \in \mathbb{C}$ one has

$$L_S(M, s) := (L_S(L/K, \overline{\chi}, s + r))_\chi \in \prod_{\chi \in \mathrm{Irr}_\mathbb{C}(G)} \mathbb{C} \cong \zeta(\mathbb{C}G),$$

where $L_S(L/K, \overline{\chi}, s + r)$ is the S-truncated Artin L-function corresponding to the character $\overline{\chi}$. We remark that since complex conjugation is continuous, for each s in $\mathbb{R}$ one has $L_S(M, s) \in \zeta(\mathbb{R}G)$ and hence $L_S(M, s)^\times \in \zeta(\mathbb{R}G)^\times$.

ii) Let $M = h^1(E)_L(1)$ and $A = \mathbb{Q}G$. Then for each $s \in \mathbb{C}$ one has

$$L_S(M, s) = (L_S(E/K, \overline{\chi}, s + 1))_\chi \in \prod_{\chi \in \mathrm{Irr}_\mathbb{C}(G)} \mathbb{C} \cong \zeta(\mathbb{C}G),$$

where $L_S(E/K, \overline{\chi}, 1)$ is the S-truncated Hasse-Weil L-function corresponding to the character $\overline{\chi}$. As in Example i) we have $L_S(M, s) \in \zeta(\mathbb{R}G)$ and $L_S(M, s)^\times \in \zeta(\mathbb{R}G)^\times$ for all s in $\mathbb{R}$.

5.5 Projective Structures

Let R be a finitely generated subring of $\mathbb{Q}$.

Definition 5.5.1. *Let $\mathfrak{A}$ be an R-order in A. A projective $\mathfrak{A}$-structure T on M is a set $\{T_v\}_{v \in S_\infty}$ where, for each v, T_v is a full projective $\mathfrak{A}$-lattice in $H_v(M)$ and for* *each prime $\ell \in \mathrm{Spec}(R)$ the image T_ℓ of $T_v \otimes \mathbb{Z}_\ell$ under the comparison isomorphism $H_v(M) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \cong H_\ell(M)$ is both independent of v and stable under the action of G_K.*

Let $\mathfrak{A}$ be an R-order in A, T a projective structure on M and $S_\ell := S(K) \cup \{v|\ell\}$. We denote by G_{S_ℓ} the Galois group of the maximal unramified outside S_ℓ extension of K. Given a profinite group H and a continuous H-module N we write $C(H, N)$ for the standard complex of continuous co-chains. Furthermore, for any continuous G_{S_ℓ}-module N we set $R\Gamma(\mathcal{O}_{K,S_\ell}, N) := C(G_{S_\ell}, N)$ and

$$R\Gamma_c(\mathcal{O}_{K,S_\ell}, N) := \mathrm{Cone}(R\Gamma(\mathcal{O}_{K,S_\ell}, N) \to \bigoplus_{v \in S_\ell} C(G_v, N))[-1]$$

where the displayed map of complexes is the restriction map induced by any choice of homomorphism $G_v \hookrightarrow G_K \twoheadrightarrow G_{S_\ell}$.

Let $V_\ell := \mathbb{Q}_\ell \otimes_{\mathbb{Z}_\ell} T_\ell$. One has a canonical conjectural isomorphism ([9, §3.4])

$$\vartheta_\ell : A_\ell \otimes_A \Xi(M) \xrightarrow{\sim} [R\Gamma_c(\mathcal{O}_{K,S_\ell}, V_\ell)] \cong A_\ell \otimes_{\mathfrak{A}_\ell} [R\Gamma_c(\mathcal{O}_{K,S_\ell}, T_\ell)].$$

We thus obtain an object $[R\Gamma_c(\mathcal{O}_{K,S_\ell}, T_\ell), \Xi(M), \vartheta_\ell]$ of $V(\mathfrak{A}_\ell) \times_{V(A_\ell)} V(A)$ which can be shown to be independent of the choice of T and S. Under the coherence hypothesis of [loc. cit., §3.3] one can define an element

$$(\Xi(M), \vartheta_\infty) := \left(\prod_\ell [R\Gamma_c(\mathcal{O}_{K,S_\ell}, T_\ell)], \Xi(M), \prod_\ell \vartheta_\ell; \vartheta_\infty \right) \in \mathbb{V}(\mathfrak{A}, \mathbb{R}).$$

We write $R\Omega(M, \mathfrak{A})$ for the class of this element under the isomorphism $\pi_0(\mathbb{V}(\mathfrak{A}, \mathbb{R})) \cong K_0(\mathfrak{A}, A_{\mathbb{R}})$.

5.6 The Equivariant Conjecture

In §2.2.5 we defined a canonical homomorphism $\widehat{\delta} : \zeta(\mathbb{R}G)^\times \to K_0(\mathbb{Z}G, \mathbb{R}G)$ for any finite group G. Let Λ be any finitely generated subring of $\mathbb{Q}$ and $\mathfrak{A}$ a Λ-order in a finite dimensional semi-simple $\mathbb{Q}$-algebra A. It is not difficult to see that one can define a map $\zeta(A_{\mathbb{R}}) \to K_0(\mathfrak{A}, A_{\mathbb{R}})$ in the same way as $\widehat{\delta}$. We also denote any such map by $\widehat{\delta}$.

Conjecture 5.6.1. [9, Conj. 4] *Let M be a motive with coefficients in A, let Λ be a finitely generated subring of $\mathbb{Q}$ and let $\mathfrak{A}$ be a Λ-order in A.*

i) The L-function $L_S(M, s)$ can be analytically continued to $s = 0$.

ii) Let $T\Omega(M, \mathfrak{A}) := \widehat{\delta}(L^{\times}(M, 0)) + R\Omega(M, \mathfrak{A})$. Then $T\Omega(M, \mathfrak{A}) = 0$.

We abbreviate this Conjecture to ETNC$(M, \mathfrak{A})$.

Remark 5.6.2. Conjecture 4 of [9] in fact says more, however in this thesis we will not be interested in these additional predictions.

5.7 Proven Cases

We list the cases in which Conjecture 5.6.1 is known to be valid.

- Let L be a finite abelian extension of $\mathbb{Q}$ and r an integer. Then in [11], [12], [13] and [21], Burns, Flach and Greither have shown that ETNC$(h^0(\operatorname{Spec} L)(r), \mathbb{Z}[\operatorname{Gal}(L/\mathbb{Q})])$ is valid.

- Let k be an imaginary quadratic extension of $\mathbb{Q}$, L an abelian extension of k, p a prime which both splits in $k/\mathbb{Q}$ and does not divide the class number of k. Then in [2] Bley has shown that the 'p part' of ETNC$(h^0(\operatorname{Spec} L)), \mathbb{Z}[\operatorname{Gal}(L/k)])$ is valid.

- In [10] Burns and Flach have shown that, for a particular family of Galois extensions $L/\mathbb{Q}$ with Galois group isomorphic to Q_8, ETNC$(h^0(\operatorname{Spec} L), \mathbb{Z}[\operatorname{Gal}(L/\mathbb{Q})])$ is valid.

- In [33] Navilarekallu has shown that, for an explicit Galois extension $L/\mathbb{Q}$ with Galois group isomorphic to A_4, ETNC$(h^0(\operatorname{Spec} L), \mathbb{Z}[\operatorname{Gal}(L/\mathbb{Q})])$ is valid.

- In [33] Navilarekallu has shown that, for an explicit Galois extension $L/\mathbb{Q}$ with Galois group isomorphic to S_3 and an explicit elliptic curve E (defined over $\mathbb{Q}$), $\mathrm{ETNC}(h^1(E)_L(1), \mathbb{Z}[\mathrm{Gal}(L/\mathbb{Q})])$ is valid.

Arithmetic Examples

Let L/K be a finite Galois extension of number fields and E an elliptic curve defined over K. In this section we consider the motives $h^0(\operatorname{Spec} L)(r)$ for $r \leq 0$ and $h^1(E)_L(1)$. We will show that for each of these motives there exists an a.t.e. whose Euler characteristic measures the failure of the validity of the ETNC. We then apply Theorem 4.3.1 to obtain explicit consequences of the ETNC in each case.

6.1 $h^0(\operatorname{Spec} L)$

We fix a Galois extension of number fields L/K with Galois group G, and a finite G-stable set of places S of L containing the set S_∞ of archimedean places and the set S_{ram} of places that ramify in L/K. We write $S(K)$, resp. $S_{ram}(K)$, for the set of places of K below those in S, resp. S_{ram}. Given a place $v \in S(K)$ we write G_v for a chosen decomposition group of v in G and I_v for the corresponding choice of inertia group. We write $\mathcal{O}$ for the ring of integers of L, $\mathcal{O}_S$ for the ring of S-integers of L and μ_L for $(\mathcal{O}^\times)_{tor}$. We recall the modules $Y_S := Y_{L,S}$, $X_S := X_{L,S}$ and the $\mathbb{R}G$-equivariant isomorphism Reg_S that were defined in §5.3 Example i). Furthermore, we recall the element $L_S(r) := L_S(h^0(\operatorname{Spec} L)(r), 0)$ of $\zeta(\mathbb{R}G)$ defined in §5.4 Example i).

Proposition 6.1.1. *There exists an a.t.e. $\tau_0 = (\epsilon_{\tau_0}, \lambda_{\tau_0}, \mathcal{L}^\times_{\tau_0})$ satisfying the following conditions:-*

- $H^0_{\tau_0} = \mathcal{O}_S^\times$, $(H^1_{\tau_0})_{tor} = Pic(\mathcal{O}_S)$ *and* $\overline{(H^1_{\tau_0})} = X_S$;
- $\lambda_{\tau_0} = Reg_S$;
- $\mathcal{L}_{\tau_0} = L_S(0)$;
- $\chi(\tau_0) = 0$ *if and only if ETNC*$(h^0(\operatorname{Spec} L), \mathbb{Z}G)$ *is valid.*

Proof See [7, Prop. 4.2].□

Remark 6.1.2. In Chapter 7 we will interpret ϵ_{τ_0} in terms of the extensions constructed by Ritter and Weiss in [38] (see Remark 7.2.5).

We set $e_0 := e_{\tau_0}$.

Corollary 6.1.3. *If ETNC($h^0(\mathrm{Spec}\, L), \mathbb{Z}G$) is valid, then each of the following claims is valid.*

i) $L_S(0) \in \mathbb{Q}G$.

ii) If p is any prime such that $(\mu_L)_p$ is a cohomologically trivial G-module, then $\mathrm{Fitt}_{\mathbb{Z}_pG}((\mu_L)_p)$ *and* $\mathrm{Fitt}_{\mathbb{Z}_pG}(H^1_{\tau_0})$ *are both defined and*

$$\mathrm{Fitt}_{\mathbb{Z}_pG}((\mu_L)_p)L_S(0) = \mathrm{Fitt}_{\mathbb{Z}_pG}(H^1_{\tau_0}).$$

iii) If G is abelian and p is any prime such that either $(\mu_L)_p$ is a cohomologically trivial G-module or $\mathbb{Z}_{(p)}Ge_0$ is relatively Gorenstein, then

$$\mathrm{Fitt}_{\mathbb{Z}_{(p)}G}((\mu_L)_p)L_S(0) = \mathrm{Fitt}_{\mathbb{Z}_{(p)}G}(H^1_{\tau_0}).$$

Proof Using the explicit formula for the order of vanishing $r(S, \chi)$ of $L_S(s, \chi)$ at $s = 0$ (see [41, Ch. I, Prop. 3.4]) it can easily be seen that $\chi \in \Upsilon(\tau_0)$ if and only if $r(S, \chi) = 0$. Thus $L_S(0) := L_S^\times(0)e_0$. The Corollary now follows directly from Theorem 4.3.1 and Proposition 6.1.1 after we note that, since μ_L is cyclic, μ_L and $\mu_L^\vee$ are isomorphic as G-modules.□

Remark 6.1.4. Corollary 6.1.3 is of particular interest when the data L/K, S and p are chosen such that:-

- There exists $\chi \in \mathrm{Irr}_{\mathbb{C}}(G)$ such that $\chi(1) > 1$ and $r(S, \chi) = 0$;
- p divides $|G|$ and $(\mu_L)_p$ is a cohomologically trivial G-module.

The following argument explains why such situations should exist.

Let n be an integer such that $n \geq 2$ and Q_{4n} denote the generalized quaternion group defined by

$$Q_{4n} =< s, t | s^{2n} = 1, t^2 = s^n, ts = s^{-1}t >$$

Let ζ be a primitive $2n^{\text{th}}$ root of unity and let V_ρ be a 2-dimensional $\mathbb{C}$-vector space with G-action given by

$$\rho(s) = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix} \qquad \rho(t) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

It is easily checked that this representation of Q_{4n} is irreducible. We write ψ for the character of ρ.

Now assume that L/K is a Galois extension of number fields such that K is totally real, L is complex and $\text{Gal}(L/K) \cong Q_{4n}$. Let $S = S_\infty \cup S_{ram}$. For any non-trivial irreducible character χ of G [41, Ch. I, Prop. 3.4] shows that

$$r(S, \chi) = \sum_{v \in S(K)} \text{dim} V_\chi^{G_v},$$

where V_χ is a $\mathbb{C}$-vector space realizing χ. Our choice of L/K and S implies that G_v is non-trivial for all $v \in S(K)$. One can check that any non-trivial subgroup H of Q_{4n} must contain an element of the form s^r, where r is an integer such that $r | 2n$ and $r \neq 2n$. Fix such a subgroup H and assume that $\text{dim} V_\rho^H \neq 0$. Then there exists $\underline{v} = (v_1, v_2) \in V_\rho$ such that $\rho(s^r)\underline{v} = \underline{v}$, i.e.

$$\begin{pmatrix} \zeta^r & 0 \\ 0 & \zeta^{-r} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix},$$

which implies $r = 2n$ or $v_1 = v_2 = 0$. Thus for all non-trivial subgroups H of Q_{4n} we have $\text{dim} V_\rho^H = 0$ and so $r(S, \psi)$=0 as required.

We now further assume that our extension L/K has order $4p$ where $p > 5$ is a prime which does not ramify in $K/\mathbb{Q}$. This implies that $\mathbb{Q}(\zeta_p) \cap K = \mathbb{Q}$. Thus if $(\mu_L)_p \neq 0$, then $p-1|4p$ which is nonsense. Thus $(\mu_L)_p = 0$ and is hence obviously cohomologically trivial.

We now formulate a non-commutative generalization of Brumer's Conjecture. If L/K is a CM extension with complex conjugation given by $j \in G$, then we write e_- for the idempotent $(1-j)/2$ of $\mathbb{Q}G$ (which is central as the subgroup of G generated by j is normal).

