ON DERIVATIVES OF ARTIN L-SERIES

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Abstract. We present a universal theory of refined Stark conjectures. We give evidence in support of these explicit and very general conjectures, including a full proof in several important cases, and explain the connection to previous conjectures of Bloch and Kato, of Lichtenbaum and of Serre and Tate. We also deduce a wide range of unconditional consequences of these results concerning the annihilation, as Galois modules, of ideal class groups by explicit elements constructed from the values of higher order derivatives of (non-abelian) Artin L-series.

1. Introduction

Let K/k be a finite Galois extension of global fields of group G. Fix a finite non-empty set of places S of k that contains the set S_∞ of archimedean places (in the number field case). Let $\mathrm{Ir}(G)$ denote the set of irreducible complex characters of G and for each ψ in $\mathrm{Ir}(G)$ write e_ψ for the primitive central idempotent $\psi(1)|G|^{-1}\sum_{g\in G}\psi(g^{-1})g$ of the complex group ring $\mathbb{C}[G]$ and $L_S(\psi,z)$ for the S-truncated Artin L-series of ψ. Then one obtains a $\mathbb{C}[G]$-valued meromorphic function of z by setting

$$\theta_{K/k,S}(z) := \sum_{\psi\in\mathrm{Ir}(G)} e_\psi L_S(\check{\psi},z) \tag{1}$$

where $\check{\psi}$ is the contragredient of ψ.

The seminal conjecture of Stark, as interpreted by Tate, predicts that the leading term at $z=0$ of $\theta_{K/k,S}(z)$ is equal, up to multiplication by an undetermined element of $\mathbb{Q}[G]$, to a regulator constructed from algebraic units. In the case that G is abelian, subsequent refinements of this conjecture (due firstly to Stark with later refinements and generalisations by, amongst others, Gross, Rubin and Tate - see below for more details) have in effect predicted that the undetermined rational factor in Stark's Conjecture should satisfy a variety of explicit integrality conditions. It has long been asked whether it is reasonable to expect analogous refinements of Stark's Conjecture in the general case. Such refinements it may be hoped could provide some much needed insight into arithmetic properties of possible groups of special units in non-abelian contexts and would also have significant advantages over the original conjecture of Stark for the purposes of obtaining evidence via computer calculations (see the discussion of Dummit in [26, §14]). However, apart from the important but very restricted cases studied by Stark [50, 52] and Chinburg [22], where such conjectures are linked to attempts to develop constructive versions of the Deligne-Serre Theorem, and the explicit but comparatively weak reinterpretation of Chinburg's 'Strong-Stark Conjecture' [23] that is given in [12], the present author is not aware of any other existing explicit predictions, let alone results, about the nature of the rational factor in the non-abelian case of Stark's Conjecture.

In this article we now present a treatment of refined Stark conjectures that appears to be definitive. For example, our central conjecture (Conjecture 2.4.1) is an explicit refinement of Stark's Conjecture that does not distinguish between abelian and non-abelian characters and yet, upon restriction to abelian characters, incorporates simultaneous refinements of the Rubin-Stark Conjecture formulated in [47, Conj. B], of the $\mathfrak{p}$-adic abelian Stark conjecture of Gross [30, Conj. 7.6], of the conjectural refined class number formulas of Gross [30, Conj. 4.1], Tate [56, $(*)$] and Aoki, Lee and Tan [1, Conj. 1.1] and of the 'guess' formulated by Gross in [30, top of p. 195]. At the same time Conjecture 2.4.1 constructs explicit annihilators of ideal class groups from the value at $z = 0$ of higher order derivatives of $\theta_{K/k,S}(z)$ and thereby extends both Brumer's Conjecture and the Brumer-Stark Conjecture (on the values of abelian L-series) to situations in which the field extension can be non-abelian and the L-series can vanish to arbitrarily high order. In all cases Conjecture 2.4.1 can also be interpreted in terms of the existence of special elements with appropriate properties.

