ON SELMER GROUPS AND REFINED STARK CONJECTURES, II

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Abstract. We prove that the higher (non-commutative) Fitting invariants of Selmer groups of the multiplicative group over Galois extensions of global fields admit a natural direct sum decomposition. Using this result, we then formulate an explicit higher-order Stark conjecture that both refines and extends the existing theory of such conjectures in two significant ways since it deals with Artin L-series that do not necessarily have 'minimal' order of vanishing at zero and also treats on an equal footing the 'boundary case' that has been excluded from previous conjectures in this area. This conjecture is, in a natural sense, best possible and we establish a direct link between it and the Tamagawa number conjecture of Bloch and Kato and also describe strong supporting evidence for it, including giving a full proof for all Galois extensions of global function fields and for all abelian extensions of $\mathbb{Q}$ and a proof, modulo a standard hypothesis on μ-invariants, of a natural analogue concerning the higher derivatives of p-adic Artin L-series. We then derive a number of consequences of these results concerning relations between derivatives of Artin L-series and the Galois structure of divisor class groups, including the proof of a natural (higher-order and non-abelian) refinement of a celebrated result of Deligne and of refinements of other results proved in several earlier articles.

1. Introduction

Let F/k be a finite Galois extension of global fields of group G. Let S be a finite non-empty set of places of k containing the set S_∞ of archimedean places (if any) and also all places that ramify in F.

Then for any finite set T of places of k that is disjoint from S the 'S-relative T-trivialized integral Selmer group' $\mathrm{Sel}_S^T(F)$ for the multiplicative group $\mathbb{G}_m$ over F is defined to be the cokernel of a canonical homomorphism of G-modules

$$\theta_{S,T} : \prod_{w \notin S \cup T} \mathbb{Z} \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(F_T^\times, \mathbb{Z}) \tag{1}$$

(see [4, Def. 2.1] where the notation $\mathcal{S}_{S,T}(\mathbb{G}_{m/F})$ is used). Here in the product w runs over all places of F that do not lie above places in $S \cup T$, $F_T^\times$ is the subgroup of $F^\times$ comprising elements u for which $u-1$ has a strictly positive valuation at each place above T and $\theta_{S,T}$ sends each element $(x_w)_w$ to the map $(u \mapsto \sum \mathrm{ord}_w(u)x_w)$ with ord_w the normalised additive valuation at w.

This group is a natural analogue for $\mathbb{G}_m$ of the integral Selmer groups of abelian varieties that are defined by Mazur and Tate in [18] and, in particular, lies in a canonical exact sequence of G-modules of the form

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$$0 \to \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Cl}_S^T(F), \mathbb{Q}/\mathbb{Z}) \to \mathrm{Sel}_S^T(F) \to \mathrm{Hom}_{\mathbb{Z}}(\mathcal{O}_{F,S,T}^\times, \mathbb{Z}) \to 0 \tag{2}$$

(see [4, Prop. 2.2]). Here $\mathrm{Cl}_S^T(F)$ is the ray class group modulo the product of all places of F above T of the subring $\mathcal{O}_{F,S}$ of F comprising elements that are integral at all places outside S and $\mathcal{O}_{F,S,T}^\times$ is the group $F_T^\times \cap \mathcal{O}_{F,S}^\times$ and both duals are endowed with the contragredient action of G.

In our earlier article [5] we considered the case of abelian extensions F/k and used certain structural properties of the G-module $\mathrm{Sel}_S^T(F)$ to formulate an explicit higher order abelian Stark conjecture. This conjecture refined the existing theory of such conjectures, due to Stark, Rubin and Popescu amongst others, and also extended the theory in two significant ways since it neither restricted to those L-series for complex characters of G that have 'minimal' order of vanishing at zero and also treated on an equal footing the 'boundary case' that had been excluded from all previous such conjectures.

In this article we shall now explain how to extend the above theory to the case of general Galois extensions by combining the approach of [5] with the formalism of non-commutative exterior powers and higher Fitting invariants that was introduced by Sano and the first author in [6].

In particular, in this way we shall first prove that the higher (non-commutative) Fitting invariants of the G-module $\mathrm{Sel}_S^T(F)$ admit a natural direct sum decomposition. For details of this result see Theorem 2.3 and its proof in §3.

By using this purely algebraic observation, we then formulate (in Conjectures 4.3 and 4.5) certain explicit predictions concerning the detailed arithmetic properties of canonical 'non-abelian Stark elements of arbitrary rank' that we introduce in §4.1.

This conjecture has the style of a non-abelian refined Rubin-Stark conjecture and simultaneously both refines and extends, in the case of abelian extensions, the central conjecture from [5] and, in the setting of general Galois extensions, the main conjecture formulated by Sano and the first author in [6] and hence also the 'universal Stark conjectures' studied by the first author in [2].

By means of explicit examples, we show our conjecture is, in a natural sense, best possible and also that it implies that Rubin-Stark elements of subfields of F/k have in general strictly stronger integrality properties than those that are predicted by the relevant cases of the refined Rubin-Stark Conjecture discussed by Kurihara, Sano and the first author in [4] (and hence, a fortiori, by the Rubin-Stark Conjecture itself)

In §5 we combine Theorem 2.3 with an extension of the methods used in [5] and [6] to establish a direct link between our conjecture and the general formalism of Tamagawa number conjectures introduced by Bloch and Kato in [1]. In this way we obtain strong supporting evidence for our conjecture, including giving a full proof of it for all abelian extensions of $\mathbb{Q}$ and for all Galois extensions of global function fields.

In §6 we describe several concrete consequences of the results of §5 concerning the Galois structure of divisor class groups.

These results include establishing, unconditionally, a natural generalisation (to Galois extensions that are non-abelian and to L-series that vanish at zero) of Deligne's proof of the Brumer-Stark Conjecture for global function fields. We also extend the results of [3]

concerning relations between derivatives of p-adic L-series and the structure of the Selmer groups of $\mathbb{G}_m$ and refine several of the main results from [2].

Finally we note that, for the reader's convenience, we have quickly recalled in an appendix the basic definitions and properties of the notions of 'presentation' and 'non-commutative higher Fitting invariant' that were introduced in [6].

2. Higher non-commutative Fitting invariants

In this section we define certain canonical central idempotents of $\mathbb{Q}[G]$ and then use them to state our main algebraic result concerning the decomposition of the higher non-commutative Fitting invariants of the Galois module $\mathrm{Sel}_S^T(F)$.

2.1. **Characters and idempotents.** We write $\widehat{G}$ for the set of irreducible complex valued characters of G and $\mathbf{1}$ for the trivial character of G.

For each non-negative integer a we write $\widehat{G}'_{a,S}$ for the (possibly empty) subset of $\widehat{G} \setminus \{\mathbf{1}\}$ comprising characters ψ that satisfy both of the following conditions

$$\text{(3)} \qquad \begin{cases} \text{(i) the set } S_\psi := \{v \in S : \psi(T_{G_v}) = |G_v| \cdot \psi(1)\} \text{ has cardinality } a, \text{ and} \\ \text{(ii) } \psi(T_{G_v}) = 0 \quad \text{for all} \quad v \in S \setminus S_\psi, \end{cases}$$

where G_v denotes the decomposition subgroup in G of any fixed place of F above v and we set $T_H := \sum_{h \in H} h \in \mathbb{Z}[G]$ for each subgroup H of G. We then define $\widehat{G}_{a,S}$ to be $\widehat{G}'_{a,S} \cup \{\mathbf{1}\}$ if $a = |S| - 1$ and to be $\widehat{G}'_{a,S}$ if $a \neq |S| - 1$.

For each such integer a we also write $\widehat{G}'_{(a),S}$ for the (possibly empty) subset of $\widehat{G} \setminus \{\mathbf{1}\}$ comprising characters ψ that satisfy the following weaker condition

$$\text{(4)} \qquad |S_\psi| \geq a,$$

and define $\widehat{G}_{(a),S}$ to be $\widehat{G}'_{(a),S} \cup \{\mathbf{1}\}$ if $a < |S|$ and to be $\widehat{G}'_{(a),S}$ if $a \geq |S|$.

We then obtain central idempotents of $\mathbb{Q}[G]$ by setting

$$e_{a,S} := \sum_{\psi \in \widehat{G}_{a,S}} e_\psi \quad \text{and} \quad e_{(a),S} := \sum_{\psi \in \widehat{G}_{(a),S}} e_\psi$$

where for ψ in $\widehat{G}$ we write e_ψ for the central idempotent $\psi(1)|G|^{-1} \sum_{g \in G} \psi(g)^{-1} g$ of $\mathbb{C}[G]$.

We finally define a subset of S by setting

$$S^a_{\min} := \begin{cases} \bigcup_{\psi \in \widehat{G}'_{a,S}} S_\psi, & \text{if } G \text{ is not trivial,} \\ S \setminus \{v^*\}, & \text{if } G \text{ is trivial and } a = |S| - 1, \\ \emptyset, & \text{if } G \text{ is trivial and } a \neq |S| - 1, \end{cases}$$

where v^* is a fixed place in S (the choice of which will not matter in the sequel).

Remark 2.1.
(i) To interpret condition (3)(i) more explicitly, note that for each ψ in $\widehat{G} \setminus \{\mathbf{1}\}$ one has $\psi(T_{G_v}) = |G_v| \cdot \psi(1)$ if and only if G_v, and hence also N_v, is contained in $\ker(\psi)$.
(ii) If ψ in $\widehat{G} \setminus \{\mathbf{1}\}$ is linear, then ψ belongs to $\widehat{G}'_{a,S}$ with $a = |S_\psi|$ since, in this case, the validity of condition (3)(ii) follows immediately from the definition of S_ψ. Thus, if G is

abelian, then $\widehat{G} = \bigcup_{a' \geq 0} \widehat{G}_{a',S}$ and there are also non-abelian extensions for which the same equality is valid. For example, if F/k is generalised dihedral, with A the abelian subgroup of index two in G, and all places in S split in F^A/k, then for every ϕ in $\widehat{A} \setminus \{\mathbf{1}\}$ one has $\mathrm{Ind}_A^G(\phi) \in \widehat{G}'_{a,S}$ with $a = |\{v \in S : G_v \subseteq \ker(\phi)\}|$.

(iii) Set $S_{\mathrm{sp}} := \{v \in S : G_v \text{ is trivial}\}$ and $r_{\mathrm{sp}} := |S_{\mathrm{sp}}|$. Then $S_{\mathrm{sp}} \subseteq S_\psi$ for all ψ in $\widehat{G} \setminus \{\mathbf{1}\}$ and so $\widehat{G}_{a,S} = \emptyset$ if $a < r_{\mathrm{sp}}$ and $\widehat{G}_{(a),S} = \widehat{G}$ if $a \leq r_{\mathrm{sp}}$. In particular, if any $\widehat{G}'_{a,S}$ with $a > 0$ contains a faithful character of G, then $a = r_{\mathrm{sp}}$, $S^a_{\min} = S_{\mathrm{sp}}$ and $\widehat{G} = \widehat{G}_{(a),S}$. (In this regard note that Schur's Lemma implies no element of $\widehat{G}$ can be faithful if G has a non-cyclic centre and that a complete classification of groups with faithful irreducible characters has been given by Gaschutz in [9]).

2.2. **Central orders and ideals.** In the sequel we write $\zeta(\Lambda)$ for the centre of a ring Λ.

Let R be a Dedekind domain of characteristic zero with quotient field F. Then for a finite group Γ we write $\xi(R[\Gamma])$ for the order in $\zeta(F[\Gamma])$ that is (additively) generated over R by the reduced norms over the semisimple algebra $F[\Gamma]$ of all finite square matrices with entries in $R[\Gamma]$.

For each non-negative integer a the a-th non-commutative Fitting invariant $\mathrm{Fit}^a_{R[\Gamma]}(Z)$ of an $R[\Gamma]$-module Z is defined by Sano and the first author in [6] to be a module over $\xi(R[\Gamma])$ and we shall often use this structure in the sequel.

If Γ is abelian, then $\xi(R[\Gamma]) = R[\Gamma]$ but in general $\xi(R[\Gamma])$ is neither contained in or contains $\zeta(R[\Gamma])$. To bound the denominators of elements of $\xi(R[\Gamma])$ one proceeds as follows.

For every natural number m and every matrix M in $\mathrm{M}_m(R[\Gamma])$ there is a unique matrix M^* in $\mathrm{M}_m(F[\Gamma])$ with $MM^* = M^*M = \mathrm{Nrd}_{F[\Gamma]}(M) \cdot \mathrm{I}_m$ and such that for every primitive central idempotent e of $F[\Gamma]$ the matrix M^*e is non-zero if and only if $\mathrm{Nrd}_{F[\Gamma]}(M)e$ is non-zero. One defines $\mathcal{H}(R[\Gamma])$ to be the ideal of $\zeta(R[\Gamma])$ comprising elements x of $\zeta(F[\Gamma])$ with the property that xM^* belongs to $\mathrm{M}_d(R[\Gamma])$ for all M in $\mathrm{M}_d(R[\Gamma])$ and all $d \geq 1$.

This ideal is such that

$$\mathcal{H}(R[\Gamma]) \cdot \xi(R[\Gamma]) = \mathcal{H}(R[\Gamma]) \subseteq \zeta(R[\Gamma]). \tag{5}$$

Motivated by this inclusion one defines the 'central pre-annihilator' of an $R[\Gamma]$-module Z to be the $\xi(R[\Gamma])$-submodule of $\zeta(F[\Gamma])$ obtained by setting

$$\mathrm{pAnn}_{R[\Gamma]}(Z) := \{x \in \zeta(F[\Gamma]) : x \cdot \mathcal{H}(R[\Gamma]) \subseteq \mathrm{Ann}_{R[\Gamma]}(Z)\}.$$

It is clear that if Γ is abelian, then $\mathrm{pAnn}_{R[\Gamma]}(Z) = \mathrm{Ann}_{R[\Gamma]}(Z)$. In addition, it is shown in [6] that in all cases one has $\mathrm{Fit}^0_{R[\Gamma]}(Z) \subseteq \mathrm{pAnn}_{R[\Gamma]}(Z)$.

Remark 2.2. The ideal $\mathcal{H}(R[\Gamma])$ defined above differs slightly from an ideal defined (using the same notation) by Johnston and Nickel in [15] since M^* differs from the 'generalized adjoint matrices' defined in loc. cit. Nevertheless, the computations made in loc. cit. give information about $\mathcal{H}(R[\Gamma])$. For example, the argument of [15, Prop. 4.8] implies that if $\mathfrak{p}$ is any prime ideal of R with residue characteristic prime to the order of the commutator subgroup of Γ, then $\mathcal{H}(R[\Gamma])_{\mathfrak{p}} = \zeta(R_{\mathfrak{p}}[\Gamma])$ and hence (by (5)) also $\xi(R[\Gamma])_{\mathfrak{p}} = \zeta(R_{\mathfrak{p}}[\Gamma])$.

2.3. **Statement of the main algebraic result.** For any set Σ and any non-negative integer a we write $\wp_a(\Sigma)$ for the set of subsets of Σ of cardinality a. We abbreviate $\wp_a(S^a_{\min})$ to $\wp^*_a(S)$ and for v in S we set $\wp_a(S,v) := \{I \in \wp^*_a(S) : v \notin I\}$.

For a normal subgroup H of G we write e_H for the central idempotent $|H|^{-1} \cdot T_H$ of $\mathbb{Q}[G]$. For v in S we write N_v denotes the normal closure of G_v in G and abbreviate e_{N_v} to e_v.

For each I in $\wp^*_a(S)$ we also define a central idempotent of $\mathbb{Q}[G]$ by setting

$$e_I := e_{\mathbf{1}} + \sum_{\psi} e_\psi \tag{6}$$

where the sum runs over $\{\psi \in \widehat{G}'_{a,S} : S_\psi = I\}$.

Finally we note that a review of the basic definitions relating to module presentations and non-commutative Fitting invariants can be found in the appendix.

We can now state the main result of this section.

Theorem 2.3. *Assume that $\mathcal{O}^\times_{F,S,T}$ is torsion-free. Then there exists a locally-quadratic presentation $\Pi = \Pi_{F/k,S,T}$ of the G-module $\mathrm{Sel}^T_S(F)$ with the property that for every non-negative integer a and every v in S there is a direct sum decomposition of $\xi(\mathbb{Z}[G])$-modules*

$$(1 - e_v + e_{\mathbf{1}})e_{(a),S} \cdot \mathrm{Fit}^a_G(\Pi) = c^a_{S,v} e_{\mathbf{1}} \cdot \mathrm{Fit}^a_G(\Pi) \oplus \bigoplus_{I \in \wp_a(S,v)} e_I \cdot \mathrm{Fit}^a_G(\Pi) \tag{7}$$

where the integer $c^a_{S,v}$ is defined by setting

$$c^a_{S,v} := \begin{cases} |\{v\} \setminus S^a_{\min}| - \min\{|S^a_{\min}|, |S \setminus S^a_{\min}|\} - \delta_{a0}, & \text{if } a < |S| \\ 0, & \text{otherwise.} \end{cases}$$

Remark 2.4. If G is abelian, then for each non-negative integer a the invariant $\mathrm{Fit}^a_G(\Pi)$ is equal to the a-th Fitting ideal of $\mathrm{Sel}^T_S(F)$ as a module over $\mathbb{Z}[G]$ (as defined by Northcott in [21]). This shows that Theorem 2.3 recovers the main algebraic result (Theorem 1.1) of our earlier article [5].

Remark 2.5. For each non-negative integer a set $\mathcal{F}_a := \mathrm{Fit}^a_G(\mathrm{Sel}^T_S(F))$ and $\mathcal{F}'_a := \mathrm{Fit}^a_G(\Pi)$ with Π as in Theorem 2.3. For each place v in S we then define $\mathfrak{n}^a_v = \mathfrak{n}^a_v(F/k)$ and $\mathfrak{m}^a_v = \mathfrak{m}^a_v(F/k)$ to be the ideal of $\xi(\mathbb{Z}[G])$ comprising elements x with

$$x \cdot (1 - e_v + e_{\mathbf{1}})e_{(a),S} \cdot \mathcal{F}_a \subseteq \xi(\mathbb{Z}[G]), \quad \text{respectively} \quad x \cdot (1 - e_v + e_{\mathbf{1}})e_{(a),S} \cdot \mathcal{F}_a \subseteq \mathcal{F}_a.$$

Then, since $\mathcal{F}'_a \subseteq \mathcal{F}_a$, Theorem 2.3 implies that for I in $\wp^*_a(S)$ one has $\mathfrak{n}^a_v \cdot e_I \cdot \mathcal{F}'_a \subseteq \xi(\mathbb{Z}[G])$ and $\mathfrak{m}^a_v \cdot e_I \cdot \mathcal{F}'_a \subseteq \mathcal{F}_a$ for all $v \in S \setminus I$ and hence also that $\mathfrak{n}^a_I \cdot e_I \cdot \mathcal{F}'_a \subseteq \xi(\mathbb{Z}[G])$ and $\mathfrak{m}^a_I \cdot e_I \cdot \mathcal{F}'_a \subseteq \mathcal{F}_a$ with $\mathfrak{n}^a_I := \sum_{v \in S \setminus I} \mathfrak{n}^a_v$ and $\mathfrak{m}^a_I := \sum_{v \in S \setminus I} \mathfrak{m}^a_v$.