Corollary 6.1.5. *Let L/K be a CM extension with complex conjugation given by $j \in G$ and $S = S_\infty \cup S_{ram}$. Let p be an odd prime such that $(\mu_L)_p$ is a cohomologically trivial G-module. Assume that all places which ramify in L/K contain j in their decomposition group and that at least one place in S has full decomposition group. If ETNC($h^0(\operatorname{Spec} L), \mathbb{Z}G$) is valid, then $\mathrm{Fitt}_{\mathbb{Z}_pG}((\mu_L)_p)$ and $\mathrm{Fitt}_{\mathbb{Z}_pG}(\mathrm{Pic}(\mathcal{O})e_-)$ are both defined and*

$$\mathrm{Fitt}_{\mathbb{Z}_pG}((\mu_L)_p)L_S(0) = \mathrm{Fitt}_{\mathbb{Z}_pG}(\mathrm{Pic}(\mathcal{O})e_-).$$

Proof Let w be a place of L with full decomposition group in G and v the place of K below w. Given our hypotheses we have an isomorphism $X_S \cong \bigoplus_{v' \in S(K), v' \neq v} \mathbb{Z}[G/G_{v'}]$. We must compute the idempotent e_0. Since X_S is the quotient of $H^1_{\tau_0}$ by a finite module we see that $e_0 = e_{X_S}$. It is clear that $e_-(\mathbb{C} \otimes X_S) = 0$ and since $\mathbb{Z}[G/<j>]$ appears as a direct summand of X_S one sees that in fact $e_0 = e_-$.

Since p is odd, $\mathbb{Z}_pGe_-$ is a projective $\mathbb{Z}_pG$-module and so $\mathrm{Tor}^1_{\mathbb{Z}_pG}(\mathbb{Z}_pGe_-, X_S) = 0$. Furthermore, we have $\mathbb{Z}_pGe_- \otimes_G X_S = 0$. Hence Proposition 3.3.3 implies that

$$\mathrm{Fitt}_{\mathbb{Z}_pG}(H^1_{\tau_0}) = \mathrm{Fitt}_{\mathbb{Z}_pG}(\mathrm{Pic}(\mathcal{O}_S), e_-).$$

Corollary 6.1.3 ii) then implies that

$$\mathrm{Fitt}_{\mathbb{Z}_pG}((\mu_L)_p)L_S(0) = \mathrm{Fitt}_{\mathbb{Z}_pG}(\mathrm{Pic}(\mathcal{O}_S), e_-) \tag{6.1}$$

and, by definition of the Fitting invariant of a module, the right hand side of (6.1) is equal to $\mathrm{Fitt}_{\mathbb{Z}_pG}(\mathrm{Pic}(\mathcal{O}_S)e_-)$.

Now, there is a natural exact sequence of G-modules $I_S \longrightarrow \mathrm{Pic}(\mathcal{O}) \longrightarrow \mathrm{Pic}(\mathcal{O}_S) \longrightarrow 0$ where I_S is the free abelian group on the non-archimedean places in S. Applying $\mathbb{Z}_pGe_- \otimes_G -$ to this sequence gives an isomorphism of $\mathbb{Z}_pGe_-$-modules $\mathbb{Z}_pGe_- \otimes_G \mathrm{Pic}(\mathcal{O}) \cong \mathbb{Z}_pGe_- \otimes_G \mathrm{Pic}(\mathcal{O}_S)$. We can thus replace the S-class group by the class group on the right of (6.1), which completes the proof of the corollary.□

We give one further application of Corollary 6.1.3.

Corollary 6.1.6. *Set* $\Lambda = \mathbb{Z}[1/2]$*. Let* L/K *be an abelian CM extension with complex conjugation given by* $j \in G$*. Assume that all places which ramify in* L/K *contain* j *in their decomposition group and let* $S = S_\infty \cup S_{ram}$*. If* $ETNC(h^0(\mathrm{Spec}\, L), \Lambda G)$ *holds, then each of the following claims is valid.*

i) $\mathrm{Fitt}_{\Lambda G}((\mu_L)_\Lambda)L_S(0) = \mathrm{Fitt}_{\Lambda G}(\mathrm{Pic}(\mathcal{O})_\Lambda e_-)$.

ii) Assume that only one non-archimedean place ramifies in L/K *and that this place has full decomposition group. Then*

$$\mathrm{Fitt}_{\mathbb{Z}G}(\mu_L)L(0) = 2^{|S_\infty|}\, \mathrm{Fitt}_{\mathbb{Z}Ge_-}(\mathbb{Z}Ge_- \otimes_G \mathrm{Pic}(\mathcal{O})).$$

Proof As in the proof of Corollary 6.1.5 we have $e_0 = e_-$. We claim that $\mathbb{Z}Ge_-$ is relatively Gorenstein. This is a consequence of the following more general fact. If G is an abelian group and $e \in \mathbb{Q}G$ is an idempotent such that $e \in \mathbb{Q}C$ for some cyclic subgroup of G, then $\mathbb{Z}Ge$ is relatively Gorenstein.

Corollary 6.1.3 iii) now implies that for all $p \in \mathrm{Spec}\,\Lambda$ we have $\mathrm{Fitt}_{\mathbb{Z}_{(p)}G}((\mu_L)_p)L_S(0) = \mathrm{Fitt}_{\mathbb{Z}_{(p)}G}(H^1_{\tau_0})$. The following two facts imply that $\mathrm{Fitt}_{\Lambda G}((\mu_L)_\Lambda)L_S(0) = \mathrm{Fitt}_{\Lambda G}(H^1_{\tau_0})$:-

- For any ΛG-module M one has $\mathrm{Fitt}_{\Lambda G}(M)\mathbb{Z}_{(p)}G = \mathrm{Fitt}_{\mathbb{Z}_{(p)}G}(M_{(p)})$ ([35, Ch. 3, Th. 3)]);
- For any ΛG lattice M one has $M = \bigcap_p M_{(p)}$.

It is clear that $e_- X_{S,\Lambda} = 0$ and so the tautological exact sequence of ΛG-modules

$$0 \longrightarrow \mathrm{Pic}(\mathcal{O}_S)_\Lambda \longrightarrow H^1_{\tau_0,\Lambda} \longrightarrow X_{S,\Lambda} \longrightarrow 0$$

implies that $H^1_{\tau_0,\Lambda} e_- = \mathrm{Pic}(\mathcal{O}_S) e_-$. This implies that

$$\mathrm{Fitt}_{\Lambda G}((\mu_L)_\Lambda) L_S(0) = \mathrm{Fitt}_{\Lambda G e_-}(\mathrm{Pic}(\mathcal{O}_S) e_-)$$

and the same argument as in the proof of Corollary 6.1.5 allows us to replace $\mathrm{Pic}(\mathcal{O}_S)$ by $\mathrm{Pic}(\mathcal{O})$, which proves claim i).

As in the proof of claim i) the fact that $\mathbb{Z}Ge_-$ is relatively Gorenstein implies that

$$\mathrm{Fitt}_{\mathbb{Z}G}(\mu_L) L_S(0) = \mathrm{Fitt}_{\mathbb{Z}G}(H^1_{\tau_0}). \tag{6.2}$$

Now, under the additional assumption in claim ii) we see that $X_S \cong \bigoplus_{w \in S_\infty} \mathbb{Z}[G/ < j >]$ and thus

$$\cdots \xrightarrow{1-j} \bigoplus_{w \in S_\infty} \mathbb{Z}G \xrightarrow{1+j} \bigoplus_{w \in S_\infty} \mathbb{Z}G \xrightarrow{1-j} \bigoplus_{w \in S_\infty} \mathbb{Z}G \longrightarrow X_S \longrightarrow 0$$

is a projective resolution of X_S. The following claims are then a direct consequence of the existence of this resolution:-

- $\mathrm{Tor}^1_{\mathbb{Z}G}(\mathbb{Z}Ge_-, X_S) = 0$;
- $0 \longrightarrow \mathbb{Z}Ge_- \otimes_G \mathrm{Pic}(\mathcal{O}_S) \longrightarrow \mathbb{Z}Ge_- \otimes_G H^1_{\tau_0} \longrightarrow \mathbb{Z}Ge_- \otimes_G X_S \longrightarrow 0$ is an exact sequence of $\mathbb{Z}Ge_-$-modules;
- $0 \longrightarrow (\mathbb{Z}Ge_-)^{|S_\infty|} \xrightarrow{2} (\mathbb{Z}Ge_-)^{|S_\infty|} \longrightarrow \mathbb{Z}Ge_- \otimes_G X_S \longrightarrow 0$ is an exact sequence of $\mathbb{Z}Ge_-$-modules;
- $\mathrm{Fitt}_{\mathbb{Z}G}(\mathbb{Z}Ge_- \otimes_G X_S) = 2^{|S_\infty|}$.

These claims show that $\mathbb{Z}Ge_- \otimes_G X_S$ satisfies the hypothesis of [18, Lemma 3] and furthermore that $\mathrm{Fitt}_{\mathbb{Z}Ge_-}(\mathbb{Z}Ge_- \otimes_G H^1_{\tau_0}) = 2^{|S_\infty|}\,\mathrm{Fitt}_{\mathbb{Z}Ge_-}(\mathbb{Z}Ge_- \otimes_G \mathrm{Pic}(\mathcal{O}_S))$. This, together with (6.2) and Lemma 3.3.2, implies that

$$\mathrm{Fitt}_{\mathbb{Z}G}(\mu_L)L_S(0) = 2^{|S_\infty|}\,\mathrm{Fitt}_{\mathbb{Z}Ge_-}(\mathbb{Z}Ge_- \otimes_G \mathrm{Pic}(\mathcal{O}_S)). \tag{6.3}$$

As in the proof of 6.1.5 we can replace $\mathrm{Pic}(\mathcal{O}_S)$ by $\mathrm{Pic}(\mathcal{O})$ in (6.3). To complete the proof of the corollary we now show that $L_S(0) = L(0)$.

Let $\chi \in \mathrm{Irr}_{\mathbb{C}}(G)$ such that $\chi(j) = -1$ and V_χ be a $\mathbb{C}$-vector space realizing χ. Let v be the prime of K which ramifies in L, I_v the inertia group of v and frob_v a choice of lifting to G of the Frobenius element of G/I_v. The components $L_S(0)_\chi$ and $L(0)_\chi$ differ by the Euler factor $(1 - \chi(\mathrm{frob}_v)|V_\chi^{I_v})$. By assumption j is in I_v and since j acts as -1 on V_χ, this Euler factor is clearly 1. Thus $L_S(0) = L(0)$ as required.□

Remark 6.1.7. Recall the definition of $T\Omega(M, \mathfrak{A})$ in Conjecture 5.6.1. We have a functorial map $K_0(\mathbb{Z}G, \mathbb{R}G) \to K_0(\mathbb{Z}[1/2]Ge_-, \mathbb{R}Ge_-)$. If the image of $T\Omega(h^0(\mathrm{Spec}\,L, \mathbb{Z}G)$ under this map is zero, then Corollary 6.1.6 i) recovers the 'Main Theorem' of Greither in [24].

6.2 $h^0(\mathrm{Spec}\,L)(r)$ with $r < 0$

We continue with the notation of §6.1. Furthermore, we fix an algebraic closure K^c of K containing L and write G_K, resp. G_L, for $\mathrm{Gal}(K^c/K)$, resp. $\mathrm{Gal}(K^c/L)$. Set $\Lambda = \mathbb{Z}[1/2]$. We recall that if p is odd, then there are canonical Chern class maps for $i = 1, 2$

$$K_{2-i-2r}(\mathcal{O}) \otimes \mathbb{Z}_p \longrightarrow H^i(\mathrm{Spec}(\mathcal{O}[1/p])_{\text{ét}}, \mathbb{Z}_p(1-r))$$

which are known to be surjective and to have finite kernel. The Quillen-Lichtenbaum conjecture asserts that these maps are in fact bijective ([28, Conj. 2.5]) and we will assume this to be valid for the remainder of this section. We remark that this conjecture is known to be true by work of Tate if $r = -1$ and $i = 2$ [43] and of Levine [27] and Merkuriev and Suslin [30] if $r = -1$ and $i = 1$. We write $\mu_{K^c}^{\otimes r}$ for the r^{th} tensor power of the roots of unity of K^c. Finally, one has $K_{1-2r}(\mathcal{O})_{tor} = H^0(G_L, \mu_{K^c}^{\otimes 1-r})$.

Proposition 6.2.1. *There exists an a.t.e.* $\tau_r = (\epsilon_{\tau_r}, \lambda_{\tau_r}, \mathcal{L}^{\times}_{\tau_r})$ *satisfying the following conditions:-*

- $H^0_{\tau_r,\Lambda} = K_{1-2r}(\mathcal{O})_\Lambda$, $(H^1_{\tau_r,\Lambda})_{tor} = K_{-2r}(\mathcal{O}_S)_\Lambda$ *and* $\overline{(H^1_{\tau_r,\Lambda})} = (\bigoplus_{L\hookrightarrow\mathbb{C}}(2\pi i)^{-r}\Lambda)^+$;
- $\lambda_{\tau_r} = Beil_r$;
- $\mathcal{L}_{\tau_r} = L_S(r)$;
- $\chi(\tau_r) = 0$ *if and only if ETNC*$(h^0(\operatorname{Spec} L)(r), \Lambda G)$ *is valid.*

Proof See [7, Prop. 4.6].□

We set $e_r := e_{\tau_r}$.