We are able to prove Conjecture 2.4.1 in a variety of important cases and also to show, by means of explicit examples, that it is the strongest possible generalisation of the previous refinements of Stark's Conjecture (in the abelian case) mentioned above. In addition, we prove that, under a very mild technical hypothesis, Conjecture 2.4.1 is a consequence of the relevant special case of the equivariant Tamagawa number conjecture. It is therefore interesting to recall that in the case of number fields, resp. global function fields, the latter conjecture is simply a natural equivariant refinement of the seminal conjecture formulated by Bloch and Kato in [6], resp. of the conjecture formulated by Lichtenbaum in [34] relating special values of Zeta functions of curves over finite fields to Weil étale cohomology.

Conjecture 2.4.1 is formulated under a restrictive hypothesis on the set S that is motivated by the approach of Rubin in [47] and therefore excludes some important cases. For this reason we formulate (as Conjecture 2.6.1) an analogous but weaker prediction in the most general possible case. We show that this general prediction refines both the question raised by Stark in [52] and the conjectures formulated by Chinburg in [22] and by the present author in [12]. For example, Conjecture 2.6.1 now predicts that (non-abelian) Stark units should encode explicit information about the structure of ideal class groups in a manner that is strikingly parallel to the well known theories for cyclotomic and elliptic units. As evidence for Conjecture 2.6.1 we prove that it is valid unconditionally for global function fields and that for number fields it is valid for all characters that validate the Strong Stark Conjecture.

In addition to establishing connections between our universal conjectural framework and earlier results and conjectures our approach also gives a wide variety of new and unconditional results that are often of a very explicit nature. To give examples of the sort of results obtained in this way we must first introduce further notation. For each set of places Σ of k we write Σ_K for the set of places of K which lie above a place in Σ. If Σ is finite, non-empty and contains S_∞ (in the number field case) we write $\mathcal{O}_{K,\Sigma}$ for the ring of Σ_K-integers in K, $U_{K,\Sigma}$ and $\mathrm{Cl}(\mathcal{O}_{K,\Sigma})$ for the unit group and ideal class group of $\mathcal{O}_{K,\Sigma}$, $Y_{K,\Sigma}$ for the free abelian group on Σ_K and $X_{K,\Sigma}$ for the kernel of the homomorphism $Y_{K,\Sigma} \to \mathbb{Z}$ which sends each element of Σ_K to 1, and we note that all of these modules have a natural action of G. In the number field case we usually abbreviate the ring of integers $\mathcal{O}_{K,S_\infty}$ and unit group U_{K,S_∞} to $\mathcal{O}_K$ and U_K respectively.

We assume henceforth that the set of places S fixed above contains all (of the finitely many) places that ramify in K/k. We also fix a finite non-empty set of places T of k that is disjoint from S and such that no non-trivial element of $U_{K,S}$ is congruent to 1 modulo all places in T_K and then define a T-modified version $\theta_{K/k,S,T}(z)$ of the function $\theta_{K/k,S}(z)$ in the usual way (see (4)). For each integer m we then define an 'm-th order (non-abelian) Stickelberger function' by setting

$$\theta^{(m)}_{K/k,S,T}(z) := \Big(\sum_{\chi\in \mathrm{Ir}(G)} z^{-m\chi(1)}e_\chi\Big)\theta_{K/k,S,T}(z). \tag{2}$$

We write $\mathrm{R}_{K,S}$ for the isomorphism of $\mathbb{R}[G]$-modules $\mathbb{R}\otimes U_{K,S} \to \mathbb{R}\otimes X_{K,S}$ which at each u in $U_{K,S}$ satisfies $\mathrm{R}_{K,S}(u) = -\sum_{\pi\in S_K}\log|u|_\pi\cdot\pi$ where $|\cdot|_\pi$ is the normalised absolute value at π. For each homomorphism ϕ in $\mathrm{Hom}_G(U_{K,S}, X_{K,S})$ we then define a $\mathbb{R}[G]$-valued regulator by setting

$$R(\phi) := \mathrm{Nrd}_{\mathbb{R}[G]}(\mathrm{R}_{K,S}^{-1}\circ(\mathbb{R}\otimes\phi)). \tag{3}$$

Here we use the following general notation: for any semisimple algebra A and any endomorphism ε of a finitely generated A-module M we choose a finitely generated A-module N such that $M\oplus N$ is a free A-module and then define $\mathrm{Nrd}_A(\varepsilon)$ to be the reduced norm of the matrix of $\varepsilon\oplus\mathrm{id}_N$ with respect to a choice of A-basis of $M\oplus N$. (It is easily checked that this definition is independent of the choices of N and of the basis of $M\oplus N$).

