ON SELMER GROUPS AND REFINED STARK CONJECTURES

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Abstract. We prove that the higher Fitting ideals of Selmer groups of the multiplicative group over abelian extensions of global fields admit a natural direct sum decomposition. Using this result, we then formulate an explicit higher-order abelian Stark conjecture that both refines and extends the existing theory of such conjectures in two significant ways since it deals with L-series that do not necessarily have 'minimal' order of vanishing at zero and also treats on an equal footing the 'boundary case' that has been excluded from all previous conjectures in this area. We show that our conjecture is best possible, establish a direct link between it and the Tamagawa number conjecture of Bloch and Kato and provide strong supporting evidence, including giving a full proof of the conjecture for all abelian extensions of $\mathbb{Q}$ and for all abelian extensions of global function fields.

1. Introduction

Let F/k be a finite abelian extension of global fields of group G. Let S be a finite non-empty set of places of k containing the set S_∞ of archimedean places (if any) and also all places that ramify in F.

Then for any finite set T of places of k that is disjoint from S the 'S-relative T-trivialized integral Selmer group' $\mathrm{Sel}_S^T(F)$ for the multiplicative group $\mathbb{G}_m$ over F is defined to be the cokernel of a canonical homomorphism of G-modules

$$\prod_w \mathbb{Z} \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(F_T^\times, \mathbb{Z})$$

(see [2, Def. 2.1] where the notation $\mathcal{S}_{S,T}(\mathbb{G}_{m/F})$ is used). Here in the product w runs over all places of F that do not lie above places in $S \cup T$, $F_T^\times$ is the subgroup of $F^\times$ comprising elements u for which $u-1$ has a strictly positive valuation at each place above T and the unlabeled arrow sends each element $(x_w)_w$ to the map $(u \mapsto \sum \mathrm{ord}_w(u)x_w)$ with ord_w the normalised additive valuation at w.

This group is a natural analogue for $\mathbb{G}_m$ of the integral Selmer groups of abelian varieties that are defined by Mazur and Tate in [12] and, in particular, lies in a canonical exact sequence of G-modules of the form

$$0 \to \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Cl}_S^T(F), \mathbb{Q}/\mathbb{Z}) \to \mathrm{Sel}_S^T(F) \to \mathrm{Hom}_{\mathbb{Z}}(\mathcal{O}_{F,S,T}^\times, \mathbb{Z}) \to 0 \tag{1}$$

(see [2, Prop. 2.2]). Here $\mathrm{Cl}_S^T(F)$ is the ray class group modulo the product of all places of F above T of the subring $\mathcal{O}_{F,S}$ of F comprising elements that are integral at all places outside

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S and $\mathcal{O}_{F,S,T}^\times$ is the group $F_T^\times \cap \mathcal{O}_{F,S}^\times$ and both duals are endowed with the contragredient action of G.

In this article we shall prove that the higher Fitting ideals, in the sense discussed by Northcott in [13], of the G-module $\mathrm{Sel}_S^T(F)$ admit a natural direct sum decomposition. For details of this result see Theorem 2.1 and its proof in §3.

By using this purely algebraic observation we shall then formulate (in Conjectures 4.3 and 4.7) certain explicit predictions concerning the detailed arithmetic properties of canonical 'Stark elements of arbitrary rank' that we introduce in §4.1.

We recall that there is a very extensive existing theory of (higher-order) refined abelian Stark conjectures that are due to Stark, to Rubin and to Popescu amongst others and are nicely surveyed, for example, by Vallières in [16, §1].

Our conjecture specialises to give refined versions of all of these earlier conjectures and, in addition, now extends the theory in two distinct ways since it doesn't restrict to those L-series for complex characters of G that have 'minimal' order of vanishing at zero and also treats on an equal footing the natural 'boundary case' that has been excluded from previous conjectures in this area.

By means of explicit examples, we show our conjecture is, in a natural sense, the best possible and also that it implies that Rubin-Stark elements of subfields of F/k have in general strictly stronger integrality properties than those that are predicted by the relevant cases of the refined Rubin-Stark Conjecture discussed by Kurihara, Sano and the first author in [2] (and hence, a fortiori, by the Rubin-Stark Conjecture itself).

Finally, in §5, we combine Theorem 2.1 with an extension of the approach developed in [2] and by the second author in [10] to establish a direct link between our conjecture and the general formalism of Tamagawa number conjectures that was introduced by Bloch and Kato in [1]. In this way we shall obtain strong supporting evidence for our conjecture, including giving a full (and unconditional) proof of it for all abelian extensions of $\mathbb{Q}$ and for all abelian extensions of global function fields.

In a subsequent article we will explain how the theory, and conjectures, developed here can be naturally extended to the setting of non-abelian Galois extensions of global fields.

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2. Statement of the main algebraic result

In this section we continue to use the notation introduced above. In addition, we write $\widehat{G}$ for the group of homomorphisms $G \to \mathbb{C}^\times$ and $\mathbf{1}$ for the trivial element of $\widehat{G}$.

2.1. We first introduce some convenient notation regarding character sets and idempotents.

For each non-negative integer a we write $\widehat{G}'_{a,S}$ for the subset of $\widehat{G} \setminus \{\mathbf{1}\}$ comprising those homomorphisms χ for which the set

$$S_\chi := \{v \in S : G_v \subseteq \ker(\chi)\}$$

has cardinality a, where we write G_v for the decomposition subgroup in G of a place v of k.

We then set

$$\widehat{G}_{a,S} := \begin{cases} \widehat{G}'_{a,S} \cup \{\mathbf{1}\}, & \text{if } a = |S|-1, \\ \widehat{G}'_{a,S}, & \text{if } a \neq |S|-1, \end{cases}$$

write $\widehat{G}_{(a),S}$ for the union of $\widehat{G}_{a',S}$ for $a' \geq a$ and define idempotents of $\mathbb{Q}[G]$ by setting

$$e_{a,S} := \sum_{\chi \in \widehat{G}_{a,S}} e_\chi \quad \text{and} \quad e_{(a),S} := \sum_{\chi \in \widehat{G}_{(a),S}} e_\chi = \sum_{a' \geq a} e_{a',S}$$

where for χ in $\widehat{G}$ we write e_χ for the idempotent $|G|^{-1} \sum_{g \in G} \chi(g)^{-1} g$ of $\mathbb{C}[G]$.

For each such a we also define a subset of S by setting

$$S^a_{\min} := \begin{cases} \bigcup_{\chi \in \widehat{G}'_{a,S}} S_\chi, & \text{if } G \text{ is not trivial}, \\ S \setminus \{v_*\}, & \text{if } G \text{ is trivial and } a = |S|-1, \\ \emptyset, & \text{if } G \text{ is trivial and } a \neq |S|-1, \end{cases}$$

where v_* is a fixed place in S (the choice of which will not matter in the sequel).

For any set Σ we write $\wp_a(\Sigma)$ for the set of subsets of Σ of cardinality a. We abbreviate $\wp_a(S^a_{\min})$ to $\wp^*_a(S)$ and for v in S we set

$$\wp_a(S, v) := \{I \in \wp^*_a(S) : v \notin I\}$$

and $e_v := e_{G_v}$ where for each subgroup H of G we write e_H for the idempotent $|H|^{-1} \sum_{g \in H} g$ of $\mathbb{Q}[G]$.

Finally, for I in $\wp^*_a(S)$ we define an idempotent of $\mathbb{Q}[G]$ by setting

$$e_I := e_{\mathbf{1}} + \sum_{\chi} e_\chi \tag{2}$$

where the sum runs over the set $\{\chi \in \widehat{G}'_{a,S} : S_\chi = I\}$.

2.2. We can now state our main algebraic result.

Theorem 2.1. *Assume that the group $\mathcal{O}^\times_{F,S,T}$ is torsion-free. Then for each non-negative integer a and each place v in S there is a direct sum decomposition*

$$(1-e_v+e_{\mathbf{1}})e_{(a),S} \cdot \mathrm{Fit}^a_G(\mathrm{Sel}^T_S(F)) = e_{\mathbf{1}} \cdot |c^a_{S,v}| \cdot \mathrm{Fit}^a_G(\mathrm{Sel}^T_S(F)) \oplus \bigoplus_{I \in \wp_a(S,v)} e_I \cdot \mathrm{Fit}^a_G(\mathrm{Sel}^T_S(F)) \tag{3}$$

where the integer $c^a_{S,v}$ is defined to be

$$\begin{cases} |\{v\} \setminus S^a_{\min}| - \min\{|S^a_{\min}|, |S \setminus S^a_{\min}|\} - \delta_{a0}, & \textit{if } a < |S|, \\ 0, & \textit{otherwise.} \end{cases}$$

To discuss consequences of this result we briefly write $\mathcal{F}_a$ in place of $\mathrm{Fit}^a_G(\mathrm{Sel}^T_S(F))$ for each non-negative integer a. For each place v in S we then define $\mathfrak{n}^a_v = \mathfrak{n}^a_v(F/k)$ and $\mathfrak{m}^a_v = \mathfrak{m}^a_v(F/k)$ to be the ideals of $\mathbb{Z}[G]$ comprising elements x with

$$x \cdot (1 - e_v + e_{\mathbf{1}})e_{(a),S} \cdot \mathcal{F}_a \subseteq \mathbb{Z}[G] \quad \text{and} \quad x \cdot (1 - e_v + e_{\mathbf{1}})e_{(a),S} \cdot \mathcal{F}_a \subseteq \mathcal{F}_a$$

respectively.

Then Theorem 2.1 implies that for I in $\wp_a^*(S)$ one has $\mathfrak{n}_v^a \cdot e_I \cdot \mathcal{F}_a \subseteq \mathbb{Z}[G]$ and $\mathfrak{m}_v^a \cdot e_I \cdot \mathcal{F}_a \subseteq \mathcal{F}_a$ for all $v \in S \setminus I$ and hence also that $\mathfrak{n}_I^a \cdot e_I \cdot \mathcal{F}_a \subseteq \mathbb{Z}[G]$ and $\mathfrak{m}_I^a \cdot e_I \cdot \mathcal{F}_a \subseteq \mathcal{F}_a$ with $\mathfrak{n}_I^a := \sum_{v \in S \setminus I} \mathfrak{n}_v^a$ and $\mathfrak{m}_I^a := \sum_{v \in S \setminus I} \mathfrak{m}_v^a$.

Thus, if we define ideals of $\mathbb{Z}[G]$ by setting

$$\mathfrak{n}_{S,T}^a(F/k) := \bigcap_{I \in \wp_a^*(S)} \mathfrak{n}_I^a \quad \text{and} \quad \mathfrak{m}_{S,T}^a(F/k) := \bigcap_{I \in \wp_a^*(S)} \mathfrak{m}_I^a,$$

then Theorem 2.1 directly implies the following result.