Thus, if we define ideals of $\xi(\mathbb{Z}[G])$ by setting

$$\mathfrak{n}^a_{S,T}(F/k) := \bigcap\nolimits_{I \in \wp^*_a(S)} \mathfrak{n}^a_I \quad \text{and} \quad \mathfrak{m}^a_{S,T}(F/k) := \bigcap\nolimits_{I \in \wp^*_a(S)} \mathfrak{m}^a_I,$$

then Theorem 2.3 implies that for each I one has

$$\mathfrak{n}^a_{S,T}(F/k) \cdot e_I \cdot \mathrm{Fit}^a_G(\Pi) \subseteq \xi(\mathbb{Z}[G]) \quad \text{and} \quad \mathfrak{m}^a_{S,T}(F/k) \cdot e_I \cdot \mathrm{Fit}^a_G(\Pi) \subseteq \mathrm{Fit}^a_G(\mathrm{Sel}^T_S(F)). \tag{8}$$

Remark 2.6. If $a \geq |S|$, then $S \setminus I$ is empty for each I in $\wp_a^*(S)$ and so the ideals $\mathfrak{n}_{S,T}^a(F/k)$ and $\mathfrak{m}_{S,T}^a(F/k)$ are both zero. However, if $a < |S|$, then an explicit computation of $\mathfrak{n}_{S,T}^a(F/k)$ and $\mathfrak{m}_{S,T}^a(F/k)$ in any general setting would be rather involved since it requires some knowledge of the higher Fitting ideals of $\mathrm{Sel}_S^T(F)$. It is, nevertheless, straightforward in this case to construct non-zero elements of $\mathfrak{n}_{S,T}^a(F/k)$ and $\mathfrak{m}_{S,T}^a(F/k)$ in a purely combinatorial way since, if t is the (finite) index of $\mathcal{H}(\mathbb{Z}[G])$ in $\zeta(\mathbb{Z}[G])$, then $\mathfrak{n}_v^a(F/k)$ and $\mathfrak{m}_v^a(F/k)$ both contain the lowest common multiple δ_v of the denominators of the coefficients of the element $t^{-1}(1 - e_v + e_{\mathbf{1}})e_{(a),S}$ of $\mathbb{Q}[G]$ and so $\mathfrak{n}_{S,T}^a(F/k)$ and $\mathfrak{m}_{S,T}^a(F/k)$ contain the lowest common multiple $\delta_{F/k}$ over I of the greatest common divisor of the integers $\{\delta_v : v \in S \setminus I\}$. The integer $\delta_{F/k}$ can be close to a generator of $\mathfrak{n}_{S,T}^a(F/k)$ and $\mathfrak{m}_{S,T}^a(F/k)$ (see, for instance, [5, Exam. 4.3 and 5.4]) and in all cases allows one to make the predictions of (8) explicit. In particular, such computations show these predictions are of interest since in many cases there exists an I in $\wp_a^*(S)$ for which one has $\delta_{F/k} \cdot e_I \notin \xi(\mathbb{Z}[G])$.

3. The proof of Theorem 2.3

In this section we prove Theorem 2.3. In particular, we assume throughout the notation and hypotheses of that result.

For convenience we also set $\mathcal{F}_a := \mathrm{Fit}_G^a(\mathrm{Sel}_S^T(F))$ for each non-negative integer a.

3.1. We start by observing that an explicit analysis of the functional equation of Artin L-series (as per [24, Chap. I, Prop. 3.4]) combines with the exact sequence (2) to imply that for each ψ in $\widehat{G}$ the order of vanishing $r_S(\psi)$ at $z = 0$ of $L_S(\psi, z)$ satisfies

$$r_S(\psi) = \dim_{\mathbb{C}}(e_{\check{\psi}}(\mathbb{C} \otimes_{\mathbb{Z}} \mathrm{Sel}_S^T(F))) = \begin{cases} \sum_{v \in S} \dim_{\mathbb{C}}(H^0(G_v, V_\psi)), & \text{if } \psi \neq \mathbf{1}, \\ |S| - 1, & \text{if } \psi = \mathbf{1}. \end{cases} \tag{9}$$

where $\check{\psi}$ denotes the contragredient of ψ.

We next record a result that will allow us to treat a special case of Theorem 2.3 (and also justifies observations made in Remark 3.2).

Lemma 3.1. *The following claims are valid for each non-negative integer a.*

(i) *For ψ in $\widehat{G}_{(a),S} \setminus \widehat{G}_{a,S}$ the space $e_\psi(\mathbb{C} \otimes_{\mathbb{Z}} \mathcal{F}_a)$ vanishes.*
(ii) *If $|S| > a + 1$, then $e_{\mathbf{1}} \cdot \mathcal{F}_a$ vanishes.*
(iii) *If $v \in S \setminus S_{\min}^a$, then $(e_v - e_{\mathbf{1}})e_{(a),S} \cdot \mathcal{F}_a$ vanishes.*

Proof. We claim first that for ψ in $\widehat{G}$, a finitely generated G-module Z and a non-negative integer a the sublattice $e_\psi \cdot \mathrm{Fit}_G^a(Z)$ of $\zeta(\mathbb{C}[G])$ will vanish whenever $n_{Z,\psi} > a \cdot \psi(1)$, where for any such module Z we set $n_\psi(Z) := \dim_{\mathbb{C}}(e_\psi(\mathbb{C} \otimes_{\mathbb{Z}} Z))$.

It is enough to prove this after localising at any given prime p and to do this we fix a free presentation Π of a finitely generated $\mathbb{Z}_{(p)}[G]$-module Z' for which there exists a surjective homomorphism of $\mathbb{Z}_{(p)}[G]$-modules $Z' \to Z$. Let M_Π be a matrix in $\mathrm{M}_{d \times d'}(\mathbb{C}[G])$ for integers d and d' with $d \geq d' \geq a$, that represents Π and let M' be any matrix that is obtained from M_Π by changing at most a of its columns. For any ψ in $\widehat{G}$ write $M_{\Pi,\psi}$ and M'_ψ for the matrices in $\mathrm{M}_{d \cdot \psi(1) \times d' \cdot \psi(1)}(\mathbb{C})$ that are respectively obtained by splitting the matrices $e_\psi \cdot M_\Pi$ and $e_\psi \cdot M'$ in $\mathrm{M}_{d \times d'}(e_\psi \cdot \mathbb{C}[G])$.

Then the column rank of $M_{\Pi,\psi}$ is $d\cdot\psi(1)-n_\psi(Z')$ and so the column rank of M'_ψ is at most $d'\cdot\psi(1)-n_\psi(Z')+a\cdot\psi(1)$. In particular, if $n_\psi(Z)>a\cdot\psi(1)$, then also $n_\psi(Z')>a\cdot\psi(1)$ and so the column rank of M'_ψ is strictly less than $d'\cdot\psi(1)$. This implies that the determinant of any $d'\cdot\psi(1)\times d'\cdot\psi(1)$ minor of M'_ψ vanishes and so, since $e_\psi\cdot\mathrm{Fit}^a_G(Z)$ is generated by the determinants of all such minors (see the discussion in the appendix), it must also vanish, as claimed.

Using this observation, and the fact that for any non-trivial ψ in $\widehat{G}_{(a),S}\setminus\widehat{G}_{a,S}$ there exists a place v in $S\setminus S_\psi$ for which $H^0(G_v,V_\psi)$ does not vanish, one derives claim (i) by noting that (9) implies for each ψ in $\widehat{G}_{a',S}$ one has

$$n_\psi(\mathrm{Sel}^T_S(F))=\begin{cases}\sum_{v\in S}\dim_{\mathbb{C}}(H^0(G_v,V_\psi))>\sum_{v\in S_\psi}\dim_{\mathbb{C}}(H^0(G_v,V_\psi)) & \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad =\sum_{v\in S_\psi}\dim_{\mathbb{C}}(V_\psi)=a\cdot\psi(1), & \text{if } \psi\neq\mathbf{1},\\ |S|-1=a'>a=a\cdot\psi(1), & \text{if } \psi=\mathbf{1},\end{cases}$$

where the second equality in the case $\psi\neq\mathbf{1}$ follows from Remark 2.1(i).

Claim (ii) is then an immediate consequence of claim (i) in the case $\psi=\mathbf{1}$ and $a'=|S|-1$.

To derive claim (iii) from claim (i) it suffices to prove that if ψ belongs to $\widehat{G}_{a,S}\setminus\{\mathbf{1}\}$, then N_v cannot be contained in $\ker(\psi)$ (and hence $e_\psi(e_v-e_{\mathbf{1}})=0$). This follows from the fact that, as v does not belong to $S^a_{\min}$, the inclusion $N_v\subseteq\ker(\psi)$ would imply that $|S_\psi|\geq 1+a$. □

Remark 3.2. The above observations can be used to make the results of Theorem 2.3 and Corollary 2.5 more explicit. To explain this we write $r=r_S(F/k)$ for the largest integer a for which one has $\widehat{G}_{(a),S}=\widehat{G}$ and note that (9) implies $r\leq|S|-1$.
(i) If $\widehat{G}_{(a),S}=\widehat{G}$, then $e_{(a),S}=1$ and this simplifies the computation of $\mathfrak{n}^a_{S,T}(F/k)$ and $\mathfrak{m}^a_{S,T}(F/k)$. In particular, if both $\widehat{G}_{(a),S}=\widehat{G}$ and $S^a_{\min}\neq S$, then Lemma 3.1(iii) implies $\mathfrak{n}^a_{S,T}(F/k)$ and $\mathfrak{m}^a_{S,T}(F/k)$ are both equal to $\xi(\mathbb{Z}[G])$ and hence contain 1.
(ii) By combining (9) with the result of Lemma 3.1(i) one can also check that both sides of (7) vanish if either $a<r$ or $a>|S|$. This shows that Theorem 2.3 is of interest only for integers a in the range $r\leq a\leq|S|$.
(iii) Assume $S\neq S^a_{\min}$. Then Lemma 3.1(iii) implies that $(e_v-e_{\mathbf{1}})\mathrm{Fit}^a_G(\mathrm{Sel}^T_S(F))$ vanishes for each v in $S\setminus S^a_{\min}$. In particular, for each such v the left hand side of (7) is equal to $e_{(a),S}\cdot\mathrm{Fit}^a_G(\mathrm{Sel}^T_S(F))$, and hence, under the conditions of remark (i), to $\mathrm{Fit}^r_G(\mathrm{Sel}^T_S(F))$ if $a=r$. In particular, in any such case one has $n^r_v(F/k)=m^r_v(F/k)=1$ and hence also $n^r_{S,T}(F/k)=m^r_{S,T}(F/k)=1$.
(iv) The case $a=r$ and $S=S^a_{\min}$ has been explicitly identified as a 'boundary case' in the theory of refined Stark conjectures and was, prior to our earlier article [5], excluded from the formulation of such conjectures. For a discussion of explicit consequences of Theorem 2.3 in the case that $S=S^a_{\min}$, $a=r$ and G is abelian see [5, Rem. 2.2(iv)].

3.2. By using Lemma 3.1 we can now quickly prove Theorem 2.3 in several special cases.

Lemma 3.3. *Theorem 2.3 is valid if $F=k$.*

Proof. In this case G is trivial and $\mathcal{F}_a$ coincides with the a-th Fitting ideal of the abelian group $\mathrm{Sel}^T_S(F)$. The validity of Theorem 2.3 in this case therefore follows directly from [5, Prop. 2.3]. □

Lemma 3.4. *Theorem 2.3 is valid if $a = 0$.*

Proof. Following Lemma 3.3 we assume both $a = 0$ and G is not trivial.

In this case Lemma 3.1(i) and (iii) combine to imply that the left hand side of (7) is equal to $e_{0,S} \cdot \mathcal{F}_0$.

In addition, $S^a_{\min} = \emptyset$ and so for each v in S one has $c^0_{S,v} = 1-0-1 = 0$ and $\wp_0(S, v) = \{\emptyset\}$. If $|S| = 1$ then (6) implies $e_\emptyset = e_{\mathbf{1}} + \sum_{\psi\in\widehat{G}'_{0,S}} e_\psi = \sum_{\psi\in\widehat{G}_{0,S}} e_\psi = e_{0,S}$, whilst if $|S| > 1$ then $e_\emptyset = e_{\mathbf{1}} + e_{0,S}$. Since $e_{\mathbf{1}} \cdot \mathcal{F}_0 = 0$ if $|S| > 1$, this shows that in both cases the right hand side of (7) is equal to $e_{0,S} \cdot \mathcal{F}_0$, as required. □

The next result deals with the case that a is 'large' (see Remark 3.2(ii)).

Proposition 3.5. *Theorem 2.3 is valid if $a \geq |S| - 1$.*

Proof. Following Remark 3.2(ii) it is enough to consider a equal to either $|S|-1$ or $|S|$. Our argument then splits into several subcases, depending on the cardinality of $S^a_{\min}$. Following Lemma 3.4 we always assume $a > 0$.

We consider first the case that $S^a_{\min} = \emptyset$ and $a > 0$, and hence that $\widehat{G}'_{a,S} = \emptyset$. In this case Lemma 3.1(i) and (ii) combine to imply $e_{(a),S} \cdot \mathcal{F}_a = e_{a,S} \cdot \mathcal{F}_a$ is equal to $e_{\mathbf{1}} \cdot \mathcal{F}_a$ if $a = |S|-1$ and vanishes if $a = |S|$. One also has $\wp_a(S, v) = \emptyset$ (since $a > 0$) and so the right hand side of (7) is equal to $c^a_{S,v} e_{\mathbf{1}} \cdot \mathcal{F}_a$. The claimed equality is thus true in this case since $c^a_{S,v}$ is equal to $1 - 0 - 0 = 1$ if $a = |S| - 1$ and to 0 if $a = |S|$.

In the remainder of the argument we assume $S^a_{\min} \neq \emptyset$ and hence that $|S^a_{\min}|$ is equal to either $|S| - 1$ (so $a = |S| - 1$) or $|S|$ (so $S^a_{\min} = S$).

We consider first the case $|S^a_{\min}| = |S| - 1 = a$ and $v \in S^a_{\min}$. In this case one has $N_v \subseteq \ker(\psi)$ for all $\psi \in \widehat{G}'_{a,S}$ so $(1 - e_v)e_{a,S} = 0$ and Lemma 3.1(i) and (ii) combine to imply the left hand side of (7) is equal to $e_{\mathbf{1}} \cdot \mathcal{F}_a$. Given this, the claimed equality follows from the fact that $\wp_a(S, v) = \emptyset$ and $c^a_{S,v} = 0 - 1 - 0 = -1$.

We next assume $|S^a_{\min}| = |S| - 1 = a$ and $v \notin S^a_{\min}$ so that $S^a_{\min} = S \setminus \{v\}$. In this case Lemma 3.1(i) and (iii) together imply that the left hand side of (7) is equal to $e_{a,S} \cdot \mathcal{F}_a$. In addition, one has $c^a_{S,v} = 1 - 1 - 0 = 0$ and the unique element of $\wp_a(S, v)$ is equal to $I := S^a_{\min}$ so $e_I = e_{a,S}$ and the right hand side of (7) is also equal to $e_{a,S} \cdot \mathcal{F}_a$.

We now consider the case $S^a_{\min} = S$ and $a = |S| - 1$. For v in S we write $N(v)$ for the subgroup generated by $N_{v'}$ as v' varies over $S \setminus \{v\}$. Then for each ψ in $\widehat{G}'_{a,S}$ one has $e_\psi e_v = 0$ if and only if $N(v) \subseteq \ker(\psi)$. This implies $(1 - e_v)e_{a,S} = (e_{N(v)} - e_{\mathbf{1}})e_{a,S}$ and hence that the left hand side of (7) is equal to $e_{N(v)}e_{a,S} \cdot \mathcal{F}_a$. The claimed equality thus follows from the fact that, in this case, $c^a_{S,v} = 0$, $\wp_a(S, v)$ has a single element $I_v := S \setminus \{v\}$ and it is straightforward to check that e_{I_v} is equal to $e_{N(v)}e_{a,S}$.

Finally, we assume $a = |S|$ and $S^a_{\min} \neq \emptyset$ (and hence that both $S^a_{\min} = S$ and G is not trivial). In this case one verifies that $e_{(a),S} = e_{a,S} = e_{G_S} - e_{\mathbf{1}}$ with N_S the normal subgroup of G generated by N_v as v varies over S. For each v in S one therefore has $(1 - e_v + e_{\mathbf{1}})e_{(a),S} = (1 - e_v)e_{N_S} = 0$, where the last equality is valid since $N_v \subseteq N_S$, and

so the left hand side of (7) vanishes. On the other hand, in this case the right hand side of (7) vanishes since for any v in S both $c^a_{S,v} = 0$ and $\wp_a(S, v)$ is empty.

This completes the proof of Theorem 2.3 in the case that $a \geq |S| - 1$. □

3.3. Following Propositions 3.3 and 3.5 and Lemma 3.4 we assume in the remainder of the argument that G is not trivial and that $0 < a < |S| - 1$.

We note that in this case Lemma 3.1(ii) implies that the term $1 - e_v + e_{\mathbf{1}}$ on the left hand side of (7) can be replaced by $1 - e_v$ and that the first summand on the right hand side of (7) can be omitted.

We next recall (from [4, §2.2]) that, since $\mathcal{O}^\times_{F,S,T}$ is assumed to be torsion-free, the G-module $\mathrm{Sel}^T_S(F)$ has a canonical transpose $\mathrm{Sel}^T_S(F)^{\mathrm{tr}}$ in the sense of Jannsen's homotopy theory of modules [14] and that there exists a canonical exact sequence

$$0 \longrightarrow \mathrm{Cl}^T_S(F) \longrightarrow \mathrm{Sel}^T_S(F)^{\mathrm{tr}} \xrightarrow{\kappa} X_{F,S} \longrightarrow 0, \tag{10}$$

with $X_{F,S}$ the submodule of the free abelian group $Y_{F,S}$ on the places of F above S comprising elements whose coefficients sum to zero.

As a final preparatory remark, we note that it suffices to prove the equality of Theorem 2.3 after tensoring with $\mathbb{Z}_p$ for every prime p.

We therefore fix a prime p and for each abelian group A write A_p in place of $\mathbb{Z}_p \otimes A$. By adapting an approach used in [4], we now construct a convenient presentation of the $\mathbb{Z}_p[G]$-module $\mathrm{Sel}^T_S(F)^{\mathrm{tr}}_p$.

