A NOTE ON SELMER GROUPS
AND REFINED STARK CONJECTURES

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Abstract. We prove that certain higher Fitting ideals of Selmer groups of the multiplicative group over abelian extensions of number fields have a natural direct sum decomposition. We derive consequences of this decomposition concerning higher-order refined Stark conjectures and, in particular, use it to formulate, and provide strong evidence for, the first (and, in a natural sense, best possible) such conjecture ‘at the boundary’.

1. Statement of the main algebraic result

Let $F/k$ be a finite abelian extension of number fields of group $G$. Let $S$ be a finite set of places of $k$ containing the set $S_\infty$ of archimedean places and all places that ramify in $F$. Then for any finite set $T$ of places of $k$ that is disjoint from $S$ the ‘$S$-relative $T$-trivialized integral Selmer group’ for $G$ over $F$ has been defined by Kurihara, Sano and the first author by setting

$$\text{Sel}_S^T(F) := \text{cok}(\prod_w \mathbb{Z} \longrightarrow \text{Hom}_\mathbb{Z}(F_T^\times, \mathbb{Z})),$$

(see [1, Def. 2.1] where the notation $S_{S,T}(\mathbb{G}_m/F)$ is used). Here in the product $w$ runs over all places of $F$ that do not lie above places in $S \cup T$, $F_T^\times$ is the subgroup of $F^\times$ comprising elements $u$ for which $u - 1$ has a strictly positive valuation at each place above $T$ and the unlabeled arrow sends each element $(x_w)_w$ to the map $(u \mapsto \sum \text{ord}_w(u)x_w)$.

We recall that this group is defined as a natural analogue for $G$ of the integral Selmer groups of abelian varieties that are defined by Mazur and Tate in [10] and, in particular, lies in a canonical exact sequence of $G$-modules of the form

$$0 \rightarrow \text{Hom}_\mathbb{Z}(\text{Cl}_S^T(F), \mathbb{Q}/\mathbb{Z}) \rightarrow \text{Sel}_S^T(F) \rightarrow \text{Hom}_\mathbb{Z}(\mathcal{O}_{F,S,T}^\times, \mathbb{Z}) \rightarrow 0.$$

(see [1, Prop. 2.2]). Here $\text{Cl}_S^T(F)$ is the ray class group modulo the product of all places of $F$ above $T$ of the subring $\mathcal{O}_{F,S}$ of $F$ comprising elements that are integral at all places outside $S$ and $\mathcal{O}_{F,S,T}^\times$ is the group $F_T^\times \cap \mathcal{O}_{F,S}^\times$ and both duals are endowed with the contragredient action of $G$.

We write $\hat{G}$ for the set of homomorphisms $G \rightarrow \mathbb{C}^\times$ and 1 for the trivial element of $\hat{G}$. For each non-negative integer $a$ we write $\hat{G}_{a,S}$ for the subset of $\hat{G}\setminus \{1\}$ comprising homomorphisms $\psi$ for which the set $S_{\psi} := \{v \in S : G_v \subseteq \ker(\psi)\}$ has cardinality $a$ and then define $\hat{G}_{a,S}$ to be $\hat{G}_{a,S} \cup \{1\}$ if $a = |S| - 1$ and to be $\hat{G}_{a,S}$ if $a \neq |S| - 1$. We also write $S_{\min}$ for the union of $S_{\psi}$ as $\psi$ ranges over $\hat{G}_{a,S}\setminus \{1\}$.

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For any set $\Sigma$ we write $\varphi_a(\Sigma)$ for the set of subsets of $\Sigma$ of cardinality $a$. We abbreviate $\varphi_a(S_{\text{min}})$ to $\varphi_a(S)$ and for $v$ in $S$ we set $\varphi_a(S,v) := \{I \in \varphi_a(S) : v \notin I\}$, write $e_v$ for the idempotent $|G_v|^{-1} \sum_{g \in G_v} g$ of $\mathbb{Q}[G]$ and $e_{a,S}$ for the idempotent $\sum_{\psi \in \hat{G}_{a,S}} e_{\psi}$. For each $I$ in $\varphi_a(S)$ we write $e_I$ for the idempotent of $\mathbb{Q}[G]$ given by the sum $e_1 + \sum_{\psi} e_{\psi}$ where for each $\psi$ in $\hat{G}$ we write $e_{\psi}$ for the associated idempotent of $\mathbb{C}[G]$ and in the sum $\psi$ runs over the set $\{\psi \in \hat{G}_{a,S} \setminus \{1\} : S_{\psi} = I\}$.

