On descent theory and main conjectures in non-commutative Iwasawa theory

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Abstract. We develop an explicit descent theory in the context of Whitehead groups of non-commutative Iwasawa algebras. We apply this theory to describe the precise connection between main conjectures of non-commutative Iwasawa theory (in the spirit of Coates, Fukaya, Kato, Sujatha and Venjakob) and the equivariant Tamagawa number conjecture. The latter result is a converse to a theorem of Fukaya and Kato and also provides an important means of both deriving explicit consequences of the main conjecture and proving special cases of the equivariant Tamagawa number conjecture.

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Introduction

There has been much interest in the study of non-commutative Iwasawa theory over the last few years. Nevertheless, there is still no satisfactory understanding of the explicit consequences for Hasse-Weil L-functions that are implied by a 'main conjecture' of the kind formulated by Coates, Fukaya, Kato, Sujatha and the second named author in [13]. Indeed, whilst explicit consequences of such a conjecture for the values (at $s = 1$) of twisted Hasse-Weil L-functions have been studied by Coates et al in [13], by Kato in [20] and by Dokchister and Dokchister in [16], all of these consequences become trivial whenever the L-functions vanish at $s = 1$. Further, the conjecture of Birch and Swinnerton-Dyer implies that these L-functions should vanish whenever the relevant component of the Mordell-Weil group has strictly positive rank and by a recent result of Mazur and Rubin [22], which is itself equivalent to a special case of an earlier result of Nekovář [24, Th. 10.7.17], this should often be the case. It is therefore of interest to understand what a main conjecture of the kind formulated in [13] predicts concerning the values of *derivatives* of Hasse-Weil L-functions at $s = 1$. In this article we take the first step towards developing such a theory by describing a general formalism of descent in non-commutative Iwasawa theory. In a subsequent article it will be shown that the results proved here can be combined with techniques developed by the first named author in [7] to derive from the main conjecture of non-commutative Iwasawa theory a variety of explicit (and highly non-trivial) congruence relations between values of derivatives of twisted Hasse-Weil L-functions. In another direction, in [8] the results of this article play a key role in the first verification of the equivariant Tamagawa number conjecture (for certain Tate motives) for a wide class of non-abelian extensions of number fields and in the proof of a long-standing conjecture of Chinburg.

However, as preparation for the above applications, we must first develop several aspects of the theory that appear themselves to be of some independent interest. These include proving a natural Weierstrass Preparation Theorem for Whitehead groups of Iwasawa algebras , defining a canonical 'characteristic series' for torsion modules over (localised) Iwasawa algebras, satisfactorily resolving the descent problem in non-commutative Iwasawa theory and formulating a main conjecture in the spirit of Coates et al that deals with interpolation properties of the 'leading terms at Artin representations' (in the sense introduced in [11]) of analytic p-adic L-functions.

In a little more detail, the main contents of this article is as follows. In §1 we recall some useful preliminaries concerning localisation of Iwasawa algebras, K-theory, virtual objects and derived categories. In §2 we state the main K-theoretical results that are proved in this article. In §3 we define a suitable notion of μ-invariant and in §4 we combine this notion with a result of Schneider and the second named author from [26] and the formalism developed by Fukaya and Kato in [17] to define canonical 'characteristic series' in non-commutative Iwasawa theory (this construction extends the notion of 'algebraic p-adic L-functions' introduced by the first named author in [6] and hence also refines the notion of 'Akashi series' introduced by Coates, Schneider and Sujatha in [12]). As a first application of these characteristic series we use them in §5 to prove an explicit formula for the 'leading terms at Artin representations ' of elements of Whitehead groups of non-commutative Iwasawa algebras: this result provides a suitable 'descent formalism' in non-commutative Iwasawa theory and in particular plays a crucial role in proving the arithmetic results discussed in the remainder of the article. In §6 we present a result of Kato that allows reduction to a convenient special class of extensions when formulating main conjectures and, in particular, shows that the main result of §5 is indeed a satisfactory resolution of the descent problem in the context of non-commutative Iwasawa theory. In §7 we formulate explicit main conjectures of non-commutative Iwasawa theory for both Tate motives and (certain) critical motives. The approach here is finer than that of [13] since we consider interpolation properties for leading terms of analytic p-adic L-functions. In §8 we combine the descent formalism described in §5 with the main results of our earlier article [11] to prove that, under suitable hypotheses, the main conjectures formulated in §7 imply the relevant special cases of the equivariant Tamagawa number conjecture formulated by Flach and the first named author in [9, Conj. 4(iv)]. These results are both a converse to the result of Fukaya and Kato in [17] which asserts that, under suitable hypotheses, the 'non-commutative Tamagawa number conjecture' of loc. cit. implies the main conjecture of Coates et al [13] and can also be used to derive explicit consequences of the main conjecture. Finally, in several appendices, we review relevant aspects of the algebraic formalism of localized K_1-groups and Bockstein homomorphisms and clarify certain normalizations used in [11].

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Part I: K-theory

1. Preliminaries

1.1. Iwasawa algebras. We fix a prime p. For any compact p-adic Lie group G we write $\Lambda(G)$ and $\Omega(G)$ for the 'Iwasawa algebras' $\varprojlim_U \mathbb{Z}_p[G/U]$ and $\varprojlim_U \mathbb{F}_p[G/U]$ where U runs over all open normal subgroups of G and the limits are taken with respect to the natural projection maps $\mathbb{Z}_p[G/U] \to \mathbb{Z}_p[G/U']$ and $\mathbb{F}_p[G/U] \to \mathbb{F}_p[G/U']$ for $U \subseteq U'$. The rings $\Lambda(G)$ and $\Omega(G)$ are both noetherian and, if G has no element of order p, they are also regular in the sense that their (left and right) global dimensions are finite. We write $Q(G)$ for the total quotient ring of $\Lambda(G)$. If $\mathcal{O}$ is any subring of $\mathbb{Q}_p^c$ that contains $\mathbb{Z}_p$, then we set $\Lambda_{\mathcal{O}}(G) := \mathcal{O} \otimes_{\mathbb{Z}_p} \Lambda(G)$ and write $Q_{\mathcal{O}}(G)$ for its total quotient ring.
We assume throughout that the following condition is satisfied

- G has a closed normal subgroup H for which the quotient group $\Gamma := G/H$ is isomorphic (topologically) to the additive group of $\mathbb{Z}_p$.

We write $\pi_\Gamma : G \to \Gamma$ for the natural projection and fix a topological generator γ of Γ. We use γ to identify $\Lambda(\Gamma)$ with the power series ring $\mathbb{Z}_p[[T]]$ in an indeterminate T (via the identification $T = \gamma - 1$).
We recall from [13, §2-§3] that there are canonical left and right denominator sets $S_{G,H}$ and $S^*_{G,H}$ of $\Lambda(G)$ where

$$S_{G,H} := \{\lambda \in \Lambda(G) : \Lambda(G)/(\Lambda(G)\cdot\lambda) \text{ is a finitely generated } \Lambda(H)\text{-module}\}$$

and $S^*_{G,H} := \bigcup_{i\geq 0} p^i S_{G,H}$. When G and H are clear from context we usually abbreviate $S_{G,H}$ to S. We also write $\mathfrak{M}_S(G)$ and $\mathfrak{M}_{S^*}(G)$ for the categories of finitely generated $\Lambda(G)$-modules M with $\Lambda(G)_S \otimes_{\Lambda(G)} M = 0$ and $\Lambda(G)_{S^*} \otimes_{\Lambda(G)} M = 0$ respectively.
For any $\mathbb{Z}_p$-module M we write $M_{\rm tor}$ for its $\mathbb{Z}_p$-torsion submodule and set $M_{\rm tf} := M/M_{\rm tor}$. We recall from [13, Prop. 2.3] that a finitely generated $\Lambda(G)$-module M belongs to $\mathfrak{M}_S(G)$, resp. $\mathfrak{M}_{S^*}(G)$, if and only if it is a finitely generated $\Lambda(H)$-module (by restriction), resp. when $M_{\rm tf}$ belongs to $\mathfrak{M}_S(G)$. This means in particular that $\mathfrak{M}_{S^*}(G)$ coincides with the category $\mathfrak{M}_H(G)$ introduced in loc. cit.

1.2. K-groups. For any ring homomorphism $R \to R'$ we write $K_0(R, R')$ for the associated relative algebraic K_0-group. We recall that this group is generated by symbols of the form (P, λ, Q) where P and Q are finitely generated projective (left) R-modules and λ is an isomorphism of R'-modules $R' \otimes_R P \to R' \otimes_R Q$ (for more details see [29, p. 215]). For any ring homomorphisms $R \to R' \to R''$ there is a natural commutative diagram of long exact sequences

$$
\begin{array}{ccccccc}
K_1(R) & \longrightarrow & K_1(R') & \xrightarrow{\partial_{R,R'}} & K_0(R,R') & \longrightarrow & K_0(R) \\
\| & & \downarrow & & \downarrow & & \| \\
K_1(R) & \longrightarrow & K_1(R'') & \xrightarrow{\partial_{R,R''}} & K_0(R,R'') & \longrightarrow & K_0(R).
\end{array} \tag{1}
$$

If G has no element of order p and Σ denotes either S or S^*, then $\Lambda(G)$ is a noetherian regular ring and $K_0(\Lambda(G),\Lambda(G)_\Sigma)$ can be identified with the Grothendieck group $K_0(\mathfrak{M}_\Sigma(G))$ of the category $\mathfrak{M}_\Sigma(G)$. To be precise we normalise this isomorphism as follows: if $g = s^{-1}h$ with $s \in \Sigma$ and $h \in \Lambda(G)\cap\Lambda(G)_\Sigma^\times$, then the element $(\Lambda(G), \mathrm{r}_g, \Lambda(G))$ of $K_0(\Lambda(G),\Lambda(G)_\Sigma)$ corresponds to $[\mathrm{cok}(\mathrm{r}_h)] - [\mathrm{cok}(\mathrm{r}_s)]$ in $K_0(\mathfrak{M}_\Sigma(G))$ where $[X]$ is the element of $K_0(\mathfrak{M}_\Sigma(G))$ associated to an object X of $\mathfrak{M}_\Sigma(G)$ and $\mathrm{r}_g, \mathrm{r}_h$ and r_s denote the automorphisms of $\Lambda(G)_\Sigma$ that are induced by right multiplication by g, h and s respectively. In particular, with respect to this isomorphism, the upper row of (1) with $R = \Lambda(G)$ and $R' = \Lambda(G)_{S^*}$ identifies with the exact sequence of [13, (24)].

If $R = \Lambda(G)$ and $R' = \Lambda(G)_{S^*}$, resp. $R = \mathbb{Z}_p[\mathcal{G}]$ and $R' = \mathbb{Q}_p^c[\mathcal{G}]$ for a finite group $\mathcal{G}$, then we usually abbreviate the connecting homomorphism $\partial_{R,R'}$ in diagram (1) to ∂_G, resp. $\partial_{\overline{G}}$.

1.3. Virtual objects. We let $\mathcal{P}_0$ denote the Picard category with unique object $\mathbf{1}_{\mathcal{P}_0}$ and $\mathrm{Aut}_{\mathcal{P}_0}(\mathbf{1}_{\mathcal{P}_0}) = 0$. For any associative unital ring R we also write $V(R)$ for the Picard category of virtual objects associated to the category of finitely generated projective R-modules and we fix a unit object $\mathbf{1}_R$ in $V(R)$. For any homomorphism of such rings $R \to R'$ we then define $V(R,R')$ to be the fibre product category in the diagram

$$
\begin{array}{ccc}
V(R,R') & \longrightarrow & \mathcal{P}_0 \\
\downarrow & & \downarrow{\scriptstyle F_2} \\
V(R) & \xrightarrow{F_1} & V(R')
\end{array}
$$

where F_2 is the (monoidal) functor sending $\mathbf{1}_{\mathcal{P}_0}$ to $\mathbf{1}_{R'}$ and $F_1(L) = R' \otimes_R L$ for each object L of $V(R)$. We regard the canonical isomorphism

$$\pi_0(V(R,R')) \cong K_0(R,R') \tag{2}$$

of [3, Lem. 5.1] (and [9, Prop. 2.5]) as an identification. In particular, for each object L of $V(R)$ and each morphism $\mu : F_1(L) \to \mathbf{1}_{R'}$ in $V(R')$ we write $[L,\mu]$ for the associated element of $K_0(R,R')$.

1.4. Euler characteristics. For any ring R we write $D(R)$ for the derived category of R-modules. If R is noetherian, then we also write $D^{\mathrm{fg},-}(R)$, resp. $D^{\mathrm{fg}}(R)$ for the full triangulated subcategory of $D(R)$ comprising complexes that are isomorphic to a bounded above, resp. bounded, complex of finitely generated R-modules and we let $D^{\mathrm{p}}(R)$ denote the full subcategory of $D^{\mathrm{fg}}(R)$ comprising complexes that are isomorphic to an object of the category $C^{\mathrm{p}}(R)$ of bounded complexes of finitely generated projective R-modules.

If Σ denotes either S or S^*, then we write $D^{\mathrm{p}}_\Sigma(\Lambda(G))$ for the full triangulated subcategory of $D^{\mathrm{p}}(\Lambda(G))$ comprising those complexes C such that $\Lambda(G)_\Sigma \otimes_{\Lambda(G)} C$ is acyclic. For each such C we write $\chi(C)$ for the *inverse* of the element of $K_0(\Lambda(G),\Lambda(G)_{S^*})$ that corresponds under (2) (with $R = \Lambda(G)$ and $R' = \Lambda(G)_{S^*}$) to the pair $([P^\bullet], \iota_{P^\bullet})$

with $[P^\bullet]$ the object of $\mathcal{V}(\Lambda(G))$ associated to any $P^\bullet$ in $C^{\mathrm{p}}(\Lambda(G))$ isomorphic to C in $D^{\mathrm{p}}(\Lambda(G))$ and $\iota_{P^\bullet}$ the morphism in $\mathcal{V}(\Lambda(G)_{S^*})$ associated to the isomorphism $\Lambda(G)_{S^*}\otimes_{\Lambda(G)}P^\bullet \cong \Lambda(G)_{S^*}\hat{\otimes}_{\Lambda(G)}C \cong 0$ in $D^{\mathrm{p}}(\Lambda(G)_{S^*})$. This element $\chi(C)$ is the inverse of the Euler characteristic $\chi_{\Lambda(G),\Lambda(G)_{S^*}}(C,t)$ that is defined in [3, Def. 5.5] with t equal to the isomorphism $\bigoplus_{i\in\mathbb{Z}} H^{2i}(\Lambda(G)_{S^*}\otimes_{\Lambda(G)} C) \cong 0 \cong \bigoplus_{i\in\mathbb{Z}} H^{2i+1}(\Lambda(G)_{S^*}\otimes_{\Lambda(G)} C)$. (We define $\chi(C)$ in terms of the inverse in order to ensure that if G has no element of order p, then the isomorphism $K_0(\Lambda(G),\Lambda(G)_{S^*}) \cong K_0(\mathfrak{M}_{S^*}(G))$ discussed in §1.2 sends $\chi(C)$ to $\sum_{i\in\mathbb{Z}}(-1)^i[H^i(C)]$.)

1.5. Wedderburn decompositions. We fix an algebraic closure $\mathbb{Q}_p^c$ of $\mathbb{Q}_p$. For any finite group $\mathcal{G}$ we write $\mathrm{Irr}(\mathcal{G})$ for the set of irreducible finite-dimensional $\mathbb{Q}_p^c$-valued characters of $\mathcal{G}$. Then the Wedderburn decomposition of the (finite dimensional semisimple) $\mathbb{Q}_p^c$-algebra $\mathbb{Q}_p^c[\mathcal{G}]$ induces a decomposition of its centre

$$\zeta(\mathbb{Q}_p^c[\mathcal{G}]) \cong \prod_{\mathrm{Irr}(\mathcal{G})} \mathbb{Q}_p^c. \tag{3}$$

The natural reduced norm map $\mathrm{Nrd}_{\mathbb{Q}_p^c[\mathcal{G}]} : K_1(\mathbb{Q}_p^c[\mathcal{G}]) \to \zeta(\mathbb{Q}_p^c[\mathcal{G}])^\times$ is bijective and we often (and without explicit comment) combine this map with (3) to regard elements of $\prod_{\mathrm{Irr}(\mathcal{G})}\mathbb{Q}_p^{c,\times}$ as elements of the Whitehead group $K_1(\mathbb{Q}_p^c[\mathcal{G}])$. In particular, we write $\partial_{\mathcal{G}} : \prod_{\mathrm{Irr}(\mathcal{G})}\mathbb{Q}_p^{c,\times} \to K_0(\mathbb{Z}_p[\mathcal{G}],\mathbb{Q}_p^c[\mathcal{G}])$ for the connecting homomorphism of relative K-theory (normalized as in (1)).

For each $\rho\in\mathrm{Irr}(\mathcal{G})$ we fix a minimal idempotent e_ρ in $\mathbb{Q}_p^c[\mathcal{G}]$ such that the left action of $\mathcal{G}$ on $\mathbb{Q}_p^c[\mathcal{G}]$ given by $x\mapsto xg^{-1}$ for $g\in\mathcal{G}$ induces an isomorphism of (left) $\mathbb{Q}_p^c[\mathcal{G}]$-modules $e_\rho\mathbb{Q}_p^c[\mathcal{G}] \cong V_{\rho^*}$ where $V_{\rho^*}\cong(\mathbb{Q}_p^c)^{n_\rho}$ is the representation space of the contragredient ρ^* of ρ over $\mathbb{Q}_p^c$. Then for each complex C in $D^{\mathrm{p}}(\mathbb{Z}_p[\mathcal{G}])$ the theory of Morita equivalence induces an identification of morphism groups

$$\begin{aligned}\mathrm{Mor}_{V(\mathbb{Q}_p^c[\mathcal{G}])}&(\mathbf{d}_{\mathbb{Q}_p^c[\mathcal{G}]}(\mathbb{Q}_p^c[\mathcal{G}]\otimes^{\mathbb{L}}_{\mathbb{Z}_p[\mathcal{G}]} C),\mathbf{1}_{\mathbb{Q}_p^c[\mathcal{G}]})\\ &\cong \prod_{\mathrm{Irr}(\mathcal{G})}\mathrm{Mor}_{V(\mathbb{Q}^c)}(\mathbf{d}_{\mathbb{Q}_p^c}(e_\rho\mathbb{Q}_p^c[\mathcal{G}]\otimes^{\mathbb{L}}_{\mathbb{Z}_p[\mathcal{G}]} C),\mathbf{1}_{\mathbb{Q}_p^c}).\end{aligned} \tag{4}$$

Details of the 'non-commutative determinants' $\mathbf{d}_{\mathbb{Q}_p^c[\mathcal{G}]}(-)$ and $\mathbf{d}_{\mathbb{Q}_p^c}(-)$ that are used here are recalled in Appendix A.

2. Statement of the main results in Part I

The first main result we prove in Part I is the following decomposition theorem for Whitehead groups.

Theorem 2.1. *If G has no element of order p, then there is a natural isomorphism of abelian groups*

$$K_0(\Omega(G))\oplus K_0(\mathfrak{M}_S(G))\oplus \mathrm{im}(K_1(\Lambda(G))\to K_1(\Lambda(G)_{S^*}))\cong K_1(\Lambda(G)_{S^*}).$$

Our proof of Theorem 2.1 will show that if $G=\Gamma$, then the above isomorphism reduces to the assertion that every element of $Q(\Gamma)^\times$ can be written uniquely in the from $p^m du$ where m is an integer, d is a quotient of distinguished polynomials and u a unit in $\Lambda(\Gamma)$ (see Remark 4.2 and §4.3). Theorem 2.1 is therefore a natural generalisation of the classical Weierstrass Preparation Theorem. (For an alternative approach to generalising the latter result see [32]).

In the remainder of Part I we apply the decomposition in Theorem 2.1 to help resolve the 'descent problem' in non-commutative Iwasawa theory. Before stating our main result in this regard we recall that for each Artin representation $\rho : G \to \mathrm{GL}_n(\mathcal{O})$ the ring homomorphism $\Lambda(G)_{S^*} \to M_n(Q(\Gamma))$ that sends each element g of G to $\rho(g)\pi_\Gamma(g)$ induces a homomorphism of groups

$$\Phi_\rho : K_1(\Lambda(G)_{S^*}) \to K_1(M_n(Q_{\mathcal{O}}(\Gamma))) \cong K_1(Q_{\mathcal{O}}(\Gamma)) \cong Q_{\mathcal{O}}(\Gamma)^\times \cong Q(\mathcal{O}[[T]])^\times \tag{5}$$

where the first isomorphism is induced by the theory of Morita equivalence, the second by taking determinants (over $Q_{\mathcal{O}}(\Gamma)$) and the third by the identification $\gamma - 1 = T$. The 'leading term' $\xi^*(\rho)$ at ρ of an element ξ of $K_1(\Lambda(G)_{S^*})$ is then defined to be the leading term at $T = 0$ of the power series $\Phi_\rho(\xi)$ (this definition can also be interpreted as a leading term at zero of a p-adic meromorphic function - see [11, Lem. 3.17]).
The problem of descent in (non-commutative) Iwasawa theory is then the following: given an element ξ of $K_1(\Lambda(G)_{S^*})$ and a finite quotient $\overline{G}$ of G, can one use knowledge of the image of ξ under the connecting homomorphism ∂_G to give an explicit formula for the image of $(\xi^*(\rho))_{\rho\in\mathrm{Irr}(\overline{G})}$ under the connecting homomorphism $\partial_{\overline{G}}$? This has been known for some time to be an important and delicate problem. To state our result we set

$$\tilde{S} := \begin{cases} S, & \text{if } G \text{ has an element of order } p, \\ S^*, & \text{otherwise.} \end{cases} \tag{6}$$

Theorem 2.2. *Let $\overline{G}$ be a finite quotient of G. Let ξ be an element of $K_1(\Lambda(G)_{S^*})$ with $\partial_G(\xi) = \chi(C)$ where C is a complex that belongs to $D^{\mathrm{p}}_{\tilde{S}}(\Lambda(G))$ and is 'semisimple at ρ' for each representation $\rho \in \mathrm{Irr}(\overline{G})$ in the sense of* [11, Def. 3.11]. *Then in $K_0(\mathbb{Z}_p[\overline{G}], \mathbb{Q}_p^c[\overline{G}])$ one has*

$$\partial_{\overline{G}}((\xi^*(\rho))_{\rho\in\mathrm{Irr}(\overline{G})}) = -[\mathbf{d}_{\mathbb{Z}_p[\overline{G}]}(\mathbb{Z}_p[\overline{G}] \otimes^{\mathbb{L}}_{\Lambda(G)} C), t(C)_{\overline{G}}]$$

with $t(C)_{\overline{G}}$ the morphism $\mathbf{d}_{\mathbb{Q}_p^c[\overline{G}]}(\mathbb{Q}_p^c[\overline{G}] \otimes^{\mathbb{L}}_{\Lambda(G)} C) \to \mathbf{1}_{\mathbb{Q}_p^c[\overline{G}]}$ that corresponds via (4) to $((-1)^{r_G(C)(\rho)} t(C_\rho))_{\rho\in\mathrm{Irr}(\overline{G})}$ where $r_G(C)(\rho)$ is the integer defined in [11, Def. 3.11] *and $t(C_\rho)$ the morphism $\mathbf{d}_{\mathbb{Q}_p^c}(e_\rho\mathbb{Q}_p^c[\overline{G}] \otimes^{\mathbb{L}}_{\Lambda(G)} C) \to \mathbf{1}_{\mathbb{Q}_p^c}$ defined in* [11, Lem. 3.13(iv)].

Remark 2.3. The hypothesis of 'semisimplicity at ρ' and the definitions of $r_G(C)(\rho)$ and $t(C_\rho)$ are recalled explicitly in §5.2.4. However, in the special case that the complex $e_\rho\mathbb{Q}_p^c[\overline{G}] \otimes^{\mathbb{L}}_{\Lambda(G)} C$ is acyclic one knows that C is automatically semisimple at ρ, that $r_G(C)(\rho) = 0$ and that $t(C_\rho)$ is simply the canonical morphism induced by property A.e) of the determinant functor described in Appendix A. In particular, if $G = \Gamma$, $C = M[0]$ for a finitely generated torsion $\Lambda(\Gamma)$-module M for which both M^Γ and M_Γ are finite and ρ is the trivial character, then the equality of Theorem 2.2 is equivalent to the classical descent formula discussed in [34, p. 318, Ex. 13.12]. Upon appropriate specialisation, Theorem 2.2 also recovers the descent formalism proved (in certain commutative cases) by Greither and the first named author in [10, §8] and is therefore related to the earlier (commutative) work of Nekovář in [24, §11].

In §6 we will prove that it suffices to consider main conjectures of non-commutative Iwasawa theory in the case that G has no element of order p. Theorem 2.2 therefore represents a satisfactory resolution of the descent problem in this context. Indeed, in Part II (§6 - §8) of this article we shall combine Theorem 2.2 with the main results of [11] to describe the precise connection between main conjectures of non-commutative

Iwasawa theory (in the spirit of Coates et al [13]) and the appropriate case of the equivariant Tamagawa number conjecture. Other important applications of Theorem 2.2 are described in [8].

3. Generalized μ-invariants

The key ingredient in our proof of Theorem 2.1 is the construction of canonical 'characteristic series' in non-commutative Iwasawa theory. In this section we prepare for this construction by generalising the classical notion of μ-invariant.

3.1. The definition. In the sequel we write $\mu_\Gamma(M)$ for the 'μ-invariant' of a finitely generated $\Lambda(\Gamma)$-module M. For each complex C in $D^{\mathrm{p}}(\Lambda(\Gamma))$ we also set

$$\mu_\Gamma(C) := \sum_{i\in\mathbb{Z}}(-1)^i\mu_\Gamma(H^i(C)).$$

Let $\rho : G \to GL_n(\mathcal{O})$ be a continuous representation of G and write $E_\rho \cong \mathcal{O}^n$ for the associated representation module, where $\mathcal{O} = \mathcal{O}_L$ denotes the ring of integers of a finite extension L of $\mathbb{Q}_p$. We denote the corresponding L-linear representation $L\otimes_{\mathcal{O}} E_\rho$ by V_ρ. We fix a uniformising parameter π of $\mathcal{O}$ and denote the residue class field of $\mathcal{O}$ by κ. We write $\bar\rho$ for the reduction of ρ modulo π and denote the associated representation space by $\overline{E_\rho}$.

For each C in $D^{\mathrm{p}}(\Lambda(G))$ we set $C(\rho^*) := \mathcal{O}^n \otimes_{\mathbb{Z}_p} C$, regarded as an object of $D^{\mathrm{p}}(\Lambda_{\mathcal{O}}(G))$ via the action $g(x\otimes_{\mathbb{Z}_p} c^i) = \rho^*(g)(x)\otimes_{\mathbb{Z}_p} g(c^i)$ for each g in G, x in $\mathcal{O}^n$ and c^i in C^i. We then set

$$C_\rho := (\Lambda_{\mathcal{O}}(\Gamma)\otimes_{\mathcal{O}}\mathcal{O}^n)\otimes^{\mathbb{L}}_{\Lambda(G)} C \cong \Lambda_{\mathcal{O}}(\Gamma)\otimes^{\mathbb{L}}_{\Lambda_{\mathcal{O}}(G)} C(\rho^*) \tag{7}$$

and also

$$\mu(C,\rho) := \mu_\Gamma(C_\rho) \in \mathbb{Z}. \tag{8}$$

3.2. Basic properties.

Lemma 3.1. *Fix a continuous representation $\rho : G \to GL_n(\mathcal{O})$.*

(i) *If $C_1 \to C_2 \to C_3 \to C_1[1]$ is an exact triangle in $D^{\mathrm{p}}_{S^*}(\Lambda(G))$, then*

$$\mu(C_2,\rho) = \mu(C_1,\rho) + \mu(C_3,\rho).$$

(ii) *If $C \in D^{\mathrm{p}}_{S^*}(\Lambda(G))$ is cohomologically perfect, then*

$$\mu(C,\rho) = \mu(\mathrm{H}(C),\rho)$$

where $\mathrm{H}(C)$ denotes the complex with zero differentials and $\mathrm{H}(C)^i = H^i(C)$ for all i.

(iii) *If $C \in D^{\mathrm{p}}_{S}(\Lambda(G))$ is cohomologically perfect, then $\mu(C,\rho) = 0$.*

(iv) *If U is any closed normal subgroup of G such that $U \subseteq H\cap\ker(\rho)$, then for any C in $D^{\mathrm{p}}(\Lambda(G))$ we have*

$$\mu(C,\rho) = \mu(\Lambda(G/U)\otimes^{\mathbb{L}}_{\Lambda(G)} C,\rho).$$

Here the first μ-invariant is formed with respect to the group G and the second with respect to G/U.