Corollary 2.2. *Assume the notation and hypotheses of Theorem 2.1. Then for every set I in $\wp_a^*(S)$ one has both*

$$\mathfrak{n}_{S,T}^a(F/k) \cdot e_I \cdot \mathrm{Fit}_G^a(\mathrm{Sel}_S^T(F)) \subseteq \mathbb{Z}[G]$$

and

$$\mathfrak{m}_{S,T}^a(F/k) \cdot e_I \cdot \mathrm{Fit}_G^a(\mathrm{Sel}_S^T(F)) \subseteq \mathrm{Fit}_G^a(\mathrm{Sel}_S^T(F)).$$

Remark 2.3. If $a \geq |S|$, then $S \setminus I$ is empty for each I in $\wp_a^*(S)$ and so the ideals $\mathfrak{n}_{S,T}^a(F/k)$ and $\mathfrak{m}_{S,T}^a(F/k)$ are both zero. However, if $a < |S|$, then an explicit computation of $\mathfrak{n}_{S,T}^a(F/k)$ and $\mathfrak{m}_{S,T}^a(F/k)$ in any general setting would be rather involved since it requires some knowledge of the higher Fitting ideals of $\mathrm{Sel}_S^T(F)$. It is, nevertheless, straightforward in this case to construct non-zero elements of $\mathfrak{n}_{S,T}^a(F/k)$ and $\mathfrak{m}_{S,T}^a(F/k)$ in a purely combinatorial way since both $\mathfrak{n}_v^a(F/k)$ and $\mathfrak{m}_v^a(F/k)$ contains the lowest common multiple of the denominators of the coefficients of the element $(1 - e_v + e_{\mathbf{1}})e_{(a),S}$ of $\mathbb{Q}[G]$. This element is often close to being a generator of $\mathfrak{n}_v^a(F/k)$ and $\mathfrak{m}_v^a(F/k)$ (see, for instance, Examples 4.5 and 5.3 below) and in all cases allows one to make the results of Theorem 2.1 and Corollary 2.2 much more explicit. In particular, such computations, taken together with Remark 3.2(iii) and (iv) below, show that the inclusions of Corollary 2.2 are of interest since in many cases there exists an I in $\wp_a^*(S)$ for which neither $\mathfrak{n}_{S,T}^a(F/k) \cdot e_I$ or $\mathfrak{m}_{S,T}^a(F/k) \cdot e_I$ is contained in $\mathbb{Z}[G]$. This observation will later play a key role in our discussion of refined Stark conjectures.

3. The proof of Theorem 2.1

We assume throughout this section the notation and hypotheses of Theorem 2.1. For convenience we also set

$$\mathcal{F}_a := \mathrm{Fit}_G^a(\mathrm{Sel}_S^T(F))$$

for each non-negative integer a.

3.1. We start by observing that an explicit analysis of the functional equation of Artin L-series (as per [15, Chap. I, Prop. 3.4]) combines with the exact sequence (1) to imply that for each χ in $\widehat{G}$ the order of vanishing $r_S(\chi)$ at $z = 0$ of $L_S(\chi, z)$ satisfies

$$r_S(\chi) = \dim_{\mathbb{C}}(e_{\check{\chi}}(\mathbb{C} \otimes_{\mathbb{Z}} \mathrm{Sel}_S^T(F))) = \begin{cases} |S_{\check{\chi}}|, & \text{if } \chi \neq \mathbf{1}, \\ |S| - 1, & \text{if } \chi = \mathbf{1}. \end{cases} \tag{4}$$

where $\check{\chi}$ denotes the contragredient of χ.

We next record a result that will allow us to treat a special case of Theorem 2.1 (and also justifies observations made in Remark 3.2).

Lemma 3.1. *The following claims are valid for each non-negative integer a.*

(i) *For χ in $\widehat{G}_{(a),S} \setminus \widehat{G}_{a,S}$ the space $e_\chi(\mathbb{C} \otimes_{\mathbb{Z}} \mathcal{F}_a)$ vanishes.*
(ii) *If $|S| > a+1$, then $e_{\mathbf{1}} \cdot \mathcal{F}_a$ vanishes.*
(iii) *If $v \in S \setminus S^a_{\min}$, then $(e_v - e_{\mathbf{1}})e_{(a),S} \cdot \mathcal{F}_a$ vanishes.*

Proof. For χ in $\widehat{G}$, a finitely generated G-module Z and a non-negative integer a it is easy to see that the sublattice $e_\chi \cdot \mathrm{Fit}^a_G(Z)$ of $\mathbb{C}[G]$ will vanish whenever $\dim_{\mathbb{C}}(e_\chi(\mathbb{C} \otimes_{\mathbb{Z}} Z)) > a$.

Using this observation, claim (i) is true since (4) implies that if $a' > a$, then for each χ in $\widehat{G}_{a',S}$ one has $\dim_{\mathbb{C}}(e_\chi(\mathbb{C} \otimes_{\mathbb{Z}} \mathrm{Sel}^T_S(F))) = a' > a$. Claim (ii) is then an immediate consequence of claim (i) in the case $\chi = \mathbf{1}$ and $a' = |S| - 1$.

To derive claim (iii) from claim (i) it suffices to prove that if χ belongs to $\widehat{G}_{a,S} \setminus \{\mathbf{1}\}$, then G_v cannot be contained in $\ker(\chi)$ (and hence $e_\chi(e_v - e_{\mathbf{1}}) = 0$). This follows from the fact that, as v does not belong to $S^a_{\min}$, the inclusion $G_v \subseteq \ker(\chi)$ would imply that $|S_\chi| \geq 1 + a$. □

Remark 3.2. The above observations can be used to make the results of Theorem 2.1 and Corollary 2.2 more explicit. To explain this we write $r = r_S(F/k)$ for the minimum of the set $\{r_S(\chi) : \chi \in \widehat{G}\}$.

(i) The formula (4) implies both that $r \leq |S| - 1$ and that $\widehat{G}$ is equal to the union of $\widehat{G}_{a',S}$ for $a' \geq r$. The latter fact implies that $e_{(a),S} = 1$ for $a \leq r$ and therefore in such a case simplifies the computation of both $\mathfrak{n}^a_{S,T}(F/k)$ and $\mathfrak{m}^a_{S,T}(F/k)$.

(ii) By combining (4) with the result of Lemma 3.1(i) one can also check that both sides of (3) vanish if either $a < r$ or $a > |S|$. This shows that Theorem 2.1 is of interest only for integers a in the range $r \leq a \leq |S|$.

(iii) Assume $S \neq S^a_{\min}$. Then Lemma 3.1(iii) implies that $(e_v - e_{\mathbf{1}}) \cdot \mathrm{Fit}^a_G(\mathrm{Sel}^T_S(F))$ vanishes for each v in $S \setminus S^a_{\min}$. In particular, for each such v the left hand side of (3) is equal to $e_{(a),S} \cdot \mathrm{Fit}^a_G(\mathrm{Sel}^T_S(F))$, and hence, by remark (i), to $\mathrm{Fit}^r_G(\mathrm{Sel}^T_S(F))$ if $a = r$. In particular, in this case one has $\mathfrak{n}^r_v(F/k) = \mathfrak{m}^r_v(F/k) = \mathbb{Z}[G]$ and hence $\mathfrak{n}^r_{S,T}(F/k) = \mathfrak{m}^r_{S,T}(F/k) = \mathbb{Z}[G]$.

(iv) Assume $S = S^a_{\min}$ and $a = r$. This is the 'boundary case' that is identified by Emmons in [5, §5.4] and has hitherto been excluded from the formulation of refined abelian Stark conjectures (see, for example, [5, 6, 7, 8, 10, 16]).

We note first that in this case $|S| > r + 1$. To see this we note that $|S| \geq r + 1$ (by remark (i)) and, in addition, that if $|S| = r + 1$, then [6, Lem. 2.2] implies the subset S_{sp} of S comprising places that split completely in F has cardinality at least r. Further, if in this case $|S_{\mathrm{sp}}| = r$, then also $S_{\mathrm{sp}} = S^r_{\min}$ and hence $S^r_{\min} \neq S$, whilst if $|S_{\mathrm{sp}}| > r$, then $S_{\mathrm{sp}} = S$ and $S^r_{\min}$ is empty, unless G is trivial, when $S^r_{\min} = S \setminus \{v_*\}$ for some $v_* \in S$ and so in both cases $S^r_{\min} \neq S$.

Then, since $|S| > r + 1$, Lemma 3.1(ii) implies $e_{\mathbf{1}} \cdot \mathrm{Fit}^r_G(\mathrm{Sel}^T_S(F))$ vanishes. This in turn combines with the observation in Remark 2.3 to imply that for each I in $\wp^*_r(S)$ the ideals $\mathfrak{n}^r_I$ and $\mathfrak{m}^r_I$ each contain the greatest common divisor d_I of $|G_v|$ as v varies over $S \setminus I$ and can actually contain proper divisors of d_I depending on the structure of $\mathrm{Fit}^r_G(\mathrm{Sel}^T_S(F))$. In particular, in all cases both $\mathfrak{n}^r_{S,T}(F/k)$ and $\mathfrak{m}^r_{S,T}(F/k)$ contains the lowest common multiple of d_I for I in $\wp^*_r(S)$. For a concrete application of this observation see Examples 4.5 and 5.3 below.

3.2. By using Lemma 3.1 we can now quickly prove Theorem 2.1 in several special cases.

Proposition 3.3. *Theorem 2.1 is valid if $F = k$.*

Proof. In this case G is trivial so $\widehat{G} = \{\mathbf{1}\}$.

In particular, if $a < |S| - 1$, then Lemma 3.1(ii) implies $\mathcal{F}_a$ vanishes and so the equality (3) is valid trivially.

If $a \geq |S|$, then (4) implies $e_{(a),S} = 0$ and so the left hand side of (3) is zero. On the other hand, in this case $c^a_{S,v}$ is defined to be 0 and $S^a_{\min}$ to be empty and so the right hand side of (3) is also zero.

We assume finally that $a = |S| - 1$ and recall $S^a_{\min}$ is then defined to be $S \setminus \{v_*\}$ for a fixed place v_* in S. We also note that in this case (4) implies $(1 - e_v + e_{\mathbf{1}})e_{(a),S} = 1$ (for each v in S) so that the left hand side of (3) is equal to $\mathcal{F}_a$.

If $a = 0$, then $S = \{v_*\}$ so $S^a_{\min}$ is empty. This in turn implies $c^a_{S,v_*} = 1 - 0 - 1 = 0$ and $\wp_a(S, v_*) = \{I\}$ with $I := \emptyset$ and, since $e_\emptyset = e_{\mathbf{1}}$, this shows that the right hand side of (3) is equal to $e_\emptyset \mathcal{F}_a = \mathcal{F}_a$, as required.

It therefore only remains to consider the case $a = |S| - 1 > 0$. In this case $S^a_{\min}$ is not empty. In particular, if $v \neq v_*$, then $v \in S^a_{\min}$ so $c^a_{S,v} = 0 - 1 - 0 = -1$ and $\wp_a(S, v) = \emptyset$, whilst if $v = v_*$, then $c^a_{S,v} = 1 - 1 - 0 = 0$ and $\wp_a(S, v) = \{I\}$ with $I := S \setminus \{v_*\}$ and $e_I = 1$. Given these facts, it is easily checked that for each choice of v the two sides of (3) agree. □

Lemma 3.4. *Theorem 2.1 is valid if $a = 0$.*

Proof. Following Proposition 3.3 we assume both $a = 0$ and G is not trivial.

In this case Lemma 3.1(i) and (iii) combine to imply that the left hand side of (3) is equal to $e_{0,S} \cdot \mathcal{F}_0$.