Finally, we write $S_{\text{sp}}$ for the subset of $S$ comprising places that split completely in $F$.

We can now state our main algebraic result.

**Theorem 1.1.** Write $r = r_{F,k,S}$ for the maximum non-negative integer such that $L_S(\psi, z)$ vanishes at $z = 0$ to order at least $r$ for each $\psi$ in $\hat{G}$. Then, if the group $\mathcal{O}_{F,S,T}$ is torsion-free, for each place $v$ in $S$ there is a direct sum decomposition

$$(2)\quad (1 - e_v + e_1) \cdot \text{Fit}_G^r(\text{Sel}_S^T(F)) = c_{S,v} e_1 \cdot \text{Fit}_G^r(\text{Sel}_S^T(F)) \oplus \bigoplus_{I \in \varphi_1(S,v)} e_I \cdot \text{Fit}_G^r(\text{Sel}_S^T(F))$$

where $c_{S,v}$ denotes the integer $|S \setminus \{S_{\text{sp}} \cup \{v\}\}| + |S_{\text{sp}}| - r$.

Set $\mathcal{F} := \text{Fit}_G^r(\text{Sel}_S^T(F))$ and for each place $v$ in $S$ write $n_v = n_v(F/k)$ and $m_v = m_v(F/k)$ for the least natural number $b$ with $b \cdot (e_v - e_1) \mathcal{F} \subseteq \mathbb{Z}[G]$ and $b \cdot (e_v - e_1) \mathcal{F} \subseteq \mathcal{F}$ respectively.

Then Theorem 1.1 implies that for each $I$ in $\varphi_1(S)$ one has $n_v \cdot e_I \mathcal{F} \subseteq \mathbb{Z}[G]$ and $m_v \cdot e_I \mathcal{F} \subseteq \mathcal{F}$ for all $v \in S \setminus I$ and hence also that $n_I \cdot e_I \mathcal{F} \subseteq \mathbb{Z}[G]$ and $m_I \cdot e_I \mathcal{F} \subseteq \mathcal{F}$ where $n_I$ and $m_I$ denote the greatest common divisor of the integers $n_v$, respectively $m_v$, for $v$ in $S \setminus I$.

Thus, if we write $n_S^T(F/k)$, resp. $m_S^T(F/k)$, for the least common multiple of the integers $n_I$, resp. $m_I$, as $I$ ranges over $\varphi_1(S)$, then Theorem 1.1 implies the following result.

**Corollary 1.2.** Assume the notation and hypotheses of Theorem 1.1. Then for each $I$ in $\varphi_1(S)$ one has

$$n_S^T(F/k) \cdot \bigoplus_{I \in \varphi_1(S)} e_I \text{Fit}_G^r(\text{Sel}_S^T(F)) \subseteq \mathbb{Z}[G]$$

and

$$m_S^T(F/k) \cdot \bigoplus_{I \in \varphi_1(S)} e_I \text{Fit}_G^r(\text{Sel}_S^T(F)) \subseteq \text{Fit}_G^r(\text{Sel}_S^T(F)).$$

**Remark 1.3.** It is not difficult to make the result of Corollary 1.2 more explicit.

(i) Assume $S \neq S_{\text{min}}$. Then Lemma 2.1(iii) below implies that $\text{Fit}_G^r(\text{Sel}_S^T(F))$ vanishes for each place $v$ in $S \setminus S_{\text{min}}$. In particular, for each such $v$ the left hand side of (2) is equal to $\text{Fit}_G^r(\text{Sel}_S^T(F))$ and one has $n_v(F/k) = m_v(F/k) = 1$ and hence also $n_S^T(F/k) = m_S^T(F/k) = 1$.