(v) *If U is any open subgroup of G, then for any continuous representation $\psi : U \to GL_n(\mathcal{O})$ and any $C \in D^{\mathrm{p}}(\Lambda(G))$ one has*

$$\mu(C, \mathrm{Ind}_U^G \psi) = \mu(\mathrm{Res}_U^G C, \psi)$$

where the first μ-invariant is formed with respect to G and the second with respect to U. Here Res_U^G denotes the restriction functor from $\Lambda(G)$- to $\Lambda(U)$-modules.

Proof. For each D in $D^{\mathrm{p}}_{S^*}(\Lambda(G))$ all of the $\Lambda(\Gamma)$-modules $H^i(D_\rho)$ are both finitely generated and torsion. Since $\mu_\Gamma(-)$ is additive on exact sequences of finitely generated torsion $\Lambda(\Gamma)$-modules, claim (i) therefore follows from the long exact sequence of cohomology of the exact triangle $C_{1,\rho} \to C_{2,\rho} \to C_{3,\rho} \to C_{1,\rho}[1]$ in $D^{\mathrm{p}}(\Lambda(\Gamma))$ that is induced by the given triangle.
If $C \cong H^i(C)[i]$ for some i, then claim (ii) is clear. The general case can then be proved by induction with respect to the cohomological length: indeed, one need only combine claim (i) together with the exact triangles given by (good) truncation.
In order to prove claim (iii) it is sufficient by claim (ii) to consider the case $C = M[0]$ with M a $\Lambda(G)$-module that is finitely generated over $\Lambda(H)$. But then $H^i(C_\rho)$ is a finitely generated $\mathbb{Z}_p$-module for all $i \in \mathbb{Z}$ and so it is clear that $\mu(C, \rho) = 0$.
In the situation of claim (iv) there is a canonical isomorphism of $\Lambda_{\mathcal{O}}(G/U)$-modules

$$\Lambda_{\mathcal{O}}(G/U) \otimes^{\mathbb{L}}_{\Lambda_{\mathcal{O}}(G)} C(\rho^*) \cong C(\rho^*)_U \cong C_U(\rho^*) \cong \big(\Lambda(G/U) \otimes^{\mathbb{L}}_{\Lambda(G)} C\big)(\rho^*),$$

from which the claim follows immediately. Similarly, in the situation of claim (v) we have a canonical isomorphism $\mathrm{Ind}_U^G\big((\mathrm{Res}_U^G C)(\psi^*)\big) \cong C(\mathrm{Ind}_U^G \psi^*)$ which corresponds to

$$\begin{array}{rcl} \Lambda_{\mathcal{O}}(G) \otimes_{\Lambda_{\mathcal{O}}(U)} \big(\mathcal{O}^n \otimes_{\mathbb{Z}_p} \mathrm{Res}_U^G C\big) & \cong & (\Lambda(G) \otimes_{\Lambda(U)} \mathcal{O}^n) \otimes_{\mathbb{Z}_p} C, \\ g \otimes (a \otimes c) & \mapsto & (g \otimes a) \otimes g(c). \end{array}$$

Now we write Γ_U for the image of U in Γ under the natural projection and obtain

$$\begin{array}{rcl} \mu(C, \mathrm{Ind}_U^G \psi) & = & \mu_\Gamma(\Lambda_{\mathcal{O}}(\Gamma) \otimes^{\mathbb{L}}_{\Lambda_{\mathcal{O}}(G)} C(\mathrm{Ind}_U^G \psi^*)) \\ & = & \mu_\Gamma(\Lambda_{\mathcal{O}}(\Gamma) \otimes^{\mathbb{L}}_{\Lambda_{\mathcal{O}}(G)} \mathrm{Ind}_U^G\big((\mathrm{Res}_U^G C)(\psi^*)\big)) \\ & = & \mu_\Gamma(\Lambda_{\mathcal{O}}(\Gamma) \otimes_{\Lambda_{\mathcal{O}}(\Gamma_U)} \big(\Lambda_{\mathcal{O}}(\Gamma_U) \otimes^{\mathbb{L}}_{\Lambda_{\mathcal{O}}(U)} (\mathrm{Res}_U^G C)(\psi^*)\big)) \\ & = & \mu_{\Gamma_U}(\Lambda_{\mathcal{O}}(\Gamma_U) \otimes^{\mathbb{L}}_{\Lambda_{\mathcal{O}}(U)} (\mathrm{Res}_U^G C)(\psi^*)) \\ & = & \mu(\mathrm{Res}_U^G C, \psi) \end{array}$$

as had to be shown. □

3.3. Module theory. In order to make a closer examination of the μ-invariant defined in (8) we recall some standard module theory.
We write $\mathrm{Jac}(\Lambda(G))$ for the Jacobson radical of $\Lambda(G)$ and $\prod_{i \in I} A_i$ for the Wedderburn decomposition of the finite dimensional semisimple $\mathbb{F}_p$-algebra $A := A(G) := \Lambda(G)/\mathrm{Jac}(\Lambda(G))$ (so I is finite). Let $R_i = Aa_i$ be a representative for the unique isomorphism class of simple A_i-modules with a_i some orthogonal primitive idempotent of $A(G)$, always assuming that $A_1 = \mathbb{F}_p = R_1$; the corresponding representations we denote by $\psi_i : G \to GL(R_i)$ for i in I. For each index i we fix an idempotent e_i of $\Lambda(G)$ which is a pre-image of a_i under the projection $\Lambda(G) \twoheadrightarrow \Lambda(G)/\mathrm{Jac}(\Lambda(G))$.
We consider the projective $\Lambda(G)$-modules $X_i := \Lambda(G)e_i$ and projective $\Omega(G)$-modules $Y_i := X_i/pX_i$. They are projective hulls of R_i since $A(G) \otimes_{\Lambda(G)} X_i = A(G) \otimes_{\Omega(G)} Y_i = R_i$.

Every finitely generated projective $\Lambda(G)$-module X, resp. $\Omega(G)$-module Y, decomposes in a unique way as a direct sum

$$X = \bigoplus_{i\in I} X_i^{\langle X,X_i\rangle}, \text{ resp. } Y = \bigoplus_{i\in I} Y_i^{\langle Y,Y_i\rangle},$$

for suitable natural numbers $\langle X, X_i\rangle$, resp. $\langle Y, Y_i\rangle$. We write $l_{R_i}(\psi)$ for the multiplicity of the occurrence of R_i in a $\mathbb{F}_p$-linear representation ψ and

$$\chi(G, M) := \prod_i |\mathrm{H}_i(G, M)|^{(-1)^i},$$

if this is finite, for the Euler-Poincaré-characteristic of a $\Lambda(G)$-module M.

LEMMA 3.2. *Let Y be a finitely generated projective $\Omega(G)$-module.*

(i) $\Lambda(\Gamma) \otimes_{\Lambda(G)} Y$ *is naturally isomorphic to* $\Omega(\Gamma) \otimes_{\Omega(G)} Y = \Omega(\Gamma)^{<Y,Y_1>}$ *and thus* $\chi(G, Y) = p^{\langle Y,Y_1\rangle}$.

(ii) $\langle Y(\psi^*), Y_1\rangle = \sum_{i\in I} l_{R_i}(\psi)\dim_{\mathbb{F}_p}(\mathrm{End}_{\Omega(G)}(R_i))\langle Y, Y_i\rangle$.

Proof. For each index i the module $\Omega(\Gamma) \otimes_{\Omega(G)} Y_i$ is isomorphic to $\Omega(\Gamma)^{n_i}$ for some natural number n_i. Since then $\mathbb{F}_p^{n_i} \cong \mathbb{F}_p \otimes_{\Omega(G)} Y_i \cong A_1 \otimes_{\Omega(G)} Y_i = a_1R_i$ is isomorphic to R_1 if $i = 1$ and is zero if $i \neq 1$, the first claim follows.
Claim (ii) is true because $\dim_{\mathbb{F}_p}(\mathrm{End}_{\Omega(G)}(R_i))\langle Y, Y_i\rangle = \dim_{\mathbb{F}_p}(\mathrm{Hom}_{\Omega(G)}(Y, R_i))$ and $\mathrm{Hom}_{\Omega(G)}(Y(\psi_i^*), R_1)$ is isomorphic to $\mathrm{Hom}_{\Omega(G)}(Y, R_i)$ (cf. [1, Prop. 4.1, Lem. 4.4]). □

3.4. THE REGULAR CASE. In this section we study the μ-invariants of §3.1 in the case that G has no element of order p.

3.4.1. *Pairings.* For a field K we write $R_K(G)$ for the Grothendieck group of the category of finite-dimensional continuous K-linear representations of G which have finite image. The tensor product induces a structure of rings on both $R_L(G)$ and $R_\kappa(G)$ and there exists a canonical surjective homomorphisms of rings $R_L(G) \twoheadrightarrow R_\kappa(G)$ that is induced by reducing modulo π any G-stable $\mathcal{O}$-lattice E of an representation V of the above type (cf. [28]).

PROPOSITION 3.3. *Assume that G has no element of order p.*

(i) *If $C \in D^{\mathrm{p}}_{S^*}(\Lambda(G))$, then for each continuous representation $\rho : G \to GL_n(\mathcal{O})$ one has*

$$\mu(C, \rho) = \sum_{i\in\mathbb{Z}} (-1)^i \mu_\Gamma\big(\Lambda(\Gamma) \otimes^{\mathbb{L}}_{\Lambda(G)} (\overline{E_\rho}^* \otimes_{\mathbb{F}_p} \mathrm{gr}(H^i(C)_{\mathrm{tor}})[0])\big)$$

where $\overline{E_\rho}^*$ denotes the contragredient module $\mathrm{Hom}_\kappa(\overline{E_\rho}, \kappa)$ while for a $\mathbb{Z}_p$-module M endowed with the p-adic filtration we denote by $\mathrm{gr}(M)$ the associated graded $\mathbb{F}_p$-module.

(ii) *The μ-invariant induces a $\mathbb{Z}$-bilinear pairing*

$$\mu(-,-) : K_0(D^{\mathrm{p}}_{S^*}(\Lambda(G))) \times R_L(G) \to \mathbb{Z}.$$

This pairing induces a $\mathbb{Z}$-bilinear pairing of the form

$$\mu(-,-) : K_0(D^{\mathrm{p}}_{S^*}(\Lambda(G))) \times R_\kappa(G) \to \mathbb{Z}.$$

(iii) *If $C \in D^{\mathrm{p}}_{S^*}(\Lambda(G))$ and $i \in I$, then the integer $\mu(C, \psi_i)$ defined by claim (ii) is divisible by $\dim_{\mathbb{F}_p}(\mathrm{End}_{\Omega(G)}(R_i))$.*

(iv) *If $C \in D^{\mathrm{p}}_{S}(\Lambda(G))$, then $\mu(C, \psi_i) = 0$ for all $i \in I$.*

Proof. To prove claim (i), we write $\mu'(C,\bar{\rho})$ for the term on the right hand side of the claimed equality. Since $\mu'(C,\bar{\rho}) = \mu'(\mathrm{H}(C),\bar{\rho})$ by definition and $\mu(C,\rho) = \mu(\mathrm{H}(C),\rho)$ by Lemma 3.1 (ii) we need only consider the case where $C \cong M[0]$ with M in $\mathfrak{M}_{S^*}(G)$. Further, since both μ-invariants are additive on exact triangles (cf. Lemma 3.1(i)), it is actually sufficient to prove the following two special cases (note that M/M_{tor} belongs to $\mathfrak{M}_S(G)$ for all M in $\mathfrak{M}_{S*}(G)$):
1.) If M is in $\mathfrak{M}_S(G)$, then both $H^i(\Lambda(\Gamma)\otimes^{\mathbb{L}}_{\Lambda(G)}(\overline{E_\rho}\otimes_{\mathbb{F}_p}\mathrm{gr}(M_{\mathrm{tor}})[0]))$ and $H^i(A_\rho)$ are finitely generated $\mathbb{Z}_p$-modules and thus $\mu(C,\rho) = 0 = \mu'(C,\bar{\rho})$.
2.) If $p^nM = 0$ for some n, we argue by induction on n. For $n = 1$ the isomorphism $\overline{E_\rho}^*\otimes_{\mathbb{F}_p}\mathrm{gr}(M_{\mathrm{tor}}) \cong \overline{E_\rho}^*\otimes_{\mathbb{F}_p} M \cong \overline{E_\rho}^*\otimes_{\mathbb{Z}_p} M$ implies the equality of the μ-invariants. For $n > 1$ one uses dévissage and again the additivity of both μ-invariants.
To prove the existence of the first pairing in claim (ii) it suffices to show that $\mu(C,\rho)$ depends only on the space V_ρ. To this end we assume that $E_{\rho'}$ is another G-stable lattice in V_ρ and we have to show that $\mu(C,\rho) = \mu(C,\rho')$. By Proposition 3.1(ii) and dévissage we may assume that $C \cong M[0]$ with $pM = 0$ and similarly that $E^*_{\rho'} \subseteq E^*_\rho$ with $\pi T = 0$ for $T := E^*_\rho/E^*_{\rho'}$. In this situation there is an exact sequence

$$0 \to M\otimes_{\mathbb{F}_p} T \to M\otimes_{\mathbb{F}_p}\overline{E^*_{\rho'}} \to M\otimes_{\mathbb{F}_p}\overline{E^*_\rho} \to M\otimes_{\mathbb{F}_p} T \to 0$$

of $\Lambda(G)$-modules. The required claim now follows from the known additivity of μ-invariants and the fact that the $\Lambda(G)$-modules $M\otimes_{\mathbb{F}_p}\overline{E^*_\rho}$ and $M\otimes_{\mathbb{F}_p}\overline{E^*_{\rho'}}$ are isomorphic to $M(\rho^*)$ and $M((\rho')^*)$ respectively. The second assertion of claim (ii) then follows from claim (i).
To prove claim (iii) we may assume that $C \cong M[0]$ with M a finitely generated $\Omega(G)$-module. After choosing a finite resolution P of M by finitely generated projective $\Omega(G)$-modules and using the additivity of $\mu(-,\psi_i)$ on short exact sequences the proof is immediately reduced to the case of a projective $\Omega(G)$-module because $\mu(P,\psi_i) = \sum_{j\in\mathbb{Z}}(-1)^j\mu(P^j,\psi_i)$. But for every projective $\Omega(G)$-module Y, considered also as a $\Lambda(G)$-module, and for each $i \in I$ we have $\mu(Y,\psi_i) = \langle Y(\psi_i^*), Y_1\rangle = \dim_{\mathbb{F}_p}(\mathrm{End}_{\Omega(G)}(R_i))\langle Y, Y_i\rangle$ by Lemma 3.2(ii).
Claim (iv) follows from Lemma 3.1(iii). □

If G has no element of order p, then Proposition 3.3(iii) allows us to define an integer $\mu^i_{\Lambda(G)}(C)$ for each complex C in $D^{\mathrm{p}}_{S^*}(\Lambda(G))$ and each index i in I by setting

$$\mu^i_{\Lambda(G)}(C) := \mu(C,\psi_i)\cdot\dim_{\mathbb{F}_p}(\mathrm{End}_{\Omega(G)}(R_i))^{-1}.$$

3.4.2. *K-groups.* We continue to assume that G has no element of order p and write $\mathfrak{D}(G)$ for the category of finitely generated $\Lambda(G)$-modules that are annihilated by a power of p. Then, by dévissage and lifting of idempotents, one obtains the following isomorphisms

$$K_0(\mathfrak{D}(G)) \cong K_0(\Omega(G)) \cong K_0(A(G)) \cong \mathbb{Z}^I \tag{9}$$

where the i-th basis vector of the free $\mathbb{Z}$-module on the right corresponds to the classes of Y_i in $K_0(\mathfrak{D}(G))$ and $K_0(\Omega(G))$. Lemma 3.2(ii) implies that if M belongs to $\mathfrak{D}(G)$, then the map in (9) sends the class of M to the vector

$$\mu(M) := (\mu^i_{\Lambda(G)}(M[0]))_{i\in I}. \tag{10}$$

The proof of the following result is a natural generalization of that given by Kato in [20, Prop. 8.6].

PROPOSITION 3.4. *If G has no element of order p, then there are natural isomorphisms*

$$\begin{aligned} K_0(\mathfrak{M}_{S^*}(G)) &\cong K_0(\mathfrak{M}_S(G)) \oplus K_0(\Omega(G)), \\ K_1(\Lambda(G)_{S^*}) &\cong K_1(\Lambda(G)_S) \oplus K_0(\Omega(G)). \end{aligned}$$

The first of these isomorphisms is induced by the embeddings of categories $\mathfrak{M}_S(G) \subset \mathfrak{M}_{S^*}(G)$ and $\mathfrak{D}(G) \subset \mathfrak{M}_{S^*}(G)$ combined with the first isomorphism in (9). The second isomorphism depends on the choice of a splitting of $K_1(\Lambda(G)_{S^*}) \twoheadrightarrow K_0(\mathfrak{D}(G)) \cong \mathbb{Z}^I$; once we have fixed an idempotent e_i for each $i \in I$ a natural choice is induced by sending the i-th basis vector of $\mathbb{Z}^I$ to the class of $f_i := 1 + (p-1)e_i$ in $K_1(\Lambda(G)_{S^*})$.

Proof. We first prove the surjectivity of the homomorphism ∂_2 in the long exact localisation sequence of K-theory

$$K_2(\Lambda(G)_{S^*}) \xrightarrow{\partial_2} K_1(\Omega(G)_S) \to K_1(\Lambda(G)_S) \to K_1(\Lambda(G)_{S^*}) \xrightarrow{\partial_1} K_0(\Omega(G)_S).$$

But, since $\Omega(G)_S$ is semi-local, the natural homomorphism $\Omega(G)_S^* \twoheadrightarrow K_1(\Omega(G)_S)$ is surjective and hence $K_1(\Omega(G)_S)$ is generated by the image of S. The surjectivity of ∂_2 thus follows from the fact that for each $f \in S$ one has $\partial_2(\{f,p\}) = [f] \in K_1(\Omega(G)_S)$, where $\{f,p\}$ denotes the symbol of f and p in $K_2(\Lambda(G)_{S^*})$ (indeed, the latter equality is proved by the argument of [18, Prop. 5]). From the above exact sequence we therefore obtain an exact sequence

$$0 \to K_1(\Lambda(G)_S) \to K_1(\Lambda(G)_{S^*}) \xrightarrow{\partial_1} K_0(\Omega(G)_S). \tag{11}$$

We next consider the composite map

$$\mathbb{Z}^I \to K_1(\Lambda(G)_{S^*}) \xrightarrow{\partial_1} K_0(\Omega(G)_S) \xrightarrow{\alpha} K_0(B(G)) \cong \mathbb{Z}^J. \tag{12}$$

Here the first map is given by sending the i-th basis vector of $\mathbb{Z}^I$ to the class of $f_i = 1 + (p-1)e_i$ (note that $\Lambda(G)/\Lambda(G)f_i \cong Y_i$), we set $B(G) := \Omega(G)_S/\mathrm{Jac}(\Omega(G)_S)$, the canonical map α is injective by [2, Chap. IX, Prop. 1.3] and the index set J parametrizes the isomorphism classes of simple modules over the semisimple Artinian ring $B(G)$. Let N be any closed normal subgroup of G which is both pro-p and open in H. Then it is straightforward to check that (12) factorizes through the composite

$$\mathbb{Z}^I \cong K_0(\Omega(G/N)) \xrightarrow{\beta} K_0(\Omega(G/N)_S) \xrightarrow{\gamma} K_0(B(G)). \tag{13}$$

Here the surjective map β comes from the exact localisation sequence and the isomorphism γ is induced from the fact that $\Omega(G)_S \twoheadrightarrow B(G)$ factors through the composite $\Omega(G)_S \twoheadrightarrow \Omega(G/N)_S \twoheadrightarrow B(G)$ by the proof of [13, Lem. 4.3] and the fact that $\mathrm{Jac}(\Omega(G/N)_S)$ is a nilpotent ideal. It follows that the map in (12) is surjective and hence that ∂_1 is surjective and α is bijective.

If $\mathfrak{D}_S(G)$ denotes the category of finitely generated $\Lambda(G)_S$-modules which are $\mathbb{Z}_p$-torsion, then we have shown that the composite map

$$K_0(\mathfrak{D}_S(G)) \cong K_0(\Omega(G)_S) \cong K_0(B(G)) = \mathbb{Z}^J \tag{14}$$

is bijective and that $|J| \leq |I|$.

By combining (11) with the surjectivity of ∂_1, the bijectivity of (14) and the assertion of Lemma 3.5(i) below we obtain an exact sequence

$$0 \to K_1(\Lambda(G)_S) \to K_1(\Lambda(G)_{S^*}) \xrightarrow{\partial_1'} K_0(\mathfrak{D}(G)) \to 0. \tag{15}$$

Further, it is straightforward to show that, with respect to the isomorphism $K_0(\mathfrak{D}(G)) \cong \mathbb{Z}^I$ of (9), this sequence is split by the map which sends the i-th basis vector of $\mathbb{Z}^I$ to the class of f_i in $K_1(\Lambda(G)_{S^*})$. This proves the final assertion of Proposition 3.4.
We next consider the following diagram with exact rows

$$
\begin{tikzcd}
0 \arrow[r] & \mathrm{im}(\iota_{S^*}) \arrow[r] \arrow[d, equal] & K_1(\Lambda(G)_S) \arrow[r] \arrow[d] & K_0(\mathfrak{M}_S(G)) \arrow[r] \arrow[d, "\delta"] & 0 \\
0 \arrow[r] & \mathrm{im}(\iota_{S^*}) \arrow[r] & K_1(\Lambda(G)_{S^*}) \arrow[r] & K_0(\mathfrak{M}_{S^*}(G)) \arrow[r] & 0,
\end{tikzcd}
\tag{16}
$$

where ι_{S^*} is the natural map $K_1(\Lambda(G)) \to K_1(\Lambda(G)_{S^*})$, δ is induced by the embedding of categories $\mathfrak{M}_S(G) \subset \mathfrak{M}_{S^*}(G)$ and [13, Prop. 3.4] implies that each row is indeed exact. By applying the snake lemma to this diagram and comparing with the sequence (15) we obtain an exact sequence of the form

$$0 \to K_0(\mathfrak{M}_S(G)) \xrightarrow{\delta} K_0(\mathfrak{M}_{S^*}(G)) \to K_0(\mathfrak{D}(G)) \to 0.$$

The first assertion of Proposition 3.4 now follows because this sequence is split by the homomorphism $K_0(\mathfrak{D}(G)) \to K_0(\mathfrak{M}_{S^*}(G))$ that is induced by the embedding of categories $\mathfrak{D}(G) \subset \mathfrak{M}_{S^*}(G)$. □

LEMMA 3.5.

(i) *The exact scalar extension functor from $\Lambda(G)$-mod to $\Lambda(G)_S$-mod identifies $\mathfrak{D}(G)$ with a full subcategory of $\mathfrak{D}_S(G)$ and induces an isomorphism $K_0(\mathfrak{D}(G)) \cong K_0(\mathfrak{D}_S(G))$.*
(ii) *The natural map $\iota : K_0(\mathfrak{D}(G)) \to K_0(\mathfrak{M}_{S^*}(G))$ is injective.*
(iii) *The natural map $K_0(\Omega(G/N)) \to K_0(\Omega(G/N)_S)$ is bijective.*

Proof. The assignment $M \mapsto (\mu^i_{\Lambda(G)}(M[0]))_{i\in I}$ induces a homomorphism

$$\mu : K_0(\mathfrak{M}_{S^*}(G)) \to \mathbb{Z}^I.$$

Now from (9) and (10) we know that $\mu \circ \iota$ is bijective whilst from Lemma 3.1(iii) we know $\delta(K_0(\mathfrak{M}_S(G))) \subseteq \ker(\mu)$ where δ is the homomorphism in diagram (16). This implies that ι is injective (so proving claim (ii)), that μ is surjective and that $|I| \le |J|$. But $|J| \le |I|$ (see just after (14)) and so $|I| = |J|$.
Since $|I| = |J|$ the isomorphisms of (9) and (13) combine to imply that the natural map $K_0(\mathfrak{D}(G)) \to K_0(\mathfrak{D}_S(G))$ is bijective (proving claim (i)).
In a similar way, claim (iii) follows by combining the equality $|I| = |J|$ together with the surjectivity of the map β in (13) and the definition of the index set J (in (12)). □

4. CHARACTERISTIC SERIES

In this section we associate a canonical 'characteristic series' to each complex in $D^{\mathrm{p}}_{\tilde{S}}(\Lambda(G))$. This construction extends the notion of 'algebraic p-adic L-functions' introduced in [6] and hence refines the 'Akashi series' introduced by Coates, Schneider and Sujatha in [12]. It will also play a key role in our proof of Theorem 2.1 (see in particular the proof of Lemma 5.7).

4.1. The definition. If M is any compact (left) $\Lambda(G)$-module, then the completed tensor product

$$\mathrm{I}_H^G(M) := \Lambda(G)\hat{\otimes}_{\Lambda(H)}\mathrm{Res}_H^G(M)$$

has a natural structure as a compact $\Lambda(G)$-module via multiplication on the left. With respect to this action, one obtains a (well-defined) endomorphism Δ_γ of $\mathrm{I}_H^G(M)$ by setting

$$\Delta_\gamma(x \otimes_{\Lambda(H)} y) := x\tilde{\gamma}^{-1} \otimes_{\Lambda(H)} \tilde{\gamma}(y)$$

for each $x \in \Lambda(G)$ and $y \in M$, where $\tilde{\gamma}$ is any lift of γ through the natural projection $G \to \Gamma$. It is easily checked that Δ_γ is independent of the precise choice of $\tilde{\gamma}$. Further, if M belongs to $\mathfrak{M}_{S^*}(G)$, then [6, Lem. 2.1] implies that

$$\delta_\gamma := \mathrm{id}_{\mathrm{I}_H^G(M)} - \Delta_\gamma$$

induces an automorphism of the (finitely generated) $\Lambda(G)_{S^*}$-module $\mathrm{I}_H^G(M)_{S^*}$.
Now the ring $\Lambda(G)_{S^*}$ is both noetherian and regular [17, Prop. 4.3.4] and so $K_1(\Lambda(G)_{S^*})$ is naturally isomorphic to the group $G_1(\Lambda(G)_{S^*})$ that is generated (multiplicatively) by symbols $\langle \alpha \mid M\rangle$ where α is an automorphism of a finitely generated $\Lambda(G)_{S^*}$-module M (cf. [29, Th. 16.11]). For each complex C in $D^{\mathrm{p}}_{S^*}(\Lambda(G))$ we may therefore define an element of $K_1(\Lambda(G)_{S^*})$ by setting

$$\mathrm{char}^*_{G,\gamma}(C) := \prod_{i\in\mathbb{Z}} \langle \delta_\gamma \mid \mathrm{I}_H^G(H^i(C))_{S^*}\rangle^{(-1)^i}.$$

For each C in $D^{\mathrm{p}}_{\tilde{S}}(\Lambda(G))$ we also define an 'equivariant multiplicative μ-invariant' in $\mathrm{im}(K_1(\Lambda(G)[\frac{1}{p}]) \to K_1(\Lambda(G)_{S^*}))$ by setting

$$\chi^\mu_G(C) := \begin{cases} \langle \sum_{i\in I} p^{\mu^i_{\Lambda(G)}(C)} e_i | \Lambda(G)_{S^*}\rangle, & \text{if } C \text{ belongs to } D^{\mathrm{p}}_{\tilde{S}}(\Lambda(G)) \setminus D^{\mathrm{p}}_{S}(\Lambda(G)),\\ 1, & \text{if } C \text{ belongs to } D^{\mathrm{p}}_{S}(\Lambda(G)) \end{cases}$$

where the integer $\mu^i_{\Lambda(G)}(C)$ is as defined at the end of §3.4.1 (this definition makes sense because if C belongs to $D^{\mathrm{p}}_{\tilde{S}}(\Lambda(G)) \setminus D^{\mathrm{p}}_{S}(\Lambda(G))$, then G has no element of order p).

Definition 4.1. For each C in $D^{\mathrm{p}}_{\tilde{S}}(\Lambda(G))$ the *characteristic series of* C is the element

$$\mathrm{char}_{G,\gamma}(C) := \chi^\mu_G(C) \cdot \mathrm{char}^*_{G,\gamma}(C)$$

of $K_1(\Lambda(G)_{S^*})$.

Remark 4.2. If $G = \Gamma$, then $\Lambda(G)_{S^*} = Q(\Gamma)$ and so there is a natural isomorphism $\iota : K_1(\Lambda(G)_{S^*}) \cong Q(\Gamma)^\times$. Further, if M is any finitely generated torsion $\Lambda(\Gamma)$-module, then $\iota(\mathrm{char}_{G,\gamma}(M[0])) = (1+T)^{-\lambda(M)}\mathrm{char}_T(M)$ where $\lambda(M)$ is the Iwasawa λ-invariant of M and $\mathrm{char}_T(M)$ is the characteristic polynomial of M with respect to the variable $T = \gamma - 1$. (For a proof of this fact see [6, Lem. 2.3]).