Corollary 6.2.2. *If ETNC*$(h^0(\operatorname{Spec} L)(r), \Lambda G)$ *is valid, then each of the following claims is valid.*

i) $L_S(r) \in \mathbb{Q}G$.

ii) If p *is any odd prime for which* $H^0(G_L, \mu_{K^c}^{\otimes r})_p$ *is a cohomologically trivial* G*-module, then* $\operatorname{Fitt}_{\mathbb{Z}_pG}(H^0(G_L, \mu_{K^c}^{\otimes r})_p)$ *and* $\operatorname{Fitt}_{\mathbb{Z}_pG}(K_{-2r}(\mathcal{O}_\mathcal{S}))$ *are both defined and*

$$\operatorname{Fitt}_{\mathbb{Z}_pG}(H^0(G_L, \mu_{K^c}^{\otimes r})_p)L_S(r) = \operatorname{Fitt}_{\mathbb{Z}_pG}(K_{-2r}(\mathcal{O}_\mathcal{S}), e_r).$$

iii) If G *is abelian and* p *is any odd prime for which either* $H^0(G_L, \mu_{K^c}^{\otimes r})_p$ *is a cohomologically trivial* G*-module or* $\mathbb{Z}_{(p)}Ge_r$ *is relatively Gorenstein, then*

$$\operatorname{Fitt}_{\mathbb{Z}_{(p)}G}(H^0(G_L, \mu_{K^c}^{\otimes r})_p)L_S(r) = \operatorname{Fitt}_{\mathbb{Z}_{(p)}Ge_r}(K_{-2r}(\mathcal{O}_\mathcal{S})).$$

Proof We first note that a similar argument to that given in the proof of Corollary 6.1.3 shows that $L_S(r) = L_S^{\times}(r)e_r$. Now, the ΛG-module $\bigoplus_{L\hookrightarrow\mathbb{C}}(2\pi i)^{-r}\Lambda$ is easily seen to be free and so we denote it by F. Since 2 is invertible in Λ this implies that F^+ is a projective ΛG-module. We have a tautological exact sequence of ΛG-modules

$$0 \longrightarrow K_{-2r}(\mathcal{O}_S)_\Lambda \longrightarrow H^1_{\tau_r,\Lambda} \longrightarrow F^+ \longrightarrow 0\,. \tag{6.4}$$

Since F^+ is projective, applying the functor $\Lambda G e_r \otimes_G -$ to (6.4) gives an exact sequence of $\Lambda G e_r$-modules

$$0 \longrightarrow \Lambda G e_r \otimes_{\Lambda G} K_{-2r}(\mathcal{O}_S)_\Lambda \longrightarrow \Lambda G e_r \otimes_{\Lambda G} H^1_{\tau_r,\Lambda} \longrightarrow \Lambda G e_r \otimes_{\Lambda G} F^+ \longrightarrow 0.$$

By definition of e_r we see that $\Lambda G e_r \otimes_{\Lambda G} F^+$ is finite. Since F^+ is projective this implies that $\Lambda G e_r \otimes_{\Lambda G} F^+$ is a finite projective $\Lambda G e_r$-module and hence zero. We thus have an isomorphism of $\Lambda G e_r$-modules $\Lambda G e_r \otimes_{\Lambda G} K_{-2r}(\mathcal{O}_S)_\Lambda \cong \Lambda G e_r \otimes_{\Lambda G} H^1_{\tau_r,\Lambda}$.

Applying Theorem 4.3.1 to Proposition 6.2.1 now gives the statement of the Corollary after we note that $H^0(G_L, \mu_{K^c}^{\otimes r}) \cong H^0(G_L, \mu_{K^c}^{\otimes r})^\vee$, since $H^0(G_L, \mu_{K^c}^{\otimes r})$ is cyclic.□

Remark 6.2.3. Corollary 6.2.2 ii) provides a natural non-commutative generalization of the Coates-Sinnott Conjecture (see [17, Conj. 1]).

6.3 $h^1(E)_L(1)$

For this section we let $S = S_\infty \cup S_{ram}$ and shall assume for simplicity that $K = \mathbb{Q}$. Let E be an elliptic curve defined over $\mathbb{Q}$ and E/E^0 be the group of connected components in the Néron model of E over $\operatorname{Spec}\mathbb{Z}$. For each prime p let E_p denote the group scheme of p-torsion on E, $\widetilde{E}_p$ the reduction of E at p and $\mathbb{F}_p$ the field of cardinality p.

We fix a prime number p and a Galois extension $L/\mathbb{Q}$ which satisfy the following conditions:-

- $p \nmid 2\mathrm{Cond}(E)|E/E^0||E(\mathbb{Q})_{tor}| \prod_{\ell \in S_{ram}(\mathbb{Q})} |\widetilde{E}_\ell(\mathbb{F}_\ell)|$;

- If ℓ is any prime at which E has bad reduction, then

$$H^0(\mathrm{Gal}(\mathbb{Q}_\ell^c/\mathbb{Q}_\ell), H^1(\mathrm{Gal}(\mathbb{Q}_\ell^c/\mathbb{Q}_\ell^{\mathrm{un}}), T_p(E)))_{tor} = 0$$

  where $T_p(E)$ is the p-adic Tate module of E and $\mathbb{Q}_\ell^{ur}$ is the maximal unramified extension of $\mathbb{Q}_\ell$;

- $L/\mathbb{Q}$ is of p-power degree and is unramified at p and at all primes of bad reduction for E.

Remark 6.3.1. In [7, Rem. 4.8] it is shown that given an elliptic curve E, there are infinitely many primes p, and for each such p infinitely many extensions $L/\mathbb{Q}$, which satisfy all of the above conditions.

We write $\mathrm{Sel}(E_{/L})$ for the 'integral Selmer group' defined by Mazur and Tate in [29, p. 720]. For each $\chi \in \mathrm{Irr}_{\mathbb{C}}(G)$ we write $\tau^*(\chi)$ for the 'modified global Galois-Gauss sum' associated to χ as defined in [7, §4.2]. We set $\Omega(E) := |\int_{\gamma^+}\omega|$ where γ^+ is a generator of the submodule of $H_1(E(\mathbb{C}), \mathbb{Z})$ that is fixed by the action of complex conjugation and ω is a Néron differential.

We define

$$L_{E,S}(1)^\times := \left(\frac{\tau^*(\chi)L_S^\times(E/K, \overline{\chi}, 1)}{\Omega(E)^{\chi(1)}}\right)_{\chi\in\mathrm{Irr}(G)} \in \zeta(\mathbb{C}G)^\times$$

and

$$L_{E,S}(1) := \left(\frac{\tau^*(\chi)L_S(E/K, \overline{\chi}, 1)}{\Omega(E)^{\chi(1)}}\right)_{\chi\in\mathrm{Irr}(G)} \in \zeta(\mathbb{C}G).$$

As $L_S^\times(h^1(E)_L(1), 1) \in \zeta(\mathbb{R}G)^\times$ it follows that $L_{E,S}(1)^\times \in \zeta(\mathbb{R}G)^\times$ and $L_{E,S}(1) \in \zeta(\mathbb{R}G)$.

Proposition 6.3.2. *Let E and L be as above. If the Tate-Shafarevich group of E over L is finite, then there exists an a.t.e. $\tau_E = (\epsilon_{\tau_E}, \lambda_{\tau_E}, \mathcal{L}_{\tau_E}^\times)$ satisfying the following conditions:-*

- $(H^0_{\tau_E})_p = E(L)_p$ *and* $(H^1_{\tau_E})_p = \mathrm{Sel}(E_{/L})_p$;

- λ_{τ_E} *is induced by the Néron-Tate height pairing;*

- $\mathcal{L}_{\tau_E} = L_{E,S}(1)$*;*

- $\chi(\tau_E) = 0$ *if and only if* $ETNC(h^1(E)_L(1), \mathbb{Z}G)$ *is valid.*

Proof See [7, Prop. 4.9].□

We set $e_E := e_{\tau_E}$.

Corollary 6.3.3. *If* $ETNC(h^1(E)_L(1), \mathbb{Z}G)$ *is valid, then each of the following claims is valid.*

i) $L_E(1) \in \mathbb{Q}G$.

ii) $\mathrm{Fitt}_{\mathbb{Z}_pG}(\mathrm{Sel}(E_{/L}))$ *is defined and* $L_E(1)e_E = \mathrm{Fitt}_{\mathbb{Z}_pG}(\mathrm{Sel}(E_{/L}))$.

iii) If G *is abelian, then* $L_E(1)\mathbb{Z}_{(p)}G = \mathrm{Fitt}_{\mathbb{Z}_{(p)}G}(\mathrm{Sel}(E_{/L}))$.

Proof This follows from Theorem 4.3.1 and Proposition 6.3.2.□

Remark 6.3.4. Corollary 6.3.3 iii) is a 'Strong Main Conjecture' of the kind that Mazur and Tate ask for in [29, Rem. after Conj. 3].

Tate Sequences

Throughout this chapter L/K is a fixed Galois extension of number fields with Galois group G. We begin by using étale cohomology to construct explicitly the extension ϵ_{τ_0} of Proposition 6.1.1. We then recall details of the construction of a Tate sequence by Ritter and Weiss in [38] and show that this has the same extension class as ϵ_{τ_0}.

For simplicity, in §7.2 we will impose the hypothesis that our base field K is totally complex. This hypothesis allows us to avoid using the explicit Artin-Verdier duality theorem.

7.1 Further Preliminaries

We first set up some notation for this chapter and recall some standard definitions and results.

7.1.1 Number Fields

Let F be a number field. We write $\mathcal{M}_F$ for the set of all places of F. Given $v \in \mathcal{M}_F$ we write F_v for the completion of F at v and U_v for the multiplicative group of integral units of F_v.

We write S_∞ for the set of archimedean places of L and S_{ram} for the places of L which ramify in L/K. Throughout this chapter S and S' will be reserved to denote finite G-stable sets of places of L which contain S_∞. We write S_f for the set of non-archimedean places in S and $S(K)$ for the places of K below those in S. We will also use the following variants of this notation; $S'(K)$, S'_f, $S'_f(K)$, $S(E)$ and $S_f(E)$, where in the latter two cases E is a subextension of L/K.

Given such a set S we write $\mathcal{O}_{L,S}$ for the ring of S integers of L. If $S = S_\infty$, then we simply write $\mathcal{O}_L$ for $\mathcal{O}_{L,S}$. Given $v \in \mathcal{M}_K$ we write $w(v)$ for a fixed place of L above v. Given $w \in \mathcal{M}_L$ we write G_w for the decomposition group of w in G.

We write $J(L)$ for the idèles of L and identify $L^\times$ with its image in $J(L)$ under the natural diagonal embedding. We write $J_S(L) := \prod_{w \in S} L_w^\times \prod_{w \notin S} U_w$ for the S-idèles of L, $C(L) := J(L)/L^\times$ for the idèle class group of L and set $C_S(L) := J(L)/L^\times \prod_{w \notin S} U_w$. We recall the definition of both $Y_{L,S}$ and $X_{L,S}$ from §5.3 Example i) and the exact sequence (5.1).

7.1.2 Class Field Theory

Given any field F we write $\mathrm{Br}(F)$ for the Brauer group of F. We recall that if F is a non-archimedean local field then there is a canonical 'local invariant isomorphism' $Br(F) \cong \mathbb{Q}/\mathbb{Z}$ ([15, Cor. to Th. 2]). If $v \in \mathcal{M}_K$ and $w \in \mathcal{M}_L$ such that w divides v, then we write c_{L_w/K_v} for the *Local Fundamental Class*. This is defined to be the inverse image of $1/|G_w|$ under the canonical isomorphism $\widehat{H}^2(G_w, L_w^\times) \xrightarrow{\sim} 1/|G_w|\mathbb{Z}/\mathbb{Z}$ induced by the local invariant isomorphism ([15, Ch. VI]). Similarly, there is a canonical isomorphism $\widehat{H}^2(G, C(L)) \xrightarrow{\sim} 1/|G|\mathbb{Z}/\mathbb{Z}$ ([loc. cit., Ch. VII]). We write $c_{L/K}$ for the *Global Fundamental Class* which is defined to be the inverse image of $1/|G|$ under this isomorphism.

Let S be a finite set of places of L containing $S_\infty \cup S_{ram}$. We write $c^{loc}_{L/K,S}$ for the *Semi-local Fundamental Class*. This is an element of $\mathrm{Ext}^2_{\mathbb{Z}G}(Y_{L,S}, \prod_{w \in S} L_w^\times)$ which we will define in §7.6.

7.1.3 Dimension Shifting

Let R be a ring and

$$0 \longrightarrow M \longrightarrow P \longrightarrow M' \longrightarrow 0 \tag{7.1}$$

a short exact sequence of R-modules in which P projective. Given an R-module N, the projectivity of P shows that the connecting homomorphism in the long exact sequence corresponding to (7.1) and the functors $\mathrm{Ext}^i_R(-,N)$ gives a canonical isomorphism $\mathrm{Ext}^i_R(M,N) \xrightarrow{\sim} \mathrm{Ext}^{i+1}_R(M',N)$ for all $i > 0$. Throughout this chapter we will refer to such isomorphisms as *dimension shifts*. In terms of Yoneda extension classes this isomorphism is obtained by 'splicing' (7.1) onto the end of any exact sequence representing an element of $\mathrm{Ext}^i_R(M,N)$ (see [26, Ch. V, §9] for a full discussion).

7.1.4 Extension Classes

Let R be a ring and r and s be integers with $r > s$. Let A be a complex of R-modules which is acyclic outside of degrees r and s. Then A defines an element of $\mathrm{Ext}^{r-s+1}_R(H^r(A), H^s(A))$ by 'truncating' in the usual way.

Lemma 7.1.1. *Let r and s be as above and A and B complexes of R-modules which are acyclic outside of degrees r and s. If $f : A \to B$ is a morphism in $\mathcal{D}(R)$, then there exists a commutative diagram of R-modules with exact rows*

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & H^s(A) & \longrightarrow & Y^s & \longrightarrow & \cdots & \quad \cdots & \longrightarrow & Y^r & \longrightarrow & H^r(A) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle H^s(f)} & & \downarrow & & & & & \downarrow & & \downarrow{\scriptstyle H^r(f)} & & \\
0 & \longrightarrow & H^s(B) & \longrightarrow & Z^s & \longrightarrow & \cdots & \quad \cdots & \longrightarrow & Z^r & \longrightarrow & H^r(B) & \longrightarrow & 0
\end{array}$$

such that the top row represents the class of A in $\mathrm{Ext}^{r-s+1}_R(H^r(A), H^s(A))$ and the bottom row represents the class of B in $\mathrm{Ext}^{r-s+1}_R(H^r(B), H^s(B))$.