In addition, $S^a_{\min} = \emptyset$ and so for each v in S one has $c^0_{S,v} = 1-0-1 = 0$ and $\wp_0(S, v) = \{\emptyset\}$. If $|S| = 1$ then (2) implies $e_\emptyset = e_{\mathbf{1}} + \sum_{\chi \in \widehat{G}'_{0,S}} e_\chi = \sum_{\chi \in \widehat{G}_{0,S}} e_\chi = e_{0,S}$ and if $|S| > 1$ then $e_\emptyset = e_{\mathbf{1}} + e_{0,S}$. Since $e_{\mathbf{1}} \cdot \mathcal{F}_0 = 0$ if $|S| > 1$ this shows that the right hand side of (3) is also equal to $e_{0,S} \cdot \mathcal{F}_0$, as required. □

The next result deals with the case that a is 'large' (see Remark 3.2(ii)).

Proposition 3.5. *Theorem 2.1 is valid if $a \geq |S| - 1$.*

Proof. Following Remark 3.2(ii) it is enough to consider a equal to either $|S|-1$ or $|S|$. Our argument then splits into several subcases, depending on the cardinality of $S^a_{\min}$. Following Lemma 3.4 we always assume $a > 0$.

We consider first the case that $S^a_{\min} = \emptyset$ and $a > 0$, and hence that $\widehat{G}'_{a,S} = \emptyset$. In this case Lemma 3.1(i) and (ii) combine to imply $e_{(a),S} \cdot \mathcal{F}_a = e_{a,S} \cdot \mathcal{F}_a$ is equal to $e_{\mathbf{1}} \cdot \mathcal{F}_a$ if $a = |S|-1$ and vanishes if $a = |S|$. One also has $\wp_a(S, v) = \emptyset$ (since $a > 0$) and so the right hand side of (3) is equal to $c^a_{S,v} e_{\mathbf{1}} \cdot \mathcal{F}_a$. The claimed equality is thus true in this case since $c^a_{S,v}$ is equal to $1 - 0 - 0 = 1$ if $a = |S| - 1$ and to 0 if $a = |S|$.

In the remainder of the argument we assume $S^a_{\min} \neq \emptyset$ and hence that $|S^a_{\min}|$ is equal to either $|S| - 1$ (so $a = |S| - 1$) or $|S|$ (so $S^a_{\min} = S$). There are then four separate cases to consider depending on whether $|S^a_{\min}| = |S| - 1$ and $v \in S^a_{\min}$, or $|S^a_{\min}| = |S| - 1$ and $v \notin S^a_{\min}$, or $S^a_{\min} = S$ and $a = |S| - 1$, or $a = |S|$ (and hence $S^a_{\min} = S$).

We consider first the case $|S^a_{\min}| = |S| - 1 = a$ and $v \in S^a_{\min}$. In this case $G_v \subseteq \ker(\chi)$ for all $\chi \in \widehat{G}'_{a,S}$ so $(1 - e_v)e_{a,S} = 0$ and Lemma 3.1(i) and (ii) combine to imply the left hand side of (3) is equal to $e_{\mathbf{1}} \cdot \mathcal{F}_a$. Given this, the claimed equality follows from the fact that $\wp_a(S, v) = \emptyset$ and $c^a_{S,v} = 0 - 1 - 0 = -1$.

We next assume $|S^a_{\min}| = |S| - 1 = a$ and $v \notin S^a_{\min}$ so that $S^a_{\min} = S \setminus \{v\}$. In this case Lemma 3.1(i) and (iii) together imply that the left hand side of (3) is equal to $e_{a,S} \cdot \mathcal{F}_a$. In addition, one has $c^a_{S,v} = 1 - 1 - 0 = 0$ and the unique element of $\wp_a(S, v)$ is equal to $I := S^a_{\min}$ so $e_I = e_{a,S}$ and the right hand side of (3) is also equal to $e_{a,S} \cdot \mathcal{F}_a$.

We now consider the case $S^a_{\min} = S$ and $a = |S| - 1$. For each v in S we write $G(v)$ for the subgroup generated by $G_{v'}$ as v' varies over $S \setminus \{v\}$. Then, in this case, for each χ in $\widehat{G}'_{a,S}$ one has $e_\chi e_v = 0$ if and only if $G(v) \subseteq \ker(\chi)$. This implies $(1 - e_v)e_{a,S} = (e_{G(v)} - e_{\mathbf{1}})e_{a,S}$ and hence that the left hand side of (3) is equal to $e_{G(v)}e_{a,S} \cdot \mathcal{F}_a$. The claimed equality thus follows from the fact that, in this case, $c^a_{S,v} = 0$, $\wp_a(S, v)$ has a single element $I_v := S \setminus \{v\}$ and it is straightforward to check that e_{I_v} is equal to $e_{G(v)}e_{a,S}$.

Finally, we assume $a = |S|$ and $S^a_{\min} \neq \emptyset$ (and hence that both $S^a_{\min} = S$ and G is not trivial). In this case one verifies that $e_{(a),S} = e_{a,S} = e_{G_S} - e_{\mathbf{1}}$ with G_S the subgroup of G generated by G_v as v varies over S. For each v in S one therefore has $(1 - e_v + e_{\mathbf{1}})e_{(a),S} = (1 - e_v)e_{G_S} = 0$, where the last equality is valid since $G_v \subseteq G_S$, and so the left hand side of (3) vanishes. On the other hand, in this case the right hand side of (3) vanishes since for any v in S both $c^a_{S,v} = 0$ and $\wp_a(S, v)$ is empty.

This completes the proof of Theorem 2.1 in the case that $a \geq |S| - 1$. □

3.3. Following Propositions 3.3 and 3.5 and Lemma 3.4 we assume in the remainder of the argument that G is not trivial and that $0 < a < |S| - 1$.

We note that in this case Lemma 3.1(ii) implies that the term $1 - e_v + e_{\mathbf{1}}$ on the left hand side of (3) can be replaced by $1 - e_v$ and that the first summand on the right hand side of (3) can be omitted.

We next recall (from [2, §2.2]) that $\mathrm{Sel}^T_S(F)$ has a canonical transpose $\mathrm{Sel}^T_S(F)^{\mathrm{tr}}$ in the sense of Jannsen's homotopy theory of modules [9], that there is a canonical exact sequence

$$0 \longrightarrow \mathrm{Cl}^T_S(F) \longrightarrow \mathrm{Sel}^T_S(F)^{\mathrm{tr}} \xrightarrow{\kappa} X_{F,S} \longrightarrow 0, \tag{5}$$

with $X_{F,S}$ the submodule of the free abelian group $Y_{F,S}$ on the set of places of F above S comprising elements whose coefficients sum to zero, and that, since $\mathcal{O}^\times_{F,S,T}$ is assumed to be torsion-free, [2, Lem. 2.8] implies that for each non-negative integer a one has

$$\mathrm{Fit}^a_G(\mathrm{Sel}^T_S(F)) = \mathrm{Fit}^a_G(\mathrm{Sel}^T_S(F)^{\mathrm{tr}})^{\#}, \tag{6}$$

where we write $x \mapsto x^{\#}$ for the $\mathbb{Q}$-linear involution of $\mathbb{Q}[G]$ that inverts elements of G.

As a final preparatory remark, we note that it suffices to prove the equality of Theorem 2.1 after tensoring with $\mathbb{Z}_p$ for every prime p.

We therefore fix a prime p and for each abelian group A write A_p in place of $\mathbb{Z}_p \otimes A$. By adapting an approach used in [2], we now construct a convenient resolution of $\mathrm{Sel}^T_S(F)^{\mathrm{tr}}_p$.

To do this we fix a place v in S and a place w of F above v, we set $d_1 := |S| - 1$, write the elements of $S \setminus \{v\}$ as $\{v_i\}_{1 \leq i \leq d_1}$ and for each index i we fix a place w_i of F above v_i.

We write $\mathbb{Z}_p[G]^{d_1}$ for the direct sum of d_1 copies of $\mathbb{Z}_p[G]$ and choose a homomorphism of $\mathbb{Z}_p[G]$-modules $\pi_1 : \mathbb{Z}_p[G]^{d_1} \to \mathrm{Sel}_S^T(F)_p^{\mathrm{tr}}$ with the property that, for each index i, the image under $(\mathbb{Z}_p \otimes_{\mathbb{Z}} \kappa) \circ \pi_1$ of the i-th element of the standard basis of $\mathbb{Z}_p[G]^{d_1}$ is $w_i - w$, where κ is the homomorphism that occurs in (5).

We also write ϖ_v for the natural projection $X_{F,S} \to Y_{F,S\setminus\{v\}}$ and fix a choice of natural number $d_{2,v}$ for which there exists a surjective homomorphism of $\mathbb{Z}_p[G]$-modules of the form $\pi_{2,v} : \mathbb{Z}_p[G]^{d_{2,v}} \to \ker(\varpi_v \circ \kappa)_p$.

Then, with these choices, the approach of [2, §5.4] shows that there is an exact sequence of $\mathbb{Z}_p[G]$-modules of the form

$$\mathbb{Z}_p[G]^{d_v} \xrightarrow{\theta_v} \mathbb{Z}_p[G]^{d_v} \xrightarrow{\pi_v} \mathrm{Sel}_S^T(F)_p^{\mathrm{tr}} \longrightarrow 0 \tag{7}$$

with $d_v := d_1 + d_{2,v}$ and $\pi_v := (\pi_1, \pi_{2,v})$.

If we write M_v for the matrix of θ_v with respect the standard basis of $\mathbb{Z}_p[G]^{d_v}$, then the equality (6) combines with the very definition of a-th Fitting ideals to imply that

$$\mathrm{Fit}_G^a(\mathrm{Sel}_S^T(F))_p = \sum_{J \in \wp_a(d_v)} m_v(J) \tag{8}$$

with $\wp_a(d_v) := \wp_a(\{i \in \mathbb{Z} : 1 \leq i \leq d_v\})$ and $m_v(J)$ denoting the ideal of $\mathbb{Z}_p[G]$ that is generated by the determinants of all $(d_v - a) \times (d_v - a)$ minors of the $d_v \times (d_v - a)$ matrix $M_v(J)^{\#}$ obtained by deleting all columns of M_v corresponding to integers in J.

Next we note that if $\chi \in \widehat{G} \setminus \{\mathbf{1}\}$, then one has

$$v \notin S_\chi \Longleftrightarrow G_v \not\subseteq \ker(\chi) \Longleftrightarrow (1 - e_v)e_\chi \neq 0$$

and hence that for each non-negative integer a one has

$$(1 - e_v)e_{a,S} = \sum_{I \in \wp_a(S,v)} (e_I - e_{\mathbf{1}}).$$

This equality combines with (8) to imply that

$$\begin{aligned}(1 - e_v)e_{(a),S} \cdot \mathrm{Fit}_G^a(\mathrm{Sel}_S^T(F))_p =& (1 - e_v)e_{a,S} \cdot \mathrm{Fit}_G^a(\mathrm{Sel}_S^T(F))_p \\ =& (\sum_{I \in \wp_a(S,v)} (e_I - e_{\mathbf{1}})) \cdot (\sum_{J \in \wp_a(d_v)} m_v(J)) \\ =& \sum_{I \in \wp_a(S,v)} (e_I - e_{\mathbf{1}}) \cdot m_v(I) \\ =& \sum_{I \in \wp_a(S,v)} (e_I - e_{\mathbf{1}}) \cdot \mathrm{Fit}_G^a(\mathrm{Sel}_S^T(F))_p.\end{aligned}$$

The first equality here is true as Lemma 3.1(i) implies that $e_{(a),S} \cdot \mathrm{Fit}_G^a(\mathrm{Sel}_S^T(F))$ is equal to $e_{a,S} \cdot \mathrm{Fit}_G^a(\mathrm{Sel}_S^T(F))$ and the third and fourth equalities follow directly from the result of Lemma 3.6 below.