Remark 4.3. In Proposition 4.7(i) we will prove that if G has no element of order p, then $\mathrm{char}_{G,\gamma}(C)$ is a 'characteristic element for C' in the sense of [13, (33)] (and see also Remark 6.2 in this regard). In [6, Th. 4.1] it is proved that this is also true if G has rank one as a p-adic Lie group. In these cases at least, we may therefore regard $\mathrm{char}_{G,\gamma}(C)$ as the canonical 'algebraic p-adic L-function' associated to C.

4.2. Basic properties.

Lemma 4.4. *If $C_1 \to C_2 \to C_3 \to C_1[1]$ is an exact triangle in $D^{\mathrm{p}}_{\tilde{S}}(\Lambda(G))$, then* $\mathrm{char}_{G,\gamma}(C_2) = \mathrm{char}_{G,\gamma}(C_1)\mathrm{char}_{G,\gamma}(C_3)$.

Proof. If G has an element of order p, then each complex C_j belongs to $D^{\mathrm{p}}_{S}(\Lambda(G))$ and so $\chi^{\mu}_G(C_j) = 1$. If G has no element of order p, then the equality $\chi^{\mu}_G(C_2) = \chi^{\mu}_G(C_1)\chi^{\mu}_G(C_3)$ follows from Lemma 3.1(i) and Proposition 3.3(iv). The equality $\mathrm{char}^*_{G,\gamma}(C_2) = \mathrm{char}^*_{G,\gamma}(C_1)\mathrm{char}^*_{G,\gamma}(C_3)$ is equivalent to that of [6, Prop. 3.1]. □

Let U be a closed subgroup of H that is normal in G and set $G_1 := G/U, H_1 := H/U$ and $S_1 := S_{G_1,H_1}$. Then there exists a natural ring homomorphism $\pi_{G_1} : \Lambda(G)_{S^*} \to \Lambda(G_1)_{S_1^*}$ and hence an induced homomorphism of groups

$$\pi_{G_1,*} : K_1(\Lambda(G)_{S^*}) \to K_1(\Lambda(G_1)_{S_1^*}).$$

Lemma 4.5. *Let G_1, H_1 and S_1 be as above. Fix C in $D^{\mathrm{p}}_{\tilde{S}}(\Lambda(G))$ and assume that either C belongs to $D^{\mathrm{p}}_{S}(\Lambda(G))$ or that G_1 has no element of order p. Then $C_1 := \Lambda(G_1)\hat{\otimes}^{\mathbb{L}}_{\Lambda(G)}C$ belongs to $D^{\mathrm{p}}_{\tilde{S}_1}(\Lambda(G_1))$ and $\pi_{G_1,*}(\mathrm{char}_{G,\gamma}(C)) = \mathrm{char}_{G_1,\gamma}(C_1)$.*

Proof. If C belongs to $D^{\mathrm{p}}_{S}(\Lambda(G))$, then C_1 belongs to $D^{\mathrm{p}}_{S_1}(\Lambda(G_1))$ and so $\pi_{G_1,*}(\chi^{\mu}_G(C)) = \pi_{G_1,*}(1) = 1 = \chi^{\mu}_{G_1}(C_1)$. If C does not belong to $D^{\mathrm{p}}_{S}(\Lambda(G))$, then neither G or G_1 has an element of order p and the equality $\pi_{G_1,*}(\chi^{\mu}_G(C)) = \chi^{\mu}_{G_1}(C_1)$ follows from Lemma 3.1(iv) and Proposition 3.3(iv). The equality $\pi_{G_1,*}(\mathrm{char}^*_{G,\gamma}(C)) = \mathrm{char}^*_{G_1,\gamma}(C_1)$ is equivalent to that of [6, Prop. 3.2]. □

In the next result we fix an open subgroup U of G and set $H_U := H \cap U$ and $\Gamma_U := U/H_U$. We use the natural isomorphism $\Gamma_U \cong HU/H$ to regard Γ_U as an open subgroup of Γ, we set $d_U := [\Gamma : \Gamma_U]$ and write γ_U for the topological generator γ^{d_U} of Γ_U. We set $S_U := S_{U,H_U}$ and note that $\Lambda(G)$, resp. $\Lambda(G)_S$, resp. $\Lambda(G)_{S^*}$ is a free $\Lambda(U)$-, resp. $\Lambda(U)_{S_U}$-, resp. $\Lambda(U)_{S_U^*}$-module, of rank $[G : U]$. In particular, restriction of scalars gives natural functors $D^{\mathrm{p}}_{S}(\Lambda(G)) \to D^{\mathrm{p}}_{S_U}(\Lambda(U))$ and $D^{\mathrm{p}}_{S^*}(\Lambda(G)) \to D^{\mathrm{p}}_{S_U^*}(\Lambda(U))$ and a natural homomorphism

$$\mathrm{res}_{U,*} : K_1(\Lambda(G)_{S^*}) \to K_1(\Lambda(U)_{S_U^*}).$$

Lemma 4.6. *Let G and U be as above and fix C in $D^{\mathrm{p}}_{\tilde{S}}(\Lambda(G))$. Then $C_1 := \mathrm{res}^G_U C$ belongs to $D^{\mathrm{p}}_{\tilde{S}}(\Lambda(G))$ and $\mathrm{res}_{U,*}(\mathrm{char}_{G,\gamma}(C)) = \mathrm{char}_{U,\gamma_U}(C_1)$.*

Proof. We prove first that $\mathrm{res}_{U,*}(\chi^{\mu}_G(C)) = \chi^{\mu}_U(C_1)$. The complex C belongs to $D^{\mathrm{p}}_{S}(\Lambda(G))$ if and only if C_1 belongs to $D^{\mathrm{p}}_{S_U}(\Lambda(U))$ and in this case $\mathrm{res}_{U,*}(\chi^{\mu}_G(C)) = 1 = \chi^{\mu}_U(C_1)$. We may therefore assume that C belongs to $D^{\mathrm{p}}_{S^*}(\Lambda(G)) \setminus D^{\mathrm{p}}_{S}(\Lambda(G))$ and hence that G (and also U) has no element of order p. By the same argument as used in the proof of Proposition 3.3(iii), we can also assume that $C = Y_a[0]$ for some index a in I. Then for each $i \in I$ one has $\mu^i_{\Lambda(G)}(C) = \langle Y_a, Y_i\rangle = \delta_{ai}$ and so

$$\mathrm{res}_{U,*}(\chi^{\mu}_G(C)) = \mathrm{res}_{U,*}(\langle pe_a|\Lambda(G)_{S^*}\rangle) = \mathrm{res}_{U,*}(\langle p|\Lambda(G)_{S^*}e_a\rangle).$$

We write $\{f_j : j \in J\}$ for the idempotents of $\Lambda(U)$ that are analogous to the idempotents e_i of $\Lambda(G)$ defined in §3.3 and $\{X(U)_j : j \in J\}$, resp. $\{Y(U)_j : j \in J\}$, for the submodules of $\Lambda(U)$, resp. $\Omega(U)$, analogous to the modules X_i, resp. Y_i, in §3.3. For each j in J we set $m_j := \langle \Lambda(G)e_a, X(U)_j\rangle = \langle Y_a, Y(U)_j\rangle$. Then the $\Lambda(U)_{S_U^*}$-module

$\Lambda(G)_{S^*}e_a$ is isomorphic to $\bigoplus_{j\in J}(\Lambda(U)_{S_U^*}f_j)^{m_j}$ and hence the last displayed expression is equal to $\langle\sum_{j\in J}p^{m_j}f_j|\Lambda(U)_{S_U^*}\rangle = \chi_U^\mu(\mathrm{res}_U^G Y_a[0])$, as required.
It remains to prove that $\mathrm{res}_{U,*}(\mathrm{char}^*_{G,\gamma}(C)) = \mathrm{char}^*_{U,\gamma_U}(C_1)$. To do this we may assume that $C = M[0]$ for a module M in $\mathfrak{M}_{S^*}(G)$ so that $\mathrm{char}^*_{G,\gamma}(C)$ is equal to $\langle\delta_\gamma \mid \mathrm{I}_H^G(M)_{S^*}\rangle$. Now the $\Lambda(U)_{S_U^*}$-module $\mathrm{I}_H^G(M)_{S^*}$ is equal to the direct sum $\bigoplus_{i=0}^{i=d_U-1}\Delta_{\gamma^i}(\mathrm{I}_{H_U}^U(M)_{S_U^*}) \cong \bigoplus_{i=0}^{i=d_U-1}\mathrm{I}_{H_U}^U(M)_{S_U^*}$ where the isomorphism identifies each translate $\Delta_{\gamma^i}(\mathrm{I}_{H_U}^U(M)_{S_U^*})$ with $\mathrm{I}_{H_U}^U(M)_{S_U^*}$ in the natural way. With respect to this decomposition δ_γ is the automorphism given by the $d_U \times d_U$ matrix

$$\begin{pmatrix} \mathrm{id} & -\mathrm{id} & 0 & \cdots & \cdots & 0 \\ 0 & \mathrm{id} & -\mathrm{id} & 0 & \cdots & 0 \\ 0 & 0 & \mathrm{id} & -\mathrm{id} & 0 \;\cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & 0 & \mathrm{id} & -\mathrm{id} \\ -\Delta_{\gamma^{d_U}} & 0 & \cdots & \cdots & 0 & \mathrm{id} \end{pmatrix}.$$

Elementary row and column operations show that this automorphism represents the same element of $K_1(\Lambda(U)_{S_U^*})$ as does the automorphism α of $\bigoplus_{i=0}^{i=d_U-1}\mathrm{I}_{H_U}^U(M)_{S_U^*}$ that acts as $\mathrm{id} - \Delta_{\gamma^{d_U}}$ on the last direct summand and as the identity on all other summands. The required result thus follows because, since $\delta_{\gamma_U} := \mathrm{id} - \Delta_{\gamma^{d_U}}$, the class of α in $K_1(\Lambda(U)_{S_U^*})$ is equal to $\langle\delta_{\gamma_U} \mid \mathrm{I}_{H_U}^U(M)_{S_U^*}\rangle =: \mathrm{char}^*_{U,\gamma_U}(C_1)$. □

4.3. The proof of Theorem 2.1. We deduce Theorem 2.1 as a consequence of the following result.

Proposition 4.7. *Assume that G has no element of order p.*

(i) *For each C in $D^{\mathrm{p}}_{S^*}(\Lambda(G))$ one has $\partial_G(\mathrm{char}_{G,\gamma}(C)) = \sum_{i\in\mathbb{Z}}(-1)^{i+1}[H^i(C)]$.*
(ii) *There exists a (unique) homomorphism $\chi_{G,\gamma}$ from $K_0(\mathfrak{M}_{S^*}(G))$ to $K_1(\Lambda(G)_{S^*})$ that simultaneously satisfies the following conditions:-*
   (a) *for each M in $\mathfrak{M}_{S^*}(G)$ one has $\chi_{G,\gamma}([M]) = \mathrm{char}_{G,\gamma}(M[1])$;*
   (b) *$\chi_{G,\gamma}$ is right inverse to ∂_G;*
   (c) *$\chi_{G,\gamma}$ respects the isomorphisms of Proposition 3.4;*
   (d) *Let U be a closed subgroup of H that is normal in G and such that $\overline{G} := G/U$ has no element of order p. Set $\overline{H} := H/U$ and $\overline{S} := S_{\overline{G},\overline{H}}$. Then there is a commutative diagram*

$$\begin{array}{ccc} K_0(\mathfrak{M}_{S^*}(G)) & \xrightarrow{\chi_{G,\gamma}} & K_1(\Lambda(G)_{S^*}) \\ \downarrow & & \downarrow \\ K_0(\mathfrak{M}_{\overline{S}^*}(\overline{G})) & \xrightarrow{\chi_{\overline{G},\gamma}} & K_1(\Lambda(\overline{G})_{\overline{S}^*}) \end{array}$$

where the vertical arrows are the natural homomorphisms.

Remark 4.8. Proposition 4.7(i) shows that $\mathrm{char}_{G,\gamma}(C)$ is a 'characteristic element for C' in the sense of [13, (33)]. The surjectivity of ∂_G (which follows directly from Proposition 4.7(ii)(b)) was first proved in [13, Prop. 3.4].

The proof of Proposition 4.7 will be the subject of §4.4. However, we now show that it implies Theorem 2.1. To do this we consider the map

$$\iota_G : K_0(\Omega(G)) \oplus K_0(\mathfrak{M}_S(G)) \oplus \mathrm{im}(K_1(\Lambda(G)) \to K_1(\Lambda(G)_{S^*})) \to K_1(\Lambda(G)_{S^*})$$

which for each M in $\Omega(G)$, N in $\mathfrak{M}_S(G)$ and u in $\mathrm{im}(K_1(\Lambda(G)) \to K_1(\Lambda(G)_{S^*}))$ satisfies $\iota_G(([M],[N],u)) = \mathrm{char}_{G,\gamma}(M[1])\mathrm{char}_{G,\gamma}(N[1])u$. Then Proposition 4.7(ii) implies that ι_G is a well-defined homomorphism which, upon restriction to the summand $K_0(\Omega(G)) \oplus K_0(\mathfrak{M}_S(G))$, gives a right inverse to the composite $K_1(\Lambda(G)_{S^*}) \to K_0(\mathfrak{M}_{S^*}(G)) \to K_0(\Omega(G)) \oplus K_0(\mathfrak{M}_S(G))$ where the first arrow is ∂_G and the second is the isomorphism of Proposition 3.4. The exactness of the lower row of (16) thus implies that ι_G is bijective. This completes the proof of Theorem 2.1.

Remark 4.9. The characteristic series for M in $\mathfrak{M}_{S^*}(G)$ and hence also the splitting $\chi_{G,\gamma}$ of ∂_G can of course be defined just in terms of modules instead of derived categories. Thus for the proof of Theorem 2.1 one can probably avoid the use of derived categories. However, since our applications involve complexes we prefer to use this language from the outset.

4.4. The proof of Proposition 4.7. In addition to proving Proposition 4.7 we shall also translate Definition 4.1 into the language of localized K_1-groups introduced by Fukaya and Kato in [17]. We therefore use the notation of Appendix A.

4.4.1. *S-acyclic complexes.* In [26, Prop. 2.2, Rem. 2.3] Schneider and the second named author have proved that for each bounded complex P of projective $\Lambda(G)$-modules in $D^{\mathrm{p}}_S(\Lambda(G))$ there exists an exact sequence of complexes in $D^{\mathrm{p}}(\Lambda(G))$

$$0 \to \mathrm{I}^G_H(P) \xrightarrow{\delta} \mathrm{I}^G_H(P) \xrightarrow{\pi} P \to 0 \tag{17}$$

where in each degree i the morphism δ, respectively π, is equal to the homomorphism $\delta_\gamma : \mathrm{I}^G_H(P^i) \to \mathrm{I}^G_H(P^i)$, respectively to the natural projection $\mathrm{I}^G_H(P^i) \to P^i$. We may therefore define a trivialization

$$t_S(P) : \mathbf{1}_{\Lambda(G)} \to \mathbf{d}_{\Lambda(G)}(\mathrm{I}^G_H(P))\mathbf{d}_{\Lambda(G)}(\mathrm{I}^G_H(P))^{-1} \to \mathbf{d}_{\Lambda(G)}(P)$$

where the first arrow is induced by the identity map on $\mathrm{I}^G_H(P)$ and the second by applying property A.d) to the exact sequence (17). By using property A.g) of the functor $\mathbf{d}_{\Lambda(G)}$ we then extend this definition to obtain for any object C of $D^{\mathrm{p}}_S(\Lambda(G))$ a canonical morphism

$$t_S(C) : \mathbf{1}_{\Lambda(G)} \to \mathbf{d}_{\Lambda(G)}(C).$$

We remark that this morphism is analogous to those that arise naturally in the context of varieties over finite fields (cf. [19, Lem. 3.5.8], [5, §3.2]).
In the following result we use the homomorphism $\mathrm{ch}_{\Lambda(G),\Sigma_C}$ defined in Appendix A.

Lemma 4.10. *For each C in $D^{\mathrm{p}}_S(\Lambda(G))$ one has* $\mathrm{ch}_{\Lambda(G),\Sigma_C}([C, t_S(C)]) = \mathrm{char}^*_{G,\gamma}(C)$.

Proof. One has $\mathrm{ch}_{\Lambda(G),\Sigma_C}([C, t_S(C)]) = \theta_{C,t_S(C)}$ where the latter element is as defined in Appendix A. To compute $\theta_{C,t_S(C)}$ explicitly we set $Q := \Lambda(G)_{S^*}$, $C_Q := Q \otimes_{\Lambda(G)} C$, $\mathrm{H}(C)_Q := Q \otimes_{\Lambda(G)} \mathrm{H}(C)$ and $\mathrm{H}(C)_{H,Q} := Q \otimes_{\Lambda(H)} \mathrm{H}(C)$ and consider the diagram

$$\begin{array}{ccccccc}
\mathbf{1}_Q & \xrightarrow{\alpha_1} & \mathbf{d}_Q(\mathrm{H}(C)_{H,Q})\mathbf{d}_Q(\mathrm{H}(C)_{H,Q})^{-1} & \xrightarrow{\alpha_2} & \mathbf{d}_Q(\mathrm{H}(C)_Q) & \xrightarrow{\alpha_3} & \mathbf{d}_Q(C_Q) \\
{\scriptstyle\alpha_5}\downarrow & & \| & & & & \| \\
\mathbf{1}_Q & \xrightarrow{\alpha_4} & \mathbf{d}_Q(\mathrm{H}(C)_{H,Q})\mathbf{d}_Q(\mathrm{H}(C)_{H,Q})^{-1} & \xrightarrow{\alpha_2} & \mathbf{d}_Q(\mathrm{H}(C)_Q) & \xrightarrow{\alpha_3} & \mathbf{d}_Q(C_Q).
\end{array}$$

In this diagram α_1 is induced by the identity map on $\mathrm{H}(C)_{H,Q}$, α_2 results from applying property A.d) to (17) with $C = \mathrm{H}(C)$, α_3 is property A.h), α_4 is induced by $\mathbf{d}_Q(\mathrm{H}(\delta_\gamma))$ and α_5 is defined so that the first square commutes. The upper row of the diagram is equal to the morphism $\mathbf{1}_Q \to \mathbf{d}_Q(C_Q)$ induced by $t_S(C)$ whilst the lower row agrees with the morphism $\mathbf{1}_Q \to \mathbf{d}_Q(C_Q)$ induced by the acyclicity of C_Q. From the commutativity of the diagram we thus deduce that the element $\theta_{C,t_S(C)}$ of $K_1(Q)$ is represented by α_5. On the other hand, a comparison of the maps α_1 and α_4 shows that α_5 represents the same element of $K_1(Q)$ as does the morphism

$$\mathbf{d}_Q(\mathrm{H}(C)_{H,Q}) \cong \prod_{i\in\mathbb{Z}} \mathbf{d}_Q(H^i(C)_{H,Q})^{(-1)^i} \to \prod_{i\in\mathbb{Z}} \mathbf{d}_Q(H^i(C)_{H,Q})^{(-1)^i} \cong \mathbf{d}_Q(\mathrm{H}(C)_{H,Q})$$

where the first and third maps use property A.h) and the second map is $\prod_{i\in\mathbb{Z}} \mathbf{d}_Q(H^i(\delta_\gamma))^{(-1)^i}$. From here we deduce the required equality

$$\theta_{\mathrm{H}(C),t_S(\mathrm{H}(C))} = \prod_{i\in\mathbb{Z}} \langle \delta_\gamma \mid H^i(C)_{H,Q}\rangle^{(-1)^i} \in K_1(Q).$$

□

4.4.2. *p-torsion complexes.* In this subsection we assume that G has no element of order p (so that $D^{\mathrm{p}}(\Lambda(G))$ identifies with $D^{\mathrm{fg}}(\Lambda(G))$).
If T is a bounded complex of finitely generated $\Omega(G)$-modules, then there is a bounded complex of finitely generated projective $\Omega(G)$-modules $\bar{P}$ that is isomorphic in $D(\Omega(G))$ (and hence also in $D^{\mathrm{p}}(\Lambda(G))$) to T. Also, following the discussion of §3.3, in each degree i there is a finitely generated projective $\Lambda(G)$-module P^i and an exact sequence of $\Lambda(G)$-modules

$$0 \to P^i \xrightarrow{p} P^i \to \bar{P}^i \to 0. \tag{18}$$

We may therefore define a morphism

$$t(T):\ \mathbf{1}_{\Lambda(G)} \to \prod_{i\in\mathbb{Z}} (\mathbf{d}_{\Lambda(G)}(P^i)\mathbf{d}_{\Lambda(G)}(P^i)^{-1})^{(-1)^i} \to \prod_{i\in\mathbb{Z}} \mathbf{d}_{\Lambda(G)}(\bar{P}^i)^{(-1)^i} = \mathbf{d}_{\Lambda(G)}(\bar{P}) \to \mathbf{d}_{\Lambda(G)}(T)$$

where the first arrow is induced by the identity map on each module P^i, the second by applying property A.d) to each of the sequences (18) and the last by the given quasi-isomorphism $\bar{P} \cong T$. If now C is any bounded complex of modules in $\mathfrak{D}(G)$, then there exists a finite length filtration of C by complexes

$$0 = C_d \subset C_{d-1} \subset \cdots C_1 \subset C_0 = C \tag{19}$$

so that each quotient complex $T_i := C_i/C_{i+1}$ belongs to $D^{\mathrm{p}}(\Omega(G))$ (and hence to $D^{\mathrm{p}}(\Lambda(G))$). This gives an identification $\mathbf{d}_{\Lambda(G)}(C) = \prod_{0\le i<d} \mathbf{d}_{\Lambda(G)}(T_i)$ and, with respect to this identification, we set

$$t(C) := \prod_{0\le i<d} t(T_i).$$

This definition is easily checked to be independent of the choice of filtration (19) and, for each i, of isomorphism $\bar{P} \cong T_i$ and resolution (18) used to define $t(T_i)$.

Lemma 4.11. *If G has no element of order p and C is any bounded complex of modules in $\mathfrak{D}(G)$, then in $K_1(\Lambda(G)_{S^*})$ one has $\mathrm{ch}_{\Lambda(G),\Sigma_{S^*}}([C, t(C)]) = \chi^{\mu}_G(C)$.*

Proof. This follows from the definition of $\mathrm{ch}_{\Lambda(G),\Sigma_{S^*}}$ and the fact that there is a resolution (18) of the form $0 \to \Lambda(G)^n \xrightarrow{d^j} \Lambda(G)^n \to \bar{P}^j \to 0$ where $n = \sum_{i=1}^{i=c}\langle \bar{P}^j, \bar{X}_i\rangle$ and d^j is given with respect to the canonical basis by the diagonal matrix with entries $f_i^{\langle \bar{P}^j,\bar{X}_i\rangle}$, $1 \le i \le n$, where the natural numbers $\langle \bar{P}^j, \bar{X}_i\rangle$ are defined via the decomposition $\bar{P}^j \cong \bigoplus_{i=1}^{i=c} \bar{X}_i^{\langle \bar{P}^j,\bar{X}_i\rangle}$. The fact that the μ-invariants give the correct multiplicities is the same as for [1, Prop. 4.8]. $\square$

4.4.3. *The proof of Proposition 4.7.* For each complex C in Σ_{S^*} we write $\mathrm{H}(C)_{\mathrm{tor}}$ and $\mathrm{H}(C)_{\mathrm{tf}}$ for the complexes with $\mathrm{H}(C)^i_{\mathrm{tor}} = H^i(C)_{\mathrm{tor}}$ and $\mathrm{H}(C)^i_{\mathrm{tf}} = H^i(C)_{\mathrm{tf}}$ in each degree i and in which all differentials are zero. There is a tautological exact sequence of complexes $0 \to \mathrm{H}(C)_{\mathrm{tor}} \to \mathrm{H}(C) \to \mathrm{H}(C)_{\mathrm{tf}} \to 0$ and hence an equality $\chi(C) = \chi(\mathrm{H}(C)) = \chi(\mathrm{H}(C)_{\mathrm{tor}}) + \chi(\mathrm{H}(C)_{\mathrm{tf}})$ in $K_0(\mathfrak{M}_{S^*}(G))$. From Definition 4.1 it is also clear that $\chi^\mu_G(C) = \chi^\mu_G(\mathrm{H}(C)_{\mathrm{tor}})$ and $\mathrm{char}^*_{G,\gamma}(C) = \mathrm{char}^*_{G,\gamma}(\mathrm{H}(C)_{\mathrm{tf}})$. Claim (i) therefore follows upon combining Lemmas 4.10 and 4.11 (with C replaced by $\mathrm{H}(C)_{\mathrm{tf}}$ and $\mathrm{H}(C)_{\mathrm{tor}}$ respectively) with the following fact: there is a commutative diagram of homomorphisms of abelian groups

$$\begin{array}{ccc} K_1(\Lambda(G),\Sigma_{S^*}) & \xrightarrow{\partial'} & K_0(\Sigma_{S^*}) \\ {\scriptstyle \mathrm{ch}_{\Lambda(G),\Sigma_{S^*}}}\downarrow & & \downarrow{\scriptstyle \iota} \\ K_1(\Lambda(G)_{S^*}) & \xrightarrow{\partial_G} & K_0(\mathfrak{M}_{S^*}(G)) \end{array}$$

where ∂' sends each class $[C,a]$ to $-[[C]]$ (cf. [17, Th. 1.3.15]) and ι sends each class $[[C]]$ to $\chi(C) = \sum_{i\in\mathbb{Z}}(-1)^i[H^i(C)]$ (cf. [17, §4.3.3]).

Regarding claim (ii) we note first that if a homomorphism $\chi_{G,\gamma}$ exists satisfying property (a), then it is automatically unique. Next we note that Lemma 4.4 implies the assignment $M \mapsto \mathrm{char}_{G,\gamma}(M[1])$ for each M in $\mathfrak{M}_{S^*}(G)$ induces a well-defined homomorphism $\chi_{G,\gamma} : K_0(\mathfrak{M}_{S^*}(G)) \to K_1(\Lambda(G)_{S^*})$ and claim (i) implies that this homomorphism is a right inverse to ∂_G. Further, for each M in $\mathfrak{D}(G)$ and N in $\mathfrak{M}_S(G)$ one has $\mathrm{char}_{G,\gamma}(M[1]) = \chi^\mu_{G,\gamma}(M[1]) \in \iota(K_0(\Omega(G)))$ and $\mathrm{char}_{G,\gamma}(N[1]) = \mathrm{char}^*_{G,\gamma}(N[1]) \in \iota(K_0(\mathfrak{M}_S(G)))$ (where the latter equality follows from Lemma 3.1(iii)) and so property (c) is satisfied. Finally, the commutativity of the diagram in (d) is a direct consequence of Lemma 4.5.

5. Descent theory

In this section we shall consider leading terms of elements of $K_1(\Lambda(G)_{S^*})$ and in particular prove Theorem 2.2. The approach of this section was initially developed by the first named author in an unpublished early version of the article [6].

We deal first with the case that C is acyclic. In this case the complex $\mathbb{Z}_p[\overline{G}] \otimes^{\mathbb{L}}_{\Lambda(G)} C$ is acyclic so $[\mathbf{d}_{\mathbb{Z}_p[\overline{G}]}(\mathbb{Z}_p[\overline{G}] \otimes^{\mathbb{L}}_{\Lambda(G)} C), t(C)_{\overline{G}}]$ is the zero element of $K_0(\mathbb{Z}_p[\overline{G}], \mathbb{Q}^c_p[\overline{G}])$ and also ξ belongs to the image of the natural map $K_1(\Lambda(G)) \to K_1(\Lambda(G)_{S^*})$. The equality of Theorem 2.2 is therefore a consequence of the following result.

LEMMA 5.1. *If u belongs to the image of the natural map $\lambda : K_1(\Lambda(G)) \to K_1(\Lambda(G)_{S^*})$, then for each finite quotient $\overline{G}$ of G the element $(u^*(\rho))_{\rho\in\mathrm{Irr}(\overline{G})}$ belongs to $\ker(\partial_{\overline{G}})$.*

Proof. Let $\mathcal{O}$ be the valuation ring of a finite extension L of $\mathbb{Q}_p$ such that all representations can be realised over $\mathcal{O}$. If $v \in K_1(\Lambda(G))$ with $u = \lambda(v)$, then $u^*(\rho) = u(\rho) = \lambda(v)(\rho) \in \mathcal{O}^\times$ for all $\rho \in \mathrm{Irr}(\overline{G})$. Thus by functoriality of K-theory and the fact that the canonical map $K_1(\Lambda_{\mathcal{O}}(\Gamma)) \to K_1(\mathcal{O})$ is equal to the 'evaluation at 0' map $\Lambda_{\mathcal{O}}(\Gamma)^\times \to \mathcal{O}^\times$, the image of v in $K_1(\mathbb{Z}_p[\overline{G}])$ under the natural projection is mapped to $(u^*(\rho))_{\rho\in\mathrm{Irr}(\overline{G})} \in K_1(L[\overline{G}])$. □

5.1. Reduction to S-acyclicity. We now reduce the general case of Theorem 2.2 to the case that C belongs to $D^{\mathrm{p}}_S(\Lambda(G))$. In this subsection we therefore assume that G has no element of order p.