Proof Without loss of generality we can assume that $s = 0$ and that A and B are both concentrated in degrees 0 to r. By definition of morphisms in the derived category, f is represented by a 'left roof' of the form $A \xleftarrow{q} X \xrightarrow{h} B$, where q is a quasi-isomorphism (cf. [22, III.2, Lemma 8]). This roof induces a commutative diagram of

R-modules with exact rows

$$
\begin{array}{ccccccccccccc}
0 & \longrightarrow & H^0(A) & \longrightarrow & A^0 & \longrightarrow & \cdots & \cdots & \longrightarrow & A^r & \longrightarrow & H^r(A) & \longrightarrow 0 \\
 & & {\scriptstyle H^0(q)}\uparrow\cong & & \uparrow & & & & & \uparrow & & {\scriptstyle H^r(q)}\uparrow\cong & \\
0 & \longrightarrow & H^0(X) & \longrightarrow & X^0/\operatorname{im} d_X^{-1} & \longrightarrow & \cdots & \cdots & \longrightarrow & \ker d_X^r & \longrightarrow & H^r(X) & \longrightarrow 0 \\
 & & {\scriptstyle H^0(h)}\downarrow & & \downarrow & & & & & \downarrow & & {\scriptstyle H^r(h)}\downarrow & \\
0 & \longrightarrow & H^0(B) & \longrightarrow & B^0 & \longrightarrow & \cdots & \cdots & \longrightarrow & B^r & \longrightarrow & H^r(B) & \longrightarrow 0.
\end{array}
\tag{7.2}
$$

If we set

$$
Y^i := \begin{cases} X^0/\operatorname{im} d_X^{-1} & \text{if } i = 0 \\ \ker d_X^r & \text{if } i = r \\ X^i & \text{if } 0 < i < r \end{cases}
$$

and $Z^i := B^i$ for all i, then the statement of the Lemma follows easily from (7.2).□

Lemma 7.1.2. *Let J be a finite group and H^0 and H^1 finitely generated J-modules. Let*

$$0 \longrightarrow H^0 \longrightarrow X^0 \longrightarrow X^1 \longrightarrow H^1 \longrightarrow 0$$

be an exact sequence of J-modules which is (Yoneda) equivalent to an exact sequence of J-modules

$$0 \longrightarrow H^0 \longrightarrow Y^0 \longrightarrow Y^1 \longrightarrow H^1 \longrightarrow 0 \tag{7.3}$$

in which Y^0 and Y^1 cohomologically trivial. Then there exist J-modules C^0 and C^1 which are both finitely generated and cohomologically trivial and an exact commutative diagram of J-modules

$$
\begin{array}{ccccccccccc}
0 & \longrightarrow & H^0 & \longrightarrow & C^0 & \longrightarrow & C^1 & \longrightarrow & H^1 & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & H^0 & \longrightarrow & X^0 & \longrightarrow & X^1 & \longrightarrow & H^1 & \longrightarrow & 0.
\end{array}
\tag{7.4}
$$

Proof By a standard argument using Yoneda extension classes (cf. [26, Ch. IV, §9]) we can construct a commutative diagram of J-modules of the form (7.4) in which C^1 is both finitely generated and free and C^0 is finitely generated. We will show that this is sufficient to imply that C^0 is cohomologically trivial. Let I be a subgroup of J. Applying the long exact sequence of Tate cohomology to (7.3) gives a canonical isomorphism

$$\widehat{H}^i(I, H^1) \xrightarrow{\sim} \widehat{H}^{i+2}(I, H^0)$$

for each i. Since the top row of (7.4) is Yoneda equivalent to (7.3) and C^1 is free, this isomorphism implies that $\widehat{H}^i(I, C^0) = 0$ for all i and for all I. Hence C^0 is cohomologically trivial as required.□

7.1.5 Miscellany

Let H be a finite group. There is an exact sequence of trivial H-modules

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0 \tag{7.5}$$

where $\mathbb{Q}$ is uniquely divisible and hence cohomologically trivial. There is an exact sequence of H-modules

$$0 \longrightarrow I_H \longrightarrow \mathbb{Z}H \longrightarrow \mathbb{Z} \longrightarrow 0 \tag{7.6}$$

where $\mathbb{Z}$ has the trivial H-action, the third map is the map which sends every element of H to 1 and I_H is defined to be the kernel of this map. We will refer to I_H as the *augmentation ideal of H*.

Let J be a subgroup of H. We write $\mathrm{Ind}_J^H(-)$, resp. $\mathrm{Res}_J^H(-)$, for the induction functor, resp. restriction functor, with respect to H and J. It is well known that 'Ind is left adjoint to Res', i.e. that there is a canonical isomorphism $\mathrm{Ext}^i_{\mathbb{Z}H}(\mathrm{Ind}_J^H N, M) \cong \mathrm{Ext}^i_{\mathbb{Z}J}(N, \mathrm{Res}_J^H M)$ for each $i \geq 0$.

For presentational purposes we will sometimes use the following notation. Let S be a finite G-stable set of places of L. Given a family of $G_{w(v)}$-modules $\{M_v\}_{v \in S(K)}$ we set

$$\prod_{S(K)}^{Ind} M_v := \prod_{v \in S(K)} \mathrm{Ind}_{G_{w(v)}}^{G} M_v.$$

7.2 ϵ_{τ_0}

In this section S is any finite G-stable set of places of L containing $S_\infty \cup S_{ram}$.

Let R be a ring. By an étale sheaf on Spec R we shall always mean a sheaf on the small étale site $(\mathrm{Spec}\, R)_{\text{ét}}$. We write $R\Gamma(R, -)$ for the derived functor of the global section functor $\Gamma(R, -) := \Gamma(\mathrm{Spec}\, R, -)$ on $(\mathrm{Spec}\, R)_{\text{ét}}$. We write $\mathbb{G}_m$ for the multiplicative sheaf on $(\mathrm{Spec}\, R)_{\text{ét}}$.

Lemma 7.2.1. *i) There are canonical isomorphisms of G-modules*

$$H^i(R\Gamma(\mathcal{O}_{L,S}, \mathbb{G}_m)) \cong \begin{cases} \mathcal{O}_{L,S}^\times & i = 0 \\ \mathrm{Pic}(\mathcal{O}_{L,S}) & i = 1 \\ \ker\left(\mathrm{Br}(L) \to \bigoplus_{w \notin S} \mathrm{Br}(L_w)\right) & i = 2 \\ \bigoplus_{w \in S_\infty} H^i(L_w, \mathbb{G}_m) & i \geq 3. \end{cases}$$

ii) $R\Gamma(\mathcal{O}_{L,S}, \mathbb{G}_m)$ can be represented by a complex of cohomologically trivial G-modules with $R\Gamma(\mathcal{O}_{L,S}, \mathbb{G}_m)^i = 0$ for all $i < 0$.

iii) Let J be a normal subgroup of G and $E := L^J$. We have a canonical isomorphism $R\,\mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}[G/J], R\Gamma(\mathcal{O}_{L,S}, \mathbb{G}_m)) \cong R\Gamma(\mathcal{O}_{E,S(E)}, \mathbb{G}_m)$.

Proof Claim i) follows directly from [31, Ch. II, Prop. 2.1, Rem. 2.2]. Claims ii) and iii) follow from a standard computation of étale cohomology (see, for example, [14, Prop. 2.1]).□

Unless stated otherwise we henceforth assume that the following hypothesis holds.

Hypothesis 7.2.2. *K is totally complex.*

Thus for each $w \in S_\infty$ one has $H^i(L_w, \mathbb{G}_m) = 0$ for all $i \neq 0$. Using the well known exact sequence of G-modules

$$0 \longrightarrow \mathrm{Br}(L) \longrightarrow \bigoplus_{w \in \mathcal{M}_L} \mathrm{Br}(L_w) \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0 \tag{7.7}$$

(cf. [15, Ch. VII, §11]) and the invariant isomorphism $Br(L_w) \cong \mathbb{Q}/\mathbb{Z}$ for each $w \in S_f$, we have a canonical identification

$$H^2(R\Gamma(\mathcal{O}_{L,S}, \mathbb{G}_m)) \cong \mathbb{Q}/\mathbb{Z} \otimes X_{L,S_f}.$$

Furthermore, as X_{L,S_f} is torsion free (7.5) induces an exact sequence of G-modules

$$0 \longrightarrow X_{L,S_f} \longrightarrow \mathbb{Q} \otimes X_{L,S_f} \xrightarrow{\pi_{L,S}} \mathbb{Q}/\mathbb{Z} \otimes X_{L,S_f} \longrightarrow 0. \tag{7.8}$$

Proposition 7.2.3. *There exists a unique morphism $\mathbb{Q} \otimes X_{L,S_f}[-2] \to R\Gamma(\mathcal{O}_{L,S}, \mathbb{G}_m)$ in $\mathcal{D}(\mathbb{Z}G)$ inducing the map $\pi_{L,S}$ on cohomology in degree two. Furthermore, this morphism fits into an exact triangle*

$$\mathbb{Q} \otimes X_{L,S_f}[-2] \longrightarrow R\Gamma(\mathcal{O}_{L,S}, \mathbb{G}_m) \longrightarrow C_{L,S} \longrightarrow \tag{7.9}$$

where $C_{L,S}$ is a complex of cohomologically trivial G-modules satisfying the following properties:-

i) $C^i_{L,S} = 0$ for all $i < 0$;

ii) $H^i(C_{L,S}) = 0$ for $i \neq 0, 1$;

iii) There exists a canonical isomorphism of G-modules $H^0(C_{L,S}) \cong \mathcal{O}^\times_{L,S}$;

iv) There exists a canonical exact sequence of G-modules

$$0 \longrightarrow \mathrm{Pic}(\mathcal{O}_{L,S}) \longrightarrow H^1(C_{L,S}) \longrightarrow X_{L,S_f} \longrightarrow 0\,;$$

v) Let J be a subgroup of G and set $E := L^J$. Applying $Res^G_J(-)$ to triangle (7.9) gives *a triangle which is canonically isomorphic to the corresponding triangle in* $\mathcal{D}(\mathbb{Z}J)$ *for* L/E *and* S*.*

vi) Let J *be a normal subgroup of* G *and set* $E := L^J$*. Applying* $R\operatorname{Hom}_{\mathbb{Z}G}(\mathbb{Z}[G/J], -)$ *to triangle* (7.9) *gives a triangle which is canonically isomorphic to the corresponding triangle in* $\mathcal{D}(\mathbb{Z}[G/J])$ *for* E/K *and* $S(E)$*.*

Proof Using [8, Lemma 7b)] and Lemma 7.2.1 we have an exact sequence of groups

$$0 \longrightarrow \operatorname{Ext}^1_{\mathbb{Z}G}(\mathbb{Q} \otimes X_{L,S_f}, \operatorname{Pic}(\mathcal{O}_{L,S})) \longrightarrow \operatorname{Hom}_{\mathcal{D}(\mathbb{Z}G)}(\mathbb{Q} \otimes X_{L,S_f}[-2], R\Gamma(\mathcal{O}_{L,S}, \mathbb{G}_m)) \longrightarrow \operatorname{Hom}_{\mathbb{Z}G}(\mathbb{Q} \otimes X_{L,S_f}, \mathbb{Q}/\mathbb{Z} \otimes X_{L,S_f}) \longrightarrow 0. \tag{7.10}$$

Since $\operatorname{Pic}(\mathcal{O}_{L,S})$ is finite and $\mathbb{Q} \otimes X_{L,S_f}$ is uniquely divisible the first Ext-group in (7.10) is zero. Thus the map $\pi_{L,S}$ lifts to a unique morphism in $\mathcal{D}(\mathbb{Z}G)$.

Since $\mathbb{Q} \otimes X_{L,S_f}$ is uniquely divisible it is a cohomologically trivial G-module. By Lemma 2.3.1 we can chose a complex of projective G-modules $P := [P^1 \longrightarrow P^2]$, with the first term placed in degree one, which is quasi-isomorphic to $\mathbb{Q} \otimes X_{L,S_f}[-2]$. Using Lemma 7.2.1 we can fix a choice of representative of $R\Gamma(\mathcal{O}_{L,S}, \mathbb{G}_m)$ in $\mathcal{D}(\mathbb{Z}G)$ such that $R\Gamma(\mathcal{O}_{L,S}, \mathbb{G}_m)^i$ is a cohomologically trivial G-module for all i and furthermore is zero for $i < 0$. Using the projectivity of the P^i one can construct a morphism of complexes $P \xrightarrow{\widetilde{\pi}_{L,S}} R\Gamma(\mathcal{O}_{L,S}, \mathbb{G}_m)$ such that the map induced on the cohomology in degree two is $\pi_{L,S}$. We set $C_{L,S} = \operatorname{Cone}(\widetilde{\pi}_{L,S})$ and thus have a tautological exact triangle

$$P \xrightarrow{\widetilde{\pi}_{L,S}} R\Gamma(\mathcal{O}_{L,S}, \mathbb{G}_m) \longrightarrow C_{L,S} \longrightarrow \cdot \tag{7.11}$$

Combining this with the fact P is isomorphic to $\mathbb{Q} \otimes X_{L,S_f}[-2]$ in $\mathcal{D}(\mathbb{Z}G)$ gives (7.9).

Claim i) follows from the definition of the mapping cone and (7.11). Claims ii), iii) and iv) are immediate from the long exact sequence of cohomology associated to (7.9). Claim v) is obvious. Finally claim vi) follows from Lemma 7.2.1 and the fact that there is a canonical isomorphism $(\mathbb{Q} \otimes X_{L,S_f})^J \cong \mathbb{Q} \otimes X_{E,S_f(E)}$.□

Definition 7.2.4. *We write* $\epsilon_{\tau_0,S}$ *for the class in* $\operatorname{Ext}^2_{\mathbb{Z}G}(H^1(C_{L,S}), \mathcal{O}^\times_{L,S})$ *defined by* *the complex $C_{L,S}$ of Proposition 7.2.3. If S is clear from context we simply write ϵ_{τ_0}.*

Remark 7.2.5. Under the assumption of Hypothesis 7.2.2 the extension class ϵ_{τ_0} is indeed the same extension class as that in Proposition 6.1.1. We shall not prove this here but we will give an indication to the reader as to how this can be checked. The existence of ϵ_{τ_0} in Proposition 6.1.1 is given by [8, Prop. 3.1] under the hypothesis that $\mathrm{Pic}(\mathcal{O}_{L,S})$ is trivial. The arguments used in the proof of [8, Prop. 3.1] can be used to construct ϵ_{τ_0} in the case $\mathrm{Pic}(\mathcal{O}_{L,S})$ is non-trivial. To show that the extension class just constructed is the same as that given by (a modified version of) [8, Prop. 3.1], one must compare the complex $R\Gamma(\mathcal{O}_{L,S}, \mathbb{G}_m)$ to the complex $\widetilde{\Psi}_S$ of loc. cit.. The key to making this comparison is noting that, under hypothesis 7.2.2, the complexes $\widetilde{R\Gamma}_c(\mathcal{O}_{L,S}, \mathbb{Z})$ and $R\Gamma_c(\mathcal{O}_{L,S}, \mathbb{Z})$ of loc. cit. are canonically isomorphic.

Corollary 7.2.6. *There exists an exact sequence of G-modules*

$$0 \longrightarrow \mathcal{O}_{L,S}^{\times} \longrightarrow E_{L,S}^{0} \longrightarrow E_{L,S}^{1} \longrightarrow H^1(C_{L,S}) \longrightarrow 0 \tag{7.12}$$

representing $\epsilon_{\tau_0,S}$, with $E_{L,S}^1$ and $E_{L,S}^2$ both finitely generated and cohomologically trivial as G-modules. Furthermore, the sequence has the following functorial properties:-

i) Let J be a subgroup of G and $E := L^J$. Applying $\mathrm{Res}_J^G(-)$ to (7.12) gives an exact sequence of J-modules with the corresponding properties for L/E and S.

ii) Let J be a normal subgroup of G and $E := L^J$. Applying $H^0(J, -)$ to the complex $[E_{L,S}^0 \to E_{L,S}^1]$ gives an exact sequence of G/J-modules with the corresponding properties for E/K and $S(E)$.