(ii) Assume $S = S_{\text{min}}$. As noted by Emmons in [4, §5.4], this is a natural ‘boundary case’ in the setting of Stark’s conjectures and has hitherto been excluded in the formulation of any refined versions (see, for example, [5, 7, 13]).

We note first that in this case $|S| > r + 1$. To see this note that if $|S| = r + 1$, then [5, Lem. 2.2] implies $|S_{\text{sp}}| \geq r$. Further, if in this case $|S_{\text{sp}}| = r$, then also $S_{\text{sp}} = S_{\text{min}}$ and hence $S_{\text{min}} \neq S$, whilst if $|S_{\text{sp}}| > r$, then $S_{\text{sp}} = S$ and $S_{\text{min}}$ is empty and so $S_{\text{min}} \neq S$. 
Then, since $|S| > r + 1$, Lemma 2.1(ii) below implies $e_1 \text{Fit}_G^r(\text{Sel}_S^r(F))$ vanishes. This in turn implies for $v \in S$ that $n_v(F/k)$ divides $|G_v|$ and can be strictly smaller depending on the structure of $\text{Fit}_G^r(\text{Sel}_S^r(F))$. In particular, in all cases $n_v^r(F/k)$ divides the lowest common multiple of $|G_v|$ for $v$ in $S_{\min}^r$. For a concrete application of this observation see Example 3.3 below.

2. THE PROOF OF THEOREM 1.1

We assume throughout this section the notation and hypotheses of Theorem 1.1.

2.1. We start by observing that an explicit analysis of the functional equation of Artin $L$-series (as per [12, Chap. I, Prop. 3.4]) combines with the exact sequence (1) to imply that for each $\psi$ in $\hat{G}$ the order of vanishing at $z = 0$ of $L_S(\psi, z)$, where $\psi$ is the contragredient of $\psi$, is equal to

$$\dim_{\mathbb{C}}(e_\psi(\mathbb{C} \otimes_{\mathbb{Z}} \text{Sel}_S^r(F))) = \begin{cases} |S_\psi|, & \text{if } \psi \neq 1, \\ |S| - 1, & \text{if } \psi = 1. \end{cases}$$

(3)

This shows that the integer $r$ defined in Theorem 1.1 is at most $|S| - 1$ and also such that $\hat{G}_{a,S}$ is empty for each $a < r$ but that $\hat{G}_{r,S}$ is empty only if both $r = |S| - 1$ and all places in $S$ split completely in $F/k$ (in which case $\hat{G}_{r+1,S} = \hat{G}$). For this reason in the sequel we set $S_{\min} := S_{\min}^r$.

We next record a result that will allow us to treat a special case of Theorem 1.1 (and also justifies observations made in Remark 1.3).

**Lemma 2.1.**

(i) For each $\psi$ in $\hat{G} \setminus \hat{G}_{r,S}$ the module $e_\psi \cdot \text{Fit}_G^r(\text{Sel}_S^r(F))$ vanishes.

(ii) If $|S| > r + 1$, then $e_1 \cdot \text{Fit}_G^r(\text{Sel}_S^r(F))$ vanishes.

(iii) If $v \in S \setminus S_{\min}$, then $(e_v - e_1) \cdot \text{Fit}_G^r(\text{Sel}_S^r(F))$ vanishes.

**Proof.** For $\psi$ in $\hat{G}$, a finitely generated $G$-module $Z$ and a non-negative integer $a$ it is easy to see that the sublattice $e_\psi \cdot \text{Fit}_G^a(Z)$ of $\mathbb{C}[G]$ will vanish whenever $\dim_{\mathbb{C}}(e_\psi(\mathbb{C} \otimes_{\mathbb{Z}} Z)) > a$.

Using this observation, claim (i) follows immediately from (3) and the definition of $\hat{G}_{r,S}$ and claim (ii) is a special case of claim (i).