Lemma 5.2. *If G has no element of order p, then it is enough to prove Theorem 2.2 in the case that C is acyclic outside at most one degree.*

Proof. We assume that the result of Theorem 2.2 is true for all complexes that are acyclic outside at most one degree. To deduce Theorem 2.2 in the general case we use induction on the number of non-zero cohomology groups of C (which we assume to be at least two). We let n denote the largest integer m for which $H^m(C)$ is non-zero. We set $C_1 := H^n(C)[-n]$ and write C_2 for the truncation of C in degrees less than n (which has fewer non-zero cohomology group than does C). Then there is an exact triangle in $D^{\mathrm{p}}_{S^*}(\Lambda(G))$ of the form

$$C_1 \to C \to C_2 \to C_1[1] \tag{20}$$

and the assumption that C is semisimple at ρ implies that C_1 and C_2 are also semisimple at ρ (for any ρ in $\mathrm{Irr}(\overline{G})$). Let ξ be an element such that $\partial_G(\xi) = \chi(C)$. If ξ_1 is such that $\partial_G(\xi_1) = \chi(C_1)$, then $\xi_2 := \xi\xi_1^{-1}$ satisfies $\partial_G(\xi_2) = \partial_G(\xi) - \partial_G(\xi_1) = \chi(C) - \chi(C_1) = \chi(C_2)$, where the last equality follows from (20). Hence, by the inductive hypothesis, one has

$$\begin{aligned}\partial_{\overline{G}}((\xi^*(\rho))_\rho &= \partial_{\overline{G}}((\xi_1^*(\rho))_\rho) + \partial_{\overline{G}}((\xi_2^*(\rho))_\rho)\\ &= -[\mathbf{d}_{\mathbb{Z}_p[\overline{G}]}(C_{1,\overline{G}}), t(C_1)_{\overline{G}}] - [\mathbf{d}_{\mathbb{Z}_p[\overline{G}]}(C_{2,\overline{G}}), t(C_2)_{\overline{G}}].\end{aligned} \tag{21}$$

where $C_{i,\overline{G}} := \mathbb{Z}_p[\overline{G}] \otimes^{\mathbb{L}}_{\Lambda(G)} C_i$ for $i = 1, 2$. Now if we set $C_{\overline{G}} := \mathbb{Z}_p[\overline{G}] \otimes^{\mathbb{L}}_{\Lambda(G)} C$, then (20) induces an exact triangle in $D^{\mathrm{p}}(\mathbb{Z}_p[\overline{G}])$

$$C_{1,\overline{G}} \to C_{\overline{G}} \to C_{2,\overline{G}} \to C_{1,\overline{G}}[1] \tag{22}$$

and, with respect to this triangle, the trivialisations $t(C_1)_{\overline{G}}, t(C)_{\overline{G}}$ and $t(C_2)_{\overline{G}}$ satisfy the 'additivity criterion' of [3, Cor. 6.6]. Indeed, for each ρ in $\mathrm{Irr}(\overline{G})$ the exact triangle (20) combines with the definition (7) of each of the complexes $C_{1,\rho}, C_\rho$ and $C_{2,\rho}$ to induce an exact triangle

$$\mathbb{Q}_p^c \otimes^{\mathbb{L}}_{\Lambda_{\mathcal{O}}(\Gamma)} C_{1,\rho} \to \mathbb{Q}_p^c \otimes^{\mathbb{L}}_{\Lambda_{\mathcal{O}}(\Gamma)} C_\rho \to \mathbb{Q}_p^c \otimes^{\mathbb{L}}_{\Lambda_{\mathcal{O}}(\Gamma)} C_{2,\rho} \to \mathbb{Q}_p^c \otimes^{\mathbb{L}}_{\Lambda_{\mathcal{O}}(\Gamma)} C_{1,\rho}[1]$$

and the cohomology sequence of this triangle gives an equality $r_G(C)(\rho) = r_G(C_1)(\rho) + r_G(C_2)(\rho)$ and a short exact sequence of complexes

$$0 \to \mathrm{H}_{\mathrm{bock}}(\triangle(C_{1,\rho},\gamma)) \to \mathrm{H}_{\mathrm{bock}}(\triangle(C_\rho,\gamma)) \to \mathrm{H}_{\mathrm{bock}}(\triangle(C_{2,\rho},\gamma)) \to 0.$$

Here we use the notation of Appendix B and write $\triangle(C_\rho,\gamma)$ for the triangle

$$\mathbb{Q}_p^c \otimes^{\mathbb{L}}_{\mathcal{O}} C_\rho \xrightarrow{\theta_{\gamma,\rho}} \mathbb{Q}_p^c \otimes^{\mathbb{L}}_{\mathcal{O}} C_\rho \to \mathbb{Q}_p^c \otimes^{\mathbb{L}}_{\Lambda_{\mathcal{O}}(\Gamma)} C_\rho \to \mathbb{Q}_p^c \otimes^{\mathbb{L}}_{\mathcal{O}} C_\rho[1] \tag{23}$$

where $\theta_{\gamma,\rho}$ is induced by multiplication by $\gamma - 1$, and we use similar notation for C_1 and C_2. This means that the criterion of [3, Cor. 6.6] is satisfied if one takes (in the notation of loc. cit.) Σ to be $\mathbb{Q}_p^c[\overline{G}]$, $P \xrightarrow{a} Q \xrightarrow{b} R \xrightarrow{c} P[1]$ to be the exact triangle

(22) (so $\ker(H^{\mathrm{ev}}a_\Sigma) = \ker(H^{\mathrm{od}}a_\Sigma) = 0$) and the trivialisations t_P, t_Q and t_R to be induced by $(-1)^{r_G(C_1)(\rho)}t(C_{1,\rho})$, $(-1)^{r_G(C)(\rho)}t(C_\rho)$ and $(-1)^{r_G(C_2)(\rho)}t(C_{2,\rho})$ respectively. From [3, Cor. 6.6] we therefore deduce that the last element in (21) is indeed equal to $-[\mathbf{d}_{\mathbb{Z}_p[\overline{G}]}(C_{\overline{G}}), t(C)_{\overline{G}}]$, as required. □

Taking account of Lemmas 5.1 and 5.2 we now assume that C is acyclic outside precisely one degree. To be specific, we assume that $C = M[0]$ with M in $\mathfrak{M}_{S^*}(\Lambda(G))$. Then there is an exact triangle of the form

$$M_{\mathrm{tor}}[0] \to M[0] \to M_{\mathrm{tf}}[0] \to M_{\mathrm{tor}}[1] \tag{24}$$

where M_{tor} belongs to $\mathfrak{D}(G)$ and M_{tf} to $\mathfrak{M}_S(\Lambda(G))$. In this case one has $t(M[0])_{\overline{G}} = t(M_{\mathrm{tf}}[0])_{\overline{G}}$ and so (by another application of [3, Cor. 6.6])

$$\begin{aligned}(25)\quad [\mathbf{d}_{\mathbb{Z}_p[\overline{G}]}(\mathbb{Z}_p[\overline{G}] \otimes^{\mathbb{L}}_{\Lambda(G)} M[0]), t(M[0])_{\overline{G}}] = \\ [\mathbf{d}_{\mathbb{Z}_p[\overline{G}]}(\mathbb{Z}_p[\overline{G}] \otimes^{\mathbb{L}}_{\Lambda(G)} M_{\mathrm{tor}}[0]), can] + [\mathbf{d}_{\mathbb{Z}_p[\overline{G}]}(\mathbb{Z}_p[\overline{G}] \otimes^{\mathbb{L}}_{\Lambda(G)} M_{\mathrm{tf}}[0]), t(M[0])_{\overline{G}}]\end{aligned}$$

with can the canonical morphism $\mathbf{d}_{\mathbb{Q}_p[\overline{G}]}(\mathbb{Q}_p[\overline{G}] \otimes^{\mathbb{L}}_{\Lambda(G)} M_{\mathrm{tor}}[0]) = \mathbf{d}_{\mathbb{Q}_p[\overline{G}]}(0) \to \mathbf{1}_{\mathbb{Q}_p[\overline{G}]}$.

Lemma 5.3. *Let N be an object of $\mathfrak{D}(G)$. If ξ is any element of $K_1(\Lambda(G)_{S^*})$ with $\partial_G(\xi) = \chi(N[0])$, then for any finite quotient $\overline{G}$ of G one has*

$$\partial_{\overline{G}}((\xi^*(\rho))_\rho) = -[\mathbf{d}_{\mathbb{Z}_p[\overline{G}]}(\mathbb{Z}_p[\overline{G}] \otimes^{\mathbb{L}}_{\Lambda(G)} N[0]), can].$$

Proof. An easy reduction (using dévissage and the additivity of Euler characteristics on exact sequences in $\mathfrak{D}(G)$) allows us to assume that N is an object of $\mathfrak{D}(G)$ which lies in an exact sequence of $\Lambda(G)$-modules of the form

$$0 \to Q \xrightarrow{d} P \to N \to 0 \tag{26}$$

where Q and P are both finitely generated and projective. (Indeed, it is actually enough to consider the case that $Q = P = \Lambda(G)e_i$ for an idempotent e_i as in §3.3 and with d equal to multiplication by p.)
To proceed we identify the subgroup $K_0(\mathfrak{D}(G))$ of $K_0(\mathfrak{M}_{S^*}(G))$ with the group $K_0(\Lambda(G), \Lambda(G)[\frac{1}{p}])$. To be compatible with the normalisations used in §1.2 we must fix this isomorphism so that for every exact sequence (26) the element $[N]$ of $K_0(\mathfrak{D}(G))$ corresponds to the element (Q, d', P) of $K_0(\Lambda(G), \Lambda(G)[\frac{1}{p}])$ with $d' := \Lambda(G)[\frac{1}{p}] \otimes_{\Lambda(G)} d$. Now since $\Lambda(G)[\frac{1}{p}] \otimes_{\Lambda(G)} N = 0$ the localisation sequence of K-theory implies that any element ξ as above belongs to the image of $K_1(\Lambda(G)[\frac{1}{p}])$ in $K_1(\Lambda(G)_{S^*})$. This implies in particular that $\xi^*(\rho) = \xi(\rho)$ for all ρ in $\mathrm{Irr}(\overline{G})$. The natural commutative diagram of connecting homomorphisms

$$\begin{array}{ccc} K_1(\Lambda(G)[\frac{1}{p}]) & \longrightarrow & K_0(\Lambda(G), \Lambda(G)[\frac{1}{p}]) \\ \big\downarrow & & \big\downarrow \\ K_1(\mathbb{Q}_p[\overline{G}]) & \xrightarrow{\partial_{\overline{G}}} & K_0(\mathbb{Z}_p[\overline{G}], \mathbb{Q}_p[\overline{G}]) \end{array}$$

also then implies that $\partial_{\overline{G}}((\xi^*(\rho))_\rho) = (Q_{\overline{G}}, d'_{\overline{G}}, P_{\overline{G}})$ with $Q_{\overline{G}} := \mathbb{Z}_p[\overline{G}] \otimes_{\Lambda(G)} Q, P_{\overline{G}} := \mathbb{Z}_p[\overline{G}] \otimes_{\Lambda(G)} P$ and $d'_{\overline{G}} := \mathbb{Q}_p[\overline{G}] \otimes_{\Lambda(G)} d$. Hence, with respect to the isomorphism (2) (with $R = \mathbb{Z}_p[\overline{G}]$ and $R' = \mathbb{Q}_p[\overline{G}]$), one has

$$\partial_{\overline{G}}((\xi^*(\rho))_\rho) = [\mathbf{d}_{\mathbb{Z}_p[\overline{G}]}(Q_{\overline{G}})\mathbf{d}_{\mathbb{Z}_p[\overline{G}]}(P_{\overline{G}})^{-1}, \tau] \tag{27}$$

with τ equal to the composite morphism

$$\mathbf{d}_{\mathbb{Q}_p[\overline{G}]}(\mathbb{Q}_p \otimes_{\mathbb{Z}_p} Q_{\overline{G}})\mathbf{d}_{\mathbb{Q}_p[\overline{G}]}(\mathbb{Q}_p \otimes_{\mathbb{Z}_p} P_{\overline{G}})^{-1} \to \mathbf{d}_{\mathbb{Q}_p[\overline{G}]}(\mathbb{Q}_p \otimes_{\mathbb{Z}_p} P_{\overline{G}})\mathbf{d}_{\mathbb{Q}_p[\overline{G}]}(\mathbb{Q}_p \otimes_{\mathbb{Z}_p} P_{\overline{G}})^{-1} = \mathbf{1}_{\mathbb{Q}_p[\overline{G}]}$$

where the first arrow is induced by $\mathbf{d}_{\mathbb{Q}_p[\overline{G}]}(d'_{\overline{G}})$. Finally we note that the (image under $\mathbb{Z}_p[\overline{G}] \otimes_{\Lambda(G)} -$ of the) sequence (26) induces an isomorphism in $D^{\rm p}(\mathbb{Z}_p[\overline{G}])$ between $\mathbb{Z}_p[\overline{G}] \otimes^{\mathbb{L}}_{\Lambda(G)} N[0]$ and the complex $Q_{\overline{G}} \xrightarrow{d_{\overline{G}}} P_{\overline{G}}$ where the first term is placed in degree -1 and $d_{\overline{G}} := \mathbb{Z}_p[\overline{G}] \otimes_{\Lambda(G)} d$ and this implies that the element on the right hand side of (27) is the inverse of $[\mathbf{d}_{\mathbb{Z}_p[\overline{G}]}(\mathbb{Z}_p[\overline{G}] \otimes^{\mathbb{L}}_{\Lambda(G)} N[0]), can]$, as required. □

Lemmas 5.1, 5.2 and 5.3 combine with (24) and (25) to reduce the proof of Theorem 2.2 to consideration of complexes in $D^{\rm p}_S(\Lambda(G))$. In the remainder of §5 we shall therefore assume that C belongs to $D^{\rm p}_S(\Lambda(G))$.

5.2. Equivariant twists. In this subsection we introduce the algebraic formalism that is key to a proper understanding of descent.

5.2.1. *The definition.* We fix an open normal subgroup U of G and set $\overline{G} := G/U$. We write

$$\Delta_{\overline{G}} : \Lambda(G) \to \Lambda(\overline{G} \times G) \cong \mathbb{Z}_p[\overline{G}] \otimes_{\mathbb{Z}_p} \Lambda(G)$$

for the (flat) ring homomorphism which sends each element σ of G to $\overline{\sigma} \otimes \sigma$ where $\overline{\sigma}$ is the image of σ in $\overline{G}$. Then for each $\Lambda(G)$-module M the induced $\Lambda(\overline{G} \times G)$-module $\Lambda(\overline{G} \times G) \otimes_{\Lambda(G),\Delta_{\overline{G}}} M$ can be identified with the module

$$\mathrm{tw}_{\overline{G}}(M) := \mathbb{Z}_p[\overline{G}] \otimes_{\mathbb{Z}_p} M$$

upon which $\overline{G}$ acts via left multiplication and each $\sigma \in G$ acts by sending $x \otimes y$ to $x\overline{\sigma}^{-1} \otimes \sigma(y)$. This construction extends to give an exact functor $C \mapsto \mathrm{tw}_{\overline{G}}(C)$ from $D^{\rm p}(\Lambda(G))$ to $D^{\rm p}(\Lambda(\overline{G} \times G))$ and for each such C we set

$$\mathrm{tw}_{\overline{G}}(C)_H := \Lambda(\overline{G} \times \Gamma) \otimes^{\mathbb{L}}_{\Lambda(\overline{G} \times G)} \mathrm{tw}_{\overline{G}}(C) \in D^{\rm p}(\Lambda(\overline{G} \times \Gamma)).$$

5.2.2. *Base change.* For each $s \in \Lambda(G)$ we write r_s and $\mathrm{r}_{\Delta_{\overline{G}}(s)}$ for the endomorphisms of $\Lambda(G)$ and $\Lambda(\overline{G} \times G)$ given by right multiplication by s and $\Delta_{\overline{G}}(s)$ respectively. Then $\mathrm{cok}(\mathrm{r}_{\Delta_{\overline{G}}(s)})$ is isomorphic as a $\Lambda(\overline{G} \times G)$-module to $\mathrm{tw}_{\overline{G}}(\mathrm{cok}(\mathrm{r}_s))$ and so is finitely generated over $\Lambda(\overline{G} \times H)$ if $\mathrm{cok}(\mathrm{r}_s)$ is finitely generated over $\Lambda(H)$. This implies that $\Delta_{\overline{G}}(S^*) \subseteq S_1^*$ where $S := S_{G,H}$ and $S_1 := S_{\overline{G} \times G, \overline{G} \times H}$ and so $\Delta_{\overline{G}}$ induces a ring homomorphism

$$\Lambda(G)_{S^*} \to \Lambda(\overline{G} \times G)_{S_1^*} \to \Lambda(\overline{G} \times \Gamma)_{S_2^*} = Q(\overline{G} \times \Gamma) \tag{28}$$

where $S_2 := S_{\overline{G} \times \Gamma, \overline{G}}$, the second arrow is the natural projection and the equality is because $\overline{G} \times \Gamma$ has rank one (as a p-adic Lie group). These maps induce a group homomorphism

$$\pi_{\overline{G} \times \Gamma} : K_1(\Lambda(G)_{S^*}) \to K_1(Q(\overline{G} \times \Gamma))$$

which forms the upper row of a natural commutative diagram of connecting homomorphisms

$$
\begin{array}{ccccc}
K_1(\Lambda(G)_{S^*}) & \longrightarrow & K_1(\Lambda(\overline{G}\times G)_{S_1^*}) & \longrightarrow & K_1(\Lambda(\overline{G}\times\Gamma)_{S_2^*}) \\
\downarrow & & \downarrow & & \downarrow \\
K_0(\Lambda(G),\Lambda(G)_{S^*}) & \longrightarrow & K_0(\Lambda(\overline{G}\times G),S_1^*) & \longrightarrow & K_0(\Lambda(\overline{G}\times\Gamma),S_2^*)
\end{array} \tag{29}
$$

where we write $K_0(\Lambda(\overline{G}\times G),S_1^*)$ and $K_0(\Lambda(\overline{G}\times\Gamma),S_2^*)$ for $K_0(\Lambda(\overline{G}\times G),\Lambda(\overline{G}\times G)_{S_1^*})$ and $K_0(\Lambda(\overline{G}\times\Gamma),\Lambda(\overline{G}\times\Gamma)_{S_2^*})$ respectively.

5.2.3. *Reduced norms.* We set $R:=\Lambda(\overline{G}\times\Gamma)$. Then the algebra $Q(R)$ identifies with the group ring $Q(\Gamma)[\overline{G}]$ and, with respect to this identification, one has

$$\zeta(Q(R))\subset\zeta(Q^c(R))=\prod_{\rho\in\mathrm{Irr}(\overline{G})}Q^c(\Gamma) \tag{30}$$

where $Q^c(R):=\mathbb{Q}_p^c\otimes_{\mathbb{Q}_p}Q(R)$ and $Q^c(\Gamma):=\mathbb{Q}_p^c\otimes_{\mathbb{Q}_p}Q(\Gamma)$. We write $x=(x_\rho)_\rho$ for the corresponding decomposition of each element x of $\zeta(Q^c(R))$.

In the next result we write $\mathrm{Nrd}_{Q(R)}:K_1(Q(R))\to\zeta(Q^c(R))^\times$ for the reduced norm map of the semisimple algebra $Q(R)$ and use the homomorphism Φ_ρ and Ore set $\tilde{S}$ defined in (5) and (6) respectively.

LEMMA 5.4. *For each ξ in $K_1(\Lambda(G)_{\tilde{S}})$ one has* $\mathrm{Nrd}_{Q(R)}(\pi_{\overline{G}\times\Gamma}(\xi))=(\Phi_\rho(\xi))_{\rho\in\mathrm{Irr}(\overline{G})}$.

Proof. It suffices to prove that, with respect to the decomposition (30), one has $\Phi_\rho(\xi)=\mathrm{Nrd}_{Q(R)}(\pi_{\overline{G}\times\Gamma}(\xi))_\rho$ for each fixed ρ in $\mathrm{Irr}(\overline{G})$. Further, since [13, Prop. 4.2, Th. 4.4] implies that the natural map $\Lambda(G)_{\tilde{S}}^\times\to K_1(\Lambda(G)_{\tilde{S}})$ is surjective, it is enough to verify this for all elements ξ of the form $\langle \mathrm{r}_s\mid\Lambda(G)_{\tilde{S}}\rangle$ with $s\in\Lambda(G)\cap\Lambda(G)_{\tilde{S}}^\times$.

To do this we fix a finite dimensional $\mathbb{Q}_p^c$-space V_ρ that corresponds to ρ and write V_{ρ^*} for the space $\mathrm{Hom}_{\mathbb{Q}_p^c}(V_\rho,\mathbb{Q}_p^c)$ that corresponds to ρ^*. Then for each $x=\sum_{\delta\in\overline{G}}c_\delta\delta$ in $Q(\Gamma)[\overline{G}]^\times=Q(R)^\times$ the argument of Ritter and Weiss in [25, §3] shows that

$$\mathrm{Nrd}_{Q(R)}(\langle \mathrm{r}_x\mid Q(R)\rangle)_\rho=\det{}_{Q^c(\Gamma)}(\alpha_x) \tag{31}$$

where α_x is the automorphism of $V_{\rho^*}\otimes_{\mathbb{Q}_p^c}Q^c(\Gamma)$ given by $\sum_{\delta\in\overline{G}}\delta^{-1}\otimes\mu(c_\delta)$ with $\mu(c_\delta)$ denoting multiplication by c_δ. Now the matrix of the action of δ^{-1} on V_{ρ^*} (with respect to a fixed $\mathbb{Q}_p^c$-basis) is the transpose of the matrix of the action of δ on V_ρ (with respect to the dual $\mathbb{Q}_p^c$-basis). Using this fact, and an explication of the role of Morita equivalence in (5), one finds that $\Phi_\rho(s)=\det_{Q^c(\Gamma)}(\alpha_{\Delta_{\overline{G}}(s)})$ for each $s\in\Lambda(G)\cap\Lambda(G)_{\tilde{S}}^\times$. Since $\pi_{\overline{G}\times\Gamma}(\langle \mathrm{r}_s\mid\Lambda(G)_{\tilde{S}}\rangle)=\langle \mathrm{r}_{\Delta_{\overline{G}}(s)}\mid Q(R)\rangle$ the claimed result is therefore a consequence of the description (31). $\square$

5.2.4. *Semisimplicity.* There are natural isomorphisms in $D^{\mathrm{p}}(\mathbb{Z}_p[\overline{G}])$ of the form

$$\mathbb{Z}_p\otimes^{\mathbb{L}}_{\Lambda(\Gamma)}\mathrm{tw}_{\overline{G}}(C)_H\cong\mathbb{Z}_p\otimes^{\mathbb{L}}_{\Lambda(G)}\mathrm{tw}_{\overline{G}}(C)\cong\mathbb{Z}_p[\overline{G}]\otimes^{\mathbb{L}}_{\Lambda(G)}C$$

and hence an exact triangle in $D(\Lambda(\overline{G}\times\Gamma))$ of the form

$$\triangle(\mathrm{tw}_{\overline{G}}(C),\gamma)\ :\ \mathrm{tw}_{\overline{G}}(C)_H\xrightarrow{\theta_\gamma}\mathrm{tw}_{\overline{G}}(C)_H\to\mathbb{Z}_p[\overline{G}]\otimes^{\mathbb{L}}_{\Lambda(G)}C\to\mathrm{tw}_{\overline{G}}(C)_H[1]$$

where θ_γ is induced by multiplication by $\gamma-\mathrm{id}\in\Lambda(\Gamma)$ on $\Lambda(\overline{G}\times\Gamma)$.

In the next result we use the terminology and notation of Appendix B. For each $\mathbb{Q}_p[\overline{G}]$-module M we also define a $\mathbb{Q}_p^c$-module $M^\rho := \mathrm{Hom}_{\mathbb{Q}_p^c[\overline{G}]}(V_\rho, \mathbb{Q}_p^c \otimes_{\mathbb{Z}_p} M)$.

Lemma 5.5.

(i) *The image of* $\triangle(\mathrm{tw}_{\overline{G}}(C), \gamma)$ *under the (exact) functor* $e_\rho\mathbb{Q}_p^c[\overline{G}] \otimes_{\mathbb{Z}_p[\overline{G}]} -$ *is naturally isomorphic to the exact triangle* $\triangle(C_\rho, \gamma)$ *defined in (23).*

(ii) *For each* ρ *in* $\mathrm{Irr}(\overline{G})$ *one has*

$$r_G(C)(\rho) = \sum_{i\in\mathbb{Z}} (-1)^{i+1} \mathrm{dim}_{\mathbb{Q}_p^c}(H^i(\mathrm{tw}_{\overline{G}}(C)_H)^{\Gamma,\rho}).$$

(iii) *The morphism* θ_γ *is semisimple if and only if* C *is semisimple at* ρ *(in the sense of* [11, Def. 3.11]*) for every* ρ *in* $\mathrm{Irr}(\overline{G})$.

(iv) *If* θ_γ *is semisimple, then*

$$r_G(C)(\rho) = \sum_{i\in\mathbb{Z}} (-1)^{i+1} i \cdot \mathrm{dim}_{\mathbb{Q}_p^c}(H^i(\mathbb{Z}_p[\overline{G}] \otimes^{\mathbb{L}}_{\Lambda(G)} C)^\rho)$$

for every ρ *in* $\mathrm{Irr}(\overline{G})$ *and, with respect to the decomposition (4), one has*

$$\beta_{\mathbb{Q}_p\otimes\triangle(\mathrm{tw}_{\overline{G}}(C),\gamma)} = (t(C_\rho))_{\rho\in\mathrm{Irr}(\overline{G})}$$

where $\mathbb{Q}_p\otimes\triangle(\mathrm{tw}_{\overline{G}}(C), \gamma)$ *is the exact triangle in* $D^{\mathrm{p}}(\Lambda(\overline{G}\times\Gamma)[\frac{1}{p}])$ *that is obtained from* $\triangle(\mathrm{tw}_{\overline{G}}(C), \gamma)$ *by scalar extension.*

Proof. For every $\Lambda(G)$-module P there is a natural isomorphism of $\Lambda(\overline{G}\times\Gamma)$-modules $\Lambda(\overline{G}\times\Gamma) \otimes_{\Lambda(\overline{G}\times G)} (\mathbb{Z}_p[\overline{G}] \otimes_{\mathbb{Z}_p} P) \cong \Lambda(\Gamma) \otimes_{\Lambda(G)} (\mathbb{Z}_p[\overline{G}] \otimes_{\mathbb{Z}_p} P)$ where the action of $\overline{G}$ on the second module is just on $\mathbb{Z}_p[\overline{G}]$ (from the left). This fact gives rise to natural isomorphisms in $D(\Lambda_{\mathbb{Q}_p^c}(\Gamma))$ of the form

$$\begin{aligned} e_\rho\mathbb{Q}_p^c[\overline{G}] \otimes_{\mathbb{Z}_p[\overline{G}]} \mathrm{tw}_{\overline{G}}(C)_H &\cong \Lambda_{\mathbb{Q}_p^c}(\Gamma) \otimes^{\mathbb{L}}_{\Lambda_{\mathbb{Q}_p^c}(G)} (e_\rho\mathbb{Q}_p^c[\overline{G}] \otimes_{\mathbb{Z}_p} C) \\ &\cong \Lambda_{\mathbb{Q}_p^c}(\Gamma) \otimes^{\mathbb{L}}_{\Lambda_{\mathbb{Q}_p^c}(G)} (V_{\rho*} \otimes_{\mathbb{Z}_p} C) \\ &\cong \mathbb{Q}_p^c \otimes_{\mathcal{O}} C_\rho. \end{aligned} \tag{32}$$

We now set $C_{\overline{G}} := \mathbb{Z}_p[\overline{G}] \otimes^{\mathbb{L}}_{\Lambda(G)} C$. Then claim (i) follows upon combining the isomorphism (32) together with the natural isomorphism in $D(\Lambda_{\mathbb{Q}_p^c}(\Gamma))$

$$e_\rho\mathbb{Q}_p^c[\overline{G}] \otimes_{\mathbb{Z}_p[\overline{G}]} C_{\overline{G}} \cong V_{\rho^*} \otimes^{\mathbb{L}}_{\Lambda(G)} C \cong \mathbb{Q}_p^c \otimes^{\mathbb{L}}_{\Lambda_{\mathcal{O}}(\Gamma)} C_\rho$$

that is induced by [7, Lem. 3.13(i)].