Proof We write d^1 for $d^1_{C_{L,S}}$. We can truncate the complex $C_{L,S}$ of Proposition 7.2.3 to get the complex $[C_{L,S} \to \ker d^1]$, which gives a tautological exact sequence of G-modules

$$0 \longrightarrow \mathcal{O}_{L,S}^{\times} \longrightarrow C_{L,S}^{0} \longrightarrow \ker d^1 \longrightarrow H^1(C_{L,S}) \longrightarrow 0. \tag{7.13}$$

We now show that $\ker d^1$ is a cohomologically trivial G-module.

Since $C_{L,S}$ is acyclic outside degrees zero and one, we have a resolution of $\ker d^1$ by cohomologically trivial G-modules

$$0 \longrightarrow \ker d^1 \longrightarrow C^1_{L,S} \longrightarrow C^2_{L,S} \longrightarrow \cdots . \tag{7.14}$$

For every subgroup J of G the cohomology groups $H^i(J, \ker d^1)$ can be computed by applying $(-)^J$ to (7.14) (cf. [34, Ch. I, §3, Prop. (1.3.9)]). Proposition 7.2.3 vi) shows that

$$0 \longrightarrow (\ker d^1)^J \longrightarrow (C^1_{L,S})^J \longrightarrow (C^2_{L,S})^J \longrightarrow \cdots$$

is exact and so $\ker d^1$ is a cohomologically trivial G-module. Since the G-modules $\mathcal{O}^{\times}_{L,S}$ and $H^1(C_{L,S})$ are finitely generated Lemma 7.1.2 shows that the existence of (7.13) implies the existence of (7.12).

Claims i) and ii) follow from Proposition 7.2.3 claims v) and vi) respectively. □

We now consider functoriality in S. Let S' be any finite G-stable set of places containing S and $j : \operatorname{Spec} \mathcal{O}_{L,S'} \to \operatorname{Spec} \mathcal{O}_{L,S}$ the open immersion induced by the canonical inclusion of rings. Let $Z := \operatorname{Spec}\left(\prod_{w\in S'\setminus S}(\mathcal{O}_L/\mathfrak{p}_w)\right)$, where $\mathfrak{p}_w$ is the prime ideal of $\mathcal{O}_L$ corresponding to w, and write $i : Z \to \operatorname{Spec} \mathcal{O}_{L,S}$ for the closed immersion induced by the canonical surjection of rings.

If $\mathbb{G}_m$ is the multiplicative sheaf on $(\operatorname{Spec} \mathcal{O}_{L,S})_{\text{ét}}$, then the restriction $j^*\mathbb{G}_m$ is equal to the multiplicative sheaf on $(\operatorname{Spec} \mathcal{O}_{L,S'})_{\text{ét}}$. We thus have an induced morphism of complexes $R\Gamma(\mathcal{O}_{L,S}, \mathbb{G}_m) \to R\Gamma(\mathcal{O}_{L,S'}, \mathbb{G}_m)$. If we set

$$R\Gamma_Z(\mathcal{O}_{L,S}, \mathbb{G}_m) := \operatorname{Cone}(R\Gamma(\mathcal{O}_{L,S}, \mathbb{G}_m) \to R\Gamma(\mathcal{O}_{L,S'}, \mathbb{G}_m))[-1],$$

then we have a tautological exact triangle

$$R\Gamma_Z(\mathcal{O}_{L,S}, \mathbb{G}_m) \longrightarrow R\Gamma(\mathcal{O}_{L,S}, \mathbb{G}_m) \longrightarrow R\Gamma(\mathcal{O}_{L,S'}, \mathbb{G}_m) \longrightarrow \cdot \tag{7.15}$$

We write $H^i_Z(\mathcal{O}_{L,S}, \mathbb{G}_m)$ for the cohomology groups of $\mathbb{G}_m$ with support in Z (see [32, Ch. III, §1, p91]) and by definition we have $H^i(R\Gamma_Z(\mathcal{O}_{L,S}, \mathbb{G}_m)) = H^i_Z(\mathcal{O}_{L,S}, \mathbb{G}_m)$.

Furthermore, it can be shown that

$$H^i_Z(\mathcal{O}_{L,S}, \mathbb{G}_m) \cong \prod_{w \in S' \setminus S} H^i_{z_w}(\mathcal{O}_{L,S}, \mathbb{G}_m),$$

where $z_w := \operatorname{Spec} \mathcal{O}_L/\mathfrak{p}_w$. By [31, Prop. 1.5] we have canonical isomorphisms

$$H^i_{z_w}(\mathcal{O}_{L,S}, \mathbb{G}_m) \cong \begin{cases} \mathbb{Z} & i = 1 \\ \mathbb{Q}/\mathbb{Z} & i = 3 \\ 0 & \textit{otherwise.} \end{cases}$$

We thus have canonical isomorphisms

$$H^i_Z(\mathcal{O}_{L,S}, \mathbb{G}_m) \cong \begin{cases} \prod_{w \in S' \setminus S} \mathbb{Z} & i = 1 \\ \prod_{w \in S' \setminus S} \mathbb{Q}/\mathbb{Z} & i = 3 \\ 0 & \textit{otherwise.} \end{cases}$$

Lemma 7.2.7. *(Functoriality in S) There exists a commutative diagram of G-modules with exact rows*

$$\begin{array}{ccccccccc} 0 \longrightarrow & \mathcal{O}^\times_{L,S} & \longrightarrow & F^0_{L,S} & \longrightarrow & F^1_{L,S} & \longrightarrow & H^1(C_{L,S}) & \longrightarrow 0 \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ 0 \longrightarrow & \mathcal{O}^\times_{L,S'} & \longrightarrow & E^0_{L,S'} & \longrightarrow & E^1_{L,S'} & \longrightarrow & H^1(C_{L,S'}) & \longrightarrow 0 \end{array} \tag{7.16}$$

where the first vertical map is the natural inclusion and the top row, resp. bottom row, represents $\epsilon_{\tau_0,S}$, resp. $\epsilon_{\tau_0,S'}$. Moreover, the right hand column fits into an exact sequence of G-modules

$$0 \longrightarrow K_{S,S'} \longrightarrow H^1(C_{L,S}) \longrightarrow H^1(C_{L,S'}) \longrightarrow Y_{L,S' \setminus S} \longrightarrow 0\ ,$$

where $K_{S,S'} := \ker(\operatorname{Pic}(\mathcal{O}_{L,S}) \to \operatorname{Pic}(\mathcal{O}_{L,S'}))$ and $Y_{L,S'\setminus S}$ is the free abelian group on $S' \setminus S$ with the obvious action of G.

Proof We can construct a commutative diagram in $\mathcal{D}(\mathbb{Z}G)$

$$\begin{array}{ccccccc}
\mathbb{Q}\otimes X_{L,S_f}[-2] & \longrightarrow & R\Gamma(\mathcal{O}_{L,S},\mathbb{G}_m) & \longrightarrow & C_{L,S} & \longrightarrow & \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathbb{Q}\otimes X_{L,S'_f}[-2] & \longrightarrow & R\Gamma(\mathcal{O}_{L,S'},\mathbb{G}_m) & \longrightarrow & C_{L,S'} & \longrightarrow & \\
\downarrow & & \downarrow & & & & \\
\mathbb{Q}\otimes Y_{L,S'\setminus S}[-2] & \longrightarrow & R\Gamma_Z(\mathcal{O}_{L,S},\mathbb{G}_m)[1] & & & & \\
\downarrow & & \downarrow & & & &
\end{array} \tag{7.17}$$

in the following way. The top horizontal triangle, resp. middle horizontal triangle, is (7.9) for S, resp. S'. The first vertical triangle is the exact triangle corresponding to the canonical exact sequence

$$0 \longrightarrow \mathbb{Q}\otimes X_{L,S_f} \longrightarrow \mathbb{Q}\otimes X_{L,S'_f} \longrightarrow \mathbb{Q}\otimes Y_{L,S'\setminus S} \longrightarrow 0.$$

The middle vertical triangle is (7.15). The top left square commutes for the following reason. The two composite morphisms from $\mathbb{Q}\otimes X_{L,S_f}[-2]$ to $R\Gamma(\mathcal{O}_{L,S'},\mathbb{G}_m)$ obtained by going either direction around the square both induce the natural map $\mathbb{Q}\otimes X_{L,S_f} \to \mathbb{Q}/\mathbb{Z}\otimes X_{L,S'_f}$ on cohomology. A similar argument to that used in the first part of the proof of Proposition 7.2.3 shows that such a morphism must be unique in $\mathcal{D}(\mathbb{Z}G)$. The rest of the diagram is then obtained by completing the top two rows to a morphism of triangles and the first two columns to a morphism of triangles.

Lemmata 7.1.1 and 7.1.2 combine with the morphism $C_{L,S} \to C_{L,S'}$ to give (7.16). The rest of Lemma 7.2.7 follows by examining the cohomology of (7.17). $\square$

7.3 The Ritter-Weiss Construction

In this section S is any finite G-stable set of places of L which contains S_∞. As in [38] we fix a finite G-stable set of places S' of L containing S such that $\mathrm{Pic}(\mathcal{O}_{L,S'}) = 0$ and $\bigcup_{w\in S'} G_w = G$ (such a set is referred to as 'large' in loc. cit.).

Let

$$W_{S'} := \prod_{S(K)}^{\mathrm{Ind}} I_{G_{w(v)}} \times \prod_{S'(K)\setminus S(K)}^{\mathrm{Ind}} \mathbb{Z}G_{w(v)}.$$

In loc. cit. the main result relies upon the construction of an exact commutative diagram of G-modules

$$\begin{array}{ccccccccc} 0 & \longrightarrow & J_S(L) & \longrightarrow & V_{S'} & \longrightarrow & W_{S'} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow \theta^0 & & \downarrow \theta^1 & & \\ 0 & \longrightarrow & C(L) & \longrightarrow & \mathcal{B} & \longrightarrow & I_G & \longrightarrow & 0 \end{array} \tag{7.18}$$

where the left-hand vertical map is the natural one, θ^0 is surjective (thus so is θ^1) and both $V_{S'}$ and $\mathcal{B}$ are cohomologically trivial ([38, Th. 1]). Moreover, the extension class of the bottom row of (7.18) corresponds to $c_{L/K}$ under the isomorphism

$$\widehat{H}^2(G, C(L)) \cong \mathrm{Ext}^2_{\mathbb{Z}G}(\mathbb{Z}, C(L)) \cong \mathrm{Ext}^1_{\mathbb{Z}G}(I_G, C(L)),$$

where the first isomorphism follows by definition of the Tate cohomology groups and the second is the dimension shift induced by (7.6). In §7.6 we will show that if S contains S_{ram}, then there is a similar dimension shift $\mathrm{Ext}^2_{\mathbb{Z}G}(Y_{L,S}, J_S(L)) \cong \mathrm{Ext}^1_{\mathbb{Z}G}(W_{S'}, J_S(L))$ under which the extension class of the top row of (7.18) corresponds to the class of $c^{loc}_{L/K,S}$.

Remarks 7.3.1. i) In [38] the authors have a more general definition of $W_{S'}$. Since we are assuming that $S_{ram} \subset S$, [loc. cit., Lemma 5] shows their definition coincides with ours.

ii) Diagram (7.18) is clearly dependent on both the choice of θ^0 and of the places $w(v)$ (indeed this is an issue which the authors of [38] consider in detail). For our purposes it will be sufficient to fix these variables in order to identify the Tate Sequence constructed in loc. cit.. For details on how different choices affect the construction of [38] see [loc. cit., Th. 2, Th. 4 and Th. 6].

If $w \notin S_\infty \cup S_{ram}$, then we write σ_w for the Frobenius element of w in G. We will need the explicit description of the map θ^1 in (7.18). It is given on each factor of the product $W_{S'}$ using the 'adjointness of Res and Ind' as follows. For $v \in S(K)$ the map is the G-homomorphism that is left adjoint to the natural inclusion $I_{G_{w(v)}} \hookrightarrow \operatorname{Res}^G_{G_{w(v)}} I_G$ of $G_{w(v)}$-modules. For $v \in S'(K) \setminus S(K)$ the map is the G-homomorphism that is left adjoint to the $G_{w(v)}$-homomorphism $\mathbb{Z}G_{w(v)} \to \operatorname{Res}^G_{G_{w(v)}} I_G$ given by $x \mapsto (\sigma_{w(v)} - 1)x$ (this follows from [38, §4]).

Let $K_0 := \ker(\theta^0)$ and $K_1 := \ker(\theta^1)$. In order to construct the Tate Sequence of [38] the authors apply the Snake Lemma to (7.18). They obtain an exact sequence of G-modules

$$0 \longrightarrow \mathcal{O}^\times_{L,S} \longrightarrow K_0 \longrightarrow K_1 \longrightarrow \operatorname{Pic}(\mathcal{O}_{L,S}) \longrightarrow 0 \tag{7.19}$$

and furthermore show that K_0 and K_1 are cohomologically trivial. They then proceed in the following manner.

For each $v \in S(K)$ we can use sequence (7.6) for both G and $G_{w(v)}$ along with the 'adjointness of Res and Ind' to construct two exact commutative diagrams of G-modules

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{Ind}^G_{G_{w(v)}} I_{G_{w(v)}} & \longrightarrow & \operatorname{Ind}^G_{G_{w(v)}} \mathbb{Z}G_{w(v)} & \longrightarrow & \operatorname{Ind}^G_{G_{w(v)}} \mathbb{Z} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & I_G & \longrightarrow & \mathbb{Z}G & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \end{array} \tag{7.20}$$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{Ind}^G_{G_{w(v)}} \mathbb{Z}G_{w(v)} & = & \operatorname{Ind}^G_{G_{w(v)}} \mathbb{Z}G_{w(v)} & \longrightarrow & 0 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & I_G & \longrightarrow & \mathbb{Z}G & \longrightarrow & \mathbb{Z} & \longrightarrow & 0. \end{array} \tag{7.21}$$

The vertical maps in (7.20) are the G-homomorphisms left adjoint to the obvious inclusions of $G_{w(v)}$-modules. The first vertical map in (7.21) is the G-homomorphism left adjoint to the $G_{w(v)}$-homomorphism $\mathbb{Z}G_{w(v)} \to \operatorname{Res}^G_{G_{w(v)}} I_G$ given by $x \mapsto (\sigma_{w(v)} - 1)x$. The middle vertical map in (7.21) is the unique map making the diagram commute. Each of the G-modules $\operatorname{Ind}^G_{G_{w(v)}} \mathbb{Z}G_{w(v)}$, $\mathbb{Z}G$, $\operatorname{Ind}^G_{G_{w(v)}} \mathbb{Q}$ and $\mathbb{Q}$ is cohomologically trivial and so we can use (7.20) and (7.21) to dimension shift Ext-groups.