We fix a place v in S and a place w of F above v, we set $d_1 := |S| - 1$, write the elements of $S \setminus \{v\}$ as $\{v_i\}_{1 \leq i \leq d_1}$ and for each index i we fix a place w_i of F above v_i. We write $\mathbb{Z}_p[G]^{d_1}$ for the direct sum of d_1 copies of $\mathbb{Z}_p[G]$ and choose a homomorphism of $\mathbb{Z}_p[G]$-modules $\pi_1 : \mathbb{Z}_p[G]^{d_1} \to \mathrm{Sel}^T_S(F)^{\mathrm{tr}}_p$ with the property that, for each index i, the image under $\kappa \circ \pi_1$ of the i-th element of the standard basis of $\mathbb{Z}_p[G]^{d_1}$ is $w_i - w$. We also write ϱ_v for the natural projection $X_{F,S} \to Y_{F,S\setminus\{v\}}$ and fix a natural number $d_{2,v}$ for which there exists a surjective homomorphism of $\mathbb{Z}_p[G]$-modules of the form $\pi_{2,v} : \mathbb{Z}_p[G]^{d_{2,v}} \to \ker(\varrho_v \circ \kappa)$.

For each v in $S \setminus S_\infty$ we write $\varpi_{v,F}$ for the element of $\mathrm{Hom}_G(\mathcal{O}^\times_{F,S,T}, \mathbb{Z}[G])$ sending u to $\sum_{g \in G} \mathrm{val}_{w_v}(g^{-1}(u))g$ with val_{w_v} the normalised additive valuation at a fixed place w_v of F above v. For each subfield E of F we write $\varpi_{v,E}$ for the restriction of $\varpi_{v,F}$ to E.

Then, in terms of this notation, the key result we need is the following. (A quick review of the basic terminology and concepts relating to presentations can be found in the appendix.)

Lemma 3.6. *There exists a locally-quadratic presentation Π_v of the G-module $\mathrm{Sel}^T_S(F)^{\mathrm{tr}}$ with all of the following properties at an arbitrary normal subgroup N of G, with $E := F^N$ and $\Gamma := G/N$.*

(i) *The transpose Π_v^{tr} of Π_v is a locally-quadratic presentation of the G-module $\mathrm{Sel}^T_S(F)$.*
(ii) *The N-coinvariants $H_0(N, \Pi_v)$ of Π_v is a locally-quadratic presentation of the Γ-module $\mathrm{Sel}^T_S(E)^{\mathrm{tr}}$.*
(iii) *For each prime p the p-primary component $\Pi_{v,p}$ of of Π_v has the form*

$$\mathbb{Z}_p[G]^{d_v} \xrightarrow{\theta_{v,p}} \mathbb{Z}_p[G]^{d_v} \xrightarrow{\pi_v} \mathrm{Sel}^T_S(F)^{\mathrm{tr}}_p \longrightarrow 0$$

with $d_v := d_1 + d_{2,v}$ *and* $\pi_v := (\pi_1, \pi_{2,v})$ *with respect to the obvious decomposition* $\mathbb{Z}_p[G]^{d_v} = \mathbb{Z}_p[G]^{d_1} \oplus \mathbb{Z}_p[G]^{d_{2,v}}$.

(iv) *There is a natural identification of* $H^0(N, \ker(\theta_{v,p}))$ *with* $\mathbb{Z}_p \otimes_{\mathbb{Z}} \mathcal{O}^\times_{E,S,T}$. *This identification combines with the canonical isomorphism* $\mathbb{Z}_p[\Gamma] \cong H^0(N, \mathbb{Z}_p[G])$ *to give a surjective restriction map*

$$\begin{aligned}\mathrm{Hom}_{\mathbb{Z}_p[\Gamma]}(\mathbb{Z}_p[\Gamma]^{d_v}, \mathbb{Z}_p[\Gamma]) &\cong \mathrm{Hom}_{\mathbb{Z}_p[\Gamma]}(H^0(N, \mathbb{Z}_p[G]^{d_v}), \mathbb{Z}_p[\Gamma]) \\ &\to \mathrm{Hom}_{\mathbb{Z}_p[\Gamma]}(H^0(N, \ker(\theta_{v,p})), \mathbb{Z}_p[\Gamma]) = \mathrm{Hom}_{\mathbb{Z}_p[\Gamma]}(\mathcal{O}^\times_{E,S,T}, \mathbb{Z}_p[\Gamma]).\end{aligned} \tag{11}$$

We write $M_{v,p}$ *for the matrix of* $\theta_{v,p}$ *with respect to the standard basis of* $\mathbb{Z}_p[G]^{d_v}$ *and* $M_{v,p,N}$ *for its image under the natural projection* $M_{d_v \times d_v}(\mathbb{Z}_p[G]) \to M_{d_v \times d_v}(\mathbb{Z}_p[\Gamma])$.

(v) *Each column of* $M_{v,p,N}$ *that corresponds to a place in* $S \setminus \{v\}$ *that splits completely in* E/k *is zero.*

(vi) *Let* Σ *be a subset of the non-archimedean places in* $S \setminus \{v\}$ *that split completely in* E/k. *For* $v' \in \Sigma$ *choose a lift* $\tilde{\varpi}_{v',E}$ *of* $\varpi_{v',E}$ *through (11) and write* $M^\Sigma_{v,p,N}$ *for the matrix that differs from* $M_{v,p,N}$ *only in that each column that corresponds to a place* v' *in* Σ *is changed from zero to the matrix of* $\tilde{w}_{v',E}$ *with respect to the standard basis of* $\mathbb{Z}_p[\Gamma]^{d_v}$. *Then there exists an exact sequence of* $\mathbb{Z}_p[\Gamma]$*-modules*

$$\mathbb{Z}_p[\Gamma]^{d_v} \to \mathbb{Z}_p[\Gamma]^{d_v} \to \mathrm{Sel}^T_{S\setminus J}(E)^{\mathrm{tr}}_p \longrightarrow 0, \tag{12}$$

in which the first arrow is induced by (left) multiplication by $M^\Sigma_{v,p,N}$. *The transpose of this presentation of* $\mathrm{Sel}^T_{S\setminus J}(E)^{\mathrm{tr}}_p$ *is a presentation of the* $\mathbb{Z}_p[\Gamma]$*-module* $\mathrm{Sel}^T_{S\setminus J}(E)_p$.

Proof. The argument of [6, §7.2.2] constructs a locally-quadratic presentation Π_v of $\mathrm{Sel}^T_S(F)$ with all of the properties stated in (i), (ii), (iii), (iv) and (v).

The fact that this presentation also has the property described in claim (vi) is proved by the argument of [3, Lem. 7.3]. □

We recall (from the appendix) that for each non-negative integer a one has

$$\mathrm{Fit}^a_G(\Pi^{\mathrm{tr}}_v) = \mathrm{Fit}^a_G(\Pi_v)^\# \tag{13}$$

where $x \mapsto x^\#$ denotes the involution of $\zeta(\mathbb{C}[G]) = \prod_{\widehat{G}} \mathbb{C}$ that sends each element $(x_\psi)_\psi$ to $(x_{\check{\psi}})_\psi$.

For each subset J of $\{i \in \mathbb{Z} : 1 \le i \le d_v\}$ we write $m_{v,p}(J)$ for the ideal of $\xi(\mathbb{Z}_p[G])$ that is generated by the reduced norms of elements in the set $M_{v,p}(J)$ of all matrices that are obtained from $M_{v,p}$ by replacing all entries in each column indexed by an integer in J by arbitrary elements of $\mathbb{Z}_p[G]$. Then the last displayed equality combines with the very definition of a-th Fitting invariants (as recalled in the appendix) to imply that

$$\mathrm{Fit}^a_G(\Pi^{\mathrm{tr}}_v)_p = \mathrm{Fit}^a_{\mathbb{Z}_p[G]}(\Pi^{\mathrm{tr}}_{v,p}) = \sum_{J \in \wp_a(d_v)} m_{v,p}(J)^\# \tag{14}$$

with $\wp_a(d_v) := \wp_a(\{i \in \mathbb{Z} : 1 \le i \le d_v\})$.

Next we note that for each ψ in $\widehat{G} \setminus \{\mathbf{1}\}$ one has

$$v \notin S_\psi \iff N_v \not\subseteq \ker(\psi) \iff (1 - e_v)e_\psi \neq 0$$

and hence that for each non-negative integer a one has

$$(1-e_v)e_{a,S} = \sum_{I\in\wp_a(S,v)} (e_I - e_{\mathbf{1}}).$$

This equality combines with (14) to imply that

$$\begin{aligned}(1-e_v)e_{(a),S}\cdot \mathrm{Fit}^a_G(M_{v,p})_p =&(1-e_v)e_{a,S}\cdot \mathrm{Fit}^a_G(M_v)_p\\ =&(\sum_{I\in\wp_a(S,v)}(e_I-e_{\mathbf{1}}))\cdot(\sum_{J\in\wp_a(d_v)} m_{v,p}(J))\\ =&\sum_{I\in\wp_a(S,v)}(e_I-e_{\mathbf{1}})\cdot m_{v,p}(I)\\ =&\sum_{I\in\wp_a(S,v)}(e_I-e_{\mathbf{1}})\cdot \mathrm{Fit}^a_G(M_{v,p})_p.\end{aligned}$$

The first equality here is true as Lemma 3.1(i) implies $e_{(a),S}\cdot \mathrm{Fit}^a_G(\Pi_v) = e_{a,S}\cdot \mathrm{Fit}^a_G(\Pi_v)$ and the third and fourth equalities follow directly from the result of Lemma 3.7 below.

To deduce the equality (7) in the case that $a < |S|-1$ it thus suffices to observe firstly that the last sum in the above formula is direct since for distinct elements I_1 and I_2 of $\wp^*_a(S)$ the idempotents $e_{I_1}-e_{\mathbf{1}}$ and $e_{I_2}-e_{\mathbf{1}}$ are orthogonal, and then that for each such I Lemma 3.1(ii) implies $(e_I-e_{\mathbf{1}})\cdot \mathrm{Fit}^a_G(M_{v,p}) = e_I\cdot \mathrm{Fit}^a_G(M_{v,p})$.

In the following result we use the correspondence $v_i \leftrightarrow i$ to identify $\wp_a(S\setminus\{v\})$ with the subset $\wp_a(d_1)$ of $\wp_a(d_v)$.

Lemma 3.7. *Take $\psi\in\widehat{G}_{a,S}\setminus\{\mathbf{1}\}$ with $v\notin S_\psi$. Then S_ψ belongs to $\wp_a(d_v)$ and for each J in $\wp_a(d_v)\setminus\{S_\psi\}$ one has $e_\psi\cdot m_{v,p}(J)^{\#}=0$.*

Proof. Set $G_\psi := G/\ker(\psi)$ and $F^\psi := F^{\ker(\psi)}$.

Then to compute $e_\psi\cdot m_{v,p}(J)$ one can replace $m_{v,p}(J)$ by its image in $\mathbb{Z}_p[G_\psi]$. Note also that the exact sequence (12) implies that each column of the image $M_{v,p,\psi}$ in $\mathbb{Z}_p[G_\psi]$ of $M_{v,p}$ that corresponds to an integer in $S_\psi\setminus J$ must vanish since the corresponding place in $S\setminus\{v\}$ splits completely in F^ψ/k.

In particular, if $J\neq S_\psi$, then the image in $\mathbb{Z}_p[G_\psi]$ of any matrix in $M_{v,p}(J)$ has at least one column of zeroes and so its reduced norm must vanish. It follows that $e_\psi\cdot m_{v,p}(J)=0$, as claimed. □

4. Refined Stark conjectures

In this section we shall formulate a conjecture that extends the existing theory of refined Stark conjectures in two significant ways in that it neither restricts to L-series that vanish at zero to order r or excludes the 'boundary case' that S is equal to $S^r_{\min}$. Our conjecture is motivated by the result of Theorem 2.3 and so we will assume throughout the notation and hypotheses of that result.

We will also use the formalism of non-commutative exterior powers developed in [6].

4.1. In order to formulate our conjecture we first define, for each non-negative integer a with $a \leq |S|$, a canonical 'Stark element of rank a' for the data $F/k, S$ and T.

To do this we use the isomorphism of $\mathbb{C}[G]$-modules

$$\lambda^a_{F,S} : e_{a,S} \cdot \bigwedge\nolimits^a_{\mathbb{C}[G]}(\mathbb{C} \cdot \mathcal{O}^\times_{F,S}) \cong e_{a,S} \cdot \bigwedge\nolimits^a_{\mathbb{C}[G]}(\mathbb{C} \cdot X_{F,S})$$

that is induced by the a-th (non-commutative) exterior power of the (Dirichlet regulator) isomorphism of $\mathbb{C}[G]$-modules

$$R_{F,S} : \mathbb{C} \cdot \mathcal{O}^\times_{F,S} \cong \mathbb{C} \cdot X_{F,S}$$

that sends each element u of $\mathcal{O}^\times_{F,S}$ to $-\sum_w \log(|u|_w) \cdot w$, where w runs over all places of F above those in S and $|\cdot|_w$ is the normalised absolute value at w.

For each such a we define a $\zeta(\mathbb{C}[G])$-valued function of a complex variable by setting

$$L^a_{S,T}(z, F/k) := \sum_{\psi \in \widehat{G}_{(a),S}} z^{-a\psi(1)} L_{S,T}(\check{\psi}, z) \cdot e_\psi.$$

In the case $a = 0$ this function recovers the 'Stickelberger function' for the data $(F/k, S, T)$ as discussed by Hayes in [12]. In general, the equality (9) implies both that $L^a_{S,T}(z, F/k)$ is holomorphic at $z = 0$ and that $L^a_{S,T}(0, F/k) = e_{a,S} \cdot L^a_{S,T}(0, F/k)$.

In the sequel we fix an ordering of S and always use it to define all exterior powers of elements that are indexed by places in S.

For each place v in S we will also fix a place w_v of F that lies above v.

4.1.1. We deal first with the case that $a < |S|$. In this case, for each I in $\wp^*_a(S)$ we define $\eta^I_{F/k,S,T}$ to be the unique element of $e_I e_{a,S} \cdot \bigwedge^a_{\mathbb{C}[G]}(\mathbb{C} \cdot \mathcal{O}^\times_{F,S})$ that satisfies

$$\lambda^a_{F,S}(\eta^I_{F/k,S,T}) = e_I \cdot \theta^a_{F/k,S,T}(0) \cdot \bigwedge\nolimits_{v \in I}(w_v - w_I), \tag{15}$$

where w_I is any choice of place of F that lies above a place in $S \setminus I$ (such a choice is possible since $|I| = a$ is assumed to be strictly less than $|S|$).

We also note that $\eta^I_{F/k,S,T}$ is independent of the choice of w_I since if w'_I is any other such place, then $w_I - w'_I$ is annihilated by $e_I e_{a,S}$. To justify the latter claim note that $w_I - w'_I$ belongs to the kernel of the natural surjective homomorphism $\pi_{S,I} : X_{F,S} \to Y_{F,I}$. In particular, if ψ is any homomorphism in $\widehat{G}'_{a,S}$ with $S_\psi = I$, then $\dim_{\mathbb{C}}(e_\psi(\mathbb{C} \cdot X_{F,S})) = \dim_{\mathbb{C}}(e_\psi(\mathbb{C} \cdot Y_{F,I})) = a \cdot \psi(1)$ so $e_\psi(\mathbb{C} \cdot \ker(\pi_{S,I})) = 0$ and hence $e_\psi(w_I - w'_I) = 0$. Since the sum of e_ψ over all such ψ is equal to $e_I(1 - e_{\mathbf{1}})$, and hence to $e_I e_{a,S}$ if $a < |S| - 1$, it suffices to show that $e_{\mathbf{1}}$ annihilates $w_I - w'_I$ if $a = |S| - 1$ and this is true since, in this case, w_I and w'_I must both lie above the unique place in $S \setminus I$.

If $a < |S| - 1$, then we define the Stark element of rank a for the data $(F/k, S, T)$ by setting

$$\eta^a_{F/k,S,T} := \sum_{I \in \wp^*_a(S)} \eta^I_{F/k,S,T} \in e_{a,S} \cdot \bigwedge\nolimits^a_{\mathbb{C}[G]}(\mathbb{C} \cdot \mathcal{O}^\times_{F,S}). \tag{16}$$

However, if $a = |S| - 1$, which we now assume, then we proceed as follows. We recall first that if, in this case, G is trivial, then we have fixed a place v_* in S and set $S^a_{\min} := S \setminus \{v_*\}$

so that $\wp_a^*(S)$ is the singleton $\{S^a_{\min}\}$. If $a = |S| - 1$ and G is not trivial, then we also now fix a place v_* of S, the precise choice of which will not matter in the sequel.

In both cases, for each set I in $\wp_a^*(S)$ we write v_I for the unique place such that $S = I \cup \{v_I\}$ (so $v_I = v_*$ if G is trivial). Then $\bigwedge^a_{\mathbb{Z}} X_{k,S}$ is a free $\mathbb{Z}$-module of rank one and both $\wedge_{v \in S \setminus \{v_*\}}(v - v_*)$ and, for any I in $\wp_a^*(S)$, also $\wedge_{v \in I}(v - v_I)$ are a basis of this module. For each such I we can therefore define $u_I \in \{\pm 1\}$ by the equality

$$u_I \cdot \bigwedge\nolimits_{v \in I}(v - v_I) = \bigwedge\nolimits_{v \in S \setminus \{v_*\}}(v - v_*). \tag{17}$$

We then define the Stark element of rank $a = |S| - 1$ for the data $(F/k, S, T)$ to be the element of $e_{a,S} \cdot \bigwedge^a_{\mathbb{C}[G]}(\mathbb{C} \cdot \mathcal{O}^\times_{F,S})$ that is obtained by setting

$$\eta^a_{F/k,S,T} := \begin{cases} ((1 - e_{\mathbf{1}}) + |\wp_a^*(S)|^{-1} e_{\mathbf{1}}) \cdot \sum_{I \in \wp_a^*(S)} u_I \cdot \eta^I_{F/k,S,T}, & \text{if } S^a_{\min} \neq \emptyset, \\ |G|^{-a} \cdot \varepsilon_k, & \text{if } S^a_{\min} = \emptyset. \end{cases} \tag{18}$$

Here we use the fact that $|\wp_a^*(S)| \neq 0$ if $S^a_{\min} \neq \emptyset$ and write ε_k for the Rubin-Stark element for $(k/k, S, T)$ that is defined with respect to the places $S \setminus \{v_*\}$. (We also note that, in each case, the element $\eta^a_{F/k,S,T}$ depends on the choice of v_* only up to a sign and so this dependence will not be indicated explicitly.)

Example 4.1. In the special case that $a = r$ (so $a < |S|$) and $S^a_{\min}$ comprises a places that split completely in F/k, then $\wp_a^*(S)$ comprises a single element $I = S^a_{\min}$. In this case one checks easily that (16), respectively (18), implies $\eta^a_{F/k,S,T}$ is equal, up to a sign, with the (non-abelian) 'Rubin-Stark element' for the data $(F/k, S, T)$, as defined by Sano and the first author in [6]. A more general version of this comparison result is proved in Proposition 5.3 below.