Claim (ii) follows by combining the equality

$$r_G(C)(\rho) = \sum_{i\in\mathbb{Z}} (-1)^{i+1} \mathrm{dim}_{\mathbb{Q}_p^c}(H^i(\mathbb{Q}_p^c \otimes_{\mathcal{O}} C_\rho)^\Gamma)$$

of [11, Lem. 3.13(ii)] with the isomorphisms of $\Lambda_{\mathbb{Q}_p^c}(\Gamma)$-modules

$$H^i(\mathbb{Q}_p^c \otimes_{\mathcal{O}} C_\rho) \cong H^i(e_\rho\mathbb{Q}_p^c[\overline{G}] \otimes_{\mathbb{Z}_p[\overline{G}]} \mathrm{tw}_{\overline{G}}(C)_H) \cong H^i(\mathrm{tw}_{\overline{G}}(C)_H)^\rho$$

that are induced by (32).

Next we note that claim (i) implies θ_γ is semisimple if and only if $\theta_{\gamma,\rho}$ is semisimple for every ρ in $\mathrm{Irr}(\overline{G})$. Claim (iii) thus follows immediately from the definition of 'semisimplicity at ρ' (in terms of $\theta_{\gamma,\rho}$).

In each degree i the exact triangle $\triangle(\mathrm{tw}_{\overline{G}}(C), \gamma)$ induces a short exact sequence

$$0 \to H^i(\mathrm{tw}_{\overline{G}}(C)_H)_\Gamma \to H^i(C_{\overline{G}}) \to H^{i+1}(\mathrm{tw}_{\overline{G}}(C)_H)^\Gamma \to 0.$$

In particular, if θ_γ is semisimple, then $\theta_{\gamma,\rho}$ is semisimple and so by applying the exact functor $M \mapsto M^\rho$ to this sequence one finds that

$$\dim_{\mathbb{Q}_p^c}(H^i(C_{\overline{G}})^\rho) = \dim_{\mathbb{Q}_p^c}(H^i(\mathrm{tw}_{\overline{G}}(C)_H)^{\Gamma,\rho}) + \dim_{\mathbb{Q}_p^c}(H^{i+1}(\mathrm{tw}_{\overline{G}}(C)_H)^{\Gamma,\rho})$$

for each integer i and hence that $\sum_{i\in\mathbb{Z}}(-1)^{i+1}i \cdot \dim_{\mathbb{Q}_p^c}(H^i(C_{\overline{G}})^\rho)$ is equal to

$$\sum_{i\in\mathbb{Z}}(-1)^{i+1}i(\dim_{\mathbb{Q}_p^c}(H^i(\mathrm{tw}_{\overline{G}}(C)_H)^{\Gamma,\rho}) + \dim_{\mathbb{Q}_p^c}(H^{i+1}(\mathrm{tw}_{\overline{G}}(C)_H)^{\Gamma,\rho}))$$
$$= \sum_{i\in\mathbb{Z}}(-1)^{i+1}\dim_{\mathbb{Q}_p^c}(H^i(\mathrm{tw}_{\overline{G}}(C)_H)^{\Gamma,\rho}).$$

This proves the explicit formula for $r_G(C)(\rho)$ in claim (iv). The explicit description of $\beta_{\mathbb{Q}_p\otimes\triangle(\mathrm{tw}_{\overline{G}}(C),\gamma)}$ in claim (iv) follows from the identification in claim (i) and the fact that $t(C_\rho)$ is, by definition, equal to $\beta_{\triangle(C_\rho,\gamma)}$. □

5.3. LEADING TERMS. We now fix an element ξ and a complex C as in Theorem 2.2. Then Lemma 5.5(iii) implies that the morphism $\theta_\gamma : \mathrm{tw}_{\overline{G}}(C)_H \to \mathrm{tw}_{\overline{G}}(C)_H$ is semisimple and so in each degree i there is a direct sum decomposition of $\Lambda(\overline{G} \times \Gamma)[\frac{1}{p}]$-modules

$$\mathbb{Q}_p \otimes_{\mathbb{Z}_p} H^i(\mathrm{tw}_{\overline{G}}(C)_H) = D_0^i \oplus D_1^i \tag{33}$$

where $D_0^i := \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \ker(H^i(\theta_\gamma)) = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} H^i(\mathrm{tw}_{\overline{G}}(C)_H)^\Gamma$ and $D_1^i := \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathrm{im}(H^i(\theta_\gamma))$. By assumption, both D_0^i and D_1^i are finitely generated (projective) $\mathbb{Q}_p[\overline{G}]$-modules and $H^i(\theta_\gamma)$ induces an automorphism of D_1^i.
The proof of the following result will occupy the rest of this section.

PROPOSITION 5.6. $\partial_{\overline{G}}(((-1)^{r_G(C)(\rho)}\xi^*(\rho))_{\rho\in\mathrm{Irr}(\overline{G})}) = \sum_{i\in\mathbb{Z}}(-1)^i\partial_{\overline{G}}(\langle H^i(\theta_\gamma) \mid D_1^i\rangle)$.

5.3.1. *The descent to $Q(R)$.* We write Σ for the subset of R consisting of those elements of $\Lambda(\Gamma)$ with non-zero image under the projection $\Lambda(\Gamma) \to \mathbb{Z}_p$. This is a multiplicatively closed Ore set in R which consists of central regular elements.

LEMMA 5.7. *For each integer i we set $M^i := (\mathrm{I}_{\overline{G}}^{\overline{G}\times\Gamma}(H^i(\mathrm{tw}_{\overline{G}}(C)_H)^\Gamma))_{S^*}$. Then the element*

$$y_\xi := \mathrm{Nrd}_{Q(R)}(\pi_{\overline{G}\times\Gamma}(\xi)\prod_{i\in\mathbb{Z}}\langle\delta_\gamma \mid M^i\rangle^{(-1)^{i+1}})$$

belongs to $\zeta(R_\Sigma)^\times \subseteq \zeta(Q(R))^\times$.

Proof. We set $X := \mathrm{tw}_{\overline{G}}(C)_H$. Then the commutative diagram (29) implies that $\partial_{\overline{G}\times\Gamma}(\pi_{\overline{G}\times\Gamma}(\xi)) = \chi(X)$ in $K_0(R, Q(R))$. But X belongs to $D_S^{\mathrm{p}}(R)$ and so [6, Th. 4.1(ii)] also implies that $\partial_{\overline{G}\times\Gamma}(\mathrm{char}_{\overline{G}\times\Gamma,\gamma}(X)) = \chi(X)$. Hence the upper row of (1) with $R' = Q(R)$ implies that there exists an element u of $K_1(R)$ with

$$\pi_{\overline{G}\times\Gamma}(\xi) = \iota_1(u)\mathrm{char}_{\overline{G}\times\Gamma,\gamma}(X) \tag{34}$$

where ι_1 is the natural homomorphism $K_1(R) \to K_1(R_\Sigma) \to K_1(Q(R))$.

For each integer i we set $N^i := (\mathrm{I}_{\overline{G}}^{\overline{G}\times\Gamma}(H^i(\mathrm{tw}_{\overline{G}}(C)_H)))_{S^*}$. Then the term $\langle\delta_\gamma \mid N^i\rangle$ occurs in the definition of $\mathrm{char}_{\overline{G}\times\Gamma,\gamma}(X) = \mathrm{char}^*_{\overline{G}\times\Gamma,\gamma}(X)$. Also, from Lemma 5.8 below, the action of δ_γ on $N^i = Q(R)\otimes_{\mathbb{Q}_p[\overline{G}]}(\mathbb{Q}_p\otimes_{\mathbb{Z}_p}H^i(\mathrm{tw}_{\overline{G}}(C)_H))$ restricts to give an automorphism of $R_\Sigma\otimes_{\mathbb{Q}_p[\overline{G}]}D_1^i$ and so (33) implies that $\langle\delta_\gamma \mid N^i\rangle$ is equal to

$$\begin{aligned}&\langle\delta_\gamma \mid Q(R)\otimes_{\mathbb{Q}_p[\overline{G}]}D_0^i\rangle\langle\delta_\gamma \mid Q(R)\otimes_{\mathbb{Q}_p[\overline{G}]}D_1^i\rangle\\ &\qquad\qquad = \langle\delta_\gamma \mid Q(R)\otimes_{\mathbb{Q}_p[\overline{G}]}D_0^i\rangle\iota_\Sigma(\langle\delta_\gamma \mid R_\Sigma\otimes_{\mathbb{Q}_p[\overline{G}]}D_1^i\rangle)\end{aligned}$$

where ι_Σ is the natural homomorphism $K_1(R_\Sigma)\to K_1(Q(R))$. Hence by combining (34) with the definition of $\mathrm{char}_{\overline{G}\times\Gamma,\gamma}(X)$ one finds that

$$\begin{aligned}(35)\quad \pi_{\overline{G}\times\Gamma}(\xi)\prod_{i\in\mathbb{Z}}\langle\delta_\gamma \mid M^i\rangle^{(-1)^{i+1}} &= \pi_{\overline{G}\times\Gamma}(\xi)\prod_{i\in\mathbb{Z}}\langle\delta_\gamma \mid Q(R)\otimes_{\mathbb{Q}_p[\overline{G}]}D_0^i\rangle^{(-1)^{i+1}}\\ &= \iota_1(u)\iota_\Sigma(\prod_{i\in\mathbb{Z}}\langle\delta_\gamma \mid R_\Sigma\otimes_{\mathbb{Q}_p[\overline{G}]}D_1^i\rangle^{(-1)^i})\in\mathrm{im}(\iota_\Sigma).\end{aligned}$$

Now R_Σ is finitely generated as a module over the commutative local ring $\Lambda(\Gamma)_\Sigma$ and so is itself a semi-local ring (cf. [15, Prop. (5.28)(ii)]). The natural homomorphism $R_\Sigma^\times\to K_1(R_\Sigma)$ is thus surjective (by [15, Th. (40.31)]) and so (35) implies that the element $\pi_{\overline{G}\times\Gamma}(\xi)\prod_{i\in\mathbb{Z}}\langle\delta_\gamma \mid M^i\rangle^{(-1)^{i+1}}$ is represented by a pair of the form $\langle \mathrm{r}_y \mid Q(R)\rangle$ with $y\in R_\Sigma^\times$. Now both y and y^{-1} are of the form $z\sigma^{-1}$ for suitable elements $z\in R\cap Q(R)^\times$ and $\sigma\in\Sigma$. Thus, to complete the proof of the lemma, it suffices to prove that for all such z and σ both $\mathrm{Nrd}_{Q(R)}(\langle\mathrm{r}_z \mid Q(R)\rangle)$ and $\mathrm{Nrd}_{Q(R)}(\langle\mathrm{r}_{\sigma^{-1}} \mid Q(R)\rangle)$ belong to $\zeta(R_\Sigma)$. But (31) implies $\mathrm{Nrd}_{Q(R)}(\langle\mathrm{r}_{\sigma^{-1}} \mid Q(R)\rangle) = (\sigma^{-d_\rho})_\rho\in\zeta(R_\Sigma)$ with $d_\rho := \dim_{\mathbb{Q}_p^c}(V_\rho)$. Also, if $\mathbb{Z}_p^c$ is the integral closure of $\mathbb{Z}_p$ in $\mathbb{Q}_p^c$ and T_ρ is any full $\mathbb{Z}_p^c$-sublattice of V_ρ, then the action of $\overline{G}$ on T_ρ induces a homomorphism $\varrho: R = \Lambda(\Gamma)[\overline{G}]\to\mathrm{M}_{d_\rho}(\mathbb{Z}_p^c\otimes_{\mathbb{Z}_p}\Lambda(\Gamma))$ and (31) implies $\mathrm{Nrd}_{Q(R)}(\langle\mathrm{r}_x \mid Q(R)\rangle) = (\det(\varrho(z)))_\rho\in(\mathbb{Q}_p^c\otimes_{\mathbb{Z}_p}\zeta(R))\cap\zeta(Q(R)) = \mathbb{Q}_p\otimes_{\mathbb{Z}_p}\zeta(R)\subseteq\zeta(R_\Sigma)$, as required. □

Lemma 5.8. *δ_γ induces an automorphism of $R_\Sigma\otimes_{\mathbb{Q}_p[\overline{G}]}D_1^i$.*

Proof. The argument of [26, Prop. 2.2, Rem. 2.3] gives a short exact sequence

$$0\to R\otimes_{\mathbb{Z}_p[\overline{G}]}D_1^i\xrightarrow{\delta_\gamma}R\otimes_{\mathbb{Z}_p[\overline{G}]}D_1^i\to D_1^i\to 0$$

and so it suffices to show that $(D_1^i)_\Sigma = 0$. But $D_1^i := \mathbb{Q}_p\otimes_{\mathbb{Z}_p}\mathrm{im}(H^i(\theta_\gamma))$ and, regarding $\mathrm{im}(H^i(\theta_\gamma))$ as a (finitely generated) module over $\Lambda(\Gamma)\subseteq R$, the decomposition (33) implies that $\mathrm{im}(H^i(\theta_\gamma))_\Gamma$ is finite. This implies that $\mathrm{im}(H^i(\theta_\gamma))$ is a finitely generated torsion $\Lambda(\Gamma)$-module whose characteristic polynomial $f(T)$ is coprime to T. It follows that $f(T)$ is invertible in R_Σ and so $(D_1^i)_\Sigma = \mathrm{im}(H^i(\theta_\gamma))_\Sigma = 0$, as required. □

5.3.2. *The proof of Proposition 5.6.* From Lemmas 5.4 and 5.7 we know that

$$(36)\qquad (\xi^*(\rho)\prod_{i\in\mathbb{Z}}(\mathrm{Nrd}_{Q(R)}(\langle\delta_\gamma \mid M^i\rangle))^*_\rho(0)^{(-1)^{i+1}})_{\rho\in\mathrm{Irr}(\overline{G})} = \pi(y_\xi)$$

where π is the natural projection $\zeta(R_\Sigma)^\times\to\zeta(\mathbb{Q}_p[\overline{G}])^\times$. But if x is in $R_\Sigma^\times$, then $\mathrm{Nrd}_{Q(R)}(\langle\mathrm{r}_x \mid Q(R)\rangle)$ belongs to $\zeta(R_\Sigma)^\times$ (see the proof of Lemma 5.7) and (31) implies $\pi(\mathrm{Nrd}_{Q(R)}(\langle\mathrm{r}_x \mid Q(R)\rangle)) = (\mathrm{Nrd}_{\mathbb{Q}_p[\overline{G}]}(\langle\mathrm{r}_{\overline{x}} \mid \mathbb{Q}_p[\overline{G}]\rangle)_\rho$ with $\overline{x}$ the image of x in $\mathbb{Q}_p[\overline{G}]^\times$. Hence (35) implies $\pi(y_\xi) = \mathrm{Nrd}_{\mathbb{Q}_p[\overline{G}]}(\overline{u})\prod_{i\in\mathbb{Z}}\mathrm{Nrd}_{\mathbb{Q}_p[\overline{G}]}(\langle H^i(\theta_\gamma) \mid D_1^i\rangle)^{(-1)^i}$ where $\overline{u}$ is the image of u under the natural composite homomorphism $K_1(R) \to K_1(\mathbb{Z}_p[\overline{G}]) \to K_1(\mathbb{Q}_p[\overline{G}])$. Since $\partial_{\overline{G}}(\overline{u}) = 0$ one therefore has

$$\partial_{\overline{G}}(\pi(y_\xi)) = \sum_{i\in\mathbb{Z}}(-1)^i\partial_{\overline{G}}(\langle H^i(\theta_\gamma) \mid D_1^i\rangle). \tag{37}$$

The equality of Proposition 5.6 now follows upon substituting (36) into (37) and then using both the explicit formula for $r_G(C)(\rho)$ given in Lemma 5.5(ii) and the following result (with $M = H^i(\mathrm{tw}_{\overline{G}}(C)_H)$ for each i).

LEMMA 5.9. *If M is any finitely generated R-module, then for every ρ in $\mathrm{Irr}(\overline{G})$ one has $(\mathrm{Nrd}_{Q(R)}(\delta_\gamma \mid \mathrm{I}_{\overline{G}}^{\overline{G}\times\Gamma}(M^\Gamma)_{S^*}))^*_\rho(0) = (-1)^{\dim_{\mathbb{Q}_p^c}(M^{\Gamma,\rho})}$.*

Proof. There are natural isomorphisms of $Q^c(\Gamma)$-modules of the form

$$\mathrm{Hom}_{\mathbb{Q}_p^c[\overline{G}]}(V_\rho, \mathrm{I}_{\overline{G}}^{\overline{G}\times\Gamma}(M^\Gamma)_{S^*}) \;\cong\; \mathrm{Hom}_{\mathbb{Q}_p^c[\overline{G}]}(V_\rho, Q^c(\Gamma)\otimes_{\mathbb{Z}_p} M^\Gamma) \;\cong\; M^{\Gamma,\rho}\otimes_{\mathbb{Q}_p^c} Q^c(\Gamma)$$

under which the induced action of δ_γ on the first module corresponds to the endomorphism $\tilde{\delta}_\gamma$ of the third module that sends $m\otimes x$ to $m\otimes(\gamma^{-1}-1)x$. It follows that $\mathrm{Nrd}_{Q(R)}(\delta_\gamma \mid \mathrm{I}_{\overline{G}}^{\overline{G}\times\Gamma}(M^\Gamma)_{S^*})_\rho$ is equal to

$$\begin{aligned}\det_{Q^c(\Gamma)}(\tilde{\delta}_\gamma \mid M^{\Gamma,\rho}\otimes_{\mathbb{Q}_p^c} Q^c(\Gamma)) &= \det_{Q^c(\Gamma)}(\gamma^{-1}-1 \mid Q^c(\Gamma))^{\dim_{\mathbb{Q}_p^c}(M^{\Gamma,\rho})}\\ &= (-T/(1+T))^{\dim_{\mathbb{Q}_p^c}(M^{\Gamma,\rho})}\end{aligned}$$

where the last equality follows from the fact that $\gamma^{-1}-1 = (1-\gamma)/\gamma = -T/(1+T)$. From this explicit formula it is clear that the first non-zero coefficient of T in the series $\mathrm{Nrd}_{Q(R)}(\delta_\gamma \mid \mathrm{I}_{\overline{G}}^{\overline{G}\times\Gamma}(M^\Gamma)_{S^*})_\rho$ is equal to $(-1)^{\dim_{\mathbb{Q}_p^c}(M^{\Gamma,\rho})}$. □

5.4. COMPLETION OF THE PROOF OF THEOREM 2.2. We set $N := U\cap H$. Then in each degree i there is a natural isomorphism of $\mathbb{Z}_p[\overline{G}]$-modules

$$H^i(\mathrm{tw}_{\overline{G}}(C)_H) \cong \mathbb{Z}_p[\overline{G}]\otimes_{\Lambda(H/N)} H^i(\Lambda(G/N)\otimes^{\mathbb{L}}_{\Lambda(G)} C).$$

But $\Lambda(G/N)\otimes^{\mathbb{L}}_{\Lambda(G)} C$ belongs to $D^{\mathrm{p}}_{S_{G/N,H/N}}(\Lambda(G/N))$ and so each module $H^i(\mathrm{tw}_{\overline{G}}(C)_H)$ is finitely generated over $\mathbb{Z}_p[\overline{G}]$. This implies that $\Delta(\mathrm{tw}_{\overline{G}}(C),\gamma)$ is an exact triangle in $D^{\mathrm{p}}(\mathbb{Z}_p[\overline{G}])$. In view of Lemma 5.5(iv) and Proposition 5.6 we may therefore deduce Theorem 2.2 by applying the following result with $\mathcal{G} = \overline{G}, R = \mathbb{Z}_p$ and $\Delta = \Delta(\mathrm{tw}_{\overline{G}}(C),\gamma)$.

PROPOSITION 5.10. *Let $\mathcal{G}$ be a finite group, R an integral domain and F the field of fractions of R. Let $\Delta : C \xrightarrow{\theta} C \to D \to C[1]$ be an exact triangle in $D^{\mathrm{p}}(R[\mathcal{G}])$. Assume that θ is semisimple and in each degree i fix an $F[\mathcal{G}][H^i(\theta)]$-equivariant direct complement W^i to $F\otimes_R \ker(H^i(\theta))$ in $F\otimes_R H^i(C)$. Then $H^i(\theta)$ induces an automorphism of the (finitely generated projective) $F[\mathcal{G}]$-module W^i, the element $\langle H^i(\theta)\rangle^* := \langle H^i(\theta) \mid W^i\rangle$ of $K_1(F[\mathcal{G}])$ is independent of the choice of W^i and in $K_0(R[\mathcal{G}],F[\mathcal{G}])$ one has*

$$\sum_{i\in\mathbb{Z}}(-1)^i\partial_{\mathcal{G}}(\langle H^i(\theta)\rangle^*) = -[\mathbf{d}_{R[\mathcal{G}]}(D),\beta_\Delta] \tag{38}$$

where $\partial_{\mathcal{G}}$ is the connecting homomorphism $K_1(F[\mathcal{G}]) \to K_0(R[\mathcal{G}],F[\mathcal{G}])$ and β_Δ is as defined in (55).

Proof. It is clear that $H^i(\theta)$ induces an automorphism of W^i and straightforward to verify that $\langle H^i(\theta)\rangle^*$ is independent of the choice of W^i. However to prove (38) we replace C by a complex P in $C^{\mathrm{p}}(R[\mathcal{G}])$ for which there exists an isomorphism $q : P \to C$ in $D^{\mathrm{p}}(R[\mathcal{G}])$ and we shall argue by induction on $|P| := \max\{i : P^i \neq 0\} - \min\{j : P^j \neq 0\}$. To do this we fix a morphism of complexes $\phi : P \to P$ such that $q \circ \phi = \theta \circ q$ in $D^{\mathrm{p}}(R[\mathcal{G}])$.
If $|P| = 0$, then $P = P^m[-m] = H^m(P)$ (and $\phi = \phi^m = H^m(\phi)$) for some integer m. In this case D identifies with the mapping cone $P^m \xrightarrow{\phi^m} P^m$ of ϕ (so the first term of this complex is placed in degree $m-1$) in such a way that the homomorphism $H^{m-1}(D) \to \ker(H^m(\theta)) \to \mathrm{cok}(H^m(\theta)) \to H^m(D)$ induced by Δ corresponds to the tautological map $\tau : \ker(\phi^m) \to \mathrm{cok}(\phi^m)$. Further, if W is a direct complement to $F \otimes_R \ker(\phi^m)$ in $F \otimes_R P^m$, then $\phi^m(W) = W$ (since ϕ is semisimple) and $[\mathbf{d}_{R[\mathcal{G}]}(D), \beta_\Delta] = (P^m, \iota^{(-1)^{m-1}}, P^m)$ with ι the composite isomorphism

$$F \otimes_R P^m = (F \otimes_R \ker(\phi^m)) \oplus W \xrightarrow{(F\otimes_R \tau, H^m(\phi))} (F \otimes_R \mathrm{cok}(\phi^m)) \oplus W \cong F \otimes_R P^m$$

where the isomorphism is induced by a choice of splitting of the tautological exact sequence $0 \to W \to F \otimes_R P^m \to F \otimes_R \mathrm{cok}(\phi^m) \to 0$ (this description of $[\mathbf{d}_{R[\mathcal{G}]}(D), \beta_\Delta]$ follows, for example, from [3, Th. 6.2]). The equality (38) is therefore valid because $-(P^m, \iota^{(-1)^{m-1}}, P^m) = (-1)^m \partial_{\mathcal{G}}(\langle \iota \mid F \otimes_R P^m\rangle) = (-1)^m \partial_{\mathcal{G}}(\langle H^m(\phi) \mid W\rangle)$.
We now assume that $|P| = n > 0$ and, to fix notation, that $\min\{j : P^j \neq 0\} = 0$. We set $C_2 := P$ and $\phi_2 = \phi$, write C_3 for the naive truncation in degree $n-1$ of P and set $C_1 := P^n[-n]$. Then one has a tautological short exact sequence of complexes

$$0 \to C_1 \to C_2 \to C_3 \to 0. \tag{39}$$

From the associated long exact cohomology sequence we deduce that $H^i(C_3) = H^i(C_2)$ if $i < n-1$ and that there are commutative diagrams of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^{n-1}(C_2) & \longrightarrow & H^{n-1}(C_3) & \longrightarrow & B^n(C_2) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle H^{n-1}(\phi_2)} & & \downarrow{\scriptstyle H^{n-1}(\phi_3)} & & \downarrow{\scriptstyle \phi^n} & & \\ 0 & \longrightarrow & H^{n-1}(C_2) & \longrightarrow & H^{n-1}(C_3) & \longrightarrow & B^n(C_2) & \longrightarrow & 0 \end{array} \tag{40}$$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B^n(C_2) & \longrightarrow & H^n(C_1) & \longrightarrow & H^n(C_2) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \phi^n} & & \downarrow{\scriptstyle H^n(\phi_1)} & & \downarrow{\scriptstyle H^n(\phi_2)} & & \\ 0 & \longrightarrow & B^n(C_2) & \longrightarrow & H^n(C_1) & \longrightarrow & H^n(C_2) & \longrightarrow & 0, \end{array} \tag{41}$$

where $B^n(C_2)$ denotes the coboundaries of C_2 in degree n. By mimicking the argument of [6, Lem. 4.4] we may change ϕ by a homotopy in order to assume that, in each degree i, the restriction of ϕ^i induces an automorphism of $F \otimes_R B^i(C_2)$. This assumption has two important consequences. Firstly, the above diagrams imply that the morphisms ϕ_1 and ϕ_3 of C_1 and C_3 that are induced by ϕ are semisimple. Secondly, if we write D_i for the mapping cone of ϕ_i for $i = 1, 2, 3$, then (39) induces short exact sequences of the form

$$0 \to D_1 \to D_2 \to D_3 \to 0$$
$$0 \to \mathrm{Z}(D_1) \to \mathrm{Z}(D_2) \to \mathrm{Z}(D_3) \to 0$$
$$0 \to \mathrm{B}(D_1) \to \mathrm{B}(D_2) \to \mathrm{B}(D_3) \to 0$$
$$0 \to \mathrm{H}(D_1) \to \mathrm{H}(D_2) \to \mathrm{H}(D_3) \to 0$$
$$0 \to \mathrm{H}_{\mathrm{bock}}(\Delta_1) \to \mathrm{H}_{\mathrm{bock}}(\Delta_2) \to \mathrm{H}_{\mathrm{bock}}(\Delta_3) \to 0.$$

Here we write $\mathrm{B}(D_i)$ and $\mathrm{Z}(D_i)$ for the complexes of coboundaries and cocycles of D_i (each with zero differentials) and Δ_i for the tautological exact triangle $C_i \xrightarrow{\phi_i} C_i \to D_i \to C_i[1]$. Now from the displayed exact sequences (and the definition of each term $[D_i, \beta_{\Delta_i}]$ in Appendix B) one has an equality

$$[\mathbf{d}_{R[\mathcal{G}]}(D_2), \beta_{\Delta_2}] = [\mathbf{d}_{R[\mathcal{G}]}(D_1), \beta_{\Delta_1}] + [\mathbf{d}_{R[\mathcal{G}]}(D_3), \beta_{\Delta_3}].$$

But the inductive hypothesis implies

$$-[\mathbf{d}_{R[\mathcal{G}]}(D_3), \beta_{\Delta_3}] = \sum_{i=0}^{i=n-1} (-1)^i \partial_{\mathcal{G}}(\langle H^i(\phi_3)\rangle^*)$$

and, since $|C_1| = 0$, our earlier argument proves

$$-[\mathbf{d}_{R[\mathcal{G}]}(D_1), \beta_{\Delta_1}] = (-1)^n \partial_{\mathcal{G}}(\langle H^n(\phi_1)\rangle^*).$$

It is also clear that $\langle H^i(\phi_3)\rangle^* = \langle H^i(\phi_2)\rangle^*$ for $i < n-1$ whilst (40) and (41) imply $\langle H^{n-1}(\phi_2)\rangle^* = \langle H^{n-1}(\phi_3)\rangle^* \langle \phi^n \mid F \otimes_R B^n(C_2)\rangle$ and $\langle H^n(\phi_1)\rangle^* = \langle \phi^n \mid F \otimes_R B^n(C_2)\rangle \langle H^n(\phi_2)\rangle^*$ respectively. The claimed description of $-[D_2, \beta_{\Delta_2}]$ thus follows upon combining the last three displayed equations. □

Remark 5.11. If $\mathcal{G}$ is abelian, then Proposition 5.10 can be reinterpreted in terms of graded determinants and in this case has been proved to within a 'sign ambiguity' by Kato in [19, Lem. 3.5.8]. (This ambiguity arises because Kato uses ungraded determinants - for more details in this regard see [19, Rem. 3.2.3(3) and 3.2.6(3),(5)] and [9, Rem. 9]).