Taking the product of (7.20) for each $v \in S(K)$, with the product of (7.21) for each $v \in S'(K) \setminus S(K)$ gives an exact commutative diagram of G-modules

$$\begin{CD}
0 @>>> W_{S'} @>>> \prod\limits_{S'(K)}^{\mathrm{Ind}} \mathbb{Z}G_{w(v)} @>>> Y_{L,S} @>>> 0 \\
@. @V{\theta^1}VV @V{\widehat{\theta}^1}VV @VVV @. \\
0 @>>> I_G @>>> \mathbb{Z}G @>>> \mathbb{Z} @>>> 0
\end{CD} \tag{7.22}$$

where we have fixed an identification of $Y_{L,S}$ with $\prod\limits_{S(K)}^{\mathrm{Ind}} \mathbb{Z}$ using our choice of place $w(v)$, for each $v \in S(K)$. Let $\widehat{K}_1 := \ker(\widehat{\theta}^1)$. Since θ^1 is surjective and the right hand vertical map in (7.22) is surjective, then so is $\widehat{\theta}^1$. Applying the Snake Lemma to (7.22) gives an exact sequence of G-modules

$$0 \longrightarrow K_1 \longrightarrow \widehat{K}_1 \longrightarrow X_{L,S} \longrightarrow 0. \tag{7.23}$$

Let R be the kernel of the map $K_1 \to \mathrm{Pic}(\mathcal{O}_{L,S})$ in (7.19). Using the Snake Lemma we now have an exact commutative diagram of G-modules

$$\begin{CD}
@. 0 @. 0 @. @. \\
@. @VVV @VVV @. @. \\
@. R @= R @. @. \\
@. @VVV @VVV @. @. \\
0 @>>> K_1 @>>> \widehat{K}_1 @>>> X_{L,S} @>>> 0 \\
@. @VVV @VVV @| @. \\
0 @>>> \mathrm{Pic}(\mathcal{O}_{L,S}) @>>> \widehat{K}_1/R @>>> X_{L,S} @>>> 0 \\
@. @VVV @VVV @. @. \\
@. 0 @. 0, @. @.
\end{CD} \tag{7.24}$$

where the map $R \to \widehat{K}_1$ is the unique map making the top square commute. It is straightforward to check that $\widehat{K}_1/R$ is the pushout of the maps $K_1 \to \widehat{K}_1$ and

$K_1 \to \mathrm{Pic}(\mathcal{O}_{L,S})$. By definition, $\widehat{K}_1/R$ is the module ∇ of [38]. The definition of R combines with (7.19) to give a tautological exact sequence

$$0 \longrightarrow \mathcal{O}_{L,S}^{\times} \longrightarrow K_1 \longrightarrow R \longrightarrow 0.$$

Splicing this sequence with the middle column of (7.24) gives an exact sequence

$$0 \longrightarrow \mathcal{O}_{L,S}^{\times} \longrightarrow K_0 \longrightarrow \widehat{K}_1 \longrightarrow \nabla \longrightarrow 0, \tag{7.25}$$

which is the Tate Sequence of [38]. We define $\widehat{K}$ to be the complex $[K_0 \to \widehat{K}_1]$ extracted from (7.25) with the first term placed in degree zero. This complex will be used in the next section.

7.4 A Reinterpretation of the Ritter-Weiss Construction

In this section the sets S and S' remain as they were in §7.3. We now interpret the construction of Ritter and Weiss in terms of morphisms of complexes. We will show in Lemma 7.4.1 that the complex $\widehat{K}$ (which defines the Tate Sequence of [38]) is equal to the kernel of a morphism $\widehat{\theta}$ in $\mathrm{Kom}(\mathbb{Z}G)$. This will give a concise construction of the Tate Sequence of loc. cit.. Now the Tate Sequence ϵ_{τ_0} constructed in §7.2 is an extension of $\mathcal{O}_{L,S}^{\times}$ by $H^1(C_{L,S})$, and this latter module fits into an exact sequence of G-modules of the form

$$0 \longrightarrow \mathrm{Pic}(\mathcal{O}_{L,S}) \longrightarrow H^1(C_{L,S}) \longrightarrow X_{L,S_f} \longrightarrow 0.$$

However, the Tate Sequence defined by $\widehat{K}$ is an extension of $\mathcal{O}_{L,S}^{\times}$ by ∇, and this latter module fits into an exact sequence of the form

$$0 \longrightarrow \mathrm{Pic}(\mathcal{O}_{L,S}) \longrightarrow \nabla \longrightarrow X_{L,S} \longrightarrow 0\,.$$

We address this discrepancy in Lemma 7.4.2. Finally, we perform a dimension shift to obtain a complex $\ker \overline{\theta}_f$ from $\widehat{K}$, which we will compare to $R\Gamma(\mathcal{O}_{L,S}, \mathbb{G}_m)$ in the next section.

The morphism $\widehat{\theta}$

Using (7.18) we define the complexes A, resp. B, to be $[V_{S'} \to W_{S'}]$, resp. $[\mathcal{B} \to I_G]$, with the first terms placed in degree zero. The maps θ^0 and θ^1 of (7.18) induce a surjective morphism of complexes $A \to B$ in Kom$(\mathbb{Z}G)$, which we denote by θ.

Lemma 7.4.1. *There exists an exact commutative diagram in Kom*$(\mathbb{Z}G)$

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \ker\theta & \longrightarrow & \widehat{K} & \longrightarrow & X_{L,S}[-1] & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \xrightarrow{\delta_A} & \widehat{A} & \longrightarrow & Y_{L,S}[-1] & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\theta} & & \downarrow{\scriptstyle\widehat{\theta}} & & \downarrow & & \\
0 & \longrightarrow & B & \xrightarrow{\delta_B} & \widehat{B} & \longrightarrow & \mathbb{Z}[-1] & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0. & &
\end{array} \tag{7.26}$$

Proof Splicing diagram (7.22) with diagram (7.18) gives an exact commutative diagram of G-modules

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & J_S(L) & \longrightarrow & V_{S'} & \longrightarrow & \prod_{S'(K)}^{\mathrm{Ind}} \mathbb{Z}G_{w(v)} & \longrightarrow & Y_{L,S} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle\theta^0} & & \downarrow{\scriptstyle\widehat{\theta}^1} & & \downarrow & & \\
0 & \longrightarrow & C(L) & \longrightarrow & \mathcal{B} & \longrightarrow & \mathbb{Z}G & \longrightarrow & \mathbb{Z} & \longrightarrow & 0.
\end{array} \tag{7.27}$$

We let $\widehat{A}$, resp. $\widehat{B}$, denote the complex $[V_{S'} \to \prod_{S'(K)}^{\mathrm{Ind}} \mathbb{Z}G_{w(v)}]$, resp. $[\mathcal{B} \to \mathbb{Z}G]$, extracted from (7.27), with the first term placed in degree zero. We write $\widehat{\theta} : \widehat{A} \to \widehat{B}$ for the surjective morphism of complexes in Kom$(\mathbb{Z}G)$ induced by θ^0 and $\widehat{\theta}^1$. We denote by $\delta_A : A \to \widehat{A}$ and $\delta_B : B \to \widehat{B}$ the obvious morphisms of complexes. It is easily checked that in Kom($\mathbb{Z}G$) both of these morphisms are injective, $\mathrm{coker}(\delta_A) = Y_{L,S}[-1]$ and $\mathrm{coker}(\delta_B) = \mathbb{Z}[-1]$. This gives the bottom two rows of (7.26). It is straightforward to show that $\widehat{K} = \ker \widehat{\theta}$ and so the rest of (7.26) follows immediately from the Snake Lemma.□

The discrepancy between ∇ and $H^1(C_{L,S})$

Lemma 7.4.1 shows that the extension class defined by the complex $\ker \widehat{\theta}$ is the class of the Tate Sequence constructed in [38]. We would like to compare this complex to $C_{L,S}$ from Proposition 7.2.3 but unfortunately the cohomology groups of these complexes are not the same. The difference is that $\overline{H^1(\ker \widehat{\theta})} = X_{L,S}$ whilst $\overline{H^1(C_{L,S})} = X_{L,S_f}$. The difference between these two modules is summarized by the canonical exact sequence of G-modules

$$0 \longrightarrow X_{L,S_f} \longrightarrow X_{L,S} \longrightarrow Y_{L,S_\infty} \longrightarrow 0$$

where, under Hypothesis 7.2.2, Y_{L,S_∞} is free. To deal with this discrepancy we amend the above construction as follows.

We first note that under Hypothesis 7.2.2 the module $\mathrm{Ind}_{G_{w(v)}}^{G} I_{G_{w(v)}}$ is zero for each $v \in S_\infty(K)$. The top row of (7.18) is thus

$$0 \longrightarrow J_S(L) \longrightarrow V_{S'} \xrightarrow{\alpha} \prod_{S_f(K)}^{\mathrm{Ind}} I_{G_{w(v)}} \times \prod_{S'(K)\backslash S(K)}^{\mathrm{Ind}} \mathbb{Z}G_{w(v)} \longrightarrow 0, \qquad (7.28)$$

and for each $v \in S_\infty(K)$ the top row of (7.20) is

$$0 \longrightarrow 0 \longrightarrow \mathrm{Ind}_{G_{w(v)}}^{G} \mathbb{Z} = \mathrm{Ind}_{G_{w(v)}}^{G} \mathbb{Z} \longrightarrow 0.$$

Now recall that we constructed the top row of (7.27) from (7.18) by splicing the exact sequence (7.28) with the top row of (7.22). Hypothesis 7.2.2 now shows (after the preceding discussion) that the second map in the top row of (7.22) can be written as

$$W_{S'} = \prod_{S_f(K)}^{\mathrm{Ind}} I_{G_{w(v)}} \times \prod_{S'(K)\setminus S(K)}^{\mathrm{Ind}} \mathbb{Z}G_{w(v)} \xrightarrow{\widetilde{\beta}} \prod_{S_f(K)}^{\mathrm{Ind}} \mathbb{Z}G_{w(v)} \times \prod_{S'(K)\setminus S(K)}^{\mathrm{Ind}} \mathbb{Z}G_{w(v)} \times \prod_{S_\infty(K)}^{\mathrm{Ind}} \mathbb{Z}\,,$$

where $\widetilde{\beta}$ is of the form $\begin{pmatrix} \beta & 0 \\ 0 & id \\ 0 & 0 \end{pmatrix}$.

We define $\widehat{A}_f$ to be the complex $[V_{S'} \to \prod_{S'_f(K)}^{\mathrm{Ind}} \mathbb{Z}G_{w(v)}]$, with the first term placed in degree zero, and whose differential is $\begin{pmatrix} \beta & 0 \\ 0 & \mathrm{id} \end{pmatrix} \circ \alpha$. Recall that $\widehat{A}$ is the complex $[V_{S'} \to \prod_{S'(K)}^{\mathrm{Ind}} \mathbb{Z}G_{w(v)}]$. We thus have a morphism of complexes $\widehat{A} \to \widehat{A}_f$, which is the identity in degree zero and the natural projection in degree one, and a commutative diagram in $\mathrm{Kom}(\mathbb{Z}G)$

$$\begin{array}{ccc} \widehat{A} & \longrightarrow & \widehat{A}_f \\ \downarrow{\scriptstyle \widehat{\theta}} & & \downarrow{\scriptstyle \widehat{\theta}_f} \\ \widehat{B} & = & \widehat{B}. \end{array}$$

If we set $\widehat{K}_{1,f} := \ker\left(\prod_{S'_f(K)}^{\mathrm{Ind}} \mathbb{Z}G_{w(v)} \to \mathbb{Z}G\right)$, then $\ker \widehat{\theta}_f$ is the complex $[K_1 \to \widehat{K}_{1,f}]$, with the first term placed in degree zero.

As in the proof of Lemma 7.4.1 we can now construct the following exact commutative diagram in $\mathrm{Kom}(\mathbb{Z}G)$

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \ker\theta & \longrightarrow & \ker\widehat{\theta}_f & \longrightarrow & X_{L,S_f}[-1] & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & \widehat{A}_f & \longrightarrow & Y_{L,S_f}[-1] & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\theta} & & \downarrow{\scriptstyle\widehat{\theta}_f} & & \downarrow & & \\
0 & \longrightarrow & B & \longrightarrow & \widehat{B} & \longrightarrow & \mathbb{Z}[-1] & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0. & &
\end{array} \tag{7.29}$$

Lemma 7.4.2. *The complex $\ker\widehat{\theta}_f$ defines an element of $\mathrm{Ext}^2_{\mathbb{Z}G}(\nabla_f, \mathcal{O}^\times_{L,S})$, where $\nabla_f := H^1(\ker\widehat{\theta}_f)$. Moreover, there is an isomorphism of G-modules $\nabla \cong \nabla_f \oplus Y_{L,S_\infty}$ and a commutative diagram of short exact sequences of G-modules*

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \mathcal{O}^\times_{L,S} & \longrightarrow & K_1 & \longrightarrow & \widehat{K}_{1,f} & \longrightarrow & \nabla_f & \longrightarrow & 0 \\
 & & \| & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}^\times_{L,S} & \longrightarrow & K_1 & \longrightarrow & \widehat{K}_1 & \longrightarrow & \nabla & \longrightarrow & 0.
\end{array} \tag{7.30}$$

The extension class of the top row of (7.30) is equal to the image of the extension class of the bottom row under the induced isomorphism $\mathrm{Ext}^2_{\mathbb{Z}G}(\nabla_f, \mathcal{O}^\times_{L,S}) \cong \mathrm{Ext}^2_{\mathbb{Z}G}(\nabla, \mathcal{O}^\times_{L,S})$.