To deduce the equality (3) in the case that $a < |S| - 1$ it thus suffices to observe firstly that the last sum in the above formula is direct since for distinct elements I_1 and I_2 of

$\wp_a^*(S)$ the idempotents $e_{I_1} - e_{\mathbf{1}}$ and $e_{I_2} - e_{\mathbf{1}}$ are orthogonal, and then that for each such I Lemma 3.1(ii) implies $(e_I - e_{\mathbf{1}}) \cdot \mathrm{Fit}_G^a(\mathrm{Sel}_S^T(F)) = e_I \cdot \mathrm{Fit}_G^a(\mathrm{Sel}_S^T(F))$.

In the following result we use the correspondence $v_i \leftrightarrow i$ to identify $\wp_a(S \setminus \{v\})$ with the subset $\wp_a(d_1)$ of $\wp_a(d_v)$.

Lemma 3.6. *Take $\chi \in \widehat{G}_{a,S} \setminus \{\mathbf{1}\}$ with $v \notin S_\chi$. Then S_χ belongs to $\wp_a(d_v)$ and for each J in $\wp_a(d_v) \setminus \{S_\chi\}$ one has $e_\chi \cdot m_v(J) = 0$.*

Proof. Set $G_\chi := G/\ker(\chi)$ and $F^\chi := F^{\ker(\chi)}$.

Then to compute $e_\chi \cdot m_v(J)$ one can replace $m_v(J)$ by its image in $\mathbb{Z}_p[G_\chi]$. Note also that the exact sequence (7) implies that each column of the image $M_v(J)_\chi^\#$ in $\mathbb{Z}_p[G_\chi]$ of $M_v(J)^\#$ that corresponds to an integer in $S_\chi \setminus J$ must vanish since the corresponding place in $S \setminus \{v\}$ splits completely in F^χ/k.

In particular, if $J \neq S_\chi$, then $M_v(J)_\chi^\#$ has at least one column of zeroes and so the determinant of any of its $(d_v - a) \times (d_v - a)$ minors must vanish.

It follows that $e_\chi \cdot m_v(J) = 0$, as claimed. □

4. Refined Stark Conjectures

Recall the integer $r = r_S(F/k)$ defined in Remark 3.2. In this section we formulate a conjecture that extends the existing theory of refined Stark conjectures in two significant ways since it doesn't restrict to L-series that vanish at zero to order r or exclude the 'boundary case' that S is equal to $S_{\min}^r$ (we recall that the possibility that such a conjecture could be formulated in this boundary case was first suggested by Dummit in [3]).

Our conjecture is motivated by the result of Theorem 2.1 and so we will assume throughout the notation and hypotheses of that result.

4.1. In order to formulate our conjecture we first define, for each non-negative integer a with $a \leq |S|$, a canonical 'Stark element of rank a' for the data $F/k, S$ and T.

To do this we use the isomorphism of $\mathbb{C}[G]$-modules

$$\lambda_{F,S}^a : e_{a,S}(\mathbb{C} \cdot \bigwedge\nolimits_{\mathbb{Z}[G]}^a \mathcal{O}_{F,S}^\times) \cong e_{a,S}(\mathbb{C} \cdot \bigwedge\nolimits_{\mathbb{Z}[G]}^a X_{F,S})$$

that is induced by the a-th exterior power of the (Dirichlet regulator) isomorphism of $\mathbb{C}[G]$-modules $\mathbb{C} \cdot \mathcal{O}_{F,S}^\times \cong \mathbb{C} \cdot X_{F,S}$ that sends each element u of $\mathcal{O}_{F,S}^\times$ to $-\sum_w \log(|u|_w) \cdot w$, where w runs over all places of F above those in S and $|\cdot|_w$ is the normalised absolute value at w.

For each such a we also set

$$\theta_{F/k,S,T}^a(z) := z^{-a} \sum_{\chi \in \widehat{G}_{(a),S}} L_{S,T}(\check{\chi}, z) \cdot e_\chi.$$

We note that (4) implies both that this $\mathbb{C}[G]$-valued function is holomorphic at $z = 0$ and that $\theta_{F/k,S,T}^a(0) = e_{a,S} \cdot \theta_{F/k,S,T}^a(0)$.

In the sequel we fix an ordering of S and use it to define all exterior powers of elements that are indexed by places in S.

For each place v in S we will also fix a place w_v of F that lies above v.

4.1.1. We deal first with the case that $a < |S|$. In this case, for each I in $\wp_a^*(S)$ we define $\eta^I_{F/k,S,T}$ to be the unique element of $e_I e_{a,S}(\mathbb{C} \cdot \bigwedge^a_{\mathbb{Z}[G]} \mathcal{O}^\times_{F,S})$ that satisfies

$$\lambda^a_{F,S}(\eta^I_{F/k,S,T}) = e_I \cdot \theta^a_{F/k,S,T}(0) \cdot \bigwedge_{v \in I} (w_v - w_I), \tag{9}$$

where w_I is any choice of place of F that lies above a place in $S \setminus I$ (such a choice is possible since $|I| = a$ is assumed to be strictly less than $|S|$).

We also note that $\eta^I_{F/k,S,T}$ is independent of the choice of w_I since if w'_I is any other such place, then $w_I - w'_I$ is annihilated by $e_I e_{a,S}$. To justify the latter claim note that $w_I - w'_I$ belongs to the kernel of the natural surjective homomorphism $\pi_{S,I} : X_{F,S} \to Y_{F,I}$. In particular, if χ is any homomorphism in $\widehat{G}'_{a,S}$ with $S_\chi = I$, then $\dim_{\mathbb{C}}(e_\chi(\mathbb{C} \cdot X_{F,S})) = \dim_{\mathbb{C}}(e_\chi(\mathbb{C} \cdot Y_{F,I})) = a$ so $e_\chi(\mathbb{C} \cdot \ker(\pi_{S,I})) = 0$ and hence $e_\chi(w_I - w'_I) = 0$. Since the sum of e_χ over all such χ is equal to $e_I(1 - e_{\mathbf{1}})$, and hence to $e_I e_{a,S}$ if $a < |S| - 1$, it suffices to show that $e_{\mathbf{1}}$ annihilates $w_I - w'_I$ if $a = |S| - 1$ and this is true since, in this case, w_I and w'_I must both lie above the unique place in $S \setminus I$.

If $a < |S| - 1$, then we define the Stark element of rank a for $F/k, S$ and T by setting

$$\eta^a_{F/k,S,T} := \sum_{I \in \wp_a^*(S)} \eta^I_{F/k,S,T} \in e_{a,S}(\mathbb{C} \cdot \bigwedge^a_{\mathbb{Z}[G]} \mathcal{O}^\times_{F,S}). \tag{10}$$

However, if $a = |S| - 1$, which we now assume, then we proceed as follows. We recall first that if, in this case, G is trivial, then we have fixed a place v_* in S and set $S^a_{\min} := S \setminus \{v_*\}$ so that $\wp_a^*(S)$ is the singleton $\{S^a_{\min}\}$. If $a = |S| - 1$ and G is not trivial, then we also now fix a place v_* of S, the precise choice of which will not matter in the sequel.

In both cases, for each set I in $\wp_a^*(S)$ we write v_I for the unique place such that $S = I \cup \{v_I\}$ (so $v_I = v_*$ if G is trivial). Then $\bigwedge^a_{\mathbb{Z}} X_{k,S}$ is a free $\mathbb{Z}$-module of rank one and both $\wedge_{v \in S \setminus \{v_*\}}(v - v_*)$ and, for any I in $\wp_a^*(S)$, also $\wedge_{v \in I}(v - v_I)$ are a basis of this module. For each such I we can therefore define $u_I \in \{\pm 1\}$ by the equality

$$u_I \cdot \wedge_{v \in I}(v - v_I) = \wedge_{v \in S \setminus \{v_*\}}(v - v_*). \tag{11}$$

We then define the Stark element of rank $a = |S| - 1$ for $F/k, S$ and T to be the element of $e_{a,S}(\mathbb{C} \cdot \bigwedge^a_{\mathbb{Z}[G]} \mathcal{O}^\times_{F,S})$ that is obtained by setting

$$\eta^a_{F/k,S,T} := \begin{cases} ((1 - e_{\mathbf{1}}) + |\wp_a^*(S)|^{-1} e_{\mathbf{1}}) \cdot \sum_{I \in \wp_a^*(S)} u_I \cdot \eta^I_{F/k,S,T}, & \text{if } S^a_{\min} \neq \emptyset, \\ |G|^{-a} \cdot \varepsilon_k, & \text{if } S^a_{\min} = \emptyset. \end{cases} \tag{12}$$

Here we use the fact that $|\wp_a^*(S)| \neq 0$ if $S^a_{\min} \neq \emptyset$ and write ε_k for the Rubin-Stark element for $(k/k, S, T)$ that is defined with respect to the places $S \setminus \{v_*\}$. (We also note that, in each case, the element $\eta^a_{F/k,S,T}$ depends on the choice of v_* only up to a sign and so this dependence will not be indicated explicitly.)

Example 4.1. In the special case that $a = r$ (so $a < |S|$) and $S^a_{\min}$ comprises a places that split completely in F/k, then $\wp_a^*(S)$ comprises a single element $I = S^a_{\min}$. In this case one checks easily that (10), respectively (12), implies $\eta^a_{F/k,S,T}$ is equal, up to a sign, with

the 'Rubin-Stark element' for the data F/k, S and T, as defined by Rubin in [14]. A more general version of this comparison result is proved in Proposition 5.4 below.

Remark 4.2. The definition of $\eta^a_{F/k,S,T}$ ensures that for every integer a and every I in $\wp^*_a(S)$ the elements $e_I(\eta^a_{F/k,S,T})$ and $\eta^I_{F/k,S,T}$ differ at most by a sign (see Lemma 5.5 below).

4.1.2. We now assume $a = |S|$. This case is somewhat exceptional since $e_{a,S} = e_{G_S} - e_{\mathbf{1}}$, with G_S the subgroup defined in the proof of Proposition 3.5, and so $e_{a,S} \neq 0$ only if the maximal extension F^{G_S} of k in F in which all places in S split completely is non-trivial. In addition, in the latter case one has $\wp^*_a(S) = \{I_a\}$ with $I_a := S$ and hence, since $S \setminus I_a$ is empty, one cannot define $\eta^a_{F/k,S,T}$ in the same way as above.