Part II: Arithmetic

For any Galois extension of fields F/E we set $G_{F/E} := \mathrm{Gal}(F/E)$. For any field E we also fix an algebraic closure E^c and abbreviate $G_{E^c/E}$ to G_E.

6. Field-theoretic preliminaries

We first introduce the class of fields for which the techniques of [13] allow one to formulate a main conjecture of non-commutative Iwasawa theory.
We fix an odd prime p and for each number field k we write $\mathcal{F}_k$ for the set of Galois extensions L of k inside $\mathbb{Q}^c$ which satisfy the following conditions

(i) L contains the cyclotomic $\mathbb{Z}_p$-extension k^{cyc} of k;
(ii) L/k is unramified outside a finite set of places;
(iii) $G_{L/k}$ is a compact p-adic Lie group.

If k is totally real, then we also let $\mathcal{F}_k^+$ denote the subset of $\mathcal{F}_k$ comprising those fields that are totally real.

The following result was explained to us by Kazuya Kato. It provides an important general reduction step and also shows that Theorem 2.2 constitutes a satisfactory resolution of the descent problem in the setting of non-commutative Iwasawa theory.

LEMMA 6.1. *For any number field k and any F in $\mathcal{F}_k$ there exists a field F' in $\mathcal{F}_k$ with $F \subseteq F'$ and such that $G_{F'/k}$ has no element of order p. If k is totally real and F belongs to $\mathcal{F}_k^+$, then one can also choose F' in $\mathcal{F}_k^+$.*

Proof. For any extension E of k we write $E(\zeta_{p^\infty})$ for the extension of E generated by all p-power roots of unity (in $\mathbb{Q}^c$).

We set $\tilde{F} := F(\zeta_{p^\infty})$ and choose a p-torsion free open normal subgroup U of $V := G_{\tilde{F}/k(\zeta_{p^\infty})}$. We let L be the extension of k in $\tilde{F}$ that corresponds to U and for each non-trivial p-torsion element σ_i of V/U we write L_i for the fixed subfield of L by σ_i. Then $L = L_i(a_i^{1/p^n})$ for some $a_i \in L_i^\times$. Let a_{ij}, $1 \le j \le s(i)$, be all conjugates of a_i over F and set L_i' denote the field generated over L_i by the set $\{a_{ij}^{1/p^n} : 1 \le j \le s(i), n \ge 1\}$. Then $\tilde{F}L_i'$ is a Galois extension of F that contains L. Furthermore G_{L_i'/L_i} is isomorphic to a subgroup of $\mathbb{Z}_p^{s(i)}$ by $\tau \mapsto (r(j))_j$ with $\tau(a_j^{1/p^n})/a_j^{1/p^n} = \zeta_{p^n}^{r(j)}$. Let F' be the composite field of $\tilde{F}$ and L_i' for all i.

The group $G_{F'/k}$ is a compact p-adic Lie group and we now prove that it has no element of order p. We note first that $G_{k(\zeta_{p^\infty})/k}$ is isomorphic to a subgroup of $\mathbb{Z}_p^\times$ and hence is p-torsion free by the assumption $p \neq 2$. Thus if $\sigma \in G_{F'/k}$ has order p, then the image of σ in $G_{\tilde{F}/k}$ is contained in V and so the image of σ in V/U coincides with σ_i for some i. Thus σ fixes all elements of L_i. But then the image of σ in G_{L_i'/L_i} is both p-torsion and also non-trivial (for its restriction to G_{L/L_i} is non-trivial). This contradicts the fact that G_{L_i'/L_i} has no element of order p. Hence $G_{F'/k}$ has no element of order p, as claimed.

Lastly we assume that F (and hence k) is totally real. Then $\tilde{F}$ is a CM field with maximal real subfield $\tilde{F}^+$ equal to the compositum of F and the maximal totally real subfield of $k(\zeta_{p^\infty})$. Also, by the above construction, the extension $F'/\tilde{F}$ is pro-p. Since p is odd, the group $G_{F'/\tilde{F}^+}$ therefore contains a unique element of order 2 and the fixed field $(F')^+$ of F' by this element is totally real, contains F, is Galois over k and such that $G_{(F')^+/k}$ has no element of order p. $\square$

In the remainder of this article we set $\Gamma_k := G_{k^{\mathrm{cyc}}/k}$, $H_{L/k} := G_{L/k^{\mathrm{cyc}}}$, $\Lambda(L/k) := \Lambda(G_{L/k})$ and $\Omega(L/k) := \Omega(G_{L/k})$ for each L in $\mathcal{F}_k$. We also fix a topological generator $\gamma_{\mathbb{Q}}$ of $\Gamma_{\mathbb{Q}}$, set $d_k := [k \cap \mathbb{Q}^{\mathrm{cyc}} : \mathbb{Q}]$ and write γ_k for the topological generator $\gamma_{\mathbb{Q}}^{d_k}$ of Γ_k.

REMARK 6.2. If $\mathcal{C}$ denotes either $\mathcal{F}_k$ or $\mathcal{F}_k^+$, then it is an ordered set (by inclusion). Lemma 6.1 implies that the subset $\mathcal{C}'$ of $\mathcal{C}$ comprising those fields F for which $G_{F/k}$ has no element of order p is cofinal. Taking account of the functorial properties of the isomorphism in Theorem 2.1 and of the results in Proposition 4.7(ii) we may therefore deduce the following extensions of these results.

- There is a natural isomorphism of abelian groups

$$\varprojlim_{F \in \mathcal{C}} K_1(\Lambda(F/k)_{S^*}) \cong \varprojlim_{F \in \mathcal{C}} K_0(\Omega(F/k)) \oplus \varprojlim_{F \in \mathcal{C}} K_0(\Lambda(F/k), \Lambda(F/k)_S) \oplus \varprojlim_{F \in \mathcal{C}} \mathrm{im}(\lambda_{F/k})$$

where $\lambda_{F/k}$ is the natural homomorphism $K_1(\Lambda(F/k)) \to K_1(\Lambda(F/k)_{S^*})$ and in each inverse limit the transition maps are induced by the homomorphism $\Lambda(F/k) \to \Lambda(F'/k)$ for each $F' \subseteq F$.

- Let $(x_F)_F$ be an element of $\varprojlim_{F\in\mathcal{C}} K_0(\Lambda(F/k), \Lambda(F/k)_{S^*})$. Then for each F in $\mathcal{C}$ we may define an element $\mathrm{char}_{\mathcal{G}_{F/k},\gamma_k}(x_F)$ of $K_1(\Lambda(F/k)_{S^*})$ in the following way: we choose F' in $\mathcal{C}'$ with $F \subseteq F'$ and let $\mathrm{char}_{G_{F/k},\gamma_k}(x_F)$ denote the image of $\mathrm{char}_{G_{F'/k},\gamma_k}(x_{F'})$ under the natural projection $K_1(\Lambda(F'/k)_{S^*}) \to K_1(\Lambda(F/k)_{S^*})$. Then Lemma 4.5 implies $\mathrm{char}_{G_{F/k},\gamma_k}(x_F)$ is independent of the precise choice of F' and Proposition 4.7 implies $\partial_{G_{F/k}}(\mathrm{char}_{G_{F/k},\gamma_k}(x_F)) = x_F$.

7. Non-commutative main conjectures

In this section we formulate explicit 'main conjectures of non-commutative Iwasawa theory' for both Tate motives and (certain) critical motives. In particular, in the setting of elliptic curves, the conjecture we formulate here is finer than that formulated by Coates et al in [13] in that we consider interpolation formulas for the leading terms (rather than values) of p-adic L-functions at Artin representations.
Henceforth we will fix an isomorphism of fields $j : \mathbb{C} \cong \mathbb{C}_p$ and often simply omit it from the notation.

7.1. Tate motives. In this subsection we fix a totally real number field k and formulate a natural main conjecture for class groups associated to each field in $\mathcal{F}_k^+$. We therefore fix a field K in $\mathcal{F}_k^+$ and a finite set of places Σ of k that contains the archimedean places and all places that ramify in K/k. We may regard each Artin representation ρ of $G_{K/k}$ as a complex representation (via j^{-1}) and then write $L_\Sigma(s, \rho^{j^{-1}})$ for the Artin L-function of ρ that is truncated by removing the Euler factors attached to places in Σ.
To formulate a main conjecture we must multiply the leading term $L^*_\Sigma(1, \rho^{j^{-1}})$ in the Taylor expansion at $s = 1$ of $L_\Sigma(s, \rho^{j^{-1}})$ by an appropriate period. To define this period we let E be any finite degree Galois extension of k with $E \subset K$ and $G_{K/E} \subseteq \ker(\rho)$. We set $E_\infty := \mathbb{R} \otimes_\mathbb{Q} E \cong \prod_{\mathrm{Hom}(E,\mathbb{C})} \mathbb{R}$ and write $\log_\infty(\mathcal{O}_E^\times)$ for the inverse image of $\mathcal{O}_E^\times \hookrightarrow E_\infty^\times$ under the (componentwise) exponential map $\exp_\infty : E_\infty \to E_\infty^\times$. Then the Dirichlet Unit Theorem implies that $\log_\infty(\mathcal{O}_E^\times)$ is a lattice in the $\mathbb{R}$-space generated by $E_0 := \{x \in E : \mathrm{Tr}_{E/\mathbb{Q}}(x) = 0\}$ and so there is a canonical isomorphism of $\mathbb{C}[G_{E/k}]$-modules $\mu_\infty : \mathbb{C} \otimes_\mathbb{Z} \log_\infty(\mathcal{O}_E^\times) \cong \mathbb{C} \otimes_\mathbb{Q} E_0$. In addition, if we write $S_p(E)$ for the set of p-adic places of E, then the composite homomorphism

$$\begin{aligned}\mathbb{Z}_p \otimes_\mathbb{Z} \log_\infty(\mathcal{O}_E^\times) &\xrightarrow{\exp_\infty} \mathbb{Z}_p \otimes_\mathbb{Z} \mathcal{O}_E^\times \\ &\to \prod_{w\in S_p(E)} U^1_{E_w} \xrightarrow{(u_w)_w \mapsto (\log_p(u_w))_w} \prod_{w\in S_p(E)} E_w \cong \mathbb{Q}_p \otimes_\mathbb{Q} E\end{aligned}$$

(where the second arrow is the natural diagonal map) factors through the inclusion $\mathbb{Q}_p \otimes_\mathbb{Q} E_0 \subset \mathbb{Q}_p \otimes_\mathbb{Q} E$ and hence induces a homomorphism $\mu_p : \mathbb{C}_p \otimes_\mathbb{Z} \log_\infty(\mathcal{O}_E^\times) \cong \mathbb{C}_p \otimes_\mathbb{Q} E_0$ of $\mathbb{C}_p[G_{E/k}]$-modules. The resulting period

$$\Omega_j(\rho) := \det_{\mathbb{C}_p}(\mu_p \circ (\mathbb{C}_p \otimes_{\mathbb{C},j} \mu_\infty)^{-1})^\rho \in \mathbb{C}_p$$

depends upon j and ρ but is independent of the choice of E.

We write $\chi_{\rm cyc}$ for the cyclotomic character $G_k \to \Gamma_k \to \mathbb{Z}_p^\times$ and for any Artin representation ρ of $G_{K/k}$ we write $\langle \rho, 1\rangle$ for the multiplicity with which the trivial representation of $G_{K/k}$ occurs in ρ.

CONJECTURE 7.1. *Assume that $G_{K/k}$ has no element of order p. Then the Galois group $X_\Sigma(K)$ of the maximal pro-p abelian extension of K that is unramified outside Σ belongs to $\mathfrak{M}_{S^*}(G_{K/k})$ and there exists an element ξ of $K_1(\Lambda(K/k)_{S^*})$ which satisfies both of the following conditions.*

(a) *At each Artin representation ρ of $G_{K/k}$ one has*

$$\xi(\rho) = (\log_p(\chi_{\rm cyc}(\gamma_k))^{\langle\rho,1\rangle} c_{\rho,k})^{-1}\Omega_j(\rho)L^*_\Sigma(1,\rho^{j^{-1}})^j$$

with $c_{\rho,k} := 1$ if either ρ is trivial or $H_{K/k} \not\subset \ker(\rho)$ and $c_{\rho,k} := 1 - \rho(\gamma_k^{-1})$ otherwise.

(b) $\partial_{G_{K/k}}(\xi) = [X_\Sigma(K)]$.

REMARK 7.2. For each Artin representation ρ of $G_{K/k}$ the '(Σ-truncated) p-adic Artin L-function' of ρ is the unique p-adic meromorphic function $L_{p,\Sigma}(\cdot,\rho) : \mathbb{Z}_p \to \mathbb{C}_p$ with the property that for each strictly negative integer n and each isomorphism $j : \mathbb{C} \cong \mathbb{C}_p$ one has $L_{p,\Sigma}(n,\rho) = L_\Sigma(n,(\rho\otimes\omega^{n-1})^{j^{-1}})^j$ where $\omega : G_\mathbb{Q} \to \mathbb{Z}_p^\times$ is the Teichmüller character. Then the 'p-adic Stark conjecture at $s = 1$', as formulated by Serre in [27] and discussed by Tate in [30, Chap. VI, §5], asserts that the term $\Omega_j(\rho)L^*_\Sigma(1,\rho^{j^{-1}})^j$ in Conjecture 7.1(a) is equal to the leading term of $L_{p,\Sigma}(s,\rho)$ at $s = 1$. See [11, Rem. 5.3] for more details.

In Conjecture 8.3 we formulate a version of Conjecture 7.1 that does not assume that $G_{K/k}$ has no element of order p.

7.2. CRITICAL MOTIVES.

7.2.1. *Preliminaries.* Let M be a critical motive over $\mathbb{Q}$ that has good ordinary reduction at p. Then its p-adic realization $V = M_p$ has a unique $\mathbb{Q}_p$-subspace $\hat{V}$ that is stable under the action of $G_{\mathbb{Q}_p}$ and such that $D^0_{dR}(\hat{V}) = t_p(V) := D_{dR}(V)/D^0_{dR}(V)$. Now let ρ be an Artin representation defined over a number field B and $[\rho]$ the corresponding Artin motive. We fix a p-adic place λ of B, set $L := B_\lambda$ and write $\mathcal{O}$ for the valuation ring of L. Then the λ-adic realisation

$$W := W_\rho := N_\lambda = V \otimes_{\mathbb{Q}_p} [\rho]^*_\lambda \tag{42}$$

of the motive $N := M(\rho^*) := M \otimes [\rho]^*$ is an L-adic representation and contains the $G_{\mathbb{Q}_p}$-subrepresentation $\hat{W} = \hat{V} \otimes_{\mathbb{Q}_p} [\rho]^*_\lambda$. The *algebraic rank* of $M(\rho^*)$ is defined as

$$r(M)(\rho) := \dim_L(H^1_f(\mathbb{Q}, W_\rho)) - \dim_L(H^3_f(\mathbb{Q}, W_\rho)). \tag{43}$$

Let Σ be a finite set of places of $\mathbb{Q}$ containing p, ∞ and all places at which M has bad reduction or which ramify in $K/\mathbb{Q}$. We fix a field K in $\mathcal{F}_\mathbb{Q}$ that is unramified outside Σ. By Υ we denote the set of those primes $\ell \neq p$ such that the ramification index of ℓ in $K/\mathbb{Q}$ is infinite.
For a B-motive N over $\mathbb{Q}$ we denote by $\Omega_\infty(N)$ and $\Omega_p(N)$ the associated complex and p-adic periods and by $R_p(N)$ and $R_\infty(N)$ the associated complex and p-adic regulators, see again [17] or [11, Th. 6.5]. We recall that $\Omega_\infty(N) \neq 0$ if N is critical and that $R_\infty(N) \neq 0$ if the (complex) height pairing of N is non-degenerate. Furthermore, for

a $\mathbb{Q}_p$-linear continuous $G_{\mathbb{Q}_p}$-representation Z we write $\Gamma(Z)$ for its Γ-factor (loc. cit.). Finally, for any L-linear continuous representation V and prime number ℓ we define an element of the polynomial ring $L[u]$ by setting

$$P_\ell(V,u) := P_{L,\ell}(V,u) := \begin{cases} \det_L(1-\varphi_\ell u|V^{I_\ell}), & \text{if } \ell \neq p, \\ \det_L(1-\varphi_p u|D_{cris}(V)), & \text{if } \ell = p, \end{cases}$$

where φ_ℓ denotes the geometric Frobenius automorphism of ℓ.
As shown by Fukaya and Kato in [17, Th. 4.2.26], the behaviour of local ϵ-factors implies that p-adic L-functions can exist only after a suitable extension of scalars. To describe this we must assume that

$$(44) \qquad \begin{cases} \text{the maximal absolutely abelian subfield } K^{ab,p} \text{ of } K \text{ in which} \\ p \text{ is unramified is finite.} \end{cases}$$

Under this hypothesis we let A denote the valuation ring of the completion at any p-adic place of the field $K^{ab,p}$. We set $\Lambda_A(K/\mathbb{Q}) := A \otimes_{\mathbb{Z}_p} \Lambda(K/\mathbb{Q})$ and $\Lambda_A(K/\mathbb{Q})_{S^*} := A \otimes_{\mathbb{Z}_p} \Lambda(K/\mathbb{Q})_{S^*}$ and write $\partial_{A,G_{K/\mathbb{Q}}} : K_1(\Lambda_A(K/\mathbb{Q})_{S^*}) \to K_0(\Lambda_A(K/\mathbb{Q}), \Lambda_A(K/\mathbb{Q})_{S^*})$ for the corresponding connecting homomorphism.

7.2.2. *Elliptic curves.* We first consider the case of the motive $M = h^1(E)(1)$ of an elliptic curve E over $\mathbb{Q}$ with good ordinary reduction at p with $K = \mathbb{Q}(E(p))$ being the extension of $\mathbb{Q}$ which arises by adjoining the p-power division points and we assume that $G_{K/\mathbb{Q}}$ does not contain any element of order p. In this situation the formulation of a (refined) main conjecture is very explicit since one can work with the dual $X(E_{/K})$ of the (p-primary) Selmer group; later we will give another formulation for general critical motives involving Selmer complexes. In the present situation one knows that the condition (44) is satisfied (cf. [13, just before Conj. 5.7]) and also that, if $Ш(E_{/K^{\ker(\rho)}})$ is finite, then

$$r(M)(\rho) = \dim_{\mathbb{C}_p}(e_{\rho^*}(\mathbb{C}_p \otimes_{\mathbb{Z}} E(K^{\ker(\rho)}))).$$

Upon combining the leading term computations of [11, Th. 6.5] with [17, Th. 4.2.22] and the general approach of [13] we are led to formulate the following conjecture.

CONJECTURE 7.3. *Fix a field K in $\mathcal{F}_{\mathbb{Q}}$ that is unramified outside a finite set of places Σ and is such that $G_{K/\mathbb{Q}}$ has no element of order p. Then, under the above conditions, the module $X(E_{/K})$ belongs to $\mathfrak{M}_{S^*}(G_{K/\mathbb{Q}})$. Further, there exists an element $\mathcal{L} = \mathcal{L}(E)$ of $K_1(\Lambda_A(K/\mathbb{Q})_{S^*})$ which satisfies both of the following conditions:*

(a) *At each Artin representation ρ of $G_{K/\mathbb{Q}}$, the value at $T = 0$ of $T^{-r(M)(\rho)}\Phi_\rho(\mathcal{L})$ is equal to*

$$(-1)^{r(M)(\rho)} \frac{L^*_{B,\Upsilon}(M(\rho^*))}{\Omega_\infty(M(\rho^*))R_\infty(M(\rho^*))} \cdot \Omega_p(M(\rho^*))R_p(M(\rho^*)) \cdot \frac{P_{L,p}(\hat{W}^*_\rho(1),1)}{P_{L,p}(\hat{W}_\rho,1)},$$

where $L^*_{B,\Upsilon}(M(\rho^*))$ is the leading coefficient at $s=0$ of the complex L-function of $M(\rho^*)$, truncated by removing Euler factors for all primes in Υ.

(b) $\partial_{A,G_{K/\mathbb{Q}}}(\mathcal{L}) = [\Lambda_A(K/\mathbb{Q}) \otimes_{\Lambda(K/\mathbb{Q})} X(E_{/K})]$.

REMARK 7.4. The interpolation formula in Conjecture 7.3(a) can of course also be stated in terms of the classical Hasse-Weil L-functions and their twists $L(E,\rho^*,s)$ in the sense of [13, (102)] (which is the same as the L-function attached to the B-motive $h^1(E) \otimes [\rho]^*$); due to the shift one now has to consider the leading term $L^*(E,\rho^*)$ of $L(E,\rho^*,s)$ at

$s=1$. Moreover, one can simplify the above expression and make it more explicit. To this end we let u in $\mathbb{Z}_p$ be the unit root of the polynomial $1-a_pX+pX^2$ where, as usual, $p+1-a_p=\#\tilde{E}_p(\mathbb{F}_p)$ with $\tilde{E}_p$ denoting the reduction of E modulo p. Furthermore we write p^{f_ρ} for the p-part of the conductor of ρ and $\epsilon_p(\rho)$ for the local ϵ-factor of ρ at the prime p. Moreover, let $d_+(\rho)$ and $d_-(\rho)$ denote the dimension of the subspace of $[\rho]_\lambda$ on which complex conjugation acts by $+1$ and -1, respectively. We denote the periods of E by

$$\Omega_+(E):=\int_{\gamma^+}\omega,\ \ \Omega_-(E):=\int_{\gamma^-}\omega$$

where ω is the Néron differential and γ^+ and γ^- denote a generator for the subspace of $\mathrm{H}_1(E(\mathbb{C}),\mathbb{Z})$ on which complex conjugation acts as $+1$ and -1 respectively. Finally, we write $R_\infty(E,\rho^*)$ and $R_p(E,\rho^*)$ for the complex and p-adic regulators of E twisted by ρ^*. Then the displayed expression in Conjecture 7.3(a) is equal to

$$(45)\quad (-1)^{\dim_{\mathbb{C}_p}(e_{\rho^*}(\mathbb{C}_p\otimes_{\mathbb{Z}}E(K^{\ker(\rho)})))}\frac{L_R^*(E,\rho^*)}{\Omega_+(E)^{d_+(\rho)}\Omega_-(E)^{d_-(\rho)}R_\infty(E,\rho^*)} \times \epsilon_p(\rho)u^{-f_\rho}R_p(E,\rho^*)\frac{P_{L,p}([\rho]_\lambda,u^{-1})}{P_{L,p}([\rho]_\lambda^*,up^{-1})}.$$

Here $L_R^*(E,\rho^*)$ is the leading coefficient at $s=1$ of the L-function $L_R(E,\rho^*,s)$ obtained from the Hasse-Weil L-function of E twisted by ρ^* by removing the Euler factors at p and at all primes ℓ at which the j-invariant j_E of E in non-integral. See [17, Rem. 4.2.27] with $u=\alpha$ for the calculation of $\Omega_p(M(\rho^*))$.

Before stating the next result we recall that the explicit interpolation formula given in the main conjecture of [13, Conj. 5.8] requires minor modification. To be precise, one must interchange all occurrences of ρ and $\hat{\rho}$ on the right hand side of the equality of [loc. cit., (107)] except for the term '$e_p(\rho)$' (for further details see the footnote at the end of §6.0 in [33]).

Proposition 7.5. *Assume the hypotheses of Conjecture 7.3. Assume also that for all Artin representations ρ of $G_{K/\mathbb{Q}}$ the 'order of vanishing part' of the Birch and Swinnerton-Dyer Conjecture for $E(\rho^*)$ holds. Then Conjecture 7.3 implies the 'main conjecture of non-commutative Iwasawa theory' of* [13, Conj. 5.8] *(modified as above).*

Proof. In this case Υ is the set comprising the prime p and all prime numbers q with $\mathrm{ord}_q(j_E)<0$ (see also [17, 4.5.3] or [33, Rem. 6.5]). In view of Remark 7.4 the only essential difference between the two conjectures is therefore that Conjecture 7.3 involves an interpolation formula for $((r(M)(\rho)!)^{-1}$ times) the value at $T=0$ of the $r(M)(\rho)$-th derivative of $\Phi_\rho(\mathcal{L})$ rather than merely for the value at $T=0$ of $\Phi_\rho(\mathcal{L})$ itself as in [13, Conj. 5.8]. (Note that the conjectured value at $T=0$ of $T^{-r(M)(\rho)}\Phi_\rho(\mathcal{L})$ should be the leading term at $T=0$ of $\Phi_\rho(\mathcal{L})$ only if $R_p(M(\rho^*))\neq 0$.)

At the outset we note that $r(\Phi_\rho(\mathcal{L}))\geq \dim_{\mathbb{C}_p}(e_{\rho^*}(E(K^{\ker(\rho)})\otimes_{\mathbb{Z}}\mathbb{C}_p))\geq 0$ because the given interpolation formula has no pole. In particular, $\mathcal{L}$ does not have ∞ as its value at any ρ. We also note that the 'order of vanishing part' of the Birch and Swinnerton-Dyer Conjecture for $E(\rho^*)$ implies that the order of vanishing of $L_R(E,\rho^*,s)$ at $s=1$ is equal to $\dim_{\mathbb{C}_p}(e_{\rho^*}(\mathbb{C}_p\otimes_{\mathbb{Z}}E(K^{\ker(\rho)})))$.

We now assume that $e_{\rho^*}(\mathbb{C}_p\otimes_{\mathbb{Z}}E(K^{\ker(\rho)}))$ vanishes. Then both $R_p(M(\rho^*))=1$ and $R_\infty(M(\rho^*))=1$. Also, the leading term $L_R^*(E,\rho^*)$ is in this case equal to the value at

$s = 1$ of $L_R(E, \rho^*, s)$. Hence, the interpolation formula (45) coincides with that given in [13, Conj. 5.8].
On the other hand, if $e_{\rho^*}(\mathbb{C}_p \otimes_{\mathbb{Z}} E(K^{\ker(\rho)})) \neq 0$, then $r(\Phi_\rho(\mathcal{L})) > 0$ and so the value of $\mathcal{L}$ at ρ is equal to 0. In addition, in this case the function $L_R(E, \rho^*, s)$ vanishes at $s = 1$ and so the interpolation formula of [13, Conj. 5.8] also implies that the value of $\mathcal{L}$ at ρ is equal to 0, as required. □

7.2.3. *The general case.* We return to the more general case discussed in §7.2.1. We fix a full Galois stable $\mathbb{Z}_p$-sublattice T of V and define a $G_{\mathbb{Q}_p}$-stable $\mathbb{Z}_p$-sublattice of $\hat{V}$ by setting $\hat{T} := T \cap \hat{V}$. As before we let $\mathbb{T}$ denote the Galois representation $\Lambda(K/\mathbb{Q}) \otimes_{\mathbb{Z}_p} T$ and set $\hat{\mathbb{T}} := \Lambda(K/\mathbb{Q}) \otimes_{\mathbb{Z}_p} \hat{T}$ similarly. Then $\hat{\mathbb{T}}$ is a $G_{\mathbb{Q}_p}$-stable $\Lambda(K/\mathbb{Q})$-submodule of $\mathbb{T}$. For the definition of the Selmer complex $SC_U := SC_U(\hat{\mathbb{T}}, \mathbb{T})$, which is originally due to Nekovář [24], we refer the reader to either [17, 4.1.2] or [11, (31)].

CONJECTURE 7.6 (General formulation for critical motives). *Fix a field K in $\mathcal{F}_{\mathbb{Q}}$ that is unramified outside a finite set of places Σ. Then, under the above conditions, the complex SC_U belongs to $D^{\mathrm{p}}_{S^*}(\Lambda(K/\mathbb{Q}))$. Further, there exists an element $\xi = \xi(U, M)$ of $K_1(\Lambda_A(K/\mathbb{Q})_{S^*})$ which satisfies both of the following conditions:*

(a) *At each Artin representation $\rho : G_{K/\mathbb{Q}} \to \mathrm{GL}_n(\mathcal{O})$ for which neither $P_{L,p}(\hat{W}_\rho, 1)$ or $P_{L,p}(W_\rho, 1)$ is equal to 0 the value at $T = 0$ of $T^{-r(M)(\rho)}\Phi_\rho(\xi)$ is equal to*

$$(-1)^{r(M)(\rho)} \frac{L^*_{B,\Sigma}(M(\rho^*))}{\Omega_\infty(M(\rho^*))R_\infty(M(\rho^*))} \cdot \Omega_p(M(\rho^*))R_p(M(\rho^*)) \cdot \Gamma(\hat{V})^{-1} \cdot \frac{P_{L,p}(\hat{W}^*_\rho(1), 1)}{P_{L,p}(\hat{W}_\rho, 1)},$$

where $L^*_{B,\Sigma}(M(\rho^*))$ is the leading coefficient of the complex L-function of $M(\rho^*)$, truncated by removing Euler factors for all primes in $\Sigma \setminus \{\infty\}$.