Proof We sketch the proof and leave the details for the reader. By comparing the middle column of (7.26) with the middle column of (7.29) we can construct an exact commutative diagram in Kom($\mathbb{Z}G$)

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \ker\theta & \longrightarrow & \ker\widehat{\theta}_f & \longrightarrow & X_{L,S_f}[-1] & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \ker\theta & \longrightarrow & \ker\widehat{\theta} & \longrightarrow & X_{L,S}[-1] & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & Y_{L,S_\infty}[-1] & = & Y_{L,S_\infty}[-1] & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0. & &
\end{array} \tag{7.31}$$

Since the top row of (7.30) is the extension defined by $\ker\widehat{\theta}_f$ and the bottom row of (7.30) is the extension defined by $\ker\widehat{\theta}$, the lemma follows from the existence of the middle column of (7.31).□

<u>A further dimension shift</u>

After Lemma 7.4.2 it is sufficient to focus our attention on the complex $\ker\widehat{\theta}_f$. Using the 'adjointness of Res and Ind', for each $v \in S_f(K)$ diagram (7.5) gives an exact commutative diagram of G-modules

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Ind}_{G_{w(v)}}^G \mathbb{Z} & \longrightarrow & \mathrm{Ind}_{G_{w(v)}}^G \mathbb{Q} & \longrightarrow & \mathrm{Ind}_{G_{w(v)}}^G \mathbb{Q}/\mathbb{Z} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Q} & \longrightarrow & \mathbb{Q}/\mathbb{Z} & \longrightarrow & 0.
\end{array} \tag{7.32}$$

Proceeding in the same way as in the proof of Lemma 7.4.1 we obtain an exact commutative diagram in $\mathrm{Kom}(\mathbb{Z}G)$

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{Q}\otimes X_{L,S_f}[-2] & \longrightarrow & \ker\overline{\theta}_f & \longrightarrow & \ker\widehat{\theta}_f & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{Q}\otimes Y_{L,S_f}[-2] & \longrightarrow & \overline{A}_f & \longrightarrow & \widehat{A}_f & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \overline{\theta}_f} & & \downarrow{\scriptstyle \widehat{\theta}_f} & & \\
0 & \longrightarrow & \mathbb{Q}[-2] & \longrightarrow & \overline{B} & \longrightarrow & \widehat{B} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{7.33}$$

where $\overline{A}_f$, resp. $\overline{B}$, is the complex $[V_{S'} \to \prod^{\mathrm{Ind}}_{S'_f(K)} \mathbb{Z}G_{w(v)} \to \mathbb{Q}\otimes Y_{L,S_f}]$, resp. $[\mathcal{B} \to \mathbb{Z}G \to \mathbb{Q}]$, with the first term placed in degree zero. The long exact sequence of cohomology corresponding to the middle column of (7.33) shows that $\ker\overline{\theta}_f$ has the same cohomology as $R\Gamma(\mathcal{O}_{L,S}, \mathbb{G}_m)$. In the next section we will show that these complexes are in fact isomorphic in $\mathcal{D}(\mathbb{Z}G)$.

7.5 The Connection to ϵ_{τ_0}

In this section S is any finite G-stable set of places of L which contains $S_\infty \cup S_{ram}$ and S' is as defined at the start of §7.3.

Lemma 7.5.1. *Let $w \in \mathcal{M}_L$. There are canonical isomorphisms of G_w-modules*

$$H^i(L_w, \mathbb{G}_m) \cong \begin{cases} L_w^\times & \text{if } i = 0 \\ 0 & \text{if } i = 1 \\ Br(L_w) & \text{if } i = 2. \end{cases}$$

If w is non-archimedean, then $H^i(L_w, \mathbb{G}_m) = 0$ for $i \geq 3$. Moreover the class of $R\Gamma(L_w, \mathbb{G}_m)$ in $\mathrm{Ext}^3_{\mathbb{Z}G_w}(\mathbb{Q}/\mathbb{Z}, L_w^\times) \cong H^2(G_w, L_w^\times)$ is the local fundamental class.

Proof See [8, Prop. 3.5.(a)].□

We consider the product $\prod_{w\in S} R\Gamma(L_w,\mathbb{G}_m)$ as a complex in $\mathcal{D}(\mathbb{Z}G)$. This is possible as the complexes $R\Gamma(L_w,\mathbb{G}_m)$ can be chosen 'compatibly' as explained in [5, §6.2.1]. Lemma 7.5.1, together with the canonical isomorphism $Br(L_w)\cong\mathbb{Q}/\mathbb{Z}$ for each $w\in S_f$, gives canonical isomorphisms of G-modules $H^0(\prod_{w\in S} R\Gamma(L_w,\mathbb{G}_m))\cong\prod_{w\in S} L_w^\times$ and $H^2(\prod_{w\in S} R\Gamma(L_w,\mathbb{G}_m))\cong\prod_{w\in S_f}\mathbb{Q}/\mathbb{Z}$. We identify $\prod_{w\in S_f}\mathbb{Q}/\mathbb{Z}$ with $\mathbb{Q}/\mathbb{Z}\otimes Y_{L,S_f}$ in the obvious way.

There is a canonical 'localization morphism' of complexes

$$R\Gamma(\mathcal{O}_{L,S},\mathbb{G}_m)\longrightarrow\bigoplus_{w\in S}R\Gamma(L_w,\mathbb{G}_m),$$

([5, §6.2]) such that the induced maps on cohomology are the natural diagonal homomorphism $\mathcal{O}_{L,S}^\times\to\prod_{w\in S}L_w^\times$ in degree zero and the natural inclusion $\mathbb{Q}/\mathbb{Z}\otimes X_{L,S_f}\to\mathbb{Q}/\mathbb{Z}\otimes Y_{L,S_f}$ in degree two. One defines

$$\widehat{R\Gamma}_c(\mathcal{O}_{L,S},\mathbb{G}_m):=\mathrm{Cone}\left(R\Gamma(\mathcal{O}_{L,S},\mathbb{G}_m)\to\bigoplus_{w\in S}R\Gamma(L_w,\mathbb{G}_m)\right)[-1]$$

and hence there exists a tautological exact triangle in $\mathcal{D}(\mathbb{Z}G)$

$$\widehat{R\Gamma}_c(\mathcal{O}_{L,S},\mathbb{G}_m)\longrightarrow R\Gamma(\mathcal{O}_{L,S},\mathbb{G}_m)\longrightarrow\prod_{w\in S}R\Gamma(L_w,\mathbb{G}_m)\longrightarrow\ . \tag{7.34}$$

Lemma 7.5.2. *There are canonical isomorphisms of G-modules*

$$H^i(\widehat{R\Gamma}_c(\mathcal{O}_{L,S},\mathbb{G}_m))\cong\begin{cases}C_S(L) & \text{if } i=1\\ \mathbb{Q}/\mathbb{Z} & \text{if } i=3\\ 0 & \text{otherwise.}\end{cases}$$

Moreover, the class defined by $\widehat{R\Gamma}_c(\mathcal{O}_{L,S},\mathbb{G}_m)[1]$ in $\mathrm{Ext}^3_{\mathbb{Z}G}(\mathbb{Q}/\mathbb{Z},C_S(L))\cong H^2(G,C_S(L))$ is the global fundamental class.

Proof The description of the cohomology is an easy exercise using the long exact sequence of cohomology of (7.34) along with the exact sequence (7.7), the natural exact sequence

$$0 \longrightarrow \mathcal{O}_{L,S}^{\times} \longrightarrow \prod_{w \in S} L_w^{\times} \longrightarrow C_S(L) \longrightarrow \mathrm{Pic}(\mathcal{O}_{L,S}) \longrightarrow 0$$

and the fact that all of the maps on cohomology are the canonical maps. The second claim is [8, Prop. 3.5(b)].□

We can now state the main result of this chapter.

Theorem 7.5.3. *There exists an isomorphism* $\ker(\overline{\theta}_f) \cong R\Gamma(\mathcal{O}_{L,S}, \mathbb{G}_m)$ *in* $\mathcal{D}(\mathbb{Z}G)$ *which induces the identity map on cohomology.*

Before we prove this result we prove the following corollary which demonstrates the relation between ϵ_{τ_0} and the Tate sequence of [38]. Recall the complex $C_{L,S}$ given by Proposition 7.2.3.

Corollary 7.5.4. *There exists an isomorphism of G-modules* $\phi : \nabla_f \cong H^1(C_{L,S})$ *which is the identity map on* $(\nabla_f)_{tor} = \mathrm{Pic}(\mathcal{O}_{L,S})$ *and induces the identity map on* $\overline{\nabla}_f = X_{L,S_f}$. *Moreover, the extension class defined by* $\ker \widehat{\theta}_f$ *is the same as the extension class* ϵ_{τ_0} *defined by* $C_{L,S}$ *under the isomorphism*

$$\mathrm{Ext}^2_{\mathbb{Z}G}(H^1(C_{L,S}), \mathcal{O}_{L,S}^{\times}) \xrightarrow{\sim} \mathrm{Ext}^2_{\mathbb{Z}G}(\nabla_f, \mathcal{O}_{L,S}^{\times})$$

induced by ϕ.

Proof We have a commutative diagram in $\mathcal{D}(\mathbb{Z}G)$

$$\begin{array}{ccccccc}
\mathbb{Q} \otimes X_{L,S_f}[-2] & \longrightarrow & \ker(\overline{\theta}_f) & \longrightarrow & \ker \widehat{\theta}_f & \longrightarrow & \\
\| & & \downarrow \cong & & & & \\
\mathbb{Q} \otimes X_{L,S_f}[-2] & \longrightarrow & R\Gamma(\mathcal{O}_{L,S}, \mathbb{G}_m) & \longrightarrow & C_{L,S} & \longrightarrow &
\end{array} \tag{7.35}$$

where the top row is the exact triangle induced by the top row of (7.33), the bottom row is (7.9) and the second vertical map is given by Theorem 7.5.3. Since the isomorphism of Theorem 7.5.3 is the identity map on cohomology, one sees that the composite maps obtained by going either direction around the left hand square of (7.35) both induce the map $\pi_{L,S}$ of (7.8) on cohomology. A similar argument to that given at the start of the proof of Proposition 7.2.3 shows that such a morphism must be unique in $\mathcal{D}(\mathbb{Z}G)$, hence the left hand square of (7.35) commutes.

Completing (7.35) to a morphism of triangles gives an isomorphism $\kappa : \ker \widehat{\theta}_f \xrightarrow{\sim} C_{L,S}$ in $\mathcal{D}(\mathbb{Z}G)$ which induces the identity map on cohomology in degree zero. If we set $\phi := H^1(\kappa)$, then the claims that ϕ is the identity on $(\nabla_f)_{tor}$ and induces the identity on $\overline{\nabla}_f$ follow directly from Theorem 7.5.3. Applying Lemma 7.1.1 to the morphism κ gives the rest of the statement of the corollary.□

Proof of Theorem 7.5.3

We need the following Lemma.

Lemma 7.5.5. *Let R be a ring and let $f : A \to B$ be an injective, resp. surjective, morphism of complexes in $Kom(R)$. Then there exists a natural quasi-isomorphism* $\mathrm{Cone}(f) \to \mathrm{coker}(f)$*, resp.* $\ker(f) \to \mathrm{Cone}(f)[-1]$.

Proof This is an easy exercise using the explicit description of the mapping cone.□

Lemma 7.5.5 shows that we have a canonical isomorphism $\ker \overline{\theta}_f \cong (\mathrm{Cone}\, \overline{\theta}_f)[-1]$ in $\mathcal{D}(\mathbb{Z}G)$. We will now construct a commutative diagram in $\mathcal{D}(\mathbb{Z}G)$

$$\begin{array}{ccccccc} \overline{A}_f & \xrightarrow{\overline{\theta}_f} & \overline{B} & \longrightarrow & \mathrm{Cone}\, \overline{\theta}_f & \longrightarrow & \\ \downarrow & & \downarrow & & \downarrow & & \\ \prod_{w \in S} R\Gamma(L_w, \mathbb{G}_m) & \longrightarrow & \widehat{R\Gamma}_c(\mathcal{O}_{L,S}, \mathbb{G}_m)[1] & \longrightarrow & R\Gamma(\mathcal{O}_{L,S}, \mathbb{G}_m)[1] & \longrightarrow & \end{array} \tag{7.36}$$

where the top row is the natural triangle and the bottom row is (7.34). A diagram chase using the long exact sequences of cohomology associated to the two horizontal triangles will then show that the right hand vertical morphism is an isomorphism and induces the identity map on cohomology. Since $\ker \overline{\theta}_f$ is canonically isomorphic to $(\mathrm{Cone}\,\overline{\theta}_f)[-1]$, this will give Theorem 7.5.3.

<u>Constructing the middle vertical morphism in (7.36)</u>

The natural map $C(L) \to C_S(L)$ induces a commutative diagram of groups

$$\begin{array}{ccc} H^2(G, C(L)) & \xrightarrow{\sim} & \mathrm{Ext}^3_{\mathbb{Z}G}(\mathbb{Q}/\mathbb{Z}, C(L)) \\ \downarrow & & \downarrow \\ H^2(G, C_S(L)) & \xrightarrow{\sim} & \mathrm{Ext}^3_{\mathbb{Z}G}(\mathbb{Q}/\mathbb{Z}, C_S(L)) \end{array} \tag{7.37}$$

where the horizontal isomorphisms are the dimension shifts induced by (7.5). The kernel of the natural map $C(L) \to C_S(L)$ is $\prod_{w \notin S} U_w$ and since $S_{ram} \subset S$ this is a cohomologically trivial G-module. The vertical maps in (7.37) are thus also isomorphisms.

Recall that we defined $\overline{B}$ to be the complex $[\mathcal{B} \to \mathbb{Z}G \to \mathbb{Q}]$, with $\mathcal{B}$ placed in degree zero, and furthermore that this complex was obtained by splicing the bottom row of (7.18) with (7.6) followed by (7.5). The bottom row of (7.18) represents the global fundamental class in $\mathrm{Ext}^1_{\mathbb{Z}G}(I_G, C(L))$ and hence $\overline{B}$ represents the global fundamental class in $\mathrm{Ext}^3_{\mathbb{Z}G}(\mathbb{Q}/\mathbb{Z}, C(L))$.

By Lemma 7.5.2 the complex $\widehat{R\Gamma}_c(\mathcal{O}_{L,S}, \mathbb{G}_m)[1]$ is quasi-isomorphic to a complex $C := [C^0 \to C^1 \to C^2]$, where the left hand object is placed in degree zero, C^i is a cohomologically trivial G-module for each i and the class of

$$0 \longrightarrow C_S(L) \longrightarrow C^0 \longrightarrow C^1 \longrightarrow C^2 \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0 \tag{7.38}$$

in $\mathrm{Ext}^3_{\mathbb{Z}G}(\mathbb{Q}/\mathbb{Z}, C_S(L))$ is the global fundamental class.

Combining all of this shows that the righthand vertical isomorphism in (7.37) maps the class defined by $\overline{B}$ to the class defined by C. The description of this map in terms of Yoneda extension classes (cf. [26, Ch. IV, §9]) immediately gives a morphism of complexes $\overline{B} \to C$ which induces the natural map $C(L) \to C_S(L)$ on cohomology in degree zero and the identity map in all other degrees. This gives the middle vertical morphism in (7.36).

Constructing the first vertical morphism in (7.36)

The complexes $\overline{A}_f$ and $\prod_{w\in S} R\Gamma(L_w, \mathbb{G}_m)$ define classes in $\mathrm{Ext}^3_{\mathbb{Z}G}(\mathbb{Q}/\mathbb{Z}\otimes Y_{L,S_f}, J_S(L))$ and $\mathrm{Ext}^3_{\mathbb{Z}G}(\mathbb{Q}/\mathbb{Z}\otimes Y_{L,S_f}, \prod_{w\in S} L_w^\times)$ respectively. We have a tautological exact sequence of G-modules

$$0 \longrightarrow \prod_{w\notin S} U_w \longrightarrow J_S(L) \longrightarrow \prod_{w\in S} L_w^\times \longrightarrow 0$$

and since $S_{ram} \subset S$ this induces an isomorphism

$$\mathrm{Ext}^3_{\mathbb{Z}G}(\mathbb{Q}/\mathbb{Z}\otimes Y_{L,S_f}, J_S(L)) \xrightarrow{\sim} \mathrm{Ext}^3_{\mathbb{Z}G}(\mathbb{Q}/\mathbb{Z}\otimes Y_{L,S_f}, \textstyle\prod_{w\in S} L_w^\times). \tag{7.39}$$

In §7.6 we will show that the classes defined by $\overline{A}_f$ and $\prod_{w\in S} R\Gamma(L_w, \mathbb{G}_m)$ correspond under this isomorphism. A similar argument to that used to construct the middle vertical morphism of (7.36) then gives the left hand vertical morphism of (7.36). Furthermore, the maps induced on cohomology are the natural projection $J_S(L) \to \prod_{w\in S} L_w^\times$ in degree zero and the identity map in all other degrees. We leave the details to the reader.