Remark 4.2. The definition of $\eta^a_{F/k,S,T}$ ensures that for every integer a and every I in $\wp_a^*(S)$ the elements $e_I(\eta^a_{F/k,S,T})$ and $\eta^I_{F/k,S,T}$ differ at most by a sign (see Lemma 5.4 below).

4.1.2. We now assume $a = |S|$. This case is somewhat exceptional since $e_{a,S} = e_{G_S} - e_{\mathbf{1}}$, with G_S the subgroup defined in the proof of Proposition 3.5, and so $e_{a,S} \neq 0$ only if the maximal extension F^{G_S} of k in F in which all places in S split completely is non-trivial. In addition, in the latter case one has $\wp_a^*(S) = \{I_a\}$ with $I_a := S$ and hence, since $S \setminus I_a$ is empty, one cannot define $\eta^a_{F/k,S,T}$ in the same way as above.

To overcome this difficulty we write v_1 for the place occurring first in the ordering of S that has been fixed and, after noting that $(1 - e_{\mathbf{1}})w_{v_1}$ belongs to $\mathbb{Q} \cdot X_{F,S}$, we define the Stark element of rank $a = |S|$ for the data $(F/k, S, T)$ to be the unique element $\eta^a_{F/k,S,T}$ of $e_{a,S} \cdot \bigwedge^a_{\mathbb{C}[G]}(\mathbb{C} \cdot \mathcal{O}^\times_{F,S})$ that satisfies

$$\lambda^a_{F,S}(\eta^a_{F/k,S,T}) = \theta^a_{F/k,S,T}(0) \cdot ((1 - e_{\mathbf{1}})w_{v_1} \wedge \bigwedge\nolimits_{v \in S \setminus \{v_1\}}(w_v - w_{v_1})). \tag{19}$$

4.2. For a non-negative integer a we define a $\xi(\mathbb{Z}[G])$-submodule of $\bigwedge^a_{\mathbb{Q}[G]} \mathrm{Hom}_G(\mathcal{O}^\times_{F,S,T}, \mathbb{Q}[G])$ by setting

$$\Phi^a_{F/k,S,T} := \xi(\mathbb{Z}[G]) \cdot \{\bigwedge\nolimits_{i=1}^{i=a} \varphi_i : \varphi_i \in \mathrm{Hom}_G(\mathcal{O}^\times_{F,S,T}, \mathbb{Z}[G])\}.$$

We also define a finitely generated $\xi(\mathbb{Z}[G])$-submodule of $\zeta(\mathbb{Q}[G])$ by setting

$$\mathcal{E}^a_{F/k,S,T} := \{\varphi(\eta^a_{F/k,S,T}) : \varphi \in \Phi^a_{F/k,S,T}\}.$$

For I in $\wp_a(S)$ we write N_I for the normal subgroup of G that is generated by the subgroups N_v for v in I.

We can now state our refined Stark conjecture in the case $a < |S|$.

Conjecture 4.3. *Assume the hypotheses of Theorem 2.3. Fix a non-negative integer a with $a < |S|$ and set $\eta^a := \eta^a_{F/k,S,T}$. For each non-negative integer b set $\Phi^b := \Phi^b_{F/k,S,T}$. Then the following claims should be valid.*

(i) *There exists a locally-quadratic presentation $\Pi = \Pi_{F/k,S,T}$ of the G-module $\mathrm{Sel}^T_S(F)$ for which for each place v in S one has*

$$(1 - e_v + e_{\mathbf{1}})e_{(a),S} \cdot \mathrm{Fit}^a_G(\Pi)^\# = c^a_{S,v} e_{\mathbf{1}} \cdot \mathcal{E}^a_{F/k,S,T} \oplus \bigoplus_{I \in \wp_a(S,v)} e_I \cdot \mathcal{E}^a_{F/k,S,T}, \tag{20}$$

where the integer $c^a_{S,v}$ is as defined in Theorem 2.3.

(ii) *For every I in $\wp^*_a(S)$ and every φ in Φ^a one has both*

$$\mathfrak{n}^a_{S,T}(F/k) \cdot e_I \cdot \varphi(\eta^a)^\# \subseteq \xi(\mathbb{Z}[G]) \text{ and } \mathfrak{m}^a_{S,T}(F/k) \cdot e_I \cdot \varphi(\eta^a)^\# \subseteq \mathrm{Fit}^a_G(\mathrm{Sel}^T_S(F)).$$

(iii) *For every $\varphi \in \Phi^a$, every I in $\wp^*_a(S)$ and every $v \in S \setminus I$ one has*

$$\varphi(\eta^a) \cdot T_I \in \mathrm{pAnn}_{\mathbb{Z}[G]}(\mathrm{Cl}^T_{S(I,v)}(F))$$

with $T_I := \sum_{g \in N_I} g$ and $S(I,v) := S_\infty \cup I \cup \{v\}$.

(iv) *For each v in $S \setminus S_\infty$ there exists a homomorphism $\tilde{\varpi}_v$ in $\mathrm{Hom}_G(\mathcal{O}^\times_{F,S,T}, \mathbb{Z}[G])$, with $\tilde{\varpi}_v = \varpi_v$ if $v \in S_{\mathrm{sp}}$, and such that the following property is satisfied. For each I in $\wp^*_a(S)$ and each subset J of $I \setminus S_\infty$ there exists a locally-quadratic presentation Π_I of the G/N_I-module $\mathrm{Sel}^T_{S\setminus J}(F^I)$ such that*

$$\xi(\mathbb{Z}[G]) \cdot \{(\varphi \wedge \textstyle\bigwedge_{v \in J} \tilde{\varpi}_v)(\eta^I)^\# : \varphi \in \Phi^{a-|J|}\} = \mathrm{Fit}^{a-|J|}_{G/N_I}(\Pi_I) \subseteq \mathrm{Fit}^{a-|J|}_{G/N_I}(\mathrm{Sel}^T_{S\setminus J}(F^I)).$$

Remark 4.4.
(i) Assume G is abelian. In this case Conjecture 4.3(i) and (ii) are together equivalent to the central conjecture (Conjecture 4.3) of our earlier article [5] and hence both refine and extend the conjectures formulated by Rubin in [23], by Emmons and Popescu in [8] and by Vallières in [25]. In particular, the first inclusion in Conjecture 4.3(ii) implies that for any x in $\mathfrak{n}^a_{S,T}(F/k)$ the element $x(\eta^a_{F/k,S,T})$ belongs to the lattice $\bigcap^a_{\mathbb{Z}[G]} \mathcal{O}^\times_{F,S,T}$ defined by Rubin. This connection also combines with [5, Exam. 4.5] to show that Conjecture 4.3 is in a natural sense best possible.
(ii) If, for any G, one has $a = |S_{\mathrm{sp}}| < |S|$, then $S^a_{\min} = S_{\mathrm{sp}}$, $\wp^*_a(S) = \{I\}$ with $I := S_{\mathrm{sp}}$, N_I is trivial, $F^I = F$, η^a is a (non-abelian) Rubin-Stark element for the data $(F/k, S, T)$ and the equality of Conjecture 4.3(iii) with J empty recovers the conjectural equality discussed by Sano and the first author in [6, Cor. 9.11(i)].

4.3. In this section we discuss the arithmetic properties of the Stark element $\eta^a_{F/k,S,T}$ in the case $a = |S|$.

In this case one has $e_{a,S} = e_{G_S} - e_{\mathbf{1}}$ and so, modulo establishing the functorial behaviour of $\eta^a_{F/k,S,T}$ under change of extension (which is straightforward by using the approach of Proposition 5.3 below), one can replace F/k by F^{G_S}/k. In this way one can assume both that all places in S split completely in F/k and that $e_{a,S} = 1 - e_{\mathbf{1}}$.

In this special case the result of Theorem 2.3 cannot usefully be applied (since, if $a = |S|$, then the argument of Proposition 3.5 shows that both sides of (7) vanish). However, we offer the following conjecture.

In the sequel we write $\epsilon : \mathbb{Z}[G] \to \mathbb{Z}$ for the homomorphism of G-modules that sends each element of G to 1.

Conjecture 4.5. *Assume F/k is a non-trivial Galois extension in which all places in S split completely and set $a := |S|$. Then for any element x of $\zeta(\mathbb{Z}[G])$ with $\epsilon(x) \in |G| \cdot \mathbb{Z}$ and any element φ of $\Phi^a_{F/k,S,T}$ one has*

$$x^{2a+1} \cdot \varphi(\eta^a_{F/k,S,T}) \in \mathrm{pAnn}_{\mathbb{Z}[G]}(\mathrm{Cl}^T_S(F)/\mathrm{N}_G(\mathrm{Cl}^T_S(F))).$$

Remark 4.6. If G is abelian, then Conjecture 4.5 implies that for each g in G the element $(g-1)^{2a+1} \cdot \varphi(\eta^a_{F/k,S,T})$ both belongs to $\mathbb{Z}[G]$, so that $(g-1)^{2a+1}(\eta^a_{F/k,S,T})$ belongs to the Rubin lattice $\bigcap^a_G \mathcal{O}^\times_{F,S,T}$, and also annihilates the module $\mathrm{Cl}^T_S(F)/\mathrm{N}_G(\mathrm{Cl}^T_S(F))$. In the case G is cyclic a finer version of this conjecture is formulated in [5, Conj. 4.7].

Remark 4.7. Since S is assumed to contain both S_∞ and all places that ramify in F/k, it is possible to satisfy the hypotheses of Conjecture 4.5 if and only if F/k is both unramified (at all non-archimedean places) and such that all archimedean places split. In particular, if F/k is any such extension of number fields, then the hypotheses of Conjecture 4.5 are satisfied with $S = S_\infty$. There has been no previous formulation of a significant 'Rubin-Stark Conjecture' for such extensions since, in earlier conjectures, only the case $a = r(= |S| - 1)$ has been considered and, as $e_{r,S} = e_{\mathbf{1}}$ in this case, the study of $L^r_{S,T}(0, F/k)$ is reduced to the analytic class number formula for k.

5. Evidence for Conjectures 4.3 and 4.5

In this section we refer to the 'leading term conjecture' formulated by the first author in [2, §6.1, Conj. LTC(F/k)]. This conjecture is an equality in the relative algebraic K_0-group of the ring inclusion $\mathbb{Z}[G] \to \mathbb{R}[G]$ and, if F is a number field, is shown in loc. cit. to be equivalent to the validity of the equivariant Tamagawa number conjecture for the pair $(h^0(\mathrm{Spec}(F)), \mathbb{Z}[G])$.

The link between this conjecture and Conjecture 4.3 is provided by the following result.

Theorem 5.1. *If* [2, §6.1, Conj. LTC(F/k)] *is valid, then so are Conjectures 4.3 and 4.5.*

In [6] it is shown that [2, §6.1, Conj. LTC(F/k)] is valid if F is either an abelian extension of $\mathbb{Q}$ or a global function field and so Theorem 5.1 leads directly to the following result.

Corollary 5.2. *Conjectures 4.3 and 4.5 are both valid if either F is an abelian extension of $\mathbb{Q}$ (and k any subfield of F) or F is a global function field.*

The proof of Theorem 5.1 will occupy the rest of this section and is obtained by combining the approach developed in [5] with aspects of the non-commutative formalism introduced in [6].

5.1. In preparation for proving Theorem 5.1, we first establish several auxiliary results.

We recall that if $a < |S|$, then for each I in $\wp_a^*(S)$ we write N_I for the normal closure of the subgroup of G generated by the decomposition subgroups G_v for v in I and we now write F^I for the fixed field of F by N_I.

For any finite group Γ, any element x in $\zeta(\mathbb{Q}[\Gamma])$ and any integer n we also write $x^{\langle n\rangle}$ for the element $\sum_{\psi\in\widehat{\Gamma}} x^{n\psi(1)}e_\psi$ of $\zeta(\mathbb{Q}[\Gamma])$.

The following result extends [25, Prop. 4.18].

Proposition 5.3. *Fix I in $\wp_a^*(S)$ for $a < |S|$ and set $\Gamma := G/N_I$. Then each place in I splits completely in F^I/k and we write $\varepsilon^I_{F^I/k,S,T}$ for the Rubin-Stark element in $\bigwedge^a_{\mathbb{C}[\Gamma]}(\mathbb{C}\cdot\mathcal{O}^\times_{F^I,S})$ defined with respect to the places in I.*

In addition, if one identifies $\bigwedge^a_{\mathbb{C}[\Gamma]}(\mathbb{C}\cdot\mathcal{O}^\times_{F^I,S})$ with a subspace of $\bigwedge^a_{\mathbb{C}[G]}(\mathbb{C}\cdot\mathcal{O}^\times_{F,S})$ in the obvious way, then $\eta^I_{F/k,S,T} = |N_I|^{\langle -a\rangle}\cdot\varepsilon^I_{F^I/k,S,T}$.

Proof. We set $\eta^I := \eta^I_{F/k,S,T}$ and $\varepsilon^I := \varepsilon^I_{F^I/k,S,T}$ and $H := N_I$. For each of the places w and $\{w_v\}_{v\in I}$ that are fixed in the definition (15) of η^I we also write w' and $\{w'_v\}_{v\in I}$ for the corresponding places of F^I that are obtained by restriction.

Then ε^I is defined by the equality

$$\lambda^a_{F^I,S}(\varepsilon^I) = \theta^a_{F^I/k,S,T}(0)\cdot\bigwedge\nolimits_{v\in I}(w'_v - w'). \tag{21}$$

In addition, identifying $\zeta(\mathbb{C}[\Gamma])$ with $e_H\cdot\zeta(\mathbb{C}[G])$ in the natural way, one has $\theta^a_{F^I/k,S,T}(0) = e_H\cdot\theta^a_{F/k,S,T}(0)$ and the analogue of the idempotent $e_{a,S}$ for F^I/k is equal to $e_He_Ie_{a,S}$ and so

$$\theta^a_{F^I/k,S,T}(0) = e_He_Ie_{a,S}\cdot\theta^a_{F/k,S,T}(0) = e_He_I\cdot\theta^a_{F/k,S,T}(0) = e_I\cdot\theta^a_{F/k,S,T}(0) \tag{22}$$

(with the latter equality since $e_He_I = e_I$).

After setting $T_H := \sum_{h\in H} h(= |H|e_H)$, and noting T_H is central in $\mathbb{Q}[G]$, one thus has

$$\begin{aligned}\lambda^a_{F,S}(\varepsilon^I) &= e_I\cdot\theta^a_{F/k,S,T}(0)\cdot\bigwedge\nolimits_{v\in I}T_H(w_v - w)\\ &= (T_H)^{\langle a\rangle}\cdot e_I\cdot\theta^a_{F/k,S,T}(0)\cdot\bigwedge\nolimits_{v\in I}(w_v - w)\\ &= |H|^{\langle a\rangle}\cdot e_I\cdot\theta^a_{F/k,S,T}(0)\cdot\bigwedge\nolimits_{v\in I}(w_v - w)\\ &= \lambda^a_{F,S}(|H|^{\langle a\rangle}\cdot\eta^I).\end{aligned}$$

Here the first equality follows by substituting (22) into (21) and then using the functoriality of the Dirichlet regulator map under change of field (as expressed by the commutativity of the diagram in [24, Chap. I, §6.5]) to explicitly compare the maps $\lambda^a_{F^I,S}$ and $\lambda^a_{F,S}$, the second is clear, the third is valid because $(T_H)^a = |H|^a e_H$ and $e_He_I = e_I$ and the last follows directly from (15).

Then, since $\lambda^a_{F,S}$ is bijective, the displayed equality implies $\varepsilon^I = |H|^a\cdot\eta^I$, as required. □

The next result clarifies Remark 4.2.

Lemma 5.4. Fix an integer a with $0 \leq a < |S|$. Then for each I in $\wp_a^*(S)$ the element $e_I(\eta^a_{F/k,S,T})$ is equal to $\eta^I_{F/k,S,T}$ if $a < |S|-1$ and to $u_I \cdot \eta^I_{F/k,S,T}$ if $a = |S|-1$.

Proof. Set $\eta^a := \eta^a_{F/k,S,T}$ and for each I in $\wp_a^*(S)$ also $\eta^I := \eta^I_{F/k,S,T}$.

If G is trivial and $a < |S|-1$, then $S^a_{\min} = \emptyset$ so $\wp_a^*(S) = \emptyset$ and there is nothing to prove. If G is trivial and $a = |S|-1$, then $\wp_a^*(S) = \{I\}$ with $I := S^a_{\min}$ and the claimed equality is true since $e_I = e_{\mathbf{1}} = 1$ and $u_I = 1$ and the definition of η^a ensures directly that $\eta^a = \eta^I$.

We can therefore assume G is not trivial. In this case, if $a < |S|-1$, then the claimed equality $e_I(\eta^a) = \eta^I$ follows directly upon combining the definition (16) of η^a with the equality $\eta^I = e_I(\eta^I)$ and the fact that for each J in $\wp_a^*(S) \setminus \{I\}$ one has

$$e_I(\eta^J) = e_I e_J(\eta^J) = (e_I - e_{\mathbf{1}})(e_J - e_{\mathbf{1}})(\eta^J) = 0$$

where the second equality is valid because in this case $e_{\mathbf{1}}(\eta^J) = 0$ and the last equality because the idempotents $e_I - e_{\mathbf{1}}$ and $e_J - e_{\mathbf{1}}$ are orthogonal (as $I \neq J$).

In the sequel we therefore assume both that G is not trivial and $a = |S|-1$. In this case the same argument as above shows the definition (18) of η^a implies that for each I in $\wp_a^*(S)$ one has $(1-e_{\mathbf{1}})e_I(\eta^a) = (1-e_{\mathbf{1}})(u_I \cdot \eta^I)$.

Since $e_{\mathbf{1}} e_I = e_{\mathbf{1}}$ it is thus enough to show that $e_{\mathbf{1}}(\eta^a) = e_{\mathbf{1}}(u_I \cdot \eta^I)$. To do this we note that Proposition 5.3 combines with the equality (17) to imply that for each J in $\wp_a^*(S)$ one has $e_{\mathbf{1}}(u_J \cdot \eta^J) = u_J \cdot e_{\mathbf{1}}(\eta^J) = |G|^{-a} \cdot \varepsilon_k$, with ε_k the element occurring in (18). One therefore has

$$|\wp_a^*(S)| \cdot e_{\mathbf{1}}(\eta^a) = e_{\mathbf{1}} \sum_{J \in \wp_a^*(S)} u_J \cdot \eta^J = |\wp_a^*(S)| \cdot |G|^{-a} \cdot \varepsilon_k = |\wp_a^*(S)| \cdot e_{\mathbf{1}}(u_I \cdot \eta^I), \tag{23}$$

and hence $e_{\mathbf{1}}(\eta^a) = e_{\mathbf{1}}(u_I \cdot \eta^I)$, as required. □

In the final result we record some general facts concerning descent from F/k to F^H/k for normal subgroups H of G.