(b) $\partial_{A,G_{K/\mathbb{Q}}}(\xi) = \chi(\Lambda_A(K/\mathbb{Q}) \otimes_{\Lambda(K/\mathbb{Q})} SC_U)$.

PROPOSITION 7.7. *If $M = h^1(E)(1)$ and $K = \mathbb{Q}(E(p))$ are as in Conjecture 7.3, then Conjecture 7.6 is equivalent to Conjecture 7.3.*

Proof. We note first that [17, Prop. 4.3.7] implies the complex SC_U belongs to $D^{\mathrm{p}}_{S^*}(\Lambda(K/\mathbb{Q}))$ precisely when the module $X(E_{/K})$ belongs to $\mathfrak{M}_{H_{K/\mathbb{Q}}}(G_{K/\mathbb{Q}}) = \mathfrak{M}_{S^*}(G_{K/\mathbb{Q}})$. Also, SC_U differs from the complex $\mathrm{SC}(\hat{\mathbb{T}}, \mathbb{T})$ in loc. cit. only by local terms which belong to $\mathfrak{M}_{S^*}(G_{K/\mathbb{Q}})$ (by [17, Prop. 4.3.6]) and have characteristic elements (denoted $\zeta(\ell, K/\mathbb{Q})$ in loc. cit.) that correspond to the Euler-factors $P_{L,\ell}(W_\rho, s)$ and whose values $P_{L,\ell}(W_\rho, 1)$ at ρ are neither 0 or ∞ (by [17, Lem. 4.2.23]). To deduce the claimed result from here one need only note that $\Gamma_{\mathbb{Q}_p}(\hat{V}) = 1$ in this case and recall (from [17, Prop. 4.3.15-18]) that the class of $\mathrm{SC}(\hat{\mathbb{T}}, \mathbb{T})$ in $K_0(\mathfrak{M}_{S^*}(G_{K/\mathbb{Q}}))$ is equal to $[X(E_{/K})]$. □

8. EQUIVARIANT TAMAGAWA NUMBERS

Let F/k be a finite Galois extension of number fields. Then for any motive M defined over k the equivariant Tamagawa number conjecture of [9, Conj. 4.1(iv)] asserts the vanishing of an element $T\Omega(M_F, \mathbb{Z}[G_{F/k}])$ of $K_0(\mathbb{Z}[G_{F/k}], \mathbb{R}[G_{F/k}])$ that is constructed from the various realisations and comparison isomorphisms associated to the motive $M_F := F \otimes_k M$. Here M_F is regarded as defined over k and endowed with a natural left action of $\mathbb{Q}[G_{F/k}]$ (via the first factor).

Now the product over all primes p and all field isomorphisms $j : \mathbb{C} \cong \mathbb{C}_p$ of the composite homomorphism

$$j_* : K_0(\mathbb{Z}[G_{F/k}], \mathbb{R}[G_{F/k}]) \to K_0(\mathbb{Z}[G_{F/k}], \mathbb{C}[G_{F/k}]) \\ \to K_0(\mathbb{Z}[G_{F/k}], \mathbb{C}_p[G_{F/k}]) \to K_0(\mathbb{Z}_p[G_{F/k}], \mathbb{C}_p[G_{F/k}])$$

is injective, where the second map is induced by j (cf. [4, Lem. 2.1]). To prove [9, Conj. 4.1(iv)] it therefore suffices to prove that $j_*(T\Omega(M_F, \mathbb{Z}[G_{F/k}])) = 0$ for every such j. This reduction has the further advantage that the element $j_*(T\Omega(M_F, \mathbb{Z}[G_{F/k}]))$ can be directly defined without assuming the 'Coherence Hypothesis' of [9, §3.3] that is necessary to define $T\Omega(M_F, \mathbb{Z}[G_{F/k}])$ (cf. [9, Rem. 8]). However, even if one assumes the standard compatibility conjectures concerning the definition of Euler factors (cf. [9, Conj. 3]), the definition of $j_*(T\Omega(M_F, \mathbb{Z}[G_{F/k}]))$ is in general still conditional, being dependent upon the conjectural existence of a fundamental exact sequence relating the motivic cohomology spaces of M_F and its Kummer dual [9, Conj. 1] and of canonical p-adic Chern class isomorphisms [9, Conj. 2]. In particular, since we are assuming here that the element $j_*(T\Omega(M_F, \mathbb{Z}[G_{F/k}]))$ is well-defined, the results that we prove in this section will not shed any new light on either of [9, Conj. 1, Conj. 2].

8.1. Tate motives. In this subsection we fix a finite Galois extension F/k of totally real number fields and write $\mathbb{Q}(1)_F$ for the motive $h^0(\mathrm{Spec}\, F)(1)$, regarded as defined over k and with coefficients $\mathbb{Q}[G_{F/k}]$. We recall that all of the conjectures necessary for the definition of $T\Omega(\mathbb{Q}(1)_F, \mathbb{Z}[G_{F/k}])$ are known to be valid and hence that $j_*(T\Omega(\mathbb{Q}(1)_F, \mathbb{Z}[G_{F/k}]))$ is defined unconditionally as an element of $K_0(\mathbb{Z}_p[G_{F/k}], \mathbb{C}_p[G_{F/k}])$. For a discussion of various explicit consequences of the vanishing of $T\Omega(\mathbb{Q}(1)_F, \mathbb{Z}[G_{F/k}])$ see [4].

Theorem 8.1. *Let K be any field which belongs to $\mathcal{F}_k^+$, contains F and is such that $G_{K/k}$ has no element of order p (such a field K exists by virtue of Lemma 6.1). If K validates Conjecture 7.1 and F validates Leopoldt's Conjecture (at p), then one has $j_*(T\Omega(\mathbb{Q}(1)_F, \mathbb{Z}[G_{F/k}])) = 0$.*

Proof. For any quotient $\mathcal{G}$ of $G_{K/k}$ we write $\Lambda(\mathcal{G})^\#(1)$ for the $\Lambda(\mathcal{G})$-module $\Lambda(\mathcal{G})$ endowed with the following action of G_k: each σ in G_k acts on $\Lambda(\mathcal{G})^\#(1)$ as right multiplication by the element $\chi_{\mathrm{cyc}}(\sigma)\bar{\sigma}^{-1}$ where $\bar{\sigma}$ denotes the image of σ in $\mathcal{G}$. If K' is the subfield of K with $G_{K'/k} = \mathcal{G}$ and $\mathcal{O}_{k,\Sigma}$ is the subring of k comprising elements that are integral at all places outside Σ then, following Fukaya and Kato [17, §2.1.1] and Nekovář [24], the compact support cohomology complex $C_{K'} := R\Gamma_{c,\text{ét}}(\mathcal{O}_{k,\Sigma}, \Lambda(\mathcal{G})^\#(1))$ is an object of $D^{\mathrm{p}}(\Lambda(\mathcal{G}))$ that lies in a canonical exact triangle in $D(\Lambda(\mathcal{G}))$ of the form

$$C_{K'} \to R\Gamma_{\text{ét}}(\mathcal{O}_{k,\Sigma}, \Lambda(\mathcal{G})^\#(1)) \to \bigoplus_{v\in\Sigma} R\Gamma_{\text{ét}}(k_v, \Lambda(\mathcal{G})^\#(1)) \to C_{K'}[1]. \tag{46}$$

In the sequel we use the following facts: there is a natural isomorphism in $D^{\mathrm{p}}(\Lambda(\mathcal{G}))$ of the form

$$\Lambda(\mathcal{G}) \otimes^{\mathbb{L}}_{\Lambda(K/k)} C_K \cong C_{K'}; \tag{47}$$

in each degree i there is a natural isomorphism $H^i(C_K) \cong \varprojlim_{K'} H^i(C_{K'})$ where K' runs over all finite degree Galois extensions K'/k with $K' \subseteq K$ and the limit is taken with respect to the natural corestriction maps; for each such K' there are natural identifications $H^i(C_{K'}) \cong H^i_{c,\text{ét}}(\mathcal{O}_{K',\Sigma}, \mathbb{Z}_p(1))$, $H^i(R\Gamma_{\text{ét}}(\mathcal{O}_{k,\Sigma}, \Lambda(\mathcal{G})^\#(1))) \cong H^i_{\text{ét}}(\mathcal{O}_{K',\Sigma}, \mathbb{Z}_p(1))$

and $H^i(R\Gamma_{\text{ét}}(k_v, \Lambda(\mathcal{G})^{\#}(1))) \cong \bigoplus_{w|v} H^i_{\text{ét}}(K'_w, \mathbb{Z}_p(1))$. In particular, by a standard computation (involving Kummer theory, class field theory and arithmetic duality) one obtains canonical identifications

$$H^i(C_{K'}) \cong \begin{cases} \ker(\lambda_{K'}), & i = 1 \\ X_\Sigma(K'), & i = 2 \\ \mathbb{Z}_p, & i = 3 \\ 0, & \text{otherwise,} \end{cases} \tag{48}$$

where $\lambda_{K'}$ is the diagonal map from $\mathcal{O}_{K'}[\frac{1}{p}]^\times \otimes \mathbb{Z}_p$ to the direct sum over $w \in S_p(K')$ of the pro-p-completion $(K'_w)^\times \hat{\otimes} \mathbb{Z}_p$ of $(K'_w)^\times$ and $X_\Sigma(K')$ is the Galois group of the maximal abelian pro-p extension of K' that is unramified outside Σ. By passing to the limit over $K' \subset K$ one finds that $H^i(C_K)$ is acyclic outside degrees 2 and 3 and that its cohomology in degrees 2 and 3 is canonically isomorphic to $X_\Sigma(K)$ and $\mathbb{Z}_p$ respectively. The first claim of Conjecture 7.1 is therefore equivalent to asserting that C_K belongs to $D^{\mathrm{p}}_{S^*}(\Lambda(K/k))$. In addition, since $\chi(C_K) = \sum_{i\in\mathbb{Z}}(-1)^i[H^i(C_K)]$, Conjecture 7.1(b) asserts that $\partial_{G_{K/k}}(\xi) = \chi(C_K) + [\mathbb{Z}_p]$ or equivalently, by Proposition 4.7(ii)(a) and (b), that

$$\partial_{G_{K/k}}(\xi') = \chi(C_K) \tag{49}$$

with $\xi' := \mathrm{char}_{G_{K/k},\gamma_k}(\mathbb{Z}_p[0]) \cdot \xi$.
In the next result we set $c_k := \log_p(\chi_{\mathrm{cyc}}(\gamma_k))$.

Lemma 8.2. *Assume Leopoldt's Conjecture is valid for F (at p) and fix an Artin representation $\rho : G_{K/k} \to \mathrm{GL}_n(\mathcal{O})$ such that V_ρ is an irreducible representation of $G_{F/k}$.*

- (i) *Conjecture 7.1(a) implies that $(\xi')^*(\rho) = c_k^{-\langle\rho,1\rangle}\Omega_j(\rho)L^*_\Sigma(1,\rho^{j^{-1}})^j$.*
- (ii) *If ρ is non-trivial, then $\mathbb{Q}_p \otimes^{\mathbb{L}}_{\Lambda(\Gamma_k)} C_{K,\rho}$ is acyclic and hence C_K is semisimple at ρ, $r_{G_{K/k}}(C_K)(\rho) = 0$ and $t(C_{K,\rho})$ is the canonical morphism.*
- (iii) *If ρ is trivial, then $\mathbb{Q}_p \otimes^{\mathbb{L}}_{\Lambda(\Gamma)} C_{K,\rho}$ is acyclic outside degrees 2 and 3 and its cohomology in degrees 2 and 3 identifies with $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathrm{cok}(\lambda_k)$ and $\mathbb{Q}_p$ respectively. Further, C_K is semisimple at ρ, $r_{G_{K/k}}(C_K)(\rho) = 1$ and $(-1)\times t(C_{K,\rho})$ is induced by the isomorphism $\beta : \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathrm{cok}(\lambda_k) \to \mathbb{Q}_p$ that sends each element $(x_v)_v$ of $\prod_{v\in S_p(k)} k_v^\times$ to $c_k^{-1}\sum_v \log_p(\mathrm{N}_v(x_v))$ with N_v the field-theoretic norm $k_v^\times \to \mathbb{Q}_p^\times$.*

Proof. Leopoldt's Conjecture implies that the determinant $\Omega_j(\rho)$ and hence also the product $(c_k^{\langle\rho,1\rangle}c_{\rho,k})^{-1}\Omega_j(\rho)L^*_\Sigma(1,\rho^{j^{-1}})^j$ in Conjecture 7.1(a) is non-zero. The latter conjecture therefore implies that $\xi^*(\rho) = \xi(\rho) = (c_k^{\langle\rho,1\rangle}c_{\rho,k})^{-1}\Omega_j(\rho)L^*_\Sigma(1,\rho^{j^{-1}})^j$. Next we observe that claim (i) is a consequence of the equality

$$\Phi_\rho(\mathrm{char}_{G_{K/k},\gamma_k}(\mathbb{Z}_p[0])) = \begin{cases} 1 - \rho(\gamma_k^{-1})(1+T)^{-1}, & \text{if } H_{K/k} \subseteq \ker(\rho), \\ 1, & \text{otherwise.} \end{cases} \tag{50}$$

Indeed, if (50) is true, then $\mathrm{char}_{G_{K/k},\gamma_k}(\mathbb{Z}_p[0])^*(\rho) = c_{\rho,k}$ and so claim (i) follows from the obvious equalities $(\xi')^*(\rho) = (\mathrm{char}_{G_{K/k},\gamma_k}(\mathbb{Z}_p[0]) \cdot \xi)^*(\rho) = \mathrm{char}_{G_{K/k},\gamma_k}(\mathbb{Z}_p[0])^*(\rho)\xi^*(\rho)$.
To prove (50) we regard $M_\rho := \Lambda(\Gamma_k) \otimes_{\mathbb{Z}_p} \mathrm{M}_n(\mathcal{O})$ as a $(\Lambda_{\mathcal{O}}(\Gamma_k), \Lambda(H_{K/k}))$-bimodule, where the (left) action of $\Lambda_{\mathcal{O}}(\Gamma_k)$ is clear and the (right) action of each element h of $H_{K/k}$ is via $x \otimes y \mapsto x \otimes y\rho(h)$. Then the definition of $\mathrm{char}_{G_{K/k},\gamma_k}(\mathbb{Z}_p[0])$ combines with

the definition of Φ_ρ to imply that

$$\begin{aligned}(51)\quad &\Phi_\rho(\mathrm{char}_{G_{K/k},\gamma_k}(\mathbb{Z}_p[0]))\\ &\qquad = \det_{Q(\mathcal{O}[[T]])}(\mathrm{id}\otimes\mathrm{id}-\mathrm{id}\otimes\theta\mid Q(\mathcal{O}[[T]])\otimes_{\Lambda_{\mathcal{O}}(\Gamma)}(M_\rho\otimes_{\Lambda(H_{K/k})}\mathbb{Z}_p))\end{aligned}$$

where θ is the endomorphism of $M_\rho\otimes_{\Lambda(H_{K/k})}\mathbb{Z}_p = (\Lambda(\Gamma_k)\otimes_{\mathbb{Z}_p}\mathrm{M}_n(\mathcal{O}))\otimes_{\Lambda(H_{K/k})}\mathbb{Z}_p$ that sends each element $(x\otimes y)\otimes z$ to $(x\gamma_k^{-1}\otimes y\rho(\tilde{\gamma}_k^{-1}))\otimes z$ (this recipe is independent of the choice of lift $\tilde{\gamma}_k$ of γ_k through $G_{K/k}\to\Gamma_k$).
Now V_ρ is irreducible and $H_{K/k}$ is normal in $G_{K/k}$ and so $M_\rho\otimes_{\Lambda(H_{K/k})}\mathbb{Q}_p$ is either canonically isomorphic to $M_\rho\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$ or vanishes depending on whether $H_{K/k}\subseteq\ker(\rho)$ or not. Thus, if $H_{K/k}\not\subseteq\ker(\rho)$, then (51) implies $\Phi_\rho(\mathrm{char}_{G_{K/k},\gamma_k}(\mathbb{Z}_p[0]))$ is the determinant of an endomorphism of the zero space and so equal to 1. On the other hand, if $H_{K/k}\subseteq\ker(\rho)$, then $n=1$ (since Γ_k is abelian and V_ρ is irreducible) and so (51) implies $\Phi_\rho(\mathrm{char}_{G_{K/k},\gamma_k}(\mathbb{Z}_p[0]))$ is the determinant of the endomorphism of $Q(\mathcal{O}[[T]])$ given by multiplication by $1-\rho(\gamma_k^{-1})(1+T)^{-1}$. The required equality (50) is therefore clear.
To prove claims (ii) and (iii) we note that in each degree i the isomorphism (47) induces an identification $H^i(\mathbb{Q}_p\otimes^{\mathbb{L}}_{\Lambda(\Gamma_k)}C_{K,\rho})\cong H^i(C_F)^\rho$. Further, if Leopoldt's Conjecture is valid for F, then $\ker(\lambda_F)$ vanishes and $\mathbb{Q}_p\otimes_{\mathbb{Z}_p}X_\Sigma(F)$ is a trivial $G_{F/k}$-module and so the explicit descriptions of (48) with $K'=F$ imply C_F is acyclic outside degrees 2 and 3 and moreover that each space $H^i(C_F)^\rho$ vanishes if ρ is non-trivial. This proves the first assertion of claim (ii) and then all remaining assertions of claim (ii) follow immediately from [11, Lem. 3.13]. Also the long exact cohomology sequence of (46) induces an identification $H^2_{c,\text{ét}}(\mathcal{O}_{k,\Sigma},\mathbb{Q}_p(1))\cong\mathrm{cok}(\lambda_k)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$ and, if we use (47) with $K'=k^{\mathrm{cyc}}$ to identify $C_{K,\rho}$ with $C_{k^{\mathrm{cyc}}}$, then [11, Lem. 3.13] implies that all remaining assertions of claim (iii) will follow if we can show that the given isomorphism β is equal to -1 times the Bockstein homomorphism $\beta^2_{\Delta_{k,c}}$ in degree 2 of the canonical exact triangle

$$\Delta_{k,c}:\mathbb{Q}_p\otimes_{\mathbb{Z}_p}\mathrm{R}\Gamma_{c,\text{ét}}(\mathcal{O}_{k,\Sigma},T_k)\xrightarrow{\gamma_k-1}\mathbb{Q}_p\otimes_{\mathbb{Z}_p}\mathrm{R}\Gamma_{c,\text{ét}}(\mathcal{O}_{k,\Sigma},T_k)\to\mathrm{R}\Gamma_{c,\text{ét}}(\mathcal{O}_{k,\Sigma},\mathbb{Q}_p(1))\to$$

where $T_k:=\Lambda(k^{\mathrm{cyc}}/k)^\#(1)$. Now the argument of [11, §3.2.1] shows that $\beta^2_{\Delta_{k,c}}$ is equal to the homomorphism $H^2_{c,\text{ét}}(\mathcal{O}_{k,\Sigma},\mathbb{Q}_p(1))\to H^3_{c,\text{ét}}(\mathcal{O}_{k,\Sigma},\mathbb{Q}_p(1))$ induced by taking cup-product with the element φ_k of $H^1_{\text{ét}}(\mathcal{O}_{k,\Sigma},\mathbb{Z}_p)=\mathrm{Hom}_{\mathrm{cts}}(M_\Sigma(k),\mathbb{Z}_p)$ obtained by composing the projection $M_\Sigma(k)\to\Gamma_k$ with the continuous homomorphism $\Gamma_k\to\mathbb{Z}_p$ that sends γ_k to 1. Since cup products commute with corestriction we therefore obtain a commutative diagram

$$\begin{array}{ccc} H^2_{c,\text{ét}}(\mathcal{O}_{k,\Sigma},\mathbb{Q}_p(1)) & \xrightarrow{\beta^2_{\Delta_{k,c}}} & H^3_{c,\text{ét}}(\mathcal{O}_{k,\Sigma},\mathbb{Q}_p(1))\\ {\scriptstyle\kappa^2}\downarrow & & \downarrow{\scriptstyle\kappa^3}\\ H^2_{c,\text{ét}}(\mathbb{Z}_{\Sigma'},\mathbb{Q}_p(1)) & \xrightarrow{d_k^{-1}\beta^2_{\Delta_{\mathbb{Q},c}}} & H^3_{c,\text{ét}}(\mathbb{Z}_{\Sigma'},\mathbb{Q}_p(1))\end{array}$$

in which Σ' is the set of rational places lying below those in Σ, $\Delta_{\mathbb{Q},c}$ denotes the exact triangle obtained from $\Delta_{k,c}$ by replacing k and Σ by $\mathbb{Q}$ and Σ' respectively, the vertical arrows are the natural corestriction maps and d_k occurs in the lower row because the restriction of $\varphi_{\mathbb{Q}}\in H^1_{\text{ét}}(\mathcal{O}_{\mathbb{Q},\Sigma},\mathbb{Z}_p)$ to $H^1_{\text{ét}}(\mathcal{O}_{k,\Sigma},\mathbb{Z}_p)$ is equal to $\varphi_k^{d_k}$ (since $\gamma_k=\gamma_{\mathbb{Q}}^{d_k}$). But, with respect to the canonical identifications $H^2_{c,\text{ét}}(\mathcal{O}_{k,\Sigma},\mathbb{Q}_p(1))\cong\mathrm{cok}(\lambda_k)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$ and $H^3_{c,\text{ét}}(\mathcal{O}_{k,\Sigma},\mathbb{Q}_p(1))\cong\mathbb{Q}_p$ (and similarly with k and Σ replaced by $\mathbb{Q}$ and Σ'), the map

κ^2 is induced by the norm maps $N_v : k_v^\times \to \mathbb{Q}_p^\times$ and κ^3 is the identity map. Thus, since $d_k \times c_\mathbb{Q} = c_k$, it is enough for us to prove that $(-1) \times \beta^2_{\Delta_{\mathbb{Q},c}}$ is induced by the homomorphism $c_\mathbb{Q}^{-1} \cdot \log_p : \mathbb{Q}_p^\times \to \mathbb{Q}_p$. To compute $\beta^2_{\Delta_{\mathbb{Q},c}}$ explicitly we use the morphism of natural exact triangles

$$\begin{array}{ccccccc} \mathrm{R\Gamma}_{\text{ét}}(\mathbb{Q}_p, T_\mathbb{Q}) & \xrightarrow{\gamma_\mathbb{Q}-1} & \mathrm{R\Gamma}_{\text{ét}}(\mathbb{Q}_p, T_\mathbb{Q}) & \longrightarrow & \mathrm{R\Gamma}_{\text{ét}}(\mathbb{Q}_p, \mathbb{Z}_p(1)) & \longrightarrow \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle\theta} & \\ \mathrm{R\Gamma}_{c,\text{ét}}(\mathbb{Z}_{\Sigma'}, T_\mathbb{Q})[1] & \xrightarrow{\gamma_\mathbb{Q}-1} & \mathrm{R\Gamma}_{c,\text{ét}}(\mathbb{Z}_{\Sigma'}, T_\mathbb{Q})[1] & \longrightarrow & \mathrm{R\Gamma}_{c,\text{ét}}(\mathbb{Z}_{\Sigma'}, \mathbb{Z}_p(1))[1] & \longrightarrow \end{array} \tag{52}$$

in which each vertical morphism is induced by the definition of compact support cohomology. Indeed, from the long exact cohomology sequences of the rows in this diagram we obtain a commutative diagram

$$\begin{array}{ccc} H^1_{\text{ét}}(\mathbb{Q}_p, \mathbb{Q}_p(1)) & \xrightarrow{\mathbb{Q}_p\otimes_{\mathbb{Z}_p} H^1(\theta)} & H^2_{c,\text{ét}}(\mathbb{Z}_{\Sigma'}, \mathbb{Q}_p(1)) \\ {\scriptstyle\beta^1}\downarrow & & \downarrow{\scriptstyle(-1)\times\beta^2_{\Delta_{\mathbb{Q},c}}} \\ H^2_{\text{ét}}(\mathbb{Q}_p, \mathbb{Q}_p(1)) & \xrightarrow{\mathbb{Q}_p\otimes_{\mathbb{Z}_p} H^2(\theta)} & H^3_{c,\text{ét}}(\mathbb{Z}_{\Sigma'}, \mathbb{Q}_p(1)). \end{array}$$

Here $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} H^1(\theta)$ identifies with the natural surjection $H^1_{\text{ét}}(\mathbb{Q}_p, \mathbb{Q}_p(1)) \cong (\mathbb{Q}_p^\times \hat{\otimes} \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to \mathrm{cok}(\lambda_\mathbb{Q}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \cong H^2_{c,\text{ét}}(\mathbb{Z}_{\Sigma'}, \mathbb{Q}_p(1))$, $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} H^2(\theta)$ is induced by the identifications $H^2_{\text{ét}}(\mathbb{Q}_p, \mathbb{Q}_p(1)) \cong \mathbb{Q}_p$ and $H^3_{c,\text{ét}}(\mathbb{Z}_{\Sigma'}, \mathbb{Q}_p(1)) \cong \mathbb{Q}_p$ and the identity map on $\mathbb{Q}_p$, β^1 is the Bockstein homomorphism in degree 1 of the image under $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} -$ of the upper row of (52) and the factor -1 occurs on the right hand vertical arrow because of the 1-shift in the lower row of (52). To complete the proof of claim (iii) it thus suffices to recall that the homomorphism β^1 is induced by $c_\mathbb{Q}^{-1} \cdot \log_p$ (for a proof of this fact see, for example, [10, p. 352]). □

Returning to the proof of Theorem 8.1 we now apply Theorem 2.2 to the conjectural equality (49). By taking into account the canonical isomorphism (47) with $K' = F$ and the explicit descriptions given in Lemma 8.2 we therefore deduce that

$$\partial_{G_{F/k}}((c_k^{-\langle\rho,1\rangle}\Omega_j(\rho)L^*_\Sigma(1,\rho^{j^{-1}})^j)_{\rho\in\mathrm{Irr}(G_{F/k})}) = -[\mathbf{d}_{\mathbb{Z}_p[G_{F/k}]}(C_F), \beta_*] \tag{53}$$

where β_* is the morphism $\mathbf{d}_{\mathbb{C}_p[G_{F/k}]}(\mathbb{C}_p[G_{F/k}] \otimes^{\mathbb{L}}_{\mathbb{Z}_p[G_{F/k}]} C_F) \to \mathbf{1}_{\mathbb{C}_p[G_{F/k}]}$ that is induced by the isomorphism

$$\mathbb{Q}_p \otimes_{\mathbb{Z}_p} H^2(C_F) \cong \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathrm{cok}(\lambda_k) \xrightarrow{\beta} \mathbb{Q}_p \cong \mathbb{Q}_p \otimes_{\mathbb{Z}_p} H^3(C_F)$$

coming from Lemma 8.2(ii) and (iii). But the proof of [11, Th. 5.5] shows that (53) is equivalent to an equality of the form $[\mathbf{d}_{\mathbb{Z}_p[G_{F/k}]}(C_F), \beta'_*] = 0$ where $\beta'_* = (\beta'_\rho)_{\rho\in\mathrm{Irr}(G_{F/k})}$ under the identification (4) and each β'_ρ is the explicit morphism described in [11, (25)]. The fact that (53) implies the vanishing of the element $j_*(T\Omega(\mathbb{Q}(1)_F, \mathbb{Z}[G_{F/k}]))$ then follows directly upon explicitly comparing the definition of $T\Omega(\mathbb{Q}(1)_F, \mathbb{Z}[G_{F/k}])$ with that of each morphism β'_ρ. This therefore completes the proof of Theorem 8.1. □

We end this subsection by noting that the above computations show that the correct generalisation of Conjecture 7.1 (to groups with an element of order p) is the following.

CONJECTURE 8.3. *Fix a totally real number field k and a field K in $\mathcal{F}_k^+$ that is unramified outside a finite set of places Σ. Then C_K belongs to $D^{\mathrm{p}}_{S^*}(\Lambda(K/k))$. Further, there exists an element ξ' of $K_1(\Lambda(K/k)_{S^*})$ which satisfies both of the following conditions.*

(a) *At each Artin representation ρ of $G_{K/k}$ one has*

$$\xi'(\rho) = \log_p(\chi_{\mathrm{cyc}}(\gamma_k))^{-\langle\rho,1\rangle}\Omega_j(\rho)L^*_\Sigma(1,\rho^{j^{-1}})^j.$$

(b) $\partial_{G_{K/k}}(\xi') = \chi(C_K)$.

This conjecture is compatible with that formulated (in the case that $G_{K/k}$ has rank one) by Ritter and Weiss in [25, §4]. For further details see [8].