To overcome this difficulty we write v_1 for the place occurring first in the ordering of S that has been fixed and, after noting that $(1 - e_{\mathbf{1}})w_{v_1}$ belongs to $\mathbb{Q} \cdot X_{F,S}$, we define the Stark element of rank $a = |S|$ for the data F/k, S and T to be the unique element $\eta^a_{F/k,S,T}$ of $e_{a,S}(\mathbb{C} \cdot \bigwedge^a_{\mathbb{Z}[G]} \mathcal{O}^\times_{F,S})$ that satisfies

$$\lambda^a_{F,S}(\eta^a_{F/k,S,T}) = \theta^a_{F/k,S,T}(0) \cdot ((1 - e_{\mathbf{1}})w_{v_1} \wedge \bigwedge_{v \in S \setminus \{v_1\}} (w_v - w_{v_1})). \tag{13}$$

4.2. In this section we formulate a precise conjecture concerning the arithmetic properties of the Stark elements $\eta^a_{F/k,S,T}$ defined above. In view of the differing definitions of these elements we deal separately with the cases $a < |S|$ and $a = |S|$.

For each non-negative integer a we set

$$\Phi_a := \bigwedge^a_{\mathbb{Z}[G]} \mathrm{Hom}_G(\mathcal{O}^\times_{F,S,T}, \mathbb{Z}[G])$$

and then define a finitely generated G-submodule of $\mathbb{Q}[G]$ by setting

$$\mathcal{E}^a_{F/k,S,T} := \{\varphi(\eta^a_{F/k,S,T}) : \varphi \in \Phi_a\}.$$

4.2.1. We consider first the case that $a < |S|$.

Conjecture 4.3. *Assume the hypotheses of Theorem 2.1 and fix a non-negative integer a with $a < |S|$. Then for each place v in S one has*

$$(1 - e_v + e_{\mathbf{1}})e_{(a),S} \cdot \mathrm{Fit}^a_G(\mathrm{Sel}^T_S(F))^\# = e_{\mathbf{1}} \cdot |c^a_{S,v}| \cdot \mathcal{E}^a_{F/k,S,T} \oplus \bigoplus_{I \in \wp_a(S,v)} e_I \cdot \mathcal{E}^a_{F/k,S,T}. \tag{14}$$

In particular, for every I in $\wp^*_a(S)$ and every φ in Φ_a one has both

$$\mathfrak{n}^a_{S,T}(F/k) \cdot e_I \cdot \varphi(\eta^a_{F/k,S,T})^\# \subseteq \mathbb{Z}[G] \ \textit{and}\ \mathfrak{m}^a_{S,T}(F/k) \cdot e_I \cdot \varphi(\eta^a_{F/k,S,T})^\# \subseteq \mathrm{Fit}^a_G(\mathrm{Sel}^T_S(F)). \tag{15}$$

Remark 4.4. If $S \neq S^r_{\min}$, then for each $v \in S \setminus S^r_{\min}$ one has $(1 - e_v + e_{\mathbf{1}}) \cdot \mathrm{Fit}^r_G(\mathrm{Sel}^T_S(F))^\# = \mathrm{Fit}^r_G(\mathrm{Sel}^T_S(F))^\#$ and $\mathfrak{n}^r_{S,T}(F/k) = \mathfrak{m}^r_{S,T}(F/k) = \mathbb{Z}[G]$ (see Remark 3.2(iii)). In this case the equality (14) with $a = r$ was first predicted by the second author in [10] and the first, respectively second, containment in (15) specialises to give the conjecture formulated under the hypothesis $S \neq S^r_{\min}$ by Vallières in [16] (and hence also the conjecture formulated by Emmons and Popescu in [6, Conj. 3.8]), respectively by the second author in [10]. In

particular, if $S^r_{\min}$ comprises r places that split completely in F/k, in which case $\eta^r_{F/k,S,T}$ is a 'Rubin-Stark element' for F/k (see Example 4.1), then the first inclusion in (15) with $a = r$ recovers the 'Rubin-Stark Conjecture' formulated by Rubin in [14]. In general, however, the first inclusion in (15) is strictly finer (even in the case $a = r$) than the Rubin-Stark Conjecture for all suitable subfields of F/k (cf. Example 5.3 below).

Example 4.5. Let $k = \mathbb{Q}(\alpha)$ with $\alpha^3 - 19\alpha + 21 = 0$ and write F for its strict Hilbert class field. Then k is a totally real non-Galois extension of $\mathbb{Q}$, F/k is unramified outside S_∞ and $\mathrm{Gal}(F/k)$ is isomorphic to $\mathbb{Z}/4\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$. In addition, if one sets $S := S_\infty$, then $r = r_S(F/k)$ is equal to 1 and $S = S^1_{\min}$ and $|G_v| = 2$ for each v in S. In this case Remark 3.2(iv) applies with $|S| = 3$ and $r = 1$ and implies $\mathfrak{n}^1_{S,T}(F/k)$ contains 2. On the other hand, Erickson has used numerical computations of Dummit and Hayes [4] to show that for some choices of T and of I in $\wp^*_1(S)$ the element $\eta^I_{F/k,S,T}$ does not belong to Rubin's lattice $\Lambda_{F/k,S,T,1}$ (see the discussion of [8, §7], and [7, §4.2] for the details), and hence that there exists φ in $\mathrm{Hom}_G(\mathcal{O}^\times_{F,S,T}, \mathbb{Z}[G])$ with $\varphi(\eta^I_{F/k,S,T}) \notin \mathbb{Z}[G]$. In particular, since the first inclusion of (15) combines with Remark 4.2 to imply that $2\cdot\varphi(\eta^I_{F/k,S,T}) = \pm 2\cdot e_I\cdot\varphi(\eta^a_{F/k,S,T}) \in \mathbb{Z}[G]$ for all such φ, this inclusion is in a natural sense best possible.

Example 4.6. Even in the case that $a > r$ the predictions of (15) refine existing conjectures concerning Stark elements. To describe an explicit example of this phenomenon (all details for which can be found in the PhD Thesis [11] of the second author) we take k to be totally real with $k \neq \mathbb{Q}$ and F to be a CM extension of k with $G = J \times P$ where $|J| = 2$ and P is non-cyclic of order p^2 for an odd prime p. We assume that the sets $\mathcal{H}_1$ and $\mathcal{H}_0$ of subgroups of G of order p that are the decomposition subgroup of precisely one, respectively of no, place in S, are both non-empty, that there exists a place v_0 in S for which G_{v_0} is a non-trivial subgroup of P that does not belong to $\mathcal{H}_1$ and that no place in $S \setminus S_\infty$ splits completely in F^J/k. Then, setting $e_- := (1-\tau)/2$ with τ the non-trivial element of J, one has $e_{0,S} = e_-(\sum_{H\in\mathcal{H}_0}(e_H - e_P)) \neq 0$, $e_{(1),S} = 1 - e_{0,S}$, $S^1_{\min} = S_1 := \{v \in S : G_v \in \mathcal{H}_1\}$ and $\wp^*_1(S) = \{I_v\}_{v\in S_1}$ with $I_v := \{v\}$. In addition, as $|S| > 2$ and any place in $S_1 \cup \{v_0\}$ splits in F^P/k, one finds that $e_-\cdot\mathcal{F}_1 \subseteq e_-\cdot I(P)$, where we set $\mathcal{F}_1 := \mathrm{Fit}^1_G(\mathrm{Sel}^T_S(F))$ and write $I(P)$ for the augmentation ideal of $\mathbb{Z}[P]$. In particular, if we fix a non-trivial element h of a subgroup in $\mathcal{H}_0$ and set $x := (1-\tau)(1-h)^{p-2}$, then $x\cdot(1 - e_{v_0} + e_{\mathbf{1}})e_{(1),S}\cdot\mathcal{F}_1 \in \mathbb{Z}[G]$ and hence $x \in \mathfrak{n}^1_{S,T}(F/k)$. For each v in S_1 this fact combines with (Remark 4.2 and) the result of Proposition 5.4 below for $\eta^{I_v}_{F/k,S,T} = \pm e_{I_v}(\eta^a_{F/k,S,T})$ to show (15) implies the Stark element ε_v for F^{G_v}/k, S and T satisfies $x(\epsilon_v) \in (F^\times)^p$, and hence $x(\epsilon_v) \in (F^{G_v,\times})^p$ if F^P contains no primitive p-th root of unity. Thus, since in this case the G-module spanned by ε_v is isomorphic to the quotient $\mathcal{A}_v := e_-(\mathbb{Z}[P/G_v]/(\sum_{\gamma\in P/G_v}\gamma))$ of $\mathbb{Z}[G]$ and the image of x in $\mathcal{A}_v$ is not divisible by p, this prediction is finer than the Rubin-Stark conjecture for F^{G_v}/k, S and T. We note, finally, that for primes $p > 3$ concrete extensions of this form can be constructed, for example, as follows. Fix primes ℓ_1 and ℓ_2 with $\ell_1 \equiv \ell_2 \equiv 1 \pmod{4p}$, $\ell_1 \equiv 2 \pmod 3$, ℓ_1 is a p-th power residue modulo ℓ_2 and ℓ_2 is not a p-th power residue modulo ℓ_1 (such as $p = 5, \ell_1 = 41$ and $\ell_2 = 101$). Then the extension of $k = \mathbb{Q}(\sqrt{3})$ generated by a root of unity of order $4\ell_1\ell_2$ contains subextensions F/k that satisfy all of

the hypotheses described above with S taken to be the set of places of k dividing either ∞, ℓ_1 or ℓ_2, S_1 the unique place of k above ℓ_1 and v_0 any place of k above ℓ_2.

4.2.2. In this section we discuss the arithmetic properties of $\eta^a_{F/k,S,T}$ in the case $a = |S|$.

In this case one has $e_{a,S} = e_{G_S} - e_{\mathbf{1}}$ and so, modulo establishing the functorial behaviour of $\eta^a_{F/k,S,T}$ under change of extension (which is straightforward by using the approach of Proposition 5.4 below), one can replace F/k by F^{G_S}/k. In this way one can assume both that all places in S split completely in F/k and that $e_{a,S} = 1 - e_{\mathbf{1}}$.

Then, since $1 - e_{\mathbf{1}}$ can be written as an integral linear combination of idempotents of the form $e_H - e_{\mathbf{1}}$ for subgroups H of G for which the quotient G/H is cyclic, a similar reduction allows one to focus on the arithmetic properties of $\eta^a_{F/k,S,T}$ in the case that G is both non-trivial and cyclic.

In this special case the result of Theorem 2.1 cannot usefully be applied (since, if $a = |S|$, then the argument of Proposition 3.5 shows that both sides of (3) vanish).

However, we offer the following precise conjecture.

Conjecture 4.7. *Assume that F/k is a non-trivial cyclic extension in which all places in S split completely and set $a := |S|$. Then, for any choice of generator g of G, one has*

$$(1 - e_{\mathbf{1}}) \cdot \mathrm{Fit}^a_G(\mathrm{Sel}^T_S(F))^{\#} = (g-1) \cdot \mathcal{E}^a_{F/k,S,T}. \tag{16}$$

In particular, for every φ in Φ_a one has

$$(g-1)^2 \cdot \varphi(\eta^a_{F/k,S,T}) \in (g-1) \cdot \mathrm{Fit}^a_G(\mathrm{Sel}^T_S(F))^{\#} \subset \mathbb{Z}[G].$$

Remark 4.8. The (conjectural) equality (16) also implies that $(g-1) \cdot \mathcal{E}^a_{F/k,S,T} \subseteq \mathbb{Z}[G]$ if and only if $e_{\mathbf{1}} \cdot \mathrm{Fit}^a_G(\mathrm{Sel}^T_S(F)) \subset \mathbb{Z}[G]$ and it can be shown that this condition is satisfied if and only if $\mathrm{Fit}^1(\mathrm{Cl}^T_S(k))$ is contained in $|G| \cdot \mathbb{Z}$.