Lemma 5.5. *For each normal subgroup H of G, with $\Gamma := G/H$, the following claims are valid.*

(i) *If $\mathrm{LTC}(F/k)$ is valid, then so is $\mathrm{LTC}(F^H/k)$.*

(ii) *Fix a prime p. If Π_p is a presentation of the $\mathbb{Z}_p[G]$-module $\mathrm{Sel}^T_S(F)^{\mathrm{tr}}_p$, then $H_0(H, \Pi_p)$ is a presentation of the $\mathbb{Z}_p[\Gamma]$-module $\mathrm{Sel}^T_S(F^H)^{\mathrm{tr}}_p$ and for every non-negative integer a one has $e_H \cdot \mathrm{Fit}^a_{\mathbb{Z}_p[G]}(\Pi_p) = \mathrm{Fit}^a_{\mathbb{Z}_p[\Gamma]}(H_0(H, \Pi_p))$.*

(iii) *The composite homomorphism of $\zeta(\mathbb{Q}[G])$-modules*

$$\rho^a_H : \bigwedge\nolimits^a_{\mathbb{Q}[G]} \mathrm{Hom}_G(\mathcal{O}^\times_{F,S,T}, \mathbb{Q}[G]) \to \bigwedge\nolimits^a_{\mathbb{Q}[G]} \mathrm{Hom}_G(\mathcal{O}^\times_{F^H,S,T}, \mathbb{Q}[G]) \to \bigwedge\nolimits^a_{\mathbb{Q}[\Gamma]} \mathrm{Hom}_\Gamma(\mathcal{O}^\times_{F^H,S,T}, \mathbb{Q}[\Gamma]),$$

where the first map is induced by restriction from $\mathcal{O}^\times_{F,S,T}$ to $\mathcal{O}^\times_{F^H,S,T}$ and the second by the natural projection $\mathbb{Q}[G] \cong \mathbb{Q}[\Gamma]$, is such that

$$e_{a,S} \cdot \rho^a_H(\Phi^a_{F/k,S,T}) = e_{a,S}|H|^{\langle a \rangle} \cdot \Phi^a_{F^H/k,S,T}.$$

Proof. Claim (i) follows directly from [2, Rem. 6.1.1(ii)].

To prove claim (ii) we recall $\mathrm{Sel}^T_S(F)^{\mathrm{tr}}$ is defined in [4, Def. 2.6] to be the cohomology in degree -1 of a complex $C_{F,S}$ that is acyclic in degrees greater than -1 and, since S contains all places that ramify in F/k, is also such that $\mathbb{Z}[G/H] \otimes^{\mathbb{L}}_{\mathbb{Z}[G]} C_{F,S}$ identifies with $C_{F^H,S}$. These facts combine to induce a natural isomorphism of Γ-modules between $\mathbb{Z}[\Gamma] \otimes_{\mathbb{Z}[G]} \mathrm{Sel}^T_S(F)^{\mathrm{tr}}$ and $\mathrm{Sel}^T_S(F^H)^{\mathrm{tr}}$ and this isomorphism implies the claimed result as a consequence of the general properties of Fitting invariants recalled in §A.1.4 below.

To prove claim (iii) we note first that the image of the composite homomorphism

$$\rho_H : \mathrm{Hom}_G(\mathcal{O}^\times_{F,S,T}, \mathbb{Z}[G]) \to \mathrm{Hom}_G(\mathcal{O}^\times_{F^H,S,T}, \mathbb{Z}[G]) \to \mathrm{Hom}_\Gamma(\mathcal{O}^\times_{F^H,S,T}, \mathbb{Z}[\Gamma]), \tag{24}$$

where the first arrow is the restriction map and the second is induced by the natural projection $\mathbb{Z}[G] \to \mathbb{Z}[\Gamma]$, is equal to $|H| \cdot \mathrm{Hom}_\Gamma(\mathcal{O}^\times_{F^H,S,T}, \mathbb{Z}[\Gamma])$. To see this it is enough to note the first arrow in the displayed homomorphism is surjective since the quotient $\mathcal{O}^\times_{F,S,T}/\mathcal{O}^\times_{F^H,S,T}$ is torsion-free, that $\mathrm{Hom}_G(\mathcal{O}^\times_{F^H,S,T}, \mathbb{Z}[G]) = \mathrm{Hom}_G(\mathcal{O}^\times_{F^H,S,T}, T_H \cdot \mathbb{Z}[G])$ and that the projection $\mathbb{Z}[G] \to \mathbb{Z}[\Gamma]$ sends T_H to $|H|$.

Since the projection map $\mathbb{Q}[G] \to \mathbb{Q}[\Gamma])$ sends $\xi(\mathbb{Z}[G])$ onto $\xi(\mathbb{Z}[\Gamma])$ the above description of $\mathrm{im}(\rho_H)$ implies that

$$\begin{aligned}\rho^a_H(\Phi^a_{F/k,S,T}) &= \xi(\mathbb{Z}[\Gamma]) \cdot \{\textstyle\bigwedge_{i=1}^{i=a} \varphi_i : \varphi_i \in |H| \cdot \mathrm{Hom}_\Gamma(\mathcal{O}^\times_{F^H,S,T}, \mathbb{Z}[\Gamma])\} \\ &= (\sum_{\psi \in \widehat{\Gamma}} |H|^{a \cdot r_S(\check{\psi})} e_\psi) \cdot \Phi^a_{F^H/k,S,T},\end{aligned}$$

where the second equality is valid because for every ψ in $\widehat{\Gamma}$ one has

$$\dim_{\mathbb{C}}(\mathrm{Hom}_{\mathbb{C}[\Gamma]}(V_\psi, \mathrm{Hom}_\Gamma(\mathcal{O}^\times_{F^H,S,T}, \mathbb{Z}[\Gamma]))) = r_S(\check{\psi}).$$

The claimed equality now follows since $r_S(\check{\psi}) = a$ for every ψ with $e_\psi \cdot e_{a,S} \neq 0$. □

5.2. We are now ready to prove Theorem 5.1. Our argument splits naturally into two cases depending on whether $a < |S|$ (and so Conjecture 4.3 is relevant) or $a = |S|$ (and Conjecture 4.5 is relevant).

We assume throughout that $\mathrm{LTC}(F/k)$ is valid.

5.2.1. We first consider claims (i) and (ii) of Conjecture 4.3. At the outset an integer a with $0 \leq a < |S|$. We also fix v in S, abbreviate $\mathrm{Fit}^a_G(\mathrm{Fit}^a_G(\Pi_v))$ to $\mathcal{F}_a$ and set $\mathcal{F}^{\mathrm{tr}}_a := \mathrm{Fit}^a_G(\Pi^{\mathrm{tr}}_v)$. We set $\Phi^a := \Phi^a_{F/k,S,T}$.

In this case Lemma 5.4 implies $e_I \cdot \mathcal{E}^a_{F/k,S,T} = e_I \cdot \{\varphi(\eta^I) : \varphi \in \Phi^a\}$ for all I in $\wp^*_a(S)$ and (13) implies $\mathcal{F}^{\mathrm{tr}}_a = \mathcal{F}^{\#}_a$. Thus, to derive the decomposition (20) from Theorem 2.3, it is enough to show that

$$e_{\mathbf{1}} \cdot \mathcal{F}^{\mathrm{tr}}_a = e_{\mathbf{1}} \cdot \{\varphi(\eta^a) : \varphi \in \Phi^a\} \tag{25}$$

and, if $\wp_a(S, v)$ is non-empty, that

$$e_I \cdot \mathcal{F}^{\mathrm{tr}}_a = e_I \cdot \{\varphi(\eta^I) : \varphi \in \Phi^a\} \quad \text{for all} \quad I \in \wp_a(S, v). \tag{26}$$

We note first that the equality (25) is obvious if $a < |S| - 1$ since both sides vanish, the left hand side by Lemma 3.1(ii) and the right hand side since $e_{\mathbf{1}}(\eta^a) = 0$.

Next we note that if $a = |S| - 1$, then Lemma 5.5(ii) implies $e_{\mathbf{1}} \cdot \mathcal{F}_a^{\mathrm{tr}} = e_{\mathbf{1}} \cdot \mathrm{Fit}^a(\mathrm{Sel}_S^T(k)^{\mathrm{tr}})$ and Lemma 5.5(iii) implies $\{\varphi(\varepsilon_k) : \varphi \in \Phi^a\}$ is equal to $|G|^a \cdot \mathcal{E}_k$ with

$$\mathcal{E}_k := \{\vartheta(\varepsilon_k) : \vartheta \in \bigwedge\nolimits_{\mathbb{Z}}^a \mathrm{Hom}_{\mathbb{Z}}(\mathcal{O}_{k,S,T}^\times, \mathbb{Z})\}.$$

In this case one also has $e_{\mathbf{1}}(\eta^a) = |G|^{-a} \cdot \varepsilon_k$ as a direct consequence of the definition (18) if $S^a_{\min} = \emptyset$ and as a consequence of (23) if $S^a_{\min} \neq \emptyset$.

To prove (25) in the case $a = |S| - 1$ it is thus enough to show that $\mathrm{Fit}^a(\mathrm{Sel}_S^T(k)^{\mathrm{tr}}) = \mathcal{E}_k$. But, in this case, the exact sequence (10) combines with the fact that $X_{k,S}$ is a free $\mathbb{Z}$-module of rank a to imply $\mathrm{Fit}^a(\mathrm{Sel}_S^T(k)^{\mathrm{tr}}) = \mathrm{Fit}^0(\mathrm{Cl}_S^T(k)) = |\mathrm{Cl}_S^T(k)| \cdot \mathbb{Z}$ whilst, by unwinding the explicit definition of ε_k, one finds that $\mathcal{E}_k = \theta^a_{k/k,S,T}(0) \cdot R^{-1}_{k,S,T} \cdot \mathbb{Z}$, with $R_{k,S,T}$ the determinant of the Dirichlet regulator isomorphism $\mathbb{R} \cdot \mathcal{O}_{k,S}^\times \cong \mathbb{R} \cdot X_{k,S}$ with respect to a choice of $\mathbb{Z}$-bases of $\mathcal{O}_{k,S,T}^\times$ and $X_{k,S}$. The required equality $\mathrm{Fit}^a(\mathrm{Sel}_S^T(k)^{\mathrm{tr}}) = \mathcal{E}_k$ is therefore equivalent, up to a sign, to the analytic class number formula of k.

Turning now to consider (26), we note first that if G is trivial, then $\wp_a(S, v)$ is non-empty only if both $a = |S| - 1$ and $v = v_*$ in which case $\wp_a(S, v) = \{I\}$ with $I := S \setminus \{v_*\}$ and both $e_I = e_{\mathbf{1}} = 1$ and $\eta^I = \varepsilon_k$. In this case, therefore, the equality (26) is equivalent to (25) and so follows (unconditionally) from the argument above.

In the sequel we therefore assume that G is not trivial. In this case we recall (from the end of the proof of Proposition 5.3) that $e_I \cdot e_{N_I} = e_I$. This equality implies (26) is valid if $e_{N_I} \cdot \mathcal{F}_a^{\mathrm{tr}} = \{\varphi(\eta^I) : \varphi \in \Phi^a\}$ or equivalently, after taking account of Lemma 5.5(ii) (with $H = N_I$) and of the final assertion of Proposition 5.3, if

$$\mathrm{Fit}^a_{G/N_I}(H_0(H, \Pi^{\mathrm{tr}})) = \{\vartheta(\varepsilon^I) : \vartheta \in \Phi^a_{F^I/k,S,T}\}$$

with $\varepsilon^I = \varepsilon^I_{F^I/k,S,T}$ the Rubin-Stark element in $\bigwedge^a_{\mathbb{C}[\Gamma_I]}(\mathbb{C} \cdot \mathcal{O}^\times_{F^I,S})$ defined with respect to the places in I. To complete the derivation of (20) it is thus enough to note that the last displayed equality is derived from the assumed validity of $\mathrm{LTC}(F^I/k)$ (which itself follows as a consequence of Lemma 5.5(iii)) by Sano and the first author in [6, Cor. 9.11(i)].

Turning now to the inclusions in Conjecture 4.3(ii) we note that for v in S and I in $\wp^*_a(S)$ the idempotents $1 - e_v + e_{\mathbf{1}}, e_I$ and $e_{(a),S}$ are all stable under the involution $x \mapsto x^\#$ of $\mathbb{Q}[G]$. The equality (20) therefore implies that for every I in $\wp_a(S, v)$ and every φ in Φ_a the element $e_I \cdot \varphi(\eta^a_{F/k,S,T})^\#$ belongs to $(1 - e_v + e_{\mathbf{1}})e_{(a),S} \cdot \mathrm{Fit}^a_G(\mathrm{Sel}_S^T(F))$.

Given this fact, the inclusions of in Conjecture 4.3(ii) are easily derived by the same argument used to derive Corollary 2.5 from Theorem 2.3.

5.2.2. We now consider claims (iii) and (iv) of Conjecture 4.3. To do this we set $N := N_I$, $E := F^I$ and $\Gamma := G/N \cong G_{E/k}$.

The action of T_I induces the field-theoretic norm map between $\mathrm{Cl}^T_{S(I,v)}(F)$ and $\mathrm{Cl}^T_{S(I,v)}(E)$ and the projection $\mathbb{Q}[G] \to \mathbb{Q}[\Gamma]$ maps $\mathcal{H}(\mathbb{Z}[G])$ to $\mathcal{H}(\mathbb{Z}[\Gamma])$. To prove $\mathrm{LTC}(F/k)$ implies Conjecture 4.3(iii) it is thus enough to show that it implies $\varphi(\eta^I) \in \mathrm{pAnn}_{\mathbb{Z}[\Gamma]}(\mathrm{Cl}^T_{S(I,v)}(E))$ for every φ in $\Phi^a_{F/k,S,T}$.

After taking account of Proposition 5.3 and Lemma 5.5(i) and (iii) it is thus enough to show that $\mathrm{LTC}(E/k)$ implies $\vartheta(\varepsilon^I_{E/k,S,T}) \in \mathrm{pAnn}_{\mathbb{Z}[G]}(\mathrm{Cl}^T_{S(I,v)}(E))$ for all ϑ in $\Phi^a_{E/k,S,T}$ and,

modulo slight differences in notation, this is proved by Sano and the first author in [6, Cor. 9.11(iii)].

We now consider claim (iv) of Conjecture 4.3. For each v in I' we write w'_v for the restriction of w_v to E and define $\nu_{v,I}$ to be the map in $\mathrm{Hom}_{\mathbb{Z}}(\mathcal{O}^\times_{E,S,T},\mathbb{Z})$ that sends u to $\mathrm{val}_{w'_v}(u)$. We then choose a pre-image θ_v of $\nu_{v,I}$ under the restriction map $\mathrm{Hom}_{\mathbb{Z}}(\mathcal{O}^\times_{F,S,T},\mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(\mathcal{O}^\times_{E,S,T},\mathbb{Z})$ (which is surjective since $\mathcal{O}^\times_{F,S,T}/\mathcal{O}^\times_{E,S,T}$ is torsion-free) and write $\tilde{\varpi}_v$ for the image of θ_v under the natural isomorphism $\mathrm{Hom}_{\mathbb{Z}}(\mathcal{O}^\times_{F,S,T},\mathbb{Z}) \cong \mathrm{Hom}_{\mathbb{Z}[G]}(\mathcal{O}^\times_{F,S,T},\mathbb{Z}[G])$. We note that if v splits completely in F/k, then one can take θ_v to be the map sending u to $\mathrm{val}_{w_v}(u)$ and for this one has $\tilde{\varpi}_v = \varpi_v$.

The image of $\tilde{\varpi}_v$ under the restriction map (24) with $H = N$ is equal to $|N| \cdot \varpi_{v,E}$. This fact combines with the results of Proposition 5.3 and Lemma 5.5(iii) to imply Conjecture 4.3(iv) is valid if for each prime p there exists a presentation Π_p of the $\mathbb{Z}_p[\Gamma]$-module $\mathrm{Sel}^T_{S\setminus J}(E)^{\mathrm{tr}}_p$ for which Π^{tr}_p is a presentation of $\mathrm{Sel}^T_{S\setminus J}(E)_p$ and, with $a' := a - |J|$, one has

$$\{(\varphi \wedge \textstyle\bigwedge_{v\in J}\varpi_{v,E})(\varepsilon^I_{E/k,S,T}) : \varphi \in \Phi^{a'}_{E/k,S,T,p}\} = \mathrm{Fit}^{a'}_{\mathbb{Z}_p[\Gamma]}(\Pi_p). \tag{27}$$

To prove this we use the resolution $\Pi_{v,p}$ constructed in Lemma 3.6 with F/k replaced by E/k and v chosen in $S \setminus I$. We recall from [6, (47)] that for each subset $\varphi_\bullet := \{\varphi_i\}_{1\le i\le a}$ of $\mathrm{Hom}_\Gamma(\mathcal{O}^\times_{E,S,T},\mathbb{Z}[\Gamma])_p$ one has

$$(\textstyle\bigwedge_{i=1}^{i=a}\varphi_i)(\varepsilon^I_{E/k,S,T}) = u \cdot \mathrm{Nrd}_{\mathbb{Q}_p[\Gamma]}(M_{v,p}(\varphi_\bullet)). \tag{28}$$

Here u is a suitable element of $\xi(\mathbb{Z}_p[\Gamma])^\times$ and $M_{v,p}(\varphi_\bullet)$ is the matrix in $\mathrm{M}_{d\times d}(\mathbb{Z}_p[\Gamma])$ with

$$M_{v,p}(\varphi_\bullet)_{ij} := \begin{cases} \tilde{\varphi}_i(b_j), & \text{if } v_i \in I, \\ M_{v,ij}, & \text{if } v_i \notin I. \end{cases}$$

where $\tilde{\varphi}_i$ is a lift of φ_i through the surjective restriction homomorphism (11).

We next recall that the matrix $M^J_{v,p}$ described in Lemma 3.6(vi) defines a presentation of $\mathrm{Sel}^T_{S\setminus J}(E)^{\mathrm{tr}}_p$ the transpose of which is a presentation of $\mathrm{Sel}^T_{S\setminus J}(E)_p$.

In particular, since all of the a' columns of $M^J_{v,p}$ that correspond to a place in $I\setminus J$ are zero (by Lemma 3.6(v)), the definition of Fitting invariant implies $\mathrm{Fit}^{a'}_{\mathbb{Z}_p[\Gamma]}(\Pi_p) = \mathrm{Fit}^{a'}_{\mathbb{Z}_p[\Gamma]}(M^J_{v,p})$ is generated over $\xi(\mathbb{Z}_p[\Gamma])$ by all terms of the form $\mathrm{Nrd}_{\mathbb{Q}_p[\Gamma]}(M_{v,p}(\varphi_\bullet))$ in which for every $v \in J$ one has $\varphi_v = \varpi_{v,E}$. With this choice of presentation of $\mathrm{Sel}^T_{S\setminus J}(E)^{\mathrm{tr}}_p$ the claimed equality (27) is thus a consequence of (28).