8.2. CRITICAL MOTIVES. In this subsection we assume the notation and hypotheses of Conjecture 7.6 and fix a subfield F of K that is both Galois and of finite degree over $\mathbb{Q}$. We set $\hat{\mathbb{T}}_F := \Lambda(F/\mathbb{Q}) \otimes_{\mathbb{Z}_p} \hat{T} \cong \Lambda(F/\mathbb{Q}) \otimes_{\Lambda(K/\mathbb{Q})} \hat{\mathbb{T}}$. We write $Z = Z_\rho$ and $\tilde{Z} = \tilde{Z}_\rho$ for the Kummer duals $W^*_\rho(1)$ and $\hat{W}^*_\rho(1)$ of W_ρ and $\hat{W}_\rho$ respectively; finally we set $\tilde{W} = \tilde{W}_\rho := W_\rho/\hat{W}_\rho$. In terms of the notation of [11] we consider the following assumption on W_ρ.

ASSUMPTION (W): For each ρ in $\mathrm{Irr}(G_{F/\mathbb{Q}})$ the space $W = W_\rho$ satisfies all of the following conditions:-

(A_1) $P_\ell(W,1)P_\ell(Z,1) \neq 0$ for all primes $\ell \neq p$,
(B_1) $P_p(W,1)P_p(Z,1) \neq 0$,
(C_1) $P_p(\tilde{W},1)P_p(\tilde{Z},1) \neq 0$ and
(D_2) $H^0_f(\mathbb{Q},W) = H^0_f(\mathbb{Q},Z) = 0$.

In the following result we write $\iota_A : K_0(\mathbb{Z}_p[G_{F/\mathbb{Q}}],\mathbb{C}_p[G_{F/\mathbb{Q}}]) \to K_0(A[G_{F/\mathbb{Q}}],\mathbb{C}_p[G_{F/\mathbb{Q}}])$ for the canonical homomorphism obtained by regarding A as a subring of $\mathbb{C}_p$. We also recall that [9, Conj. 4(iii)] (which is a natural equivariant version of the Deligne-Beilinson Conjecture) for the motive M_F, regarded as defined over $\mathbb{Q}$ and with an action of $\mathbb{Q}[G_{F/\mathbb{Q}}]$, implies that the element $j_*(T\Omega(M_F,\mathbb{Z}[G_{F/\mathbb{Q}}]))$ belongs to the image of the natural map $K_0(\mathbb{Z}_p[G_{F/\mathbb{Q}}],\mathbb{Q}_p[G_{F/\mathbb{Q}}]) \to K_0(\mathbb{Z}_p[G_{F/\mathbb{Q}}],\mathbb{C}_p[G_{F/\mathbb{Q}}])$.

THEOREM 8.4. *Assume that*

- *Assumption* (W) *is valid;*
- *the complex* SC_U *is semi-simple at all ρ in* $\mathrm{Irr}(G_{F/\mathbb{Q}})$;
- *an ϵ-isomorphism*

$$\epsilon_{p,\mathbb{Z}_p[G_{F/\mathbb{Q}}]}(\hat{\mathbb{T}}_F) : \mathbf{1}_{\mathbb{Z}_p[G_{F/\mathbb{Q}}]} \to \mathbf{d}_{\mathbb{Z}_p[G_{F/\mathbb{Q}}]}(\mathrm{R}\Gamma(\mathbb{Q}_p,\hat{\mathbb{T}}_F))\mathbf{d}_{\mathbb{Z}_p[G_{F/\mathbb{Q}}]}(\hat{\mathbb{T}}_F)$$

  in the sense of [17, Conj. 3.4.3] *exists;*
- *Conjecture 7.6 is valid for the motive M and the extension $K/\mathbb{Q}$.*

Then $\iota_A(j_*(T\Omega(M_F,\mathbb{Z}[G_{F/\mathbb{Q}}])))$ vanishes. Further, if $j_*(T\Omega(M_F,\mathbb{Z}[G_{F/\mathbb{Q}}]))$ belongs to the image of the natural map $K_0(\mathbb{Z}_p[G_{F/\mathbb{Q}}],\mathbb{Q}_p[G_{F/\mathbb{Q}}]) \to K_0(\mathbb{Z}_p[G_{F/\mathbb{Q}}],\mathbb{C}_p[G_{F/\mathbb{Q}}])$, then $j_*(T\Omega(M_F,\mathbb{Z}[G_{F/\mathbb{Q}}]))$ vanishes.

Proof. We fix an element ξ as in Conjecture 7.6. Since SC_U is semisimple at each ρ in $\mathrm{Irr}(G_{F/\mathbb{Q}})$, the obvious analogue of Theorem 2.2 with A in place of $\mathbb{Z}_p$ combines with Conjecture 7.6(b) to imply that

$$\partial_{G_{F/\mathbb{Q}}}((\xi^*(\rho))_{\rho\in\mathrm{Irr}(G_{F/\mathbb{Q}})}) = -\iota_A([\mathbf{d}_{\mathbb{Z}_p[G_{F/\mathbb{Q}}]}(\mathrm{SC}_U(\hat{\mathbb{T}}_F,\mathbb{T}_F)),t(\mathrm{SC}_U(\hat{\mathbb{T}}_F,\mathbb{T}_F))_{G_{F/\mathbb{Q}}}]).$$

After unwinding the identification (2), this means that there exists a morphism in $V(A[G_{F/\mathbb{Q}}])$

$$\psi : \mathbf{1}_{A[G_{F/\mathbb{Q}}]} \to \mathbf{d}_{A[G_{F/\mathbb{Q}}]}(A[G_{F/\mathbb{Q}}] \otimes_{\mathbb{Z}_p[G_{F/\mathbb{Q}}]} \mathrm{SC}_U(\hat{\mathbb{T}}_F, \mathbb{T}_F))$$

such that

$$(\xi^*(\rho)^{-1})_{\rho\in\mathrm{Irr}(G_{F/\mathbb{Q}})} = t(\mathrm{SC}_U(\hat{\mathbb{T}}_F, \mathbb{T}_F))_{G_{F/\mathbb{Q}}} \circ \psi_{\mathbb{C}_p[G_{F/\mathbb{Q}}]} \\ \in \mathrm{Aut}_{V(\mathbb{C}_p[G_{F/\mathbb{Q}}])}(\mathbf{1}_{\mathbb{C}_p[G_{F/\mathbb{Q}}]}) \cong K_1(\mathbb{C}_p[G_{F/\mathbb{Q}}])$$

under the identification (4). After recalling the explicit definition of the morphism $t(\mathrm{SC}_U(\hat{\mathbb{T}}_F, \mathbb{T}_F))_{G_{F/\mathbb{Q}}}$ given in Theorem 2.2 and then taking inverses we obtain a morphism in $V(A[G_{F/\mathbb{Q}}])$

$$\psi^{-1} : \mathbf{1}_{A[G_{F/\mathbb{Q}}]} \to \mathbf{d}_{A[G_{F/\mathbb{Q}}]}(A[G_{F/\mathbb{Q}}] \otimes_{\mathbb{Z}_p[G_{F/\mathbb{Q}}]} \mathrm{SC}_U(\hat{\mathbb{T}}_F, \mathbb{T}_F))^{-1}$$

such that

$$(-1)^{r_G(\mathrm{SC}_U)(\rho)}\xi^*(\rho) = t(\mathrm{SC}_U(\rho^*))^{-1} \circ \psi^{-1}(\rho) \in \mathrm{Aut}_{V(\mathbb{C}_p)}(\mathbf{1}_{\mathbb{C}_p}) \cong \mathbb{C}_p^\times$$

for all ρ in $\mathrm{Irr}(G_{F/\mathbb{Q}})$. Here we write $\psi^{-1}(\rho)$ for the ρ-component of the morphism $\mathbf{1}_{\mathbb{C}_p[G_{F/\mathbb{Q}}]} \to \mathbf{d}_{\mathbb{C}_p[G_{F/\mathbb{Q}}]}(\mathbb{C}_p[G_{F/\mathbb{Q}}] \otimes_{\mathbb{Z}_p[G_{F/\mathbb{Q}}]} \mathrm{SC}_U(\hat{\mathbb{T}}_F, \mathbb{T}_F))^{-1}$ that is induced by ψ^{-1}. This is equivalent to asserting the existence of a morphism in $V(A[G_{F/\mathbb{Q}}])$

$$\psi' : \mathbf{1}_{A[G_{F/\mathbb{Q}}]} \to \mathbf{d}_{A[G_{F/\mathbb{Q}}]}(A[G_{F/\mathbb{Q}}] \otimes_{\mathbb{Z}_p[G_{F/\mathbb{Q}}]} \mathrm{R}\Gamma_c(U, \mathbb{T}_F))^{-1}$$

such that for all ρ in $\mathrm{Irr}(G_{F/\mathbb{Q}})$ the composite morphism

$$(54)\quad \mathbf{1}_{\mathbb{C}_p} \xrightarrow{\psi'(\rho)_{\mathbb{C}_p}} \mathbf{d}_L(\mathrm{R}\Gamma_c(U, W_\rho))^{-1}_{\mathbb{C}_p} \xrightarrow{\beta(\rho)\epsilon(\hat{\mathbb{T}})^{-1}(\rho)} \mathbf{d}_L(\mathrm{SC}_U(\hat{W}_\rho, W_\rho))^{-1}_{\mathbb{C}_p} \\ \xrightarrow{t(\mathrm{SC}_U(\rho^*))^{-1}_{\mathbb{C}_p}} \mathbf{1}_{\mathbb{C}_p}$$

corresponds to $(-1)^{r_G(\mathrm{SC}_U)(\rho)}\xi^*(\rho)$. In this displayed expression we write $\epsilon(\hat{\mathbb{T}})(\rho)$ for $V_{\rho^*} \otimes_{\mathbb{Z}_p[G_{F/\mathbb{Q}}]} \epsilon_{p,\mathbb{Z}_p[G_{F/\mathbb{Q}}]}(\hat{\mathbb{T}}_F)$ and $\beta(\rho)$ for $V_{\rho^*} \otimes_{\mathbb{Z}_p[G_{F/\mathbb{Q}}]} (\mathbb{Z}_p[G_{F/\mathbb{Q}}] \otimes_{\Lambda(G)} \beta) \cong V_{\rho^*} \otimes_{\Lambda(G)} \beta$ with β the morphism $\mathbf{d}_\Lambda(\mathbb{T}^+)_{\tilde{\Lambda}} \cong \mathbf{d}_\Lambda(\hat{\mathbb{T}})_{\tilde{\Lambda}}$ that is defined in [11, (35)], and all underlying identifications are as explained in [11, §6].

Now the hypothesis that SC_U is semisimple at ρ combines with the assumption (W), the duality isomorphism $H^3_f(\mathbb{Q}, W) \cong H^0_f(\mathbb{Q}, Z)$ and the results of [11, Lem. 6.7 and Lem. 3.13(ii)] to imply that the algebraic rank $r(M)(\rho)$ defined in (43) is equal to $r_G(\mathrm{SC}_U)(\rho)$, that [11, Condition (F)] is satisfied and that the value at $T = 0$ of $T^{-r(M)(\rho)}\Phi_\rho(\xi)$ is equal to the leading term $\xi^*(\rho)$. Conjecture 7.6(a) therefore gives an explicit formula for $\xi^*(\rho)$. Taking this formula into account, one can compare the composite morphism (54) to the first displayed morphism after [11, Lem. 6.8]. After unwinding the proof of [11, Th. 6.5] (for which we use assumption (W)) this comparison shows that

$$\psi'(\rho) = \vartheta_\lambda(M(\rho^*))_{\mathbb{C}_p} \circ \zeta_K(M(\rho^*))_{\mathbb{C}_p}$$

for all ρ in $\mathrm{Irr}(G_{F/\mathbb{Q}})$, where $\vartheta_\lambda(M(\rho^*))_{\mathbb{C}_p}$ and $\zeta_K(M(\rho^*))_{\mathbb{C}_p}$ are the morphisms that occur in [11, Conj. 4.1]. Finally we note that the validity of the last displayed equality (for all ρ in $\mathrm{Irr}(G_{F/\mathbb{Q}})$) is equivalent to asserting that the element $\iota_A(j_*(T\Omega(M_F, \mathbb{Z}[G_{F/\mathbb{Q}}])))$ vanishes (by the very definition of the latter element). This proves the first claim of the theorem.

The second claim of Theorem 8.4 will now follow if we can show that the natural composite homomorphism

$$K_0(\mathbb{Z}_p[G_{F/\mathbb{Q}}], \mathbb{Q}_p[G_{F/\mathbb{Q}}]) \to K_0(\mathbb{Z}_p[G_{F/\mathbb{Q}}], \mathbb{C}_p[G_{F/\mathbb{Q}}]) \xrightarrow{\iota_A} K_0(A[G_{F/\mathbb{Q}}], \mathbb{C}_p[G_{F/\mathbb{Q}}])$$

is injective. To do this we write F for the field of fractions of A (so $F \subset \mathbb{C}_p$). Then, since the natural scalar extension map $K_1(F[G_{F/\mathbb{Q}}]) \to K_1(\mathbb{C}_p[G_{F/\mathbb{Q}}])$ is injective, the exact commutative diagram (1) with $R = A[G_{F/\mathbb{Q}}], R' = F[G_{F/\mathbb{Q}}]$ and $R'' = \mathbb{C}_p[G_{F/\mathbb{Q}}]$ implies that the natural homomorphism $K_0(A[G_{F/\mathbb{Q}}], F[G_{F/\mathbb{Q}}]) \to K_0(A[G_{F/\mathbb{Q}}], \mathbb{C}_p[G_{F/\mathbb{Q}}])$ is also injective. It therefore suffices to prove that the natural homomorphism $K_0(\mathbb{Z}_p[G_{F/\mathbb{Q}}], \mathbb{Q}_p[G_{F/\mathbb{Q}}]) \to K_0(A[G_{F/\mathbb{Q}}], F[G_{F/\mathbb{Q}}])$ is injective. But, since $A/\mathbb{Z}_p$ is unramified, this is an immediate consequence of a result of M. Taylor [31, Chap. 8, Th. 1.1]. Indeed, one need only note that the groups $K_0(\mathbb{Z}_p[G_{F/\mathbb{Q}}], \mathbb{Q}_p[G_{F/\mathbb{Q}}])$ and $K_0(A[G_{F/\mathbb{Q}}], F[G_{F/\mathbb{Q}}])$ are naturally isomorphic to the Grothendieck groups $K_0T(\mathbb{Z}_p[G_{F/\mathbb{Q}}])$ and $K_0T(A[G_{F/\mathbb{Q}}])$ which occur in loc. cit. □

Remark 8.5. If $M = h^1(A)$ for an abelian variety A that has good ordinary reduction at p and is such that the Tate-Shafarevich group $Ш(A_{/F})$ of A over F is finite, then the vanishing of $j_*(T\Omega(M_F, \mathbb{Z}[G_{F/\mathbb{Q}}]))$ implies the 'p-part' of a Birch and Swinnerton-Dyer type formula (see, for example, [33, §3.1]). However, Conjecture 7.6 does *not* itself imply that $Ш(A_{/F})$ is finite.

Remark 8.6. Explicit consequences of Conjecture 7.6 for the values (at $s = 1$) of twisted Hasse-Weil L-functions have been described by Coates et al in [13], by Kato in [20] and by Dokchister and Dokchister in [16]. However, all of the consequences described in [13, 16, 20] become trivial when the L-functions vanish. One of the key advantages of Theorem 8.4 is that in many of these cases it can be combined with the approach of [7] to show that Conjecture 7.6 implies a variety of explicit (and highly non-trivial) congruence relations between values of *derivatives* of twisted Hasse-Weil L-functions. Such explicit (conjectural) congruences will be considered elsewhere.

Remark 8.7. Following Theorem 8.4 it is of some interest to study elements in K-theory of the form $T\Omega(h^1(E_{/F})(1), \mathbb{Z}[G_{F/\mathbb{Q}}])$ with E an elliptic curve over $\mathbb{Q}$ and $F/\mathbb{Q}$ a finite non-abelian Galois extension. The study of such elements is however still very much in its infancy. Indeed, the only explicit computation that we are currently aware of is the following. Let E be the elliptic curve $y^2 + y = x^3 - x^2 - 10x - 20$ (this is the curve 11A1 in the sense of Cremona [14]). Then, with F equal to the splitting field of the polynomial $x^3 - 4x - 1$, the group $G_{F/\mathbb{Q}}$ is dihedral of order 6 and Navilarekallu [23] has proved numerically that if $Ш(E_{/F})$ is trivial, then the element $T\Omega(h^1(E_{/F})(1), \mathbb{Z}[G_{F/\mathbb{Q}}])$ vanishes.

Appendix A. Determinant functors

In this appendix we recall the formalism of determinant functors introduced by Fukaya and Kato in [17] and used in [11] (see also [33])

For any ring R we write $B(R)$ for the category of bounded complexes of (left) R-modules, $C(R)$ for the category of bounded complexes of finitely generated (left) R-modules, $P(R)$ for the category of finitely generated projective (left) R-modules, $C^{\mathrm{p}}(R)$ for the category of bounded (cohomological) complexes of finitely generated projective (left) R-modules. By $D^{\mathrm{p}}(R)$ we denote the category of perfect complexes as full triangulated subcategory of the derived category $D^{\mathrm{b}}(R)$ of $B(R)$. We write $(C^{\mathrm{p}}(R), \mathrm{quasi})$ and $(D^{\mathrm{p}}(R), \mathrm{is})$ for the subcategory of quasi-isomorphisms of $C^{\mathrm{p}}(R)$ and isomorphisms of $D^{\mathrm{p}}(R)$, respectively. For each complex $C = (C^\bullet, d_C^\bullet)$ and each integer r we define the r-fold shift $C[r]$ of C by setting $C[r]^i = C^{i+r}$ and $d^i_{C[r]} = (-1)^r d_C^{i+r}$ for each integer i.

We first recall that for any (associative unital) ring R there exists a Picard category $\mathcal{C}_R$ and a determinant functor $\mathbf{d}_R : (C^{\mathrm{p}}(R), \mathrm{quasi}) \to \mathcal{C}_R$ with the following properties (for objects C, C' and C'' of $\mathrm{C}^{\mathrm{p}}(R)$)

A.d) [1] If $0 \to C' \to C \to C'' \to 0$ is a short exact sequence of complexes, then there is a canonical isomorphism $\mathbf{d}_R(C) \cong \mathbf{d}_R(C')\mathbf{d}_R(C'')$. which we take as an identification.

A.e) If C is acyclic, then the quasi-isomorphism $0 \to C$ induces a canonical isomorphism $\mathbf{1}_R \to \mathbf{d}_R(C)$.

A.f) For any integer r one has $\mathbf{d}_R(C[r]) = \mathbf{d}_R(C)^{(-1)^r}$.

A.g) the functor $\mathbf{d}_R$ factorizes over the image of $C^{\mathrm{p}}(R)$ in $D^{\mathrm{p}}(R)$ and extends (uniquely up to unique isomorphisms) to $(D^{\mathrm{p}}(R), \mathrm{is})$.

A.h) For each C in $D^{\mathrm{b}}(R)$ we write $\mathrm{H}(C)$ for the complex which has $\mathrm{H}(C)^i = H^i(C)$ in each degree i and in which all differentials are 0. If $\mathrm{H}(C)$ belongs to $D^{\mathrm{p}}(R)$ (in which case one says that C is *cohomologically perfect*), then C belongs to $D^{\mathrm{p}}(R)$ and there are canonical isomorphisms

$$\mathbf{d}_R(C) \cong \mathbf{d}_R(\mathrm{H}(C)) \cong \prod_{i\in\mathbb{Z}} \mathbf{d}_R(H^i(C))^{(-1)^i}.$$

(For an explicit description of the first isomorphism see [21, §3] or [3, Rem. 3.2].)

A.i) If R' is any further ring and Y an (R', R)-bimodule that is both finitely generated and projective as an R'-module, then the functor $Y \otimes_R - : P(R) \to P(R')$ extends to a commutative diagram

$$\begin{array}{ccc} (D^{\mathrm{p}}(R), \mathrm{is}) & \xrightarrow{\mathbf{d}_R} & \mathcal{C}_R \\ {\scriptstyle Y\otimes^{\mathbb{L}}_R -}\big\downarrow & & \big\downarrow{\scriptstyle Y\otimes_R -} \\ (D^{\mathrm{p}}(R'), \mathrm{is}) & \xrightarrow{\mathbf{d}_{R'}} & \mathcal{C}_{R'}. \end{array}$$

In particular, if $R \to R'$ is a ring homomorphism and C is in $\mathrm{D}^{\mathrm{p}}(R)$, then we often write $\mathbf{d}_R(C)_{R'}$ in place of $R' \otimes_R \mathbf{d}_R(C)$.

In [17] a *localized* K_1-group was defined for any full subcategory Σ of $C^{\mathrm{p}}(R)$ which satisfies the following four conditions:

(i) $0 \in \Sigma$,

(ii) if C, C' are in $C^{\mathrm{p}}(R)$ and C is quasi-isomorphic to C', then $C \in \Sigma \Leftrightarrow C' \in \Sigma$,

(iii) if $C \in \Sigma$, then also $C[n] \in \Sigma$ for all $n \in \mathbb{Z}$,

(iv′) if C' and C'' belong to Σ, then $C' \oplus C''$ belongs to Σ.

DEFINITION A.1. (Fukaya-Kato) Assume that Σ satisfies (i), (ii), (iii) and (iv′). The *localized* K_1*-group* $K_1(R, \Sigma)$ is defined to be the (multiplicatively written) abelian group which has as generators symbols of the form $[C, a]$ for each $C \in \Sigma$ and morphism $a : \mathbf{1}_R \to \mathbf{d}_R(C)$ in $\mathcal{C}_R$ and the following relations:

(0) $[0, \mathrm{id}_{\mathbf{1}_R}] = 1$,

---

[1] The listing starts with d) to be compatible with the notation of [33].

(1) $[C', \mathbf{d}_R(f) \circ a] = [C, a]$ if $f : C \to C'$ is an quasi-isomorphism with C (and thus C') in Σ,
(2) if $0 \to C' \to C \to C'' \to 0$ is an exact sequence in Σ, then

$$[C, a] = [C', a'] \cdot [C'', a'']$$

where a is the composite of $a' \cdot a''$ with the isomorphism induced by property A.d),
(3) $[C[1], a^{-1}] = [C, a]^{-1}$.

We now assume given a left denominator set S of R and let $R_S := S^{-1}R$ denote the corresponding localization and Σ_S the full subcategory of $C^{\mathrm{p}}(R)$ consisting of all complexes C such that $R_S \otimes_R C$ is acyclic. For any C in Σ_S and any morphism $a : \mathbf{1}_R \to \mathbf{d}_R(C)$ in $\mathcal{C}_R$ we write $\theta_{C,a}$ for the element of $K_1(R_S)$ which corresponds under the canonical isomorphism $K_1(R_S) \cong \mathrm{Aut}_{\mathcal{C}_{R_S}}(\mathbf{1}_{R_S})$ to the composite

$$\mathbf{1}_{R_S} \to \mathbf{d}_{R_S}(R_S \otimes_R C) \to \mathbf{1}_{R_S}$$

where the first arrow is induced by a and the second by the fact that $R_S \otimes_R C$ is acyclic. Then it can be shown that the assignment $[C, a] \mapsto \theta_{C,a}$ induces an isomorphism of groups

$$\mathrm{ch}_{R,\Sigma_S} : K_1(R, \Sigma_S) \cong K_1(R_S)$$

(cf. [17, Prop. 1.3.7]). Hence, if Σ is any subcategory of Σ_S we also obtain a composite homomorphism

$$\mathrm{ch}_{R,\Sigma} : K_1(R, \Sigma) \to K_1(R, \Sigma_S) \cong K_1(R_S).$$

In particular, we often use this construction in the following case: $C \in \Sigma_S$ and Σ is equal to smallest full subcategory Σ_C of $C^{\mathrm{p}}(R)$ that contains C and also satisfies the conditions (i), (ii), (iii) and (iv$'$) that are described above.

Appendix B. Bockstein homomorphisms

Let A be a noetherian regular ring and assume given an exact triangle in $D^{\mathrm{p}}(A)$

$$\Delta : \quad C \xrightarrow{\theta} C \to D \to C[1].$$

For each integer i we define the Bockstein homomorphism in degree i of Δ to be the composite homomorphism

$$\beta^i_\Delta : H^i(D) \to \ker(H^{i+1}(\theta)) \to H^{i+1}(C) \to \mathrm{cok}(H^{i+1}(\theta)) \to H^{i+1}(D)$$

where the first and fourth maps occur in the long exact sequence of cohomology of Δ and the second and third are tautological and write

$$\mathrm{H}_{\mathrm{bock}}(\Delta) : \cdots \xrightarrow{\beta^{i-1}_\Delta} H^i(D) \xrightarrow{\beta^i_\Delta} H^{i+1}(D) \xrightarrow{\beta^{i+1}_\Delta} \cdots$$

for the associated complex (with $H^i(D)$ is placed in degree i). The morphism θ is said to be 'semisimple' if the tautological map $\ker(H^i(\theta)) \to \mathrm{cok}(H^i(\theta))$ is bijective in each degree i. This condition is equivalent to asserting the acyclicity of $\mathrm{H}_{\mathrm{bock}}(\Delta)$. Hence, if true, there is a composite morphism in $V(A)$ of the form

$$\beta_\Delta : \mathbf{d}_A(D) \to \mathbf{d}_A(\mathrm{H}(D)) \to \mathbf{d}_A(\mathrm{H}_{\mathrm{bock}}(\Delta)) \to \mathbf{1}_{V(A)} \tag{55}$$

where the first map is as in A.h), the second is the obvious map (induced by the fact that the complexes $\mathrm{H}(D)$ and $\mathrm{H}_{\mathrm{bock}}(\Delta)$ agree termwise) and the third is induced by property A.e) and the fact that $\mathrm{H}_{\mathrm{bock}}(\Delta)$ is acyclic.

Appendix C. Sign conventions in [11]

Due to a difference of conventions, which unfortunately had not been noticed by the authors, the following sign conflict has arisen: in [17] for a discrete valuation ring $\mathcal{O}$ with field of fractions L an element $c \in \mathcal{O} \setminus \{0\} \subseteq L^\times$ corresponds to the class $[\mathcal{O} \xrightarrow{c} \mathcal{O}, \mathrm{id}]$ in $K_1(L)$, where the complex is concentrated in degree 0 and 1 (while it is implicitly concentrated in degrees -1 and 0 in [11, Rem. 2.4]). With this convention, Fukaya and Kato must define the connecting homomorphism as $[C, a] \mapsto -[[C]]$ in order to ensure that the connecting homomorphism $L^\times = K_1(L) \to K_0(\Sigma_{\mathcal{O}\setminus\{0\}}) \cong \mathbb{Z}$ coincides with the valuation ord_L (cf. [17, Rem. 1.3.16]). It follows that in [11, Rem. 2.4] the correct formula is

$$\mathrm{ord}_L(c) = -\mathrm{length}_{\mathcal{O}}(\mathrm{A}),$$

if we identify A with the complex $A[0]$. For the same reason, the signs in [11, Prop. 3.19] are incorrect, the correct versions being

$$\chi_{\mathrm{add}}(G, C(\rho^*)) = -\mathrm{ord}_L(\mathcal{L}^*(\rho))$$

and

$$\chi_{\mathrm{mult}}(G, C(\rho^*)) = |\mathcal{L}^*(\rho)|_p^{[L:\mathbb{Q}_p]}.$$

We note also that $\mathcal{L} := [C, a] \in K_1(\Lambda, \Sigma_{S^*})$ is a characteristic element of $-[[C]] = [[C[1]]]$ (rather than of $[[C]]$) in $K_0(\Sigma_{S^*})$ due to the normalisation of the connecting homomorphism in [17]. For a similar reason we have to add a sign in the formulae of [loc. cit., (38) and (39)] to obtain the corrected versions

$$\mathcal{L}_{U,\beta} := \mathcal{L}_{U,\beta}(M) : \mathbf{1}_\Lambda \to \mathbf{d}_\Lambda(\mathrm{SC}_U(\hat{\mathbb{T}}, \mathbb{T}))^{-1} \tag{56}$$

and

$$\mathcal{L}_{\beta} := \mathcal{L}_{\beta}(M) : \mathbf{1}_\Lambda \to \mathbf{d}_\Lambda(\mathrm{SC}(\hat{\mathbb{T}}, \mathbb{T}))^{-1} \tag{57}$$

Also in the following convention we need a shift by one: we write $\mathcal{L}_{U,\beta}$ and $\mathcal{L}_\beta$ for the elements $[\mathrm{SC}_U[1], \mathcal{L}_{U,\beta}]$ and $[\mathrm{SC}[1], \mathcal{L}_\beta]$ of $K_1(\Lambda(G), \Sigma_{\mathrm{SC}_U})$ and $K_1(\Lambda(G), \Sigma_{\mathrm{SC}})$ respectively. Finally, in the first displayed formula after [11, Lem. 6.8] one has to replace $t(\mathrm{SC}_U(\rho^*))_{\tilde{L}}$ by $t(\mathrm{SC}_U(\rho^*)[1])_{\tilde{L}} = t(\mathrm{SC}_U(\rho^*))_{\tilde{L}}^{-1}$.
The authors would like to apologise for these oversights.

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