5. Evidence for Conjectures 4.3 and 4.7

In this section we refer to the 'Leading Term Conjecture' formulated by Kurihara, Sano and the first author in [2, Conj. 3.6]. We recall that if F is a number field, then it is shown in loc. cit. that this conjecture is equivalent to the validity of the equivariant Tamagawa number conjecture for the pair $(h^0(\mathrm{Spec}(F)), \mathbb{Z}[G])$.

5.1. The main evidence that we can offer in support of Conjectures 4.3 and 4.7 is the following result (which will be proved in §5.3).

Theorem 5.1. *If* [2, Conj. 3.6] *is valid for F/k, then so are Conjectures 4.3 and 4.7.*

As already discussed in [2], earlier work of the first author, of the first author and Greither and of Flach can be used to show the validity of [2, Conj. 3.6] in the case that F is either an abelian extension of $\mathbb{Q}$ or a global function field. The last result therefore has the following immediate consequence.

Corollary 5.2. *Conjectures 4.3 and 4.7 are valid if either F is an abelian extension of $\mathbb{Q}$ (and k is any subfield of F) or if F and k are global function fields.*

By means of an explicit application of this result we consider an infinite family of extensions containing the extension $\mathbb{Q}(\sqrt{5}, \sqrt{-7}, \sqrt{-11})/\mathbb{Q}$ that is (numerically) investigated extensively by Erickson in [7, 8].

Example 5.3. Let p_1, p_2 and p_3 be distinct prime numbers satisfying

$$p_1 \equiv -p_2 \equiv -p_3 \equiv 1 \pmod 4$$

and such that the respective Legendre symbols satisfy

$$\left(\frac{p_1}{p_2}\right) = \left(\frac{p_2}{p_3}\right) = -\left(\frac{p_3}{p_1}\right) = -1.$$

Then, setting $F := \mathbb{Q}(\sqrt{p_1}, \sqrt{-p_2}, \sqrt{-p_3})$, one checks that the abelian extension $F/\mathbb{Q}$ satisfies the hypotheses of Conjecture 4.3 with S equal to the set $\{\infty, p_1, p_2, p_3\}$ of places that ramify in F, $S^1_{\min} = S$, $S^2_{\min} = \{\infty, p_3\}$ and $S^3_{\min} = \emptyset$. One also has $r_S(F/\mathbb{Q}) = 1$, $e_{1,S} = 1 - e_H, e_{2,S} = e_H - e_{\mathbf{1}}$ and $e_{3,S} = e_{\mathbf{1}}$ with $H := \mathrm{Gal}(F/\mathbb{Q}(\sqrt{p_1}))$.

Corollary 5.2 implies that Conjecture 4.3 is valid in this case. In addition, for any finite set of primes T that is disjoint from S and such that $\mathcal{O}^\times_{F,S,T}$ is torsion-free one checks easily (using the observations in Remark 3.2(iv)) that the ideals $\mathfrak{n}^1_{S,T}(F/\mathbb{Q})$ and $\mathfrak{m}^1_{S,T}(F/\mathbb{Q})$ both contain 2. In particular, since in this case two places in S have decomposition subgroups of order 4, this combines with Proposition 5.4 below to show that the first, respectively second, integrality prediction in (15) implies Rubin-Stark elements of subfields of $F/\mathbb{Q}$ have stronger integrality properties than are predicted by the relevant case of the Rubin-Stark Conjecture, respectively of the refined Rubin-Stark Conjecture discussed in [2, Th. 1.5(i)].

5.2. In preparation for proving Theorem 5.1, we now establish several auxiliary results.

The first result extends [16, Prop. 4.18] (and hence also [10, Lem. 4.3]). If $a < |S|$, then for each I in $\wp^*_a(S)$ we write D_I for the subgroup of G that is generated by the groups G_v for v in I and F^I for the fixed field of F by D_I.

Proposition 5.4. *Fix I in $\wp^*_a(S)$ for $a < |S|$. Fix a subgroup H of G that contains D_I, write E for the fixed field of F by H, set $\Gamma := G/H$ and identify $\mathbb{C} \cdot \bigwedge^a_{\mathbb{Z}[\Gamma]} \mathcal{O}^\times_{E,S}$ with a subspace of $\mathbb{C} \cdot \bigwedge^a_{\mathbb{Z}[G]} \mathcal{O}^\times_{F,S}$ in the obvious way.*

Then each place in I splits completely in E/k and, if one writes $\varepsilon^I_{E/k,S,T}$ for the Rubin-Stark element in $\mathbb{C} \cdot \bigwedge^a_{\mathbb{Z}[\Gamma]} \mathcal{O}^\times_{E,S}$ that is defined with respect to the places in I, then one has $e_H(\eta^I_{F/k,S,T}) = |H|^{-a} \cdot \varepsilon^I_{E/k,S,T}$ and hence also $\eta^I_{F/k,S,T} = |D_I|^{-a} \cdot \varepsilon^I_{F^I/k,S,T}$ with $F^I := F^{D_I}$.

Proof. We set $\eta^I := \eta^I_{F/k,S,T}$ and $\varepsilon^I_E := \varepsilon^I_{E/k,S,T}$. For each of the places w and $\{w_v\}_{v\in I}$ that are fixed in the definition (9) of η^I we also write w' and $\{w'_v\}_{v\in I}$ for the corresponding places of E that are obtained by restriction.

Then ε^I_E is defined by the equality

$$\lambda^a_{E,S}(\varepsilon^I_E) = \theta^a_{E/k,S,T}(0) \cdot \bigwedge_{v\in I}(w'_v - w'). \tag{17}$$

In addition, identifying $\mathbb{C}[\Gamma]$ with $e_H\mathbb{C}[G]$ in the natural way, one has $\theta^a_{E/k,S,T}(0) = e_H\theta^a_{F/k,S,T}(0)$ and the analogue of the idempotent $e_{a,S}$ for E/k is equal to $e_He_Ie_{a,S}$ and so

$$\theta^a_{E/k,S,T}(0) = e_He_Ie_{a,S}\cdot\theta^a_{F/k,S,T}(0) = e_He_I\cdot\theta^a_{F/k,S,T}(0). \tag{18}$$

Setting $T_H := \sum_{h\in H} h(=|H|e_H)$, one therefore has

$$\begin{aligned}\lambda^a_{F,S}(\varepsilon^I_E) &= e_He_I\cdot\theta^a_{F/k,S,T}(0)\cdot\bigwedge_{v\in I}T_H(w_v-w)\\ &= (T_H)^a\cdot e_He_I\cdot\theta^a_{F/k,S,T}(0)\cdot\bigwedge_{v\in I}(w_v-w)\\ &= |H|^a\cdot e_He_I\cdot\theta^a_{F/k,S,T}(0)\cdot\bigwedge_{v\in I}(w_v-w)\\ &= \lambda^a_{F,S}(|H|^a\cdot e_H(\eta^I)).\end{aligned}$$

Here the first equality follows by substituting (18) into (17) and then using the functoriality of the Dirichlet regulator map under change of field (as expressed by the commutativity of the diagram in [15, Chap. I, §6.5]) to compare the maps $\lambda^a_{E,S}$ and $\lambda^a_{F,S}$, the second is clear, the third is valid because $(T_H)^a = |H|^ae_H$ and the last follows directly from (9).

Then, since $\lambda^a_{F,S}$ is bijective, the displayed equality implies that $\varepsilon^I_E = |H|^a\cdot e_H(\eta^I)$, as claimed.

To deduce the equality $\varepsilon^I_{F^I/k,S,T} = |D_I|^a\cdot\eta^I$ from this it is now enough to note that if $H = D_I$, then $E = F^I$ and $e_{D_I}(\eta^I) = e_{D_I}(e_I(\eta^I)) = e_I(\eta^I) = \eta^I$. The second equality here is valid because

$$e_I = \begin{cases} e_{a,S}\cdot e_{D_I} & \text{if } a = |S|-1,\\ e_{\mathbf{1}} + e_{a,S}\cdot e_{D_I} & \text{if } a \neq |S|-1\end{cases}$$

and hence in every case $e_{D_I}\cdot e_I = e_I$. □

The next result clarifies Remark 4.2.

Lemma 5.5. *Fix an integer a with $0 \le a < |S|$. Then for each I in $\wp^*_a(S)$ the element $e_I(\eta^a_{F/k,S,T})$ is equal to $\eta^I_{F/k,S,T}$ if $a < |S|-1$ and to $u_I\cdot\eta^I_{F/k,S,T}$ if $a = |S|-1$.*

Proof. Set $\eta^a := \eta^a_{F/k,S,T}$ and for each I in $\wp^*_a(S)$ also $\eta^I := \eta^I_{F/k,S,T}$.

If G is trivial and $a < |S|-1$, then $S^a_{\min} = \emptyset$ so $\wp^*_a(S) = \emptyset$ and there is nothing to prove. If G is trivial and $a = |S|-1$, then $\wp^*_a(S) = \{I\}$ with $I := S^a_{\min}$ and the claimed equality is true since $e_I = e_{\mathbf{1}} = 1$ and $u_I = 1$ and the definition of η^a ensures directly that $\eta^a = \eta^I$.

We can therefore assume G is not trivial. In this case, if $a < |S|-1$, then the claimed equality $e_I(\eta^a) = \eta^I$ follows directly upon combining the definition (10) of η^a with the equality $\eta^I = e_I(\eta^I)$ and the fact that for each J in $\wp^*_a(S)\setminus\{I\}$ one has

$$e_I(\eta^J) = e_Ie_J(\eta^J) = (e_I - e_{\mathbf{1}})(e_J - e_{\mathbf{1}})(\eta^J) = 0$$

where the second equality is valid because in this case $e_{\mathbf{1}}(\eta^J) = 0$ and the last equality because the idempotents $e_I - e_{\mathbf{1}}$ and $e_J - e_{\mathbf{1}}$ are orthogonal (as $I \neq J$).

In the sequel we therefore assume both that G is not trivial and $a = |S| - 1$. In this case the same argument as above shows the definition (12) of η^a implies that for each I in $\wp_a^*(S)$ one has $(1 - e_{\mathbf{1}})e_I(\eta^a) = (1 - e_{\mathbf{1}})(u_I \cdot \eta^I)$.