Showing the left hand square of (7.36) commutes

We will need the following Lemma.

Lemma 7.5.6. *Let R be a ring and let $\mathcal{A}, \mathcal{B} \in \mathcal{D}(R)$. There is a spectral sequence*

$$E_2^{p,q} = \prod_{i\in\mathbb{Z}} \mathrm{Ext}^p_R(H^i(\mathcal{A}), H^{q+i}(\mathcal{B})) \Longrightarrow H^{p+q}(RHom(\mathcal{A}, \mathcal{B})).$$

Proof See [44, Ch. III, 4.6.10].□

When dealing with spectral sequences we will refer to the objects $E_r^{p,q}$, for fixed r, as lying on the r^{th} *sheet*. Given $n \in \mathbb{Z}$ we refer to the objects lying on the diagonal

$p+q=n$ as lying in *degree* n.

Consider the spectral sequence of Lemma 7.5.6 for

$$\mathcal{A}=\overline{A}_f \qquad \mathcal{B}=\widehat{R\Gamma}_c(\mathcal{O}_{L,S},\mathbb{G}_m)[1].$$

In degree $p+q=0$ the sequence converges to $\mathrm{Hom}_{\mathcal{D}(\mathbb{Z}G)}(\mathcal{A},\mathcal{B})$. Using Lemma 7.5.1, Lemma 7.5.2 and the definition of $\overline{A}_f$ we see that in degree zero the group $E_2^{p,q}$ is trivial unless $(p,q)=(0,0)$ or $(p,q)=(2,-2)$. Thus in degree zero the only two contributions on the second sheet of the spectral sequence are

$$\begin{aligned} E_2^{0,0} &= \mathrm{Hom}_{\mathbb{Z}G}\Big(J_S(L),C_S(L)\Big)\times \mathrm{Hom}_{\mathbb{Z}G}\Big(\textstyle\prod_{w\in S} Br(L_w),\mathbb{Q}/\mathbb{Z}\Big)\\ \text{and}\quad E_2^{2,-2} &= \mathrm{Ext}^2_{\mathbb{Z}G}\Big(\mathbb{Q}/\mathbb{Z}\otimes Y_{L,S_f},C_S(L)\Big).\end{aligned}$$

We first show that $E_2^{2,-2}=0$.

Using the 'adjointness of Ind and Res' we have

$$E_2^{2,-2}\cong \prod_{v\in S_f(K)} \mathrm{Ext}^2_{\mathbb{Z}G_{w(v)}}\Big(\mathbb{Q}/\mathbb{Z},\mathrm{Res}^G_{G_{w(v)}}C_S(L)\Big).$$

Using the exact sequence (7.5) we can dimension shift to obtain

$$\begin{aligned}\mathrm{Ext}^2_{\mathbb{Z}G_{w(v)}}\Big(\mathbb{Q}/\mathbb{Z},\mathrm{Res}^G_{G_{w(v)}}C_S(L)\Big) &\cong \mathrm{Ext}^1_{\mathbb{Z}G_{w(v)}}\Big(\mathbb{Z},\mathrm{Res}^G_{G_{w(v)}}C_S(L)\Big)\\ &\cong \widehat{H}^1\Big(G_{w(v)},\mathrm{Res}^G_{G_{w(v)}}C_S(L)\Big),\end{aligned}$$

where the second isomorphism follows from the definition of the Tate cohomology groups. Using the exact sequence (7.38) and the exact sequence (7.5), we can dimension shift to obtain

$$\widehat{H}^1\Big(G_{w(v)},\mathrm{Res}^G_{G_{w(v)}}C_S(L)\Big)\cong \widehat{H}^{-2}(G_{w(v)},\mathbb{Q}/\mathbb{Z})\cong \widehat{H}^{-1}(G_{w(v)},\mathbb{Z}).$$

This latter group is then easily seen to be zero and so $E_2^{2,-2}=0$.

We must now be careful as the spectral sequence does not converge on the second sheet in degree zero. On the second sheet all the morphisms are zero and so the objects on the third sheet are the same as those on the second. On the third sheet in degree zero we have the morphism $d_3^{0,0} : E_3^{0,0} \to E_3^{3,-2}$. In the same way we proved $E_2^{2,-2} = 0$ it can be shown that $E_3^{3,-2}$ is isomorphic to $\prod_{v \in S_f(K)} \mathbb{Z}/|G_{w(v)}|\mathbb{Z}$, which is non-trivial. By definition of the spectral sequence, if $r \neq 3$ and $(p,q) \neq (3,-2)$, then $d_r^{p,q} = 0$ for all $r \geq 3$ and for all (p,q) such that $p+q=0$. Hence the spectral sequence converges in degree zero on the third sheet, i.e.

$$\mathrm{Hom}_{\mathcal{D}(\mathbb{Z}G)}(\mathcal{A},\mathcal{B}) \cong E_3^{0,0} \cong \ker(d_3^{0,0}) \times \mathrm{Hom}_{\mathbb{Z}G}\Big(\prod_{w \in S} Br(L_w), \mathbb{Q}/\mathbb{Z}\Big).$$

Since $\ker(d_3^{0,0})$ is a subset of $\mathrm{Hom}_{\mathbb{Z}G}\Big(J_S(L), C_S(L)\Big)$, we have shown that two morphisms from $\mathcal{A}$ to $\mathcal{B}$ are equal if and only if the maps induced on cohomology are equal. By construction the left hand square of (7.36) commutes on cohomology and thus must commute in $\mathcal{D}(\mathbb{Z}G)$.□

7.6 The Semi-local Fundamental Class

In this section S is any finite G-stable set of places containing $S_\infty \cup S_{ram}$. We first recall the definition of $c_{L/K,S}^{loc}$ and then show that, under suitable isomorphisms of Ext-groups, the extension classes defined by both the top row of (7.18) and $\prod_{w \in S} R\Gamma(L_w^\times, \mathbb{G}_m)$ correspond to $c_{L/K,S}^{loc}$.

The definition of $c^{loc}_{L/K,S}$

We have the following sequence of isomorphisms and inclusions

$$\begin{aligned}
\mathrm{Ext}^2_{\mathbb{Z}G}(Y_{L,S}, J_S(L)) &\cong \mathrm{Ext}^2_{\mathbb{Z}G}(Y_{L,S}, \prod_{w\in S} L_w^\times) \\
&\cong \prod_{v,v'\in S(K)} \mathrm{Ext}^2_{\mathbb{Z}G}(\mathrm{Ind}^G_{G_{w(v)}} \mathbb{Z}, \mathrm{Ind}^G_{G_{w(v')}} L^\times_{w(v')}) \\
&\supset \prod_{v\in S(K)} \mathrm{Ext}^2_{\mathbb{Z}G}(\mathrm{Ind}^G_{G_{w(v)}} \mathbb{Z}, \mathrm{Ind}^G_{G_{w(v)}} L^\times_{w(v)}) \\
&\cong \prod_{v\in S(K)} \mathrm{Ext}^2_{\mathbb{Z}G_{w(v)}}(\mathbb{Z}, \mathrm{Res}^G_{G_{w(v)}} \mathrm{Ind}^G_{G_{w(v)}} L^\times_{w(v)}) \\
&\supset \prod_{v\in S(K)} \mathrm{Ext}^2_{\mathbb{Z}G_{w(v)}}(\mathbb{Z}, L^\times_{w(v)}).
\end{aligned}$$

The first line follows from the fact that if v' is unramified, then $U_{w(v')}$ is a cohomologically trivial G-module. The second and third lines are obvious. The fourth lines follows from the 'adjointness of Res and Ind'. The fifth line follows because $L^\times_{w(v)}$ is a direct factor of $\mathrm{Res}^G_{G_{w(v)}} \mathrm{Ind}^G_{G_{w(v)}} L^\times_{w(v)}$ as a $G_{w(v)}$-module. One defines $c^{loc}_{L/K,S}$ to be the pre-image of the element $(c_{L_{w(v)}/K_v})_{v\in S(K)}$ of $\prod_{v\in S(K)} \mathrm{Ext}^2_{\mathbb{Z}G_{w(v)}}(\mathbb{Z}, L^\times_{w(v)})$ under this composite map. Using Shapiro's Lemma one can show that $c^{loc}_{L/K,S}$ is independent of the choice of the $w(v)$.

In terms of Yoneda extensions one can construct $c^{loc}_{L/K,S}$ in the following way. Let

$$0 \longrightarrow L^\times_{w(v)} \longrightarrow C^0_{w(v)} \longrightarrow C^1_{w(v)} \longrightarrow \mathbb{Z} \longrightarrow 0$$

be an exact sequence of $G_{w(v)}$-modules representing $c_{L_{w(v)}/K_v}$. An extension class representing $c^{loc}_{L/K,S}$ is given by first applying the functor $\mathrm{Ind}^G_{G_{w(v)}}(-)$ to each such sequence and than taking the direct sum over all places v in $S(K)$.

Determining the extension class of the top row of (7.18)

We recall that the top row of (7.18) is the exact sequence of G-modules

$$E := [0 \longrightarrow J_S(L) \longrightarrow V_{S'} \longrightarrow W_{S'} \longrightarrow 0] \tag{7.40}$$

and that we defined A to be the complex $[V_{S'} \to W_{S'}]$. In [38, §1] the authors construct E in the following way. They construct a family of exact sequences

$$\begin{aligned}
E_v &:= [0 \longrightarrow L_{w(v)}^{\times} \longrightarrow V_{w(v)} \longrightarrow I_{G_{w(v)}} \longrightarrow 0] && \text{if } v \in S(K) \\
E_v &:= [0 \longrightarrow U_{w(v)} \longrightarrow U_{w(v)} \oplus \mathbb{Z}G_{w(v)} \longrightarrow \mathbb{Z}G_{w(v)} \longrightarrow 0] && \text{if } v \in S'(K) \setminus S(K) \\
E_v &:= [0 \longrightarrow U_{w(v)} = U_{w(v)} \longrightarrow 0 \longrightarrow 0] && \text{if } v \notin S'(K)
\end{aligned} \tag{7.41}$$

and then define $E := \prod_{v \in \mathcal{M}_K} \mathrm{Ind}_{G_{w(v)}}^{G} E_v$ (where the notation should be self-explanatory). Since E_v has the trivial extension class for each $v \notin S(K)$ it is clear that the extension class of E in $\mathrm{Ext}^1_{\mathbb{Z}G}(W_{S'}, J_S(L))$ is determined by the extension class of $\prod_{v \in S(K)} \mathrm{Ind}_{G_{w(v)}}^{G} E_v$ in $\mathrm{Ext}^1_G(\prod_{v \in S(K)} \mathrm{Ind}_{G_{w(v)}}^{G} I_{G_{w(v)}}, \prod_{w \in S} L_w^{\times})$.

For each $v \in S(K)$ we let T_v be the exact sequence of $G_{w(v)}$-modules

$$0 \longrightarrow I_{G_{w(v)}} \longrightarrow \mathbb{Z}G_{w(v)} \longrightarrow \mathbb{Z} \longrightarrow 0.$$

The exact sequence of G-modules $\prod_{v \in S(K)} \mathrm{Ind}_{G_{w(v)}}^{G} T_v$ induces a dimension shift

$$\mathrm{Ext}^1_{\mathbb{Z}G}(\overset{\mathrm{Ind}}{\prod_{v \in S(K)}} I_{G_{w(v)}}, \prod_{w \in S} L_w^{\times}) \xrightarrow{\sim} \mathrm{Ext}^2_{\mathbb{Z}G}(Y_{L,S}, \prod_{w \in S} L_w^{\times}). \tag{7.42}$$

Let E' denote the exact sequence obtained by splicing $\prod_{v \in S(K)} \mathrm{Ind}_{G_{w(v)}}^{G} E_v$ with $\prod_{v \in S} \mathrm{Ind}_{G_{w(v)}}^{G} T_v$. In terms of Yoneda extensions the isomorphism (7.42) sends the class of $\prod_{v \in S(K)} \mathrm{Ind}_{G_{w(v)}}^{G} E_v$ to the class of E'. One can also construct this class in another way. For each $v \in S(K)$ we define an exact sequence

$$E'_v := [0 \longrightarrow L_{w(v)}^{\times} \longrightarrow V_{w(v)} \longrightarrow \mathbb{Z}G_{w(v)} \longrightarrow \mathbb{Z} \longrightarrow 0]$$

by splicing E_v with T_v. We leave it to the reader to check that $\prod_{v \in S(K)} \mathrm{Ind}_{G_{w(v)}}^{G} E'_v$ has the same extension class as E'.

The second construction of E', along with our discussion on the construction of $c_{L/K,S}^{loc}$ above, shows that the class of E' in $\mathrm{Ext}^2_{\mathbb{Z}G}(Y_{L,S}, \prod_{w \in S} L_w^{\times})$ is $c_{L/K,S}^{loc}$. Hence the extension class of E is also $c^{loc}_{L/K,S}$ (in the appropriate Ext-group).

We now leave it to the reader to check that both the complexes $\widehat{A}_f$ and $\overline{A}_f$ represent $c^{loc}_{L/K,S}$ in the appropriate Ext-groups (this is a consequence of the fact that $\widehat{A}_f$ and $\overline{A}_f$ were both obtained from A by dimension shifts of the kind just discussed).

Determining the class of $\prod_{w\in S} R\Gamma(L_w^\times, \mathbb{G}_m)$

By Lemma 7.5.1 the complex $R\Gamma(L_w^\times, \mathbb{G}_m)$ represents c_{L_w/K_v}, where v is the place of K below w. Since the complexes $R\Gamma(L_w^\times, \mathbb{G}_m)$ were chosen compatibly (see the discussion immediately following the proof of 7.5.1) a similar argument to that above shows that $\prod_{w\in S} R\Gamma(L_w^\times, \mathbb{G}_m)$ represents $c^{loc}_{L/K,S}$ in $\mathrm{Ext}^3_{\mathbb{Z}G}(\mathbb{Q}/\mathbb{Z} \otimes Y_{L,S} \prod_{w\in S} L_w^\times)$. We leave it to the reader to check the details.

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