For each place v of k we fix a place w of K above v and write G_w for the decomposition subgroup of w in G. The following result is proved in Corollary 4.1.5.

Theorem A *Let K be a finite real abelian extension of $\mathbb{Q}$ and k any subfield of K and assume that S contains at least one non-archimedean place v.*

Then $\theta^{([k:\mathbb{Q}])}_{K/k,S,T}(z)$ is holomorphic at $z=0$ and for every ϕ in $\mathrm{Hom}_G(U_{K,S}, X_{K,S})$ and every g in G_w the product $(g-1)\theta^{([k:\mathbb{Q}])}_{K/k,S,T}(0)R(\phi)$ belongs to $\mathbb{Z}[G]$ and annihilates $\mathbb{Z}[\frac{1}{2}]\otimes\mathrm{Cl}(\mathcal{O}_K)$.

In Example 2.5.3 we will show that Theorem A gives both a natural generalisation and a new proof of the main annihilation result of Rubin in [46]. Our approach also proves a natural analogue of Theorem A for abelian extensions of imaginary quadratic fields (cf. Remark 4.1.4(ii)).

As far as we are aware, the next result is the first to use the values of higher order derivatives of Artin L-series to construct explicit annihilators of the G-module $\mathrm{Cl}(\mathcal{O}_K)$ for any non-abelian extension of totally real fields K/k. This result will be proved in Corollary 4.3.5.

Theorem B *Let K/k be a finite Galois extension of totally real fields and assume that S contains at least one non-archimedean place.*

(i) *Then $\theta^{([k:\mathbb{Q}])}_{K/k,S,T}(z)$ is holomorphic at $z=0$.*

(ii) *For every ϕ in $\mathrm{Hom}_G(U_{K,S}, X_{K,S})$ and every non-trivial ψ in $\mathrm{Ir}(G)$ that is rational valued the product $\psi(1)^{-2}|G|^3\theta^{([k:\mathbb{Q}])}_{K/k,S,T}(0)R(\phi)e_\psi$ belongs to $\mathbb{Z}[G]$ and annihilates $\mathrm{Cl}(\mathcal{O}_K)$.*

(iii) *If every element of $\mathrm{Ir}(G)$ is rational valued and S contains at least two non-archimedean places, then the product $|G|^3\theta^{([k:\mathbb{Q}])}_{K/k,S,T}(0)R(\phi)$ belongs to $\mathbb{Z}[G]$ and annihilates $\mathrm{Cl}(\mathcal{O}_K)$.*

We recall that all elements of $\mathrm{Ir}(G)$ are rational valued if G is a symmetric group of any degree, the quaternion group of order eight or a group of exponent two. In general we will show that the validity for all primes p of the p-adic Stark Conjecture at $s=1$ of Serre and Tate implies the validity of a stronger and more general version of Theorem B(ii) (see Remark 4.3.2) and that if, as conjectured by Iwasawa, certain μ-invariants also vanish, then a stronger version of Theorem B(iii) is valid without any condition on the elements of $\mathrm{Ir}(G)$ (see Corollary 4.1.6). The latter results constitute natural non-abelian generalisations of a refined version of the main result of Büyükboduk in [8] (see Remark 4.1.7).

We now assume that K/k is an extension of global function fields. We write $Z_{K/k,S,T}(t)$ for the Zeta function obtained by changing variable in $\theta_{K/k,S,T}(z)$ from z to $t := \mathrm{char}(k)^{-z}$ and recall that if m is any integer for which the function

$$Z^m_{K/k,S,T}(t) := (1-t)^{-m} Z_{K/k,S,T}(t)$$

is holomorphic at $t=1$, then $m < |S|$ (cf. (5) and Lemma 2.7.1). The following rather remarkable result will be proved in Corollary 4.3.7.