Since $e_{\mathbf{1}}e_I = e_{\mathbf{1}}$ it is thus enough to show that $e_{\mathbf{1}}(\eta^a) = e_{\mathbf{1}}(u_I \cdot \eta^I)$. To do this we note that Proposition 5.4 combines with the equality (11) to imply that for each J in $\wp_a^*(S)$ one has $e_{\mathbf{1}}(u_J \cdot \eta^J) = u_J \cdot e_{\mathbf{1}}(\eta^J) = |G|^{-a} \cdot \varepsilon_k$, with ε_k the element occurring in (12). One therefore has

$$|\wp_a^*(S)| \cdot e_{\mathbf{1}}(\eta^a) = e_{\mathbf{1}} \sum_{J \in \wp_a^*(S)} u_J \cdot \eta^J = |\wp_a^*(S)| \cdot |G|^{-a} \cdot \varepsilon_k = |\wp_a^*(S)| \cdot e_{\mathbf{1}}(u_I \cdot \eta^I), \tag{19}$$

and hence $e_{\mathbf{1}}(\eta^a) = e_{\mathbf{1}}(u_I \cdot \eta^I)$, as required. □

In the final result we record some general facts concerning descent from F/k to F^H/k for subgroups H of G.

Lemma 5.6. *For each subgroup H of G the following claims are valid.*

(i) *If* [2, Conj. 3.6] *is valid for F/k, then it is also valid for F^H/k.*

(ii) *For each non-negative integer a one has $e_H \cdot \mathrm{Fit}_G^a(\mathrm{Sel}_S^T(F)^{\mathrm{tr}}) = \mathrm{Fit}_{G/H}^a(\mathrm{Sel}_S^T(F^H)^{\mathrm{tr}})$.*

(iii) *The image of the composite homomorphism*

$$\mathrm{Hom}_G(\mathcal{O}_{F,S,T}^\times, \mathbb{Z}[G]) \to \mathrm{Hom}_G(\mathcal{O}_{F^H,S,T}^\times, \mathbb{Z}[G]) \to \mathrm{Hom}_{G/H}(\mathcal{O}_{F^H,S,T}^\times, \mathbb{Z}[G/H]),$$

where the first arrow is the restriction map and the second is induced by the natural projection $\mathbb{Z}[G] \to \mathbb{Z}[G/H]$, is equal to $|H| \cdot \mathrm{Hom}_{G/H}(\mathcal{O}_{F^H,S,T}^\times, \mathbb{Z}[G/H])$.

Proof. Claim (i) is proved in [2, Rem. 3.2 and Prop. 3.4].

To prove claim (ii) we recall that $\mathrm{Sel}_S^T(F)^{\mathrm{tr}}$ is defined in [2, Def. 2.6] to be the cohomology in degree -1 of a complex $C_{F,S}$ that is acyclic in degrees greater than -1 and, since S contains all places that ramify in F/k, is also such that $\mathbb{Z}[G/H] \otimes_{\mathbb{Z}[G]}^{\mathbb{L}} C_{F,S}$ identifies with $C_{F^H,S}$. These facts combine to induce an isomorphism of G/H-modules between $\mathbb{Z}[G/H] \otimes_{\mathbb{Z}[G]} \mathrm{Sel}_S^T(F)^{\mathrm{tr}}$ and $\mathrm{Sel}_S^T(F^H)^{\mathrm{tr}}$ and, given this, the equality of claim (ii) follows from the standard functorial behaviour of higher Fitting ideals under ring extension.

To prove claim (iii) it is enough to note the first arrow in the displayed homomorphism is surjective since the quotient $\mathcal{O}_{F,S,T}^\times / \mathcal{O}_{F^H,S,T}^\times$ is torsion-free, that $\mathrm{Hom}_G(\mathcal{O}_{F^H,S,T}^\times, \mathbb{Z}[G]) = \mathrm{Hom}_G(\mathcal{O}_{F^H,S,T}^\times, T_H \cdot \mathbb{Z}[G])$ and that the projection $\mathbb{Z}[G] \to \mathbb{Z}[G/H]$ sends T_H to $|H|$. □

5.3. We are now ready to prove Theorem 5.1. Our argument splits naturally into two cases depending on whether $a < |S|$ (and so Conjecture 4.3 is relevant) or $a = |S|$ (and Conjecture 4.7 is relevant).

We assume throughout that [2, Conj. 3.6] is valid for F/k.

5.3.1. We first fix an integer a with $0 \leq a < |S|$. We again abbreviate $\mathrm{Fit}_G^a(\mathrm{Sel}_S^T(F))$ to $\mathcal{F}_a$ and in addition now set $\mathcal{F}_a^{\mathrm{tr}} := \mathrm{Fit}_G^a(\mathrm{Sel}_S^T(F)^{\mathrm{tr}})$.

In this case Lemma 5.5 implies that $e_I \cdot \mathcal{E}_{F/k,S,T}^a = e_I \cdot \{\varphi(\eta^I) : \varphi \in \Phi_a\}$ for all I in $\wp_a^*(S)$ and (6) implies $\mathcal{F}_a^{\mathrm{tr}} = \mathcal{F}_a^{\#}$. Thus, to derive the decomposition (14) from Theorem 2.1, it is

enough to show that

$$e_{\mathbf{1}} \cdot \mathcal{F}_a^{\mathrm{tr}} = e_{\mathbf{1}} \cdot \{\varphi(\eta^a) : \varphi \in \Phi_a\} \tag{20}$$

and, if $\wp_a(S, v)$ is non-empty, that

$$e_I \cdot \mathcal{F}_a^{\mathrm{tr}} = e_I \cdot \{\varphi(\eta^I) : \varphi \in \Phi_a\} \quad \text{for all} \quad I \in \wp_a(S, v). \tag{21}$$

We note first that the equality (20) is obvious if $a < |S| - 1$ since both sides vanish, the left hand side by Lemma 3.1(ii) and the right hand side since $e_{\mathbf{1}}(\eta^a) = 0$.

Next we note that if $a = |S| - 1$, then Lemma 5.6(ii) implies $e_{\mathbf{1}} \cdot \mathcal{F}_a^{\mathrm{tr}} = e_{\mathbf{1}} \cdot \mathrm{Fit}^a(\mathrm{Sel}_S^T(k)^{\mathrm{tr}})$ and Lemma 5.6(iii) implies $\{\varphi(\varepsilon_k) : \varphi \in \Phi_a\}$ is equal to $|G|^a \cdot \mathcal{E}_k$ with

$$\mathcal{E}_k := \{\vartheta(\varepsilon_k) : \vartheta \in \textstyle\bigwedge_{\mathbb{Z}}^a \mathrm{Hom}_{\mathbb{Z}}(\mathcal{O}_{k,S,T}^\times, \mathbb{Z})\}.$$

In this case one also has $e_{\mathbf{1}}(\eta^a) = |G|^{-a} \cdot \varepsilon_k$ as a direct consequence of the definition (12) if $S_{\min}^a = \emptyset$ and as a consequence of (19) if $S_{\min}^a \neq \emptyset$.

To prove (20) in the case $a = |S| - 1$ it is thus enough to show that $\mathrm{Fit}^a(\mathrm{Sel}_S^T(k)^{\mathrm{tr}}) = \mathcal{E}_k$. But, in this case, the exact sequence (5) combines with the fact that $X_{k,S}$ is a free $\mathbb{Z}$-module of rank a to imply $\mathrm{Fit}^a(\mathrm{Sel}_S^T(k)^{\mathrm{tr}}) = \mathrm{Fit}^0(\mathrm{Cl}_S^T(k)) = |\mathrm{Cl}_S^T(k)| \cdot \mathbb{Z}$ whilst, by unwinding the explicit definition of ε_k, one finds that $\mathcal{E}_k = \theta_{k/k,S,T}^a(0) \cdot R_{k,S,T}^{-1} \cdot \mathbb{Z}$, with $R_{k,S,T}$ the determinant of the Dirichlet regulator isomorphism $\mathbb{R} \cdot \mathcal{O}_{k,S}^\times \cong \mathbb{R} \cdot X_{k,S}$ with respect to a choice of $\mathbb{Z}$-bases of $\mathcal{O}_{k,S,T}^\times$ and $X_{k,S}$. The required equality $\mathrm{Fit}^a(\mathrm{Sel}_S^T(k)^{\mathrm{tr}}) = \mathcal{E}_k$ is therefore equivalent, up to a sign, to the analytic class number formula of k.

Turning now to consider (21), we note first that if G is trivial, then $\wp_a(S, v)$ is non-empty only if both $a = |S| - 1$ and $v = v_*$ in which case $\wp_a(S, v) = \{I\}$ with $I := S \setminus \{v_*\}$ and both $e_I = e_{\mathbf{1}} = 1$ and $\eta^I = \varepsilon_k$. In this case, therefore, the equality (21) is equivalent to (20) and so follows (unconditionally) from the argument above.

In the sequel we therefore assume that G is not trivial. We also note that, in this case, if $a = r$, then (21) has been derived from the assumed validity of [2, Conj. 3.6] for F^I/k (which itself follows from Lemma 5.6(i)) by the second author in [10, Th. 4.5 and Rem. 4.7].

To derive (21) from [2, Conj. 3.6] in the general case we recall (from the end of the proof of Proposition 5.4) that $e_I \cdot e_{D_I} = e_I$. This equality implies (21) is valid if $e_{D_I} \cdot \mathcal{F}_a^{\mathrm{tr}} = \{\varphi(\eta^I) : \varphi \in \Phi_a\}$ or equivalently, after taking account of Lemma 5.6(ii) (with $H = D_I$) and of the final assertion of Proposition 5.4, if

$$\mathrm{Fit}_{G/D_I}^a(\mathrm{Sel}_S^T(F^I)^{\mathrm{tr}}) = \{\vartheta(\varepsilon^I) : \vartheta \in \textstyle\bigwedge_{\mathbb{Z}[G/D_I]}^a \mathrm{Hom}_{G/D_I}(\mathcal{O}_{F^I,S,T}^\times, \mathbb{Z}[G/D_I])\}$$

where ε^I denotes the Rubin-Stark element for $(F^I/k, S, T)$ defined with respect to the places in I. To complete the derivation of (14) it is thus enough to note that the last displayed equality is derived from the assumed validity of [2, Conj. 3.6] for F^I/k by Kurihara, Sano and the first author in [2, Th. 7.5].

Turning now to the inclusions (15) we note that for v in S and I in $\wp_a^*(S)$ the idempotents $1 - e_v + e_{\mathbf{1}}, e_I$ and $e_{(a),S}$ are all stable under the involution $x \mapsto x^\#$ of $\mathbb{Q}[G]$. The equality (14) therefore implies that for every I in $\wp_a(S, v)$ and every φ in Φ_a the element $e_I \cdot \varphi(\eta_{F/k,S,T}^a)^\#$

belongs to $(1 - e_v + e_{\mathbf{1}})e_{(a),S} \cdot \mathrm{Fit}^a_G(\mathrm{Sel}^T_S(F))$. Given this fact, the inclusions of (15) are easily derived by the same argument used to derive Corollary 2.2 from Theorem 2.1.

This completes the proof that Conjecture 4.3 is a consequence of [2, Conj. 3.6].

5.3.2. In this final section we set $a := |S|$ and assume throughout the notation and hypotheses of Conjecture 4.7. In particular, the group G is now cyclic with generator g.

We fix a prime p and observe it suffices to show [2, Conj. 3.6] implies that the equality (16) is valid after tensoring with $\mathbb{Z}_p$. To do this we shall adapt the approach of [2] (as already used in §3.3).