This completes the proof that $\mathrm{LTC}(F/k)$ implies Conjecture 4.3.

5.3. In this section we adapt an argument from [6, §12.4] to show Conjecture 4.5 is a consequence of $\mathrm{LTC}(F/k)$. We assume throughout the notation and hypotheses of Conjecture 4.5 (so that $a = |S|$) and now list the elements of the (ordered) set S as $\{v_i\}_{1\le i\le a}$ and abbreviate w_{v_i} to w_i. We also set $\eta^a := \eta^a_{F/k,S,T}$ and $e := e_{a,S} = 1 - e_{\mathbf{1}}$.

At the outset we note that the argument of [6, Prop. 12.4] implies it is enough to prove the claimed containment for elements of the form $\varphi = \bigwedge_{i=1}^{i=a}\varphi_i$ where the subset $\{\varphi_i\}_{1\le i\le a}$ of $\mathrm{Hom}_G(\mathcal{O}^\times_{F,S,T},\mathbb{Z}[G])$ spans a free $\mathbb{Q}[G]e$-module of rank a.

We also fix a prime p and set $\mathcal{A} := \mathbb{Z}_p[G]e$, $A := \mathbb{Q}_p[G]e$, $\mathcal{X} := \mathrm{Hom}_{\mathbb{Z}}(X_{F,S}, \mathbb{Z})_p$, $\mathfrak{X} := \mathrm{Sel}^T_S(F)_p$ and $\mathcal{U} := \mathrm{Hom}_G(\mathcal{O}^\times_{F,S,T}, \mathbb{Z}[G])_p$. Then the approach of [4, §5.4] shows that there is an exact sequence of $\mathbb{Z}_p[G]$-modules of the form

$$0 \to \mathcal{X} \xrightarrow{\iota} \mathbb{Z}_p[G]^c \xrightarrow{\theta} \mathbb{Z}_p[G]^c \xrightarrow{\pi} \mathfrak{X} \longrightarrow 0 \tag{29}$$

that has the following property. With $C_{F,S}$ as in the proof of Lemma 5.5(ii), the complex $D := R\,\mathrm{Hom}_{\mathbb{Z}}(C_{F,S}, \mathbb{Z})_p$ identifies with the complex $\mathbb{Z}_p[G]^c \xrightarrow{\theta} \mathbb{Z}_p[G]^c$, where the first term is placed in degree one, in such a way that $H^1(D)$ and $H^2(D)$ are identified with $\mathcal{X}$ and $\mathfrak{X}$ via the maps ι and π in the sequence above.

Now, since all places in S split completely in F/k, there is a direct sum decomposition of G-modules

$$X_{F,S} = \ker(\epsilon) \cdot w_1 \oplus \bigoplus_{1<i\leq a} \mathbb{Z}[G] \cdot (w_i - w_1).$$

By taking linear duals this decomposition induces an isomorphism of $\mathbb{Z}_p[G]$-modules

$$\mathcal{X} = \mathcal{A} \cdot w_1^* \oplus \bigoplus_{1<i\leq a} \mathbb{Z}_p[G] \cdot w_i^* \cong \mathcal{A} \oplus \mathbb{Z}_p[G]^{a-1} \tag{30}$$

where $w_1^* \in \mathrm{Hom}_{\mathbb{Q}[G]}(\mathbb{Q} \cdot X_{F,S}, \mathbb{Q}[G])$ maps $e \cdot w_1$ to 1 and each element w_i for $i > 1$ to zero and $w_i^* \in \mathrm{Hom}_{\mathbb{Q}[G]}(\mathbb{Q} \cdot X_{F,S}, \mathbb{Q}[G])$ for $i > 1$ maps $e \cdot w_1$ to zero and $w_j - w_1$ to 1 if $j = i$ and to zero otherwise.

For a $\mathbb{Z}_p[G]$-module M we endow the tensor product $M\langle\mathcal{A}\rangle := \mathcal{A} \otimes_{\mathbb{Z}_p} M$ with the natural multiplication action of $\mathcal{A}$ and the diagonal action of G and then define $\mathcal{A}$-modules $M^{\mathcal{A}} := H^0(G, M\langle\mathcal{A}\rangle)$ and $M_{\mathcal{A}} := H_0(G, M\langle\mathcal{A}\rangle) = \mathcal{A} \otimes_{\mathbb{Z}_p[G]} M$. The map $A \otimes_{\mathbb{Z}_p} M \mapsto \mathbb{Q}_p \otimes_{\mathbb{Z}_p} M$ that sends $x \otimes m$ to $x(m)$ for $x \in A$ and $m \in M$ identifies $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} M^{\mathcal{A}}$ with $e(\mathbb{Q}_p \otimes_{\mathbb{Z}_p} M)$ and, if M (and hence $M^{\mathcal{A}}$) is torsion-free, we use this map to identify $M^{\mathcal{A}}$ as a sublattice of $e(\mathbb{Q}_p \otimes_{\mathbb{Z}_p} M)$.

In particular, with respect to these identifications, one finds that $\mathbb{Z}_p[G]^{\mathcal{A}} = |G| \cdot \mathcal{A}$ and that the natural exact sequence $0 \to \mathbb{Z}_p\langle\mathcal{A}\rangle \to \mathbb{Z}_p[G]\langle\mathcal{A}\rangle \to \mathcal{A}\langle\mathcal{A}\rangle \to 0$ induces an exact sequence of $\mathcal{A}$-modules

$$0 \to |G| \cdot \mathcal{A} \to \mathcal{A}^{\mathcal{A}} \to H^2(G, \mathbb{Z}_p) \to 0 \tag{31}$$

where the action of $\mathcal{A}$ on $H^2(G, \mathbb{Z}_p)$ is induced by the trivial action of G (and the fact that $H^2(G, \mathbb{Z}_p)$ is annihilated by $|G|$).

We now consider the following exact commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{X}^{\mathcal{A}} & \xrightarrow{\nu^{-1}\circ\iota^{\mathcal{A}}} & \mathcal{A}^c & \xrightarrow{\theta_{\mathcal{A}}} & \mathcal{A}^c & \xrightarrow{\pi_{\mathcal{A}}} & \mathfrak{X}_{\mathcal{A}} \to 0 \\ & & & & \uparrow{\scriptstyle \kappa^1} & & \uparrow{\scriptstyle \kappa^2} & & \\ & & & & \mathcal{A}^a & \xrightarrow{0} & \mathcal{A}^a. & & \end{array}$$

Here ν is the isomorphism $\mathcal{A}^c = \mathbb{Z}_p[G]^c_{\mathcal{A}} \cong \mathbb{Z}_p[G]^{c,\mathcal{A}}$ induced by the action of T_G and the upper row is obtained by applying the left, respectively, right, exact functors $M \mapsto M^{\mathcal{A}}$ and

$M \mapsto M_{\mathcal{A}}$ to the exact sequence (29). We then write κ^1 and κ^2 for the homomorphisms of $\mathcal{A}$-modules with

$$\kappa^i(b_j) = \begin{cases} (\nu^{-1} \circ \iota^{\mathcal{A}})(|G| \cdot w_1^*), & \text{if } i = j = 1, \\ (\nu^{-1} \circ \iota^{\mathcal{A}})(|G| \cdot w_j^*), & \text{if } i = 1 \text{ and } 1 < j \leq a, \\ \tilde{\varphi}_j, & \text{if } i = 2, \end{cases}$$

where $\tilde{\varphi}_j$ is any choice of element in $\mathcal{A}^c$ that maps under the composite homomorphism $\mathcal{A}^c \to \mathfrak{X}_{\mathcal{A}} \to \mathcal{U}_{\mathcal{A}}$ to $1 \otimes \varphi_j$.

The upper row of the above diagram implies $D_{\mathcal{A}} := \mathcal{A} \otimes^{\mathbb{L}}_{\mathbb{Z}[G]} D$ identifies with $\mathcal{A}^c \xrightarrow{\theta_{\mathcal{A}}} \mathcal{A}^c$, where the first term is in degree one, in such a way that $H^1(D_{\mathcal{A}})$ and $H^2(D_{\mathcal{A}})$ are identified with $\mathcal{X}^{\mathcal{A}}$ and $\mathfrak{X}_{\mathcal{A}}$ by using the maps $\nu^{-1} \circ \iota^{\mathcal{A}}$ and $\pi_{\mathcal{A}}$. The homomorphisms κ^1 and κ^2 then together constitute a morphism κ between $C := \mathcal{A}^a[-1] \oplus \mathcal{A}^a[-2]$ and $D_{\mathcal{A}}$, the mapping cone of which is acyclic outside degrees one and two and has cohomology in these respective degrees isomorphic to $H^2(G, \mathbb{Z}_p)$ (as a consequence of the sequence (31)) and $\mathfrak{X}_{\mathcal{A}}/\mathcal{E}$, with $\mathcal{E}$ the $\mathcal{A}$-module generated by $\{\tilde{\varphi}_i\}_{1 \leq i \leq a}$.

Now our assumption on the maps φ_i implies $\mathfrak{X}_{\mathcal{A}}/\mathcal{E}$ is finite and, given this, the argument of [6, Prop. 12.9] shows LTC(F/k) implies $\varphi(\eta^a)$ is a characteristic element of Cone(κ).

In particular, since any element x of $\zeta(\mathbb{Z}[G])^0$ annihilates $H^2(G, \mathbb{Z}_p)$, and Cone(κ) is represented by a complex of the form $\mathcal{A}^a \to \mathcal{A}^a \oplus \mathcal{A}^c \to \mathcal{A}^c$, the result of Lemma 5.6 below implies $x^{2a} \cdot \varphi(\eta^a)$ belongs to $\mathrm{pAnn}_{\mathcal{A}}(\mathfrak{X}_{\mathcal{A}}/\mathcal{E})$.

To conclude we note that the exact sequence $0 \to \mathrm{Cl}^T_S(F) \to \mathfrak{X} \to \mathcal{U} \to 0$ induces an exact sequence of $\mathcal{A}$-modules $\mathrm{Tor}^1_{\mathbb{Z}_p[G]}(\mathcal{A}, \mathcal{U}) \to \mathrm{Cl}^T_S(F)_{p,\mathcal{A}} \to \mathfrak{X}_{\mathcal{A}}/\mathcal{E} \to \mathcal{U}_{\mathcal{A}}/\mathcal{E} \to 0$.

Since $\mathrm{Tor}^1_{\mathbb{Z}_p[G]}(\mathcal{A}, \mathcal{U})$ is annihilated by the image of $\zeta(\mathbb{Z}[G])^0$ in $\mathcal{A}$ this sequence implies that $\zeta(\mathbb{Z}[G])^0 \cdot \mathrm{pAnn}_{\mathcal{A}}(\mathfrak{X}_{\mathcal{A}}/\mathcal{E})$, and hence also $x^{2a+1} \cdot \varphi(\eta^a)$, is contained in $\mathrm{pAnn}_{\mathcal{A}}(\mathrm{Cl}^T_S(F)_{p,\mathcal{A}})$. The claimed result now follows because $\mathrm{Cl}^T_S(F)_{p,\mathcal{A}}$ identifies with $\mathrm{Cl}^T_S(F)_p / \mathrm{N}_G(\mathrm{Cl}^T_S(F)_p)$.

Lemma 5.6. *Let C be a complex of $\mathcal{A}$-modules of the form $\mathcal{A}^{d_0} \xrightarrow{\theta^0} \mathcal{A}^{d_1} \xrightarrow{\theta^1} \mathcal{A}^{d_2}$ where the first term is placed in degree zero. Assume C is acyclic outside degrees one and two and has finite cohomology in these degrees. Then for any characteristic element ξ of C, any x in $\zeta(\mathbb{Z}[G])^0$ and any element y of $\zeta(\mathcal{A})$ that annihilates $H^1(C)$ the product $\xi \cdot (xy)^{d_0}$ belongs to* $\mathrm{Fit}^0_{\mathcal{A}}(H^2(C))$.

Proof. We set $B^1 := \mathrm{im}(\theta^0)$, $Z^1 := \ker(\theta^1)$ and $B^2 := \mathrm{im}(\theta^1)$. The tautological sequences $0 \to B^1 \to Z^1 \to H^1(C) \to 0$ and $0 \to Z^1 \to P^1 \to B^2 \to 0$ induce exact sequences

$$\mathrm{Hom}_{\mathcal{A}}(Z^1, \mathcal{A}^{d_0}) \to \mathrm{Hom}_{\mathcal{A}}(B^1, \mathcal{A}^{d_0}) \to \mathrm{Ext}^1_{\mathcal{A}}(H^1(C), \mathcal{A}^{d_0})$$

and

$$\mathrm{Hom}_{\mathcal{A}}(\mathcal{A}^{d_1}, \mathcal{A}^{d_0}) \to \mathrm{Hom}_{\mathcal{A}}(Z^1, \mathcal{A}^{d_0}) \to \mathrm{Ext}^1_{\mathcal{A}}(B^2, \mathcal{A}^{d_0}).$$

We note that y annihilates $\mathrm{Ext}^1_{\mathcal{A}}(H^1(C), \mathcal{A}^{d_0})$ and that $\mathrm{Ext}^1_{\mathcal{A}}(B^2, \mathcal{A}^{d_0})$ embeds into $\mathrm{Ext}^1_{\mathbb{Z}_p[G]}(B^2, \mathcal{A}^{d_0}) \cong \mathrm{Ext}^2_{\mathbb{Z}_p[G]}(B^2, \mathbb{Z}_p^{d_0})$ and so is annihilated by x.

Given these facts, the above sequences imply there exists θ in $\mathrm{Hom}_{\mathcal{A}}(\mathcal{A}^{d_1}, \mathcal{A}^{d_0})$ whose restriction to B^1 is equal to xy times the inverse ι of the (bijective) map $\theta^0 : \mathcal{A}^{d_0} \to B^1$.

For any such map we write $\tilde{\theta} : \mathcal{A}^{d_1} \to \mathcal{A}^{d_0} \oplus \mathcal{A}^{d_2}$ for the homomorphism sending y to $(\theta(y), \theta^1(y))$ and $M_{\tilde{\theta}}$ for the matrix of $\tilde{\theta}$ with respect to the standard bases.

In a similar way, since both Ext-groups in the above sequences are finite, we can fix a homomorphism of A-modules $\theta' : A^{d_1} \to A^{d_0}$ which on $\mathbb{Q}_p \cdot B^1$ is equal to $\mathbb{Q}_p \otimes \iota$. We write $\hat{\theta} : A^{d_1} \to A^{d_0} \oplus A^{d_2}$ for the homomorphism sending y to $(\theta'(y), \theta^1(y))$ and $M_{\hat{\theta}}$ for the matrix of $\hat{\theta}$ with respect to the standard bases.

It is straightforward to check $\mathrm{Nrd}_A(M_{\hat{\theta}})$ is a characteristic element for C and hence that $\xi = u \cdot \mathrm{Nrd}_A(M_{\hat{\theta}})$ for some u in $\xi(\mathcal{A})^\times$. The claimed result then follows because

$$\begin{aligned}\mathrm{Nrd}_A(M_{\tilde{\theta}}) \cdot \mathrm{Nrd}_A(M_{\hat{\theta}})^{-1} &= \mathrm{Nrd}_A((\mathbb{Q}_p \otimes \tilde{\theta}\mid_{\mathbb{Q}_p \cdot B^1}) \circ (\hat{\theta}\mid_{\mathbb{Q}_p \cdot B^1})^{-1}) \\ &= \mathrm{Nrd}_A((\mathbb{Q}_p \otimes xy \cdot \iota) \circ (\mathbb{Q}_p \otimes \iota)^{-1}) = (xy)^{d_0}\end{aligned}$$

and because $\mathrm{Nrd}_A(M_{\tilde{\theta}})$ belongs to $\mathrm{Fit}^0_{\mathcal{A}}(H^2(C))$ as $H^2(C) = \mathrm{cok}(\theta^1)$ is a quotient of $\mathrm{cok}(\tilde{\theta})$. □

6. Divisor class groups

In the section we derive several consequences of Theorem 5.1 concerning relations between the values at zero of derivatives of Artin L-series and the Galois structure of divisor class groups.

Throughout we write $r = r_{F/k,S}$ for the greatest integer a such that $\widehat{G}_{(a),S} = \widehat{G}$.

6.1. $\mathscr{L}$-invariants. We first show that the predictions in Conjecture 4.3 can be reinterpreted in terms of the general formalism introduced by the first author in [2].

To do this we associate to each homomorphism θ in $\mathrm{Hom}_G(\mathcal{O}^\times_{F,S,T}, X_{F,S})$ a $\zeta(\mathbb{R}[G])$-valued '$\mathscr{L}$-invariant' by setting

$$\mathscr{L}(\theta) := \mathrm{Nrd}_{\mathbb{R}[G]}((\mathbb{R} \otimes_{\mathbb{Z}} \theta) \circ (R_{F,S})^{-1}).$$

For each integer with $0 \leq a < |S|$ and each I in $\wp^*_a(S)$ we fix a place w_I of F lying above a place in $S \setminus I$ and consider the diagram

$$\begin{array}{ccc} & & \mathrm{Hom}_G(\mathcal{O}^\times_{F,S,T}, \mathbb{Z}[G]^a) \\ & & \Big\downarrow \pi_{I,*} \\ \mathrm{Hom}_G(\mathcal{O}^\times_{F,S,T}, X_{F,S}) & \xrightarrow{\pi_{S,I,*}} & \mathrm{Hom}_G(\mathcal{O}^\times_{F,S,T}, Y_{F,I}) \end{array}$$

where π_I is the surjective homomorphism $\mathbb{Z}[G]^a \to Y_{F,I}$ that maps each element b_v of the standard basis $\{b_v\}_{v \in I}$ of $\mathbb{Z}[G]^a$ to $w_v - w_I$ and $\pi_{S,I}$ is the natural projection $X_{F,S} \to Y_{F,I}$. We write $\mathrm{Hom}^I_G(\mathcal{O}^\times_{F,S,T}, X_{F,S})$ for the full pre-image of $\mathrm{im}(\pi_{I,*})$ under the map $\pi_{S,I,*}$.