To derive claim (iii) from claim (i) it suffices to prove that for each $\psi$ in $\hat{G} \setminus \{1\}$ with $G_v \subseteq \ker(\psi)$ one has $\psi \notin \hat{G}_{r,S}$. This follows from the fact that, as $v$ does not belong to $S_{\min}$, the inclusion $G_v \subseteq \ker(\psi)$ implies $|S_\psi| > r$. \hfill $\square$

We can now quickly prove Theorem 1.1 in the case that $|S| = r + 1$.

As already noted in Remark 1.3(ii), in this case one has either $S_{\min} = S_{sp}$ and hence $|S_{sp}| = r$ or $S_{\min} = \emptyset$ and $S_{sp} = S$.

We assume first that $S_{\min} = S_{sp}$ and fix $v$ in $S$.

If $v \in S_{\min}$, then $e_v = 1$ and so the left hand side of (2) is equal to $e_1 \text{Fit}_G^r(\text{Sel}_S^r(F))$. This is in turn clearly equal to the right hand side of (2) since in this case $c_{S,v} = \left(|S| - |S_{sp}| + |S_{sp}|-r\right) = 1$ whilst $v$ belongs to $I_\psi := \{v \in S : G_v \subseteq \ker(\psi)\}$ for all $\psi$ in $\hat{G}_{r,S} \setminus \{1\}$ so $\varphi_r(S,v) = \emptyset$. 

On the other hand, if \( v \in S \setminus S_{\text{min}} \), then Lemma 2.1(iii) implies the left hand side of (2) is equal to \( \text{Fit}_G^r(\text{Sel}^T_S(F)) \) and this is equal to the right hand side as \( c_{S,v} = |S| - |S| + |S - r| = 0 \) and \( \varphi_r(S,v) = S_{\text{min}} = S_{\text{sp}} \) consists of a single element \( I \) for which one has \( e_I = 1 \).

We assume finally that \( S_{\text{min}} = \emptyset \) and hence that \( S = S_{\text{sp}} \). In this case the claimed equality is true since for each \( v \) in \( S \) one has \( e_v = 1 \), \( c_{S,v} = |S| - |S| - r = 1 \) and \( \varphi_r(S,v) = \emptyset \) and so the left and right hand sides of (2) are both equal to \( e_1 \text{Fit}_G^r(\text{Sel}^T_S(F)) \).

This completes the proof of Theorem 1.1 in the case that \( |S| = r + 1 \).

2.2. In the remainder of the argument we will thus assume that \( |S| > r + 1 \).

We note that in this case Lemma 2.1(ii) implies that the term \( 1 - e_v + e_1 \) on the left hand side of (2) can be replaced by \( 1 - e_v \) and that the first summand on the right hand side of (2) can be omitted.

We next recall (from [1, §2.2]) that \( \text{Sel}^T_S(F) \) has a canonical transpose \( \text{Sel}^T_S(F)^{tr} \) in the sense of Jannsen’s homotopy theory of modules [8], that there is a canonical exact sequence

\[
0 \rightarrow \text{Cl}^T_S(F) \rightarrow \text{Sel}^T_S(F)^{tr} \xrightarrow{r} X_{F,S} \rightarrow 0,
\]

with \( X_{F,S} \) the submodule of the free abelian group \( Y_{F,S} \) on the set of places of \( F \) above \( S \) comprising elements whose coefficients sum to zero, and that, since \( \mathcal{O}_{F,S,T}^\infty \) is assumed to be torsion-free, [1, Lem. 2.8] implies

\[
\text{Fit}_G^r(\text{Sel}^T_S(F)) = \text{Fit}_G^r(\text{Sel}^T_S(F)^{tr})^\#,
\]

with \( x \mapsto x^\# \) the \( \mathbb{Z} \)-linear involution on \( \mathbb{Z}[G] \) that inverts elements of \( G \).

As a final preparatory remark, we note that it suffices to prove the equality of Theorem 1.1 after tensoring with \( \mathbb{Z}_p \) for every prime \( p \).

We therefore fix a prime \( p \) and for each abelian group \( A \) write \( A_p \) in place of \( \mathbb{Z}_p \otimes A \). By adapting an approach used in [1], we now construct a convenient resolution of \( \text{Sel}^T_S(F)^{tr}_p \).

We fix a place \( v \) in \( S \) and a place \( w \) of \( F \) above \( v \), we set \( d_1 := |S| - 1 \), write the elements of \( S \setminus \{ v \} \) as \( \{ v_i \}_{1 \leq i \leq d_1} \) and for each index \( i \) we fix a place \( w_i \) of \( F \) above \( v_i \). We write \( \mathbb{Z}_p[G]^{d_1} \) for the direct sum of \( d_1 \) copies of \( \mathbb{Z}_p[G] \) and choose a homomorphism of \( \mathbb{Z}_p[G] \)-modules \( \pi_1 : \mathbb{Z}_p[G]^{d_1} \rightarrow \text{Sel}^T_S(F)^{tr} \) with the property that, for each index \( i \), the image under \( \kappa \circ \pi_1 \) of the \( i \)-th element of the standard basis of \( \mathbb{Z}_p[G]^{d_1} \) is \( w_i - w \). We also write \( \varpi_v \) for the natural projection \( X_{F,S} \rightarrow Y_{F,S \setminus \{ v \}} \) and fix a natural number \( d_{2,v} \) for which there exists a surjective homomorphism of \( \mathbb{Z}_p[G] \)-modules of the form \( \pi_{2,v} : \mathbb{Z}_p[G]^{d_{2,v}} \rightarrow \ker(\varpi_v \circ \kappa) \).

Then, with these choices, the approach of [1] shows that there is an exact sequence of \( \mathbb{Z}_p[G] \)-modules of the form

\[
\mathbb{Z}_p[G]^{d_v} \xrightarrow{\theta_v} \mathbb{Z}_p[G]^{d_v} \xrightarrow{\pi_v} \text{Sel}^T_S(F)^{tr}_p \rightarrow 0
\]

with \( d_v := d_1 + d_{2,v} \) and \( \pi_v := (\pi_1, \pi_{2,v}) \).

If we write \( M_v \) for the matrix of \( \theta_v \) with respect the standard basis of \( \mathbb{Z}_p[G]^{d_v} \), then the equality (5) combines with the very definition of \( r \)-th Fitting ideals to imply that

\[
\text{Fit}_G^r(\text{Sel}^T_S(F))_p = \sum_{J \in \mathcal{P}_r(d_v)} m_v(J)
\]
with \( \varphi_r(d_v) := \varphi_r(\{i \in \mathbb{Z} : 1 \leq i \leq d_v\}) \) and \( m_v(J) \) denoting the ideal of \( \mathbb{Z}_p[G] \) that is generated by the determinants of all \((d_v - r) \times (d_v - r)\) minors of the \(d_v \times (d_v - r)\) matrix \( M_v(J)^\# \) obtained by deleting all columns of \( M_v \) corresponding to integers in \( J \).

Next we note that if \( \psi \in \hat{G} \setminus \{1\} \), then one has
\[
v \notin I_\psi \iff G_v \not\subseteq \ker(\psi) \iff (1 - e_v)e_\psi \neq 0
\]
and hence that
\[
(1 - e_v)e_{r,S} = \sum_{I \in \varphi_r(S,v)} (e_I - e_1).
\]
This equality combines with (7) to imply that
\[
(1 - e_v) \cdot \text{Fit}_G^r(\text{Sel}_S^T(F))_p = (1 - e_v)e_{r,S} \cdot \text{Fit}_G^r(\text{Sel}_S^T(F))_p
\]
\[
= (\sum_{I \in \varphi_r(S,v)} (e_I - e_1)) \cdot (\sum_{J \in \varphi_r(d_v)} m_v(J))
\]
\[
= \sum_{I \in \varphi_r(S,v)} (e_I - e_1) \cdot m_v(I)
\]
\[
= \sum_{I \in \varphi_r(S,v)} (e_I - e_1) \cdot \text{Fit}_G^r(\text{Sel}_S^T(F))_p.
\]

The first equality here is true as Lemma 2.1(i) implies \( \text{Fit}_G^r(\text{Sel}_S^T(F)) = e_{r,S} \cdot \text{Fit}_G^r(\text{Sel}_S^T(F)) \) and the second and third equalities follow directly from the result of Lemma 2.2 below.

To deduce the equality (2) in the case that \( |S| > r + 1 \) it thus suffices to observe firstly that the last sum in the above formula is direct since for distinct elements \( I_1 \) and \( I_2 \) of \( \varphi_r(S) \) the idempotents \( e_{I_1} - e_1 \) and \( e_{I_2} - e_1 \) are orthogonal, and then that for each such \( I \) Lemma 2.1(ii) implies \( (e_I - e_1) \cdot \text{Fit}_G^r(\text{Sel}_S^T(F)) = e_I \cdot \text{Fit}_G^r(\text{Sel}_S^T(F)) \).

In the following result we use the correspondence \( v_i \leftrightarrow i \) to identify \( \varphi_r(S \setminus \{v\}) \) with the subset \( \varphi_r(d_1) \) of \( \varphi_r(d_v) \).

**Lemma 2.2.** Take \( \psi \in \hat{G} \setminus \{1\} \) with \( v \notin I_\psi \). Then \( I_\psi \) belongs to \( \varphi_r(d_v) \) and for each \( J \) in \( \varphi_r(d_v) \setminus \{I_\psi\} \) one has \( e_\psi \cdot m_v(J) = 0 \).

**Proof.** Set \( G_\psi := G/\ker(\psi) \) and \( F_\psi := F^{\ker(\psi)} \).

Then to compute \( e_\psi \cdot m_v(J) \) one can replace \( m_v(J) \) by its image in \( \mathbb{Z}[G_\psi] \). Note also that the exact sequence (6) implies that each column of the image \( M_v(J)^\#_\psi \) in \( \mathbb{Z}[G_\psi] \) of \( M_v(J)^\# \) that corresponds to an integer in \( I_\psi \setminus J \) must vanish since the corresponding place in \( S \setminus \{v\} \) splits completely in \( F_\psi/k \).

In particular, if \( J \neq I_\psi \), then \( M_v(J)^\#_\psi \) has at least one column of zeroes and so the determinant of any of its \((d_v - r) \times (d_v - r)\) minors must vanish. It follows that \( e_\psi \cdot m_v(J) = 0 \), as claimed. \( \square \)

3. **Refined Stark Conjectures**

In this section we formulate, and provide strong supporting evidence for, a conjecture that extends the existing theory of refined Stark conjectures in that it does not exclude the ‘boundary case’ that \( S \) is equal to \( S_{\min} \) (we note that the possibility that such a conjecture
could be formulated in this case was first suggested by Dummit in [2]). Our conjecture is motivated by the result of Theorem 1.1 and so we will assume throughout the notation and hypotheses of that result.

3.1. Before formulating the conjecture we recall that in [5, §3] Emmons and Popescu define a canonical ‘regulator’ isomorphism of $\mathbb{C}[G]$-modules

$$\mathcal{R}_{F/k,S} : e_r,S(C \cdot \bigwedge^r_G \mathcal{O}_{F,S}^\times) \cong \mathbb{C}[G]e_r,S.$$  

We further recall that for each $I$ in $\mathfrak{S}_I(S)$ Vallières [13] defines $\eta^I_{F/k,S,T}$ to be the unique element of $(e_I - e_1)(C \cdot \bigwedge^r_G \mathcal{O}_{F,S}^\times)$ with $\mathcal{R}_{F/k,S}(\eta^I_{F/k,S,T}) = \theta^*_{F/k,S,T}(0) \cdot (e_I - e_1)$, where $\theta^*_{F/k,S,T}(0)$ is the leading term at zero of the $S$-truncated $T$-modified $\mathbb{C}[G]$-valued $L$-function that is associated to $F/k$. We finally recall that the ‘Emmons-Popescu evaluator’ $\eta^I_{F/k,S,T}$ that is defined in [5] is equal to the sum $\sum_{I \in \mathfrak{S}_I(S)} \eta^I_{F/k,S,T}$.

We can now state our conjecture.

**Conjecture 3.1.** Assume the hypotheses of Theorem 1.1 and set $\eta := \eta^I_{F/k,S,T}$, $\eta^I := \eta^I_{F/k,S,T}$ and $\Phi := \bigwedge^r_G \text{Hom}_G(\mathcal{O}_{F,S,T}^\times, \mathbb{Z}[G])$. Then for each $v$ in $S$ one has

$$(8) \quad (1-e_v + e_1) \cdot \text{Fit}^r_G(\text{Sel}_S^r(F))^\# = c_{S,v}e_1 \cdot \{\varphi(\eta_{k,S,T}) : \varphi \in \Phi \} \oplus \bigoplus_{I \in \mathfrak{S}_I(S,v)} \{\varphi(\eta^I) : \varphi \in \Phi \}.$$  

In particular, for each $\varphi$ in $\Phi$ one has both

$$(9) \quad n_S^T(F/k) : \varphi(\eta) \in \mathbb{Z}[G] \quad \text{and} \quad m_S^T(F/k) : \varphi(\eta) \in \text{Fit}^r_G(\text{Sel}_S^T(F)).$$  

**Remark 3.2.** If $S \neq S_{\text{min}}$, then for each $v \in S \setminus S_{\text{min}}$ one has $(1-e_v + e_1) \cdot \text{Fit}^r_G(\text{Sel}_S^T(F)) = \text{Fit}^r_G(\text{Sel}_S^T(F))$ and $n_S^T(F/k) = m_S^T(F/k) = 1$ (see Remark 1.3(ii)). In this case the equality (8) was first predicted by the second author in [9] and the first, respectively second, containment in (9) specialises to give the conjecture formulated (under the hypothesis $S \neq S_{\text{min}}$) by Emmons and Popescu in [5, Conj. 3.8], respectively by the second author in [9]. In particular, if $S_{\text{min}}$ comprises $r$ places that split completely in $F/k$ (in which case $\eta^I_{F/k,S,T}$ is a ‘Rubin-Stark element’ for $F/k$), then Conjecture 3.1 recovers the conjectures formulated by Rubin in [11].

**Example 3.3.** Let $k = \mathbb{Q}(\alpha)$ with $\alpha^3 - 19\alpha + 21 = 0$ and write $F$ for its strict Hilbert class field. Then $k$ is a totally real non-Galois extension of $\mathbb{Q}$, $F/k$ is unramified outside $S_\infty$ and $\text{Gal}(F/k)$ is isomorphic to $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. In addition, if one sets $S := S_\infty$, then the minimal order of vanishing at $z = 0$ of the functions $L_S(\psi, z)$ for $\psi \in \tilde{G}$ is one and $S = S_{\text{min}}^1$ and $|G_v| = 2$ for each $v$ in $S$. In this case Remark 1.3(ii) applies with $|S| = 3$ and $r = 1$ and shows that $n_S^T(F/k)$ is either 1 or 2. Since in this case Erickson has used numerical computations of Dummit and Hayes [3] to show that for some choice of $T$ the element $\eta^I_{F/k,S,T}$ does not belong to Rubin’s lattice $\Lambda_{F/k,S,T,1}$ (see the discussion of [7, §7], and [6, §4.2] for the details), and hence that there exists a homomorphism $\varphi$ in $\Phi$ such that $\varphi(\eta^I_{F/k,S,T}) \notin \mathbb{Z}[G]$, this shows that the first containment predicted by (9) is in a natural sense best possible.
3.2. The main evidence that we can offer in support of Conjecture 3.1 is the following result.

**Theorem 3.4.** Conjecture 3.1 is implied by the validity of the equivariant Tamagawa number conjecture for the pair \( h^0(\text{Spec}(F), \mathbb{Z}[G]) \).

**Proof.** We assume first that \( |S| = r + 1 \) and hence, following Remark 1.3(ii), that either \( S_{\text{min}} = S_{\text{sp}} \) and \( |S_{\text{sp}}| = r \) or \( S_{\text{min}} = \emptyset \) and \( S_{\text{sp}} = S \).

If \( S_{\text{min}} = S_{\text{sp}} \) and \( |S_{\text{sp}}| = r \), then \( \eta \) is a Rubin-Stark element for \( F/k \) and \( I = S_{\text{min}} \) is the only element of \( \varphi^*_r(S) \) so \( \eta' = \eta \). Given these facts, (8) follows directly upon combining Theorem 1.1 with [1, Th. 1.5].

On the other hand, if \( S_{\text{min}} = \emptyset \), then \( \varphi_r(S,v) = \emptyset \) and so Theorem 1.1 implies that the claimed equality is true since \( e_1 \cdot \text{Fit}_G^r(\text{Sel}^T_S(F)) = e_1 \cdot \{ \varphi(\eta_{k/k,S,T}) : \varphi \in \Phi \} \) as an easy special case of [9, Th. 4.5].

In the remainder of the argument we assume \( |S| > r + 1 \). By the same argument as at the beginning of §2.2, it therefore suffices to prove (8) after removing all terms involving the idempotent \( e_1 \).

The key fact that we shall use is that for each \( I \) in \( \varphi^*_r(S) \) one has

\[
(10) \quad e_I \cdot \text{Fit}_G^r(\text{Sel}^T_S(F))^\# = e_I \cdot \text{Fit}_G^r(\text{Sel}^T_S(F)^\{\eta\}) = \{ \varphi(\eta^I) : \varphi \in \Phi \}.
\]

The first equality here follows directly from (5) and the second can be derived from results of the second author in [9]. To be precise, to obtain the second equality one need only combine [9, Rem. 4.7] with the equality in [9, Th. 4.5], noting that the latter result does not depend on the hypothesis \( S \neq S_{\text{min}} \) since, after fixing \( I \), the derivation given in loc. cit. works with any place in \( S \setminus I \) as a substitute for the specified place \( v_0 \). (Note, however, that the supplementary inclusion that is proved in [9, Th. 4.5], but which is not required here, does depend on the hypothesis \( S \neq S_{\text{min}} \). We also note that, whilst the idempotent \( e_1 \) defined in loc. cit. differs (by a summand of \( e_1 \)) from that defined here, this difference does not matter since \( e_1 \cdot \text{Fit}_G^r(\text{Sel}^T_S(F)) \) vanishes in this case.

The equality (8) is now obtained by simply combining (10) with Theorem 1.1 and (9) follows by combining (10) with Corollary 1.2 and the fact that \( \varphi(\eta) = \sum_{I \in \varphi_r(S)} \varphi(\eta^I). \)

Since the equivariant Tamagawa number conjecture for \( h^0(\text{Spec}(F), \mathbb{Z}[G]) \) is known to be true when \( F \) is an abelian extension of \( \mathbb{Q} \) (by work of Greither and the first author, and of Flach), the last result has the following direct consequence.

**Corollary 3.5.** Conjecture 3.1 is valid if \( F \) is an abelian extension of \( \mathbb{Q} \) (and \( k \) is any subfield of \( F \)).

**Example 3.6.** Let \( p_1, p_2 \) and \( p_3 \) be distinct prime numbers with \( p_1 \equiv -p_2 \equiv -p_3 \equiv 1 \) (mod 4) and such that the respective Legendre symbols satisfy \( (p_1/p_2) = (p_2/p_3) = -(p_3/p_1) = -1 \). Then, setting \( F := \mathbb{Q}(\sqrt{p_1}, \sqrt{-p_2}, \sqrt{-p_3}) \), the abelian extension \( F/\mathbb{Q} \) satisfies the hypotheses of Conjecture 3.1 with \( r = 1 \) and both \( S \) and \( S_{\text{min}} \) equal to the set \( \{ \infty, p_1, p_2, p_3 \} \) of places that ramify in \( F \). Corollary 3.5 therefore implies that Conjecture 3.1 is valid in this case. In addition, for any finite set of primes \( T \) that is disjoint from \( S \) one checks easily in this case that \( n^T_S(F/\mathbb{Q}) \) is either 1 or 2.
REFERENCES


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