We set $U := \mathcal{O}^\times_{F,S,T}$ and $\mathfrak{X} := \mathrm{Sel}^T_S(F)^{\mathrm{tr}}$ and use the surjective homomorphism of G-modules $\kappa : \mathfrak{X} \to X_{F,S}$ that occurs in (5).

We choose a homomorphism of G-modules $\pi_1 : \mathbb{Z}[G]^a \to \mathfrak{X}$ with the property that for each index i the image under $\kappa \circ \pi_1$ of the i-th element of the standard basis of $\mathbb{Z}[G]^a$ is $(g-1)w_{v_1}$ if $i = 1$ and is $w_{v_i} - w_{v_1}$ if $i > 1$ where v_i is the i-th place in the ordering of S that has been fixed. We note $\{(g-1)w_1\} \cup \{w_v - w_1\}_{v \in S \setminus \{v_1\}}$ is a set of generators for the G-module $X_{F,S}$ and hence that the map $\kappa \circ \pi_1$ is surjective. We also fix a choice of natural number d for which there exists a surjective homomorphism of G-modules $\pi_2 : \mathbb{Z}[G]^d \to \ker(\kappa)$.

Then, with these choices, the approach of [2, §5.4] shows that there is an exact sequence of $\mathbb{Z}_p[G]$-modules

$$0 \to U_p \xrightarrow{\iota} \mathbb{Z}_p[G]^c \xrightarrow{\theta} \mathbb{Z}_p[G]^c \xrightarrow{\pi} \mathfrak{X}_p \longrightarrow 0 \tag{22}$$

in which $c := a + d$ and, with respect to the identification $\mathbb{Z}_p[G]^c = \mathbb{Z}_p[G]^a \oplus \mathbb{Z}_p[G]^d$, one has $\pi = \mathbb{Z}_p \otimes_{\mathbb{Z}} (\pi_1, \pi_2)$.

In addition, the complex $C_{F,S}$ that is used in the proof of Lemma 5.6(ii), and is described in detail in [2, §2], is such that $\mathbb{Z}_p \otimes_{\mathbb{Z}} C_{F,S}$ identifies with the complex $\mathbb{Z}_p[G]^c \xrightarrow{\theta} \mathbb{Z}_p[G]^c$, where the first term is placed in degree -2, in such a way that $H^{-2}(\mathbb{Z}_p \otimes_{\mathbb{Z}} C_{F,S})$ and $H^{-1}(\mathbb{Z}_p \otimes_{\mathbb{Z}} C_{F,S})$ are identified with U_p and $\mathfrak{X}_p$ via the maps ι and π in (22).

We now set $e := e_{a,S} = 1 - e_{\mathbf{1}}$ and write $\mathcal{A}$ and A for the algebras $\mathbb{Z}_p[G]e$ and $\mathbb{Q}_p[G]e$. For a $\mathbb{Z}_p[G]$-module M we endow the tensor product $M\langle\mathcal{A}\rangle := \mathcal{A} \otimes_{\mathbb{Z}_p} M$ with the natural multiplication action of $\mathcal{A}$ and the diagonal action of G and then define $\mathcal{A}$-modules $M^{\mathcal{A}} := H^0(G, M\langle\mathcal{A}\rangle)$ and $M_{\mathcal{A}} := H_0(G, M\langle\mathcal{A}\rangle) = \mathcal{A} \otimes_{\mathbb{Z}_p[G]} M$. The map $A \otimes_{\mathbb{Z}_p} M \mapsto \mathbb{Q}_p \otimes_{\mathbb{Z}_p} M$ that sends $x \otimes m$ to $x(m)$ for $x \in A$ and $m \in M$ identifies $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} M^{\mathcal{A}}$ with $e(\mathbb{Q}_p \otimes_{\mathbb{Z}_p} M)$ and, if M (and hence $M^{\mathcal{A}}$) is torsion-free, we use this map to identify $M^{\mathcal{A}}$ as a sublattice of $e(\mathbb{Q}_p \otimes_{\mathbb{Z}_p} M)$.

By applying the left, respectively, right, exact functors $M \mapsto M^{\mathcal{A}}$ and $M \mapsto M_{\mathcal{A}}$ to the exact sequence (22) one obtains an exact commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & U_p^{\mathcal{A}} & \xrightarrow{\iota^{\mathcal{A}}} & \mathbb{Z}_p[G]^{\mathcal{A},c} & \xrightarrow{\theta^{\mathcal{A}}} & \mathbb{Z}_p[G]^{\mathcal{A},c} & & \\ & & & & {\scriptstyle\nu}\uparrow & & \uparrow{\scriptstyle\nu} & & \\ & & & & \mathcal{A}^c & \xrightarrow{\theta_{\mathcal{A}}} & \mathcal{A}^c & \xrightarrow{\pi_{\mathcal{A}}} & \mathfrak{X}_{p,\mathcal{A}} \to 0 \end{array} \tag{23}$$

in which ν is the isomorphism induced by the action of $\sum_{g \in G} g$ (and the facts that $\mathbb{Z}_p[G]_{\mathcal{A}} = \mathcal{A}$ and $\mathbb{Z}_p[G]\langle\mathcal{A}\rangle$ is a cohomologically-trivial G-module).

This diagram implies the complex $C_{\mathcal{A}} := \mathcal{A} \otimes^{\mathbb{L}}_{\mathbb{Z}[G]} C_{F,S}$ is represented by $\mathcal{A}^c \xrightarrow{\theta_{\mathcal{A}}} \mathcal{A}^c$, with the first term placed in degree -2, in such a way that $H^{-2}(C_{\mathcal{A}})$ and $H^{-1}(C_{\mathcal{A}})$ are identified with $U_p^{\mathcal{A}}$ and $\mathfrak{X}_{p,\mathcal{A}}$ by using the maps $\nu^{-1} \circ \iota^{\mathcal{A}}$ and $\pi_{\mathcal{A}}$.

Next we note that the present hypotheses on S imply the quotient of $X_{F,S,\mathcal{A}}$ by its torsion subgroup is a free $\mathcal{A}$-module of rank a with basis $\underline{b} := \{(g-1)w_1\} \cup \{e(w_v - w_1)\}_{v \in S \setminus \{v_1\}}$.

This fact combines with the exactness of the lower row in (23) to imply that, with respect to the natural decomposition $\mathcal{A}^c = \mathcal{A}^a \oplus \mathcal{A}^d$, one has $\mathrm{im}(\theta_{\mathcal{A}}) \subseteq \mathcal{A}^d$. This inclusion then in turn combines with the fact that the lower row of (23) constitutes a presentation of the $\mathcal{A}$-module $\mathfrak{X}_{p,\mathcal{A}}$ to imply that

$$(24)\quad e \cdot \mathrm{Fit}^a_G(\mathrm{Sel}^T_S(F))^{\#}_p = e \cdot \mathrm{Fit}^a_G(\mathfrak{X})_p = \mathrm{Fit}^a_{\mathcal{A}}(\mathfrak{X}_{p,\mathcal{A}}) = \{\det(M_{\varphi_\bullet}) : \varphi_\bullet \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{A}^c, \mathcal{A})^a\}.$$

Here the first equality follows from (6) and the second from the standard behaviour of higher Fitting ideals under ring extension and for $\varphi_\bullet = (\varphi_1, \varphi_2, \cdots, \varphi_a)$ we write $M_{\varphi_\bullet}$ for the matrix in $\mathrm{M}_c(\mathcal{A})$ with i-th column equal, if $1 \leq i \leq a$, to the matrix of φ_i with respect to the standard basis of $\mathcal{A}^c$ and, if $a < i \leq c$, to the i-th column of the matrix of $\theta_{\mathcal{A}}$ with respect to the standard basis of $\mathcal{A}^c$.

Next we note that the definition (13) of $\eta^a := \eta^a_{F/k,S,T}$ implies

$$\lambda^a_{F,S}((g-1) \cdot \eta^a) = (g-1) \cdot \lambda^a_{F,S}(\eta^a) = \theta^a_{F/k,S,T}(0) \cdot ((g-1)w_{v_1} \wedge \bigwedge_{v \in S \setminus \{v_1\}} (w_v - w_{v_1})).$$

Given this equality, and the fact that $\pi_{\mathcal{A}}$ sends the standard basis of $\mathcal{A}^a \subset \mathcal{A}^a \oplus \mathcal{A}^d = \mathcal{A}^c$ to the elements in the (ordered) basis $\underline{b}$, the argument of [2, Th. 7.5] shows [2, Conj. 3.6] predicts that for each $\varphi_\bullet = (\varphi_1, \varphi_2, \cdots, \varphi_a)$ in $\mathrm{Hom}_{\mathcal{A}}(\mathcal{A}^c, \mathcal{A})^a$ one has

$$(25)\qquad (g-1)(\bigwedge_{i=1}^{i=a} \rho_\nu(\varphi_i))(\eta^a) = |G|^a \cdot \det(M_{\varphi_\bullet})$$

where ρ_ν denotes the restriction map $\mathrm{Hom}_{\mathcal{A}}(\nu^{-1} \circ \iota^{\mathcal{A}}, \mathcal{A})$. We note that the factor $|G|^a$ arises in this formula because, with respect to the natural identifications $\mathbb{Q}_p[G]^c_{\mathcal{A}} = \mathbb{Q}_p[G]^{c,\mathcal{A}} = A^c$, the map $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \nu$ is multiplication by $|G|$.

To derive the required equality (16) from (24) and (25) it suffices to show that

$$(26)\quad \{(\bigwedge_{i=1}^{i=a} \vartheta_i)(\eta^a) : \vartheta_i \in \mathrm{Hom}_G(U, \mathbb{Z}[G])_p\} = |G|^a \cdot \{(\bigwedge_{i=1}^{i=a} \rho_\nu(\varphi_i))(\eta^a) : \varphi_i \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{A}^c, \mathcal{A})\}.$$

To verify this we note that, as the cokernel of the map ι in (22) is torsion-free, the map $\rho := \mathrm{Hom}_{\mathbb{Z}_p[G]}(\iota, \mathbb{Z}_p[G])$ is surjective and so the left hand side of (26) is equal to

$$\{(\bigwedge_{i=1}^{i=a} \rho(\varpi_i))(\eta^a) : \varpi_i \in \mathrm{Hom}_{\mathbb{Z}_p[G]}(\mathbb{Z}_p[G]^c, \mathbb{Z}_p[G])\}.$$

To show that this set is also equal to the right hand side of (26) (and hence that the latter equality is valid) the key point to note is that, with respect to the identification of $\mathbb{Z}_p[G]^{\mathcal{A}}$ as a sublattice of A that has been fixed above, one has $\mathbb{Z}_p[G]^{\mathcal{A}} = |G| \cdot \mathcal{A}$ and hence

that the subset of $\mathrm{Hom}_{\mathcal{A}}(\mathbb{Z}_p[G]^{\mathcal{A},c}, \mathcal{A})$ comprising homomorphisms induced by elements of $\mathrm{Hom}_{\mathbb{Z}_p[G]}(\mathbb{Z}_p[G]^c, \mathbb{Z}_p[G])$ is equal to $|G| \cdot \mathrm{Hom}_{\mathcal{A}}(\mathbb{Z}_p[G]^{\mathcal{A},c}, \mathcal{A})$.

This completes the proof that [2, Conj. 3.6] implies Conjecture 4.7.

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