Lemma 6.1. *For each a with $a < |S|$ and each I in $\wp^*_a(S)$ one has*

$$\begin{aligned}(32)\quad \{(\textstyle\bigwedge_{v \in I} \varphi_v)(\eta^I) : \varphi_v \in \mathrm{Hom}_G(\mathcal{O}^\times_{F,S,T}, \mathbb{Z}[G])\} &= \\ \xi(\mathbb{Z}[G])e_I \cdot \{L^a_{S,T}(0, F/k) \cdot \mathscr{L}(\theta) : \theta \in \mathrm{Hom}^I_G(\mathcal{O}^\times_{F,S,T}, X_{F,S})\}\end{aligned}$$

Proof. For each subset $\varphi_\bullet := \{\varphi_v\}_{v\in I}$ of $\mathrm{Hom}_G(\mathcal{O}^\times_{F,S,T}, \mathbb{Z}[G])$ the definition (15) of η^I implies

$$\begin{aligned}(\textstyle\bigwedge_{v\in I}\varphi_v)(\eta^I) &= (\textstyle\bigwedge_{v\in I}\varphi_v)((\lambda^a_{F,S})^{-1}(e_I\cdot L^a_{S,T}(0,F/k)\cdot \textstyle\bigwedge_{v'\in I}(w_{v'}-w_I)))\\ &= e_I\cdot L^a_{S,T}(0,F/k)\cdot (\textstyle\bigwedge_{v\in I}\varphi_v)(\textstyle\bigwedge^a_{\mathbb{R}[G]}R^{-1}_{F,S})(\textstyle\bigwedge_{v'\in I}(w_{v'}-w_I))\\ &= e_I\cdot L^a_{S,T}(0,F/k)\cdot \mathrm{Nrd}_{\mathbb{R}[G]}(M(\varphi_\bullet)),\end{aligned} \tag{33}$$

where $M(\varphi_\bullet)$ is the matrix $((\varphi_v\circ R^{-1}_{F,S})(w_{v'}-w_I))_{v,v'}$ in $M_{a\times a}(\mathbb{R}[G])$ and the last equality follows from [6, Prop. 2.6].

For each $v\in I$ we now write θ_v for the composite homomorphism $\mathcal{O}^\times_{F,S,T}\to\mathbb{Z}[G]\to X_{F,S}$ where the first arrow is φ_v and the second sends the trivial element of G to w_v-w_I, and we then define $\theta_{\varphi_\bullet}$ to be the sum $\sum_{v\in I}\theta_v$.

We note $\{e_I(w_v-w_I)\}_{v\in I}$ is an $\mathbb{R}[G]e_I$-basis of $e_I(\mathbb{R}\cdot X_{F,S})$ and that, with respect to this basis, the matrix of the endomorphism obtained by restricting $(\mathbb{R}\otimes_{\mathbb{Z}}\theta_{\varphi_\bullet})\circ(R_{F,S})^{-1}$ to $e_I(\mathbb{R}\cdot X_{F,S})$ is equal to $e_I\cdot M(\varphi_\bullet)$. It follows that

$$e_I\cdot\mathrm{Nrd}_{\mathbb{R}[G]}(M(\varphi_\bullet)) = e_I\cdot\mathrm{Nrd}_{\mathbb{R}[G]}((\mathbb{R}\otimes_{\mathbb{Z}}\theta)\circ(R_{F,S})^{-1}) = e_I\cdot\mathscr{L}(\theta_{\varphi_\bullet})$$

and hence, by (33), that

$$(\textstyle\bigwedge_{v\in I}\varphi_v)(\eta^I) = e_I\cdot L^a_{S,T}(0,F/k)\cdot\mathscr{L}(\theta_{\varphi_\bullet}).$$

Next we note e_I annihilates $\ker(\pi_{S,I,*})$ and hence that for any homomorphisms θ and θ' in $\mathrm{Hom}_G(\mathcal{O}^\times_{F,S,T}, X_{F,S})$ one has $e_I\cdot\mathscr{L}(\theta) = e_I\cdot\mathscr{L}(\theta')$ whenever $\pi_{S,I,*}(\theta)=\pi_{S,I,*}(\theta')$.

To deduce the claimed equality from the last displayed equality it is thus enough to note that $\mathrm{Hom}^I_G(\mathcal{O}^\times_{F,S,T}, X_{F,S})$ is equal to the subset of $\mathrm{Hom}_G(\mathcal{O}^\times_{F,S,T}, X_{F,S})$ comprising all homomorphisms θ for which there exists a homomorphism $\theta_{\varphi_\bullet}$ of the above sort for which one has $\pi_{S,I,*}(\theta)=\pi_{S,I,*}(\theta_{\varphi_\bullet})$. □

Remark 6.2. If $S^a_{\min}$ comprises a places that split completely in F/k, then $\wp^*_a(S)$ is the singleton $\{I\}$ with $I := S^a_{\min}$ and one has $\mathrm{Hom}^I_G(\mathcal{O}^\times_{F,S,T}, X_{F,S}) = \mathrm{Hom}_G(\mathcal{O}^\times_{F,S,T}, X_{F,S})$ since the G-module $Y_{F,I}$ is free of rank a.

6.2. Global functions fields. We assume in this section that k is a global function field.

In this case Conjecture 4.3 is known to be valid (by Corollary 5.2) and we shall now derive from this fact a natural higher-order non-abelian generalisation of a result of Deligne concerning the Galois structure of divisor class groups.

To do this we define the T-modified degree zero divisor class group $\mathrm{Pic}^{T,0}(F)$ of F to be the cokernel of the natural divisor map $\mathrm{div}^0_{F,T}$ from $F^\times_T$ to the group of divisors of F of degree zero with support outside T.

The map $\mathrm{div}_{F,T}$ is injective (as $F^\times_T$ is torsion-free since T is non-empty) and that the G-module $\mathrm{Pic}^{T,0}(F)$ is an extension of $\mathrm{Pic}^0(F)$ by a finite group.

For $a=|S|-1$ we set $c_{a,S} := (1-e_{\mathbf{1}}) + |\wp^*_a(S)|e_{\mathbf{1}}$ and for $a<|S|-1$ we set $c_{a,S}=1$.

Theorem 6.3. *Let F/k be a finite Galois extension of global function fields. Then for every non-negative integer a with $a<|S|$ and every φ in $\bigcap_{I\in\wp^*_a(S)}\mathrm{Hom}^I_G(\mathcal{O}^\times_{F,S,T}, X_{F,S})$ one has*

$$L^a_{S,T}(0,F/k)\cdot\mathscr{L}(\varphi)\cdot\mathfrak{m}^a_{S,T}(F/k)\subseteq c_{a,S}\cdot\mathrm{Fit}^a_G(\mathrm{Pic}^{T,0}(F)^\vee)^\#.$$

Proof. Since $e_{a,S} = c_{a,S} \cdot \sum_{I \in \wp_a^*(S)} e_I$ one has $L^a_{S,T}(0, F/k) = c_{a,S} \cdot \sum_{I \in \wp_a^*(S)} e_I \cdot L^a_{S,T}(0, F/k)$ and so it suffices to show that for every I in $\wp_a^*(S)$ and every φ in $\mathrm{Hom}_G^I(\mathcal{O}^\times_{F,S,T}, X_{F,S})$ one has $e_I \cdot L^a_{S,T}(0, F/k) \cdot \mathscr{L}(\varphi) \cdot \mathfrak{m}^a_{S,T}(F/k) \subseteq \mathrm{Fit}^a_G(\mathrm{Pic}^{T,0}(F)^\vee)^\#$.

Thus, as (the second inclusion of) Conjecture 4.3(ii) is known to be valid in this case, the result of Lemma 6.1 reduces us to showing that $\mathrm{Fit}^a_G(\mathrm{Sel}^T_S(F)) \subseteq \mathrm{Fit}^a_G(\mathrm{Pic}^{T,0}(F)^\vee)$. To do this it is enough to prove $\mathrm{Pic}^{T,0}(F)^\vee$ is isomorphic to a quotient of $\mathrm{Sel}^T_S(F)$.

We write θ_T for the homomorphism defined as in (1) but with S taken to be empty. Then there is clearly a natural surjection from $\mathrm{Sel}^T_S(F)$ to $\mathrm{cok}(\theta_T)$ and so it suffices to prove that the latter group is naturally isomorphic to $\mathrm{Pic}^{T,0}(F)^\vee$.

We now write div_T for the (injective) divisor map from $F_T^\times$ to the group $\mathrm{Div}_T(F)$ of divisors of F with support outside T and note that $\mathrm{Pic}^{T,0}(F)$ identifies with the (finite) torsion subgroup of $\mathrm{cok}(\mathrm{div}_T)$.

It is then enough to note that the functor $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Z})$ applies to the tautological short exact sequence $0 \to F_T^\times \to \mathrm{Div}_T(F) \to \mathrm{cok}(\mathrm{div}_T) \to 0$ to give an exact sequence

$$\prod_{w \notin T_F} \mathbb{Z} \xrightarrow{\theta_T} \mathrm{Hom}_{\mathbb{Z}}(F_T^\times, \mathbb{Z}) \to \mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{cok}(\mathrm{div}_T), \mathbb{Z}) \to 0$$

and that $\mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{cok}(\mathrm{div}_T), \mathbb{Z})$ identifies with

$$\mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{cok}(\mathrm{div}_T)_{\mathrm{tor}}, \mathbb{Z}) = \mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{Pic}^{T,0}(F), \mathbb{Z}) = \mathrm{Pic}^{T,0}(F)^\vee.$$

□

Remark 6.4. Theorem 6.3 is a natural extension of the Brumer-Stark Conjecture for F/k, as proved by Deligne in [24, Chap. V]. To see this note that if $a = 0$, then $S^a_{\min}$ is empty and Lemma 3.1(iii) implies that for v in S one has $(1 - e_v - e_{\mathbf{1}})e_{(a),S} \cdot \mathrm{Fit}^a_G(\mathrm{Sel}^T_S(F)) = \mathrm{Fit}^a_G(\mathrm{Sel}^T_S(F))$. This implies both that $c_{a,S} = 1$ and $\mathfrak{m}^0_{S,T}(F/k) = \xi(\mathbb{Z}[G])$ and hence $1 \in \mathfrak{m}^0_{S,T}(F/k)$. Since in this case $L^a_{S,T}(0, F/k) \cdot \mathscr{L}(\varphi) = L_{S,T}(0, F/k)$, Theorem 6.3 implies $L_{S,T}(0, F/k) \in \mathrm{pAnn}_{\mathbb{Z}[G]}(\mathrm{Pic}^{T,0}(F)^\vee)^\#$, or equivalently $L_{S,T}(0, F/k) \in \mathrm{pAnn}_{\mathbb{Z}[G]}(\mathrm{Pic}^{T,0}(F))$. To recover the Brumer-Stark Conjecture from the latter containment one need only note that if G is abelian, then $\mathrm{pAnn}_{\mathbb{Z}[G]}(\mathrm{Pic}^{T,0}(F)) = \mathrm{Ann}_{\mathbb{Z}[G]}(\mathrm{Pic}^{T,0}(F))$.

Remark 6.5. If $\widehat{G}_{(a),S} = \widehat{G}$ and $S^a_{\min} \neq S$, then Remark 3.2(i) implies $\mathfrak{m}^a_{S,T}(F/k) = \xi(\mathbb{Z}[G])$ contains 1 and so can be omitted from the inclusion in Theorem 6.3.

Remark 6.6. One can also combine the known validity of Conjecture 4.3(i) in this case with the argument of Lemma 6.1 to give an explicit formula for $(1-e_v+e_{\mathbf{1}})e_{(a),S} \cdot \mathrm{Fit}^a_G(\Pi_{F/k,S,T})^\#$ in terms of $L^a_{S,T}(0, F/k)$ multiplied by a set of suitable $\mathscr{L}$-invariants. Since the derivation of such formulas is straightforward we leave details to the reader.

6.3. CM fields. In this section we extend results of the first author in [3] concerning the arithmetic properties of the values at zero of derivatives of p-adic Artin L-series.

To do this we assume k to be totally real and F a Galois CM extension of k. We write τ for the complex conjugation in G and for each G-module M we write $M^\pm$ for the submodule comprising elements upon which τ acts as multiplication by ± 1.

We fix an embedding $\mathbb{Q}^c \to \mathbb{C}_p$ and identify $\widehat{G}$ with the set of irreducible $\mathbb{C}_p$-valued characters of G. For each non-negative integer a we write $\widehat{G}^{\pm}_{a,S}$ and $\widehat{G}^{\pm}_{(a),S}$ for the subsets of $\widehat{G}_{a,S}$ and $\widehat{G}_{(a),S}$ comprising characters ψ for which one has $\psi(\tau) = \pm\psi(1)$.

We also write $\phi_{F,\Sigma} : \mathcal{O}^{\times,-}_{F,\Sigma} \to Y^-_{F,\Sigma}$ for the homomorphism of G-modules that sends each ϵ to $\sum_w \mathrm{ord}_w(\epsilon)\cdot w$ where w runs over all non-archimedean places of F. The scalar extension $\mathbb{Q}_p \otimes_{\mathbb{Z}} \phi_{F,\Sigma}$ is bijective and so for any homomorphism θ in $\mathrm{Hom}_{\mathbb{Z}_p[G]}(\mathcal{O}^{\times,-}_{F,\Sigma,p}, Y^-_{F,\Sigma,p})$ we can define a $\zeta(\mathbb{Q}_p[G])$-valued $\mathscr{L}$-invariant by setting

$$\mathscr{L}_p(\theta) := \mathrm{Nrd}_{\mathbb{Q}_p[G]}(\theta \circ (\mathbb{Q}_p \otimes_{\mathbb{Z}} \phi_{F,\Sigma})^{-1}).$$

We also write $\lambda_{F,\Sigma,p} : \mathcal{O}^{\times,-}_{F,\Sigma,p} \to Y^-_{F,\Sigma,p}$ for the homomorphism of $\mathbb{Z}_p[G]$-modules sending each u in $\mathcal{O}^{\times,-}_{F,\Sigma}$ to $\sum_{w\in\Sigma_F} \log_p||u||_{w,p}\cdot w$, where $||\cdot||_{w,p}$ is the local p-adic absolute value on $F_w^\times$ defined by Gross in [11, §1].

For each ψ in $\widehat{G}^+$ we write $L_{p,S,T}(\psi, s)$ for the T-modified S-truncated Deligne-Ribet p-adic Artin L-series of ψ (as discussed by Greenberg in [10]). We also write ω_k for the Teichmüller character $G_k \to \mathbb{Z}_p^\times$ and then for each non-negative integer a we define a $\zeta(\mathbb{C}_p[G])$-valued function by setting

$$L^a_{p,S,T}(s, F/k) := \sum_{\psi\in\widehat{G}^-_{(a),S}} e_\psi \cdot s^{-\psi(1)a} L_{p,S,T}(s, \check{\psi}\omega_k).$$

We abbreviate $L^0_{p,S,T}(s, F/k)$ to $L_{p,S,T}(s, F/k)$ and observe that this function constitutes a natural $\zeta(\mathbb{C}_p[G])$-valued (T-modified, S-truncated) p-adic Artin L-series for F/k.

We write $r_p = r_{p,F/k}$ for the number of p-adic places in k that split completely in F/k and $\mu_p(F)$ for the Iwasawa-theoretic μ-invariant of the cyclotomic $\mathbb{Z}_p$-extension F^{cyc} of F.

Theorem 6.7. *For each non-negative integer a the function $L^a_{p,S,T}(s, F/k)$ is holomorphic at $s = 0$. If $a < r_p$, then $L^a_{p,S,T}(0, F/k) = 0$.*

In addition, if $a \geq r_p$ and either p is prime to $[F : k]$ or $\mu_p(F)$ vanishes, then the following claims are valid.

(i) *For any element x of $\zeta(\mathbb{Q}[G])$ with $x \cdot e_{(a),S} \in \xi(\mathbb{Z}[G])$ one has*

$$x \cdot L^a_{p,S,T}(0, F/k) \in \mathscr{L}_p(\lambda^{\mathrm{Gr}}_{F,S}) \cdot \mathrm{Fit}^{a-r_p}_G(\mathrm{Cl}^T(F)^\vee)^{\#}_p.$$

In particular, one has $L^{r_p}_{p,S,T}(0, F/k) \in \mathscr{L}_p(\lambda^{\mathrm{Gr}}_{F,S}) \cdot \mathrm{pAnn}_{\mathbb{Z}[G]}(\mathrm{Cl}^T(F))_p$.

(ii) *For each I in $\wp^*_a(S)$ one has*

$$L^a_{p,S,T}(0, F/k) \cdot T_I \in \mathscr{L}_p(\lambda^{\mathrm{Gr}}_{F,S}) \cdot \mathrm{pAnn}_{\mathbb{Z}[G]}(\mathrm{Cl}^T_{S\setminus I}(F))_p$$

with $T_I := \sum_{g\in N_I} g$.

Proof. The first assertion is valid if for ψ in $\widehat{G}^-_{(a),S}$ the function $L_{p,S,T}(s, \check{\psi}\omega_k)$ vanishes at $s = 0$ to order at least $\psi(1)a$. This is true since [3, Th. 5.2(i)] combines with the equality (9) to imply $L_{p,S,T}(s, \check{\psi}\omega_k)$ vanishes at $s = 0$ to order at least $\psi(1)\cdot r_S(\psi) \geq \psi(1)\cdot|S_\psi| \geq \psi(1)\cdot a$.

In addition, if $a < r_p$, then $\widehat{G}^-_{(a),S} = \widehat{G}^-_{(r_p),S}$ and so the same argument also shows that $L^a_{p,S,T}(0, F/k) = 0$.

Next we note that $\mathrm{Sel}^T_{S_\infty}(F)^-$ is a quotient of $\mathrm{Sel}^T_{S'}(F)^-$ and that the exact sequence (10) combines with the vanishing of X^-_{F,S_∞} to imply $\mathrm{Cl}^T(F)^{\vee,-}$ is equal to $\mathrm{Sel}^T_{S_\infty}(F)^-$. This shows it is enough to prove claim (i) after replacing $\mathrm{Cl}^T(F)^\vee$ by $\mathrm{Sel}^T_{S'}(F)^-$.

To do this we write S_p^{sp} for the set of p-adic places that split completely in F/k (so $r_p = |S_p^{\mathrm{sp}}|$) and set $S' := S \setminus S^{\mathrm{sp}}$ and $a' := a - r_p$. We note $\widehat{G}^-_{a,S} = \widehat{G}^-_{a',S'}$, $\widehat{G}^-_{(a),S} = \widehat{G}^-_{(a'),S'}$ and that the assignment $I \mapsto I'$ induces a bijection between $\wp^*_a(S)$ and $\wp^*_{a'}(S')$ so that $e_I = e_{I'}$ and $e_{(a),S} = e_{(a'),S'}$.

In particular, since $S_\infty \subseteq S'$ we can apply Lemma 3.6 to the data $(F/k, S', T)$ and with v chosen to be an element of S_∞ to deduce that there exists a locally-quadratic presentation Π' of the G-module $\mathrm{Sel}^T_{S'}(F)^{\mathrm{tr}}$ such that for every I in $\wp^*_a(S)$ one has $e_I \cdot \mathrm{Fit}^a_G(\Pi')_p \subseteq e_{(a),S}\mathrm{Fit}^a_G(\Pi')^-_p$ and hence $x \cdot e_I \cdot \mathrm{Fit}^a_G(\Pi')_p \subseteq \mathrm{Fit}^a_G(\Pi')_p$ for any element x with $x \cdot e_{(a),S} \in \xi(\mathbb{Z}[G])$.