Theorem C *Let K/k be a finite Galois extension of global function fields and write r for the largest integer for which $Z^r_{K/k,S,T}(t)$ is holomorphic at $t=1$. If $r < |S|-1$, then $|G|^{3+r}Z^r_{K/k,S,T}(1)$ is a non-zero element of $\mathbb{Z}[G]$ that annihilates $\mathrm{Cl}(\mathcal{O}_{K,S})$.*

In addition to the cases considered in Theorems A, B and C we shall also discuss in detail several other special cases of our general conjectural approach including families of anti-cyclotomic dihedral extensions (Proposition 3.6.1), extensions with group isomorphic to the alternating group of order twelve (Example 4.3.4) and, for any algebraic closure $\mathbb{Q}^c$ of $\mathbb{Q}$, irreducible complex representations of $\mathrm{Gal}(\mathbb{Q}^c/\mathbb{Q})$ that are of degree two and such that the associated L-series vanishes to order one at $z = 0$ (Proposition 12.2.1).

The main contents of this article is as follows. In §2 we state the central conjectures of this article (as Conjectures 2.4.1 and 2.6.1). In §3 we prove Conjecture 2.4.1 refines Stark's Conjecture and both refines and generalises the Rubin-Stark Conjecture and Brumer's Conjecture. We also reinterpret Conjecture 2.4.1 in terms of the existence of suitable special elements and show that it implies the strongest possible generalisation of Brumer's Conjecture to non-abelian Galois extensions. In §4 we state our main results and derive several interesting consequences of these results including Theorems A, B and C. In §5 we prove an important reduction step concerning Conjecture 2.4.1. In §6 we recall the natural equivariant leading term conjecture and develop methods for computing explicitly the Euler characteristic in this conjectural formula. In §7 and §8 we prove that, under a mild technical hypothesis, the natural leading term conjecture implies Conjecture 2.4.1. In §9 we explain the link between Conjectures 2.4.1 and 2.6.1 and the p-adic Stark Conjecture at $s = 1$ of Serre and Tate. In §10 we prove Conjecture 2.4.1 for all quadratic extensions and for an infinite family of quaternion extensions of $\mathbb{Q}$. In §11 and §12 we assume that K/k is an extension of function fields or validates the Strong-Stark Conjecture and prove both Conjecture 2.6.1 and a natural weakening of Conjecture 2.4.1. In §12 we also prove that Conjecture 2.6.1 both refines and generalises the Stark-Chinburg Conjecture. Finally, in an appendix we recall the relevant properties of the formalism of refined Euler characteristics that is used throughout this article.

In the rest of this article we will use the following notation. For any finite group Γ and any ring R we abbreviate groups of the form $\mathrm{Hom}_\Gamma(A, B)$ and $\mathrm{Hom}_R(C, D)$ to $[A, B]_G$ and $[C, D]_R$ respectively. We write $\mathfrak{Z}(A)$ for the centre of a ring A. We use the fact that if A is semisimple, then for any endomorphism, resp. automorphism, ε of a finitely generated A-module the reduced norm $\mathrm{Nrd}_A(\varepsilon)$ defined just after (3) belongs to $\mathfrak{Z}(A)$, resp. $\mathfrak{Z}(A)^\times$. If F/E is any Galois extension of fields, then we set $G_{F/E} := \mathrm{Gal}(F/E)$.

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2. Statement of the conjectures

2.1. In this section we fix a finite non-empty set S of places of k that contains S_∞ and all places which ramify in K/k. We set $n := |S| - 1$ and label, and thereby order, the elements of S as $\{v_i : 0 \le i \le n\}$. (This labeling is convenient but does not effect the validity of our main results.) For each integer i with $0 \le i \le n$ we fix a place

w_i of K which lies above v_i and write G_i for the decomposition subgroup of w_i in G. More generally, for each place v of K we fix a place w of K above v and write G_w for the decomposition subgroup of w in G. For every integer m with $1 \le m \le n$ we also set $S_m := \{v_i : 1 \le i \le m\}$ and for every intermediate field E of K/k we abbreviate Y_{E,S_m} to $Y_{E,m}$. We fix a finite set of places T of k that is disjoint from S