To prove claim (i) is thus enough to show, under the given hypotheses, $L^a_{p,S,T}(0, F/k) \cdot e_I$ belongs to $\mathscr{L}_p(\lambda^{\mathrm{Gr}}_{F,S}) \cdot e_I \cdot \mathrm{Fit}^{a-r_p}_G(\Pi')_p$. But, after fixing such an I and setting $H := N_I$, $E := F^I$ and $\Gamma := G/N_I = \mathrm{Gal}(E/k)$, then the identification $\zeta(\mathbb{Q}_p[G])e_H = \zeta(\mathbb{Q}_p[\Gamma])$ gives identifications $L^a_{p,S,T}(0, F/k) \cdot e_H = L^a_{p,S,T}(0, E/k)$, $\mathscr{L}_p(\lambda^{\mathrm{Gr}}_{F,S}) \cdot e_H = \mathscr{L}_p(\lambda^{\mathrm{Gr}}_{F^E,S})$ and $e_H \cdot \mathrm{Fit}^{a-r_p}_G(\Pi')_p = \mathrm{Fit}^{a-r_p}_{\mathbb{Z}_p[\Gamma]}(H_0(H, \Pi'))_p$ and so claim (i) will follow if one can show that

$$L^a_{p,S,T}(0, E/k) \in \mathscr{L}_p(\lambda^{\mathrm{Gr}}_{E,S}) \cdot \mathrm{Fit}^{a-r_p}_{\mathbb{Z}_p[\Gamma]}(M_{p,H}) \tag{34}$$

where M_p is the matrix of Π'_p with respect to the standard basis of $\mathbb{Z}_p[G]^{d_v}$.

We next combine the equality of [3, (42)] with that proved just before [3, (40)] (the proof of which relies crucially on the validity, modulo the assumption that either p is prime to $[F : k]$ or $\mu_p(F)$ vanishes, of the main conjecture of non-commutative Iwasawa theory for F^{cyc}/k, as proved independently by Ritter and Weiss [22] and Kakde [16]). In this way we deduce that

$$L^a_{p,S,T}(0, E/k) = u \cdot \mathscr{L}_p(\lambda^{\mathrm{Gr}}_{E,S}) \cdot \mathrm{Nrd}_{\mathbb{Q}_p[\Gamma]}(M^{I\setminus\Sigma}_{p,H})$$

where u is a unit of $\xi(\mathbb{Z}_p[\Gamma])$ and $M^{I\setminus\Sigma}_{p,H}$ is the matrix constructed from $M_{p,H}$ by the method of Lemma 3.6(vi).

This equality directly implies the required containment (34) since Lemma 3.6(v) implies that all columns of $M_{p,H}$ that correspond to places in $I \setminus \Sigma$ are zero and hence that $\mathrm{Nrd}_{\mathbb{Q}_p[\Gamma]}(M^{I\setminus\Sigma}_{p,H})$ belongs to $\mathrm{Fit}^{a-r_p}_{\mathbb{Z}_p[\Gamma]}(M_{p,H})$.

Turning to claim (ii) we note (10) identifies $\mathrm{Cl}^T_{S\setminus I}(F)^{\vee,-}$ with a submodule of $\mathrm{Sel}^T_{S\setminus I}(F)^-$ and hence that it is enough to prove the assertion of claim (ii) after replacing the term $\mathrm{pAnn}_{\mathbb{Z}[G]}(\mathrm{Cl}^T_{S\setminus I}(F))_p = \mathrm{pAnn}_{\mathbb{Z}[G]}(\mathrm{Cl}^T_{S\setminus I}(F)^\vee)^\#_p$ by $\mathrm{pAnn}_{\mathbb{Z}[G]}(\mathrm{Sel}^T_{S\setminus I}(F)^-)^\#$.

To do this we note Lemma 3.6(vi) implies $M^{I\setminus\Sigma}_{p,H}$ corresponds to a presentation of the $\mathbb{Z}_p[\Gamma]$-module $\mathrm{Sel}^T_{S\setminus I}(E)^{\mathrm{tr}}_p$, the transpose of which is a presentation of $\mathrm{Sel}^T_{S\setminus I}(E)_p$, and hence that $\mathrm{Nrd}_{\mathbb{Q}_p[\Gamma]}(M^{I\setminus\Sigma}_{p,H})$ belongs to

$$\mathrm{Fit}^0_{\mathbb{Z}_p[\Gamma]}(\mathrm{Sel}^T_{S\setminus I}(E)^{\mathrm{tr}}_p) = \mathrm{Fit}^0_{\mathbb{Z}_p[\Gamma]}(\mathrm{Sel}^T_{S\setminus I}(E)_p)^\# \subseteq \mathrm{pAnn}_{\mathbb{Z}[\Gamma]}(\mathrm{Sel}^T_{S\setminus I}(E)_p)^\#.$$

Given this, claim (ii) follows from (34) and the fact that T_I induces a homomorphism of G-modules $\mathrm{Cl}^T_{S\setminus I}(F) \to \mathrm{Cl}^T_{S\setminus I}(E)$ and hence an inclusion $\mathrm{pAnn}_{\mathbb{Z}[\Gamma]}(\mathrm{Sel}^T_{S\setminus I}(E)_p)^{\#} \cdot T_I \subseteq \mathrm{pAnn}_{\mathbb{Z}[G]}(\mathrm{Sel}^T_{S\setminus I}(F)^-)^{\#}$. □

Remark 6.8.
(i) The final assertion of Theorem 6.7(i) recovers the result of [3, Cor. 3.10(iii)].
(ii) If $\widehat{G}_{(a),S} = \widehat{G}$, then $e_{(a),S} = 1$ and one can take $x = 1$ in Theorem 6.7(i).
(iii) If G is abelian, then $L^a_{p,S,T}(s, F/k)$ is equal to $(a!)^{-1} \cdot (\mathrm{d}/\mathrm{d}s)^a \left(L_{p,S,T}(s, F/k)\right)$ and also $\mathrm{pAnn}_{\mathbb{Z}[G]}(\mathrm{Cl}^T(F))$ is the annihilator ideal of $\mathrm{Cl}^T(F)$ in $\mathbb{Z}[G]$.
(iv) In [13] Iwasawa conjectures that $\mu_p(F)$ should always vanish. This is known to be true if F is a p-power degree Galois extension of a field that is abelian over $\mathbb{Q}$.

6.4. General number fields. The predictions included in the following result constitute a significant strengthening of the main conjecture formulated by the first author in [2].

We use the element $c_{a,S}$ of $\zeta(\mathbb{Q}[G])$ defined just before Theorem 6.3.

Proposition 6.9. *Fix a finite Galois extension of number fields F/k for which Conjecture 4.3 is valid and a homomorphism θ in $\bigcap_{I\in\wp^*_a(S)} \mathrm{Hom}^I_G(\mathcal{O}^\times_{F,S,T}, X_{F,S})$.*

(i) *For any integer a with $0 \le a < |S|$ one has*
$$L^a_{S,T}(0, F/k) \cdot \mathscr{L}(\theta) \cdot \mathfrak{m}^a_{S,T}(F/k) \subseteq c_{a,S} \cdot \mathrm{Fit}^a_G(\mathrm{Sel}^T_{S_\infty}(F))^{\#}.$$
(ii) *For any integer a with $0 \le a < |S|$, any I in $\wp^*_a(S)$ and any $v \in S \setminus I$ one has*
$$L^a_{S,T}(0, F/k) \cdot \mathscr{L}(\theta) \cdot T_I \in \mathrm{pAnn}_{\mathbb{Z}[G]}(\mathrm{Cl}^T_{S(I,v)}(F))$$
with $T_I := \sum_{g\in N_I} g$ and $S(I, v) := S_\infty \cup I \cup \{v\}$.

Proof. By the same argument as used in the proof of Theorem 6.3 it is enough to prove the inclusion of claim (i) after replacing $L^a_{S,T}(0, F/k)$ by $e_I \cdot L^a_{S,T}(0, F/k)$ for $I \in \wp^*_a(S)$ and then omitting the term $c_{a,S}$.

The resulting inclusion then follows by combining the assumed validity of Conjecture 4.3(ii) with Lemma 6.1 and the fact that $\mathrm{Sel}^T_{S_\infty}(F)$ is isomorphic to a quotient of $\mathrm{Sel}^T_S(F)$ (as a consequence of the exact triangle in [4, Prop. 2.4(ii)]).

Claim (ii) follows directly upon combining Conjecture 4.3(iii) with Lemma 6.1. □

Remark 6.10.
(i) The exact sequence (2) implies $\mathrm{Fit}^a_G(\mathrm{Sel}^T_{S_\infty}(F)) \subseteq \mathrm{Fit}^a_G(\mathrm{Hom}_{\mathbb{Z}}(\mathcal{O}^\times_{F,S_\infty,T}, \mathbb{Z}))$.
(ii) One can also combine Conjecture 4.3(i) with the argument of Lemma 6.1 to derive an explicit conjectural formula for $(1 - e_v + e_{\mathbf{1}})e_{(a),S} \cdot \mathrm{Fit}^a_G(\Pi_{F/k,S,T})^{\#}$ in terms of $L^a_{S,T}(0, F/k)$ multiplied by a set of suitable $\mathscr{L}$-invariants.
(iii) Assume now that $a = r$ (so $a < |S|$) and $S^a_{\min}$ comprises a places that split completely in F/k. Then $\wp^*_a(S) = \{I\}$ with $I := S^a_{\min}$, the subgroup N_I is trivial so $T_I = 1$, $\mathfrak{m}^a_{S,T}(F/k) = \xi(\mathbb{Z}[G])$ (by Remark 3.2(i)) and so contains 1 and $\mathrm{Hom}^I_G(\mathcal{O}^\times_{F,S,T}, \mathbb{Z}[G]) = \mathrm{Hom}_G(\mathcal{O}^\times_{F,S,T}, \mathbb{Z}[G])$ (by Remark 6.2). Hence, in this case, Proposition 6.9(ii) predicts that $L^r_{S,T}(0, F/k) \cdot \mathscr{L}(\theta) \in \mathrm{pAnn}_{\mathbb{Z}[G]}(\mathrm{Cl}^T_{S(I,v)}(F))$ for all $\theta \in \mathrm{Hom}_G(\mathcal{O}^\times_{F,S,T}, \mathbb{Z}[G])$ and thus refines the central conjecture of [2]. In particular, the argument of [2, Prop. 3.5.1 and Rem.

3.5.2(ii)] shows this prediction (in the case $a = r = 0$) refines the 'non-abelian Brumer-Stark Conjecture' formulated by Nickel in [20]. At present, however, we are not aware of the precise connection between this case of Proposition 6.9(ii) and the 'Galois-Brumer-Stark Conjecture' formulated by Dejou and Roblot in [7].

Appendix A. Higher non-commutative Fitting invariants

For the convenience of the reader, in this section we quickly recall the definition and relevant properties of the higher non-commutative Fitting invariants that are introduced by Sano and the first author in [6].

A.1. We fix a Dedekind domain R of characteristic zero and write F for its quotient field. We fix a finite group Γ and use the R-order $\xi(R[\Gamma])$ in $\zeta(F[\Gamma])$ discussed in §2.2.

A.1.1. Let M be a matrix in $\mathrm{M}_{d\times d'}(F[\Gamma])$ with $d \geq d'$. Then for any integer t with $0 \leq t \leq d'$ and any $\varphi = (\varphi_i)_{1\leq i\leq t}$ in $\mathrm{Hom}_\Gamma(R[\Gamma]^d, R[\Gamma])^t$ we write $\mathrm{Min}^{d'}_\varphi(M)$ for the set of all $d' \times d'$ minors of the matrices $M(J, \varphi)$ that are obtained from M by choosing any t-tuple of integers $J = \{i_1, i_2, \cdots, i_t\}$ with $1 \leq i_1 < i_2 < \cdots < i_t \leq d'$, and setting

$$M(J,\varphi)_{ij} := \begin{cases} \varphi_a(b_i), & \text{if } j = i_a \text{ with } 1 \leq a \leq t \\ M_{ij}, & \text{otherwise.} \end{cases} \tag{35}$$

For any non-negative integer a the 'a-th (non-commutative) Fitting invariant of M' is defined to be the ideal of $\xi(R[\Gamma])$ obtained by setting

$$\mathrm{Fit}^a_{R[\Gamma]}(M) := \xi(R[\Gamma]) \cdot \{\mathrm{Nrd}_{F[\Gamma]}(N) : N \in \mathrm{Min}^{d'}_\varphi(M),\ \varphi \in \mathrm{Hom}_{R[\Gamma]}(R[\Gamma]^d, R[\Gamma])^t,\ t \leq a\}.$$

A.1.2. A 'free presentation' Π of a finitely generated $R[\Gamma]$-module Z is an exact sequence of $R[\Gamma]$-modules of the form

$$R[\Gamma]^d \xrightarrow{\theta} R[\Gamma]^{d'} \to Z \to 0.$$

The a-th Fitting invariant $\mathrm{Fit}^a_{R[\Gamma]}(\Pi)$ of Π is defined to be $\mathrm{Fit}^a_{R[\Gamma]}(M_\theta)$ with M_θ the matrix of θ with respect to the standard bases of $R[\Gamma]^d$ and $R[\Gamma]^{d'}$.

A.1.3. A finitely generated $R[\Gamma]$-module P is said to be 'locally-free' if for all prime ideals $\mathfrak{p}$ of R the localisation $P_{(\mathfrak{p})}$ of P at $\mathfrak{p}$ is a free $R_{(\mathfrak{p})}[\Gamma]$-module. (If no prime divisor of $|\Gamma|$ is invertible in R, then a result of Swan implies that a finitely generated $R[\Gamma]$-module is locally-free if and only if it is projective).

A 'locally-free presentation' Π of a finitely generated $R[\Gamma]$-module Z is an exact sequence of $R[\Gamma]$-modules of the form

$$P' \xrightarrow{\theta} P \to Z \to 0 \tag{36}$$

in which P' and P are both locally-free. Such a presentation is said to be 'locally-quadratic' if the localised modules $P'_{(\mathfrak{p})}$ and $P_{(\mathfrak{p})}$ are free of the same rank for any (and therefore every) prime ideal $\mathfrak{p}$.

By localising the sequence Π at a prime ideal $\mathfrak{p}$ of R one obtains a free resolution $\Pi_{(\mathfrak{p})}$ of the $R_{(\mathfrak{p})}[\Gamma]$-module $Z_{(\mathfrak{p})}$ and one then defines the a-th Fitting invariant of Π by setting

$$\mathrm{Fit}^a_{R[\Gamma]}(\Pi) := \bigcap\nolimits_{\mathfrak{p}} \mathrm{Fit}^a_{R_{(\mathfrak{p})}[\Gamma]}(\Pi_{(\mathfrak{p})}),$$

where the intersection is over all prime ideals $\mathfrak{p}$ of R and takes place in $\zeta(F[\Gamma])$. In the case that $R = \mathbb{Z}$ we usually abbreviate $\mathrm{Fit}^a_{R[\Gamma]}(\Pi)$ to $\mathrm{Fit}^a_{\Gamma}(\Pi)$.

This notion extends the definition of non-commutative Fitting invariant of presentations given (in the case $a = 0$) by Nickel in [19].

A.1.4. The 'transpose' Π^{tr} of Π is the exact sequence

$$\mathrm{Hom}_R(P, R) \xrightarrow{\mathrm{Hom}_R(\theta,R)} \mathrm{Hom}_R(P', R) \to \mathrm{cok}(\mathrm{Hom}_R(\theta, R)) \to 0$$

(where the linear duals are endowed with the contragredient action of Γ) and is a locally-free presentation of $\mathrm{cok}(\mathrm{Hom}_R(\theta, R))$. The presentation Π^{tr} is locally-quadratic if and only if Π is locally-quadratic.

It is shown in [6] that for each non-negative integer a one has

$$\mathrm{Fit}^a_{R[\Gamma]}(\Pi^{\mathrm{tr}}) = \mathrm{Fit}^a_{R[\Gamma]}(\Pi)^{\#}$$

where $x \mapsto x^{\#}$ denotes the involution of $\zeta(F[\Gamma])$ induced by restricting the F-linear anti-involution of $F[\Gamma]$ that inverts elements of Γ. If F is a subset of $\mathbb{C}$, then $\zeta(F[G])$ identifies with a subalgebra of $\prod_{\widehat{G}} \mathbb{C}$ and one has $((x_\psi)_\psi)^{\#} = (x_{\check{\psi}})_\psi$.

For any normal subgroup Δ of Γ the Δ-coinvariants $H_0(\Delta, \Pi)$ of Π is the exact sequence of $R[\Gamma/\Delta]$-modules

$$H_0(\Delta, P') \xrightarrow{H_0(\Delta,\theta)} H_0(\Delta, P) \to H_0(\Delta, Z) \to 0$$

and is a locally-free presentation of $H_0(\Delta, Z)$. If Π is locally-quadratic, then $H_0(\Delta, \Pi)$ is a locally-quadratic presentation of $R[\Gamma/\Delta]$-modules.

With respect to the natural identification $e_\Delta \cdot \zeta(F[\Gamma]) = \zeta(F[\Gamma/\Delta])$ one has

$$e_\Delta \cdot \mathrm{Fit}^a_{R[\Gamma]}(\Pi) = \mathrm{Fit}^a_{R[\Gamma/\Delta]}(H_0(\Delta, \Pi)).$$

A.1.5. The 'a-th Fitting invariant' of a finitely generated $R[\Gamma]$-module Z is defined to be the ideal of $\xi(R[\Gamma])$ obtained by setting

$$\mathrm{Fit}^a_{R[\Gamma]}(Z) := \sum\nolimits_{\Pi} \mathrm{Fit}^a_{R[\Gamma]}(\Pi),$$

where in the sum Π runs over all locally-free presentations of finitely generated $R[\Gamma]$-modules Z' for which there exists a surjective homomorphism of $R[\Gamma]$-modules of the form $Z' \to Z$. In the case that $R = \mathbb{Z}$ we usually abbreviate $\mathrm{Fit}^a_{R[\Gamma]}(Z)$ to $\mathrm{Fit}^a_{\Gamma}(Z)$.

If Γ is abelian (in which case $\xi(R[\Gamma]) = R[\Gamma]$), then $\mathrm{Fit}^a_{R[\Gamma]}(Z)$ coincides with the a-th Fitting ideal of the $R[\Gamma]$-module Z, as discussed by Northcott [21].

In the general case, the invariants $\mathrm{Fit}^a_{R[\Gamma]}(Z)$ form an increasing sequence of ideals of $\xi(R[\Gamma])$ with $\mathrm{Fit}^a_{R[\Gamma]}(Z) = \xi(R[\Gamma])$ for any large enough a and $\mathrm{Fit}^0_{R[\Gamma]}(Z)$ contained in the pre-annihilator module $\mathrm{pAnn}_{R[\Gamma]}(Z)$ defined in §2.2.

Further basic properties of these invariants are established in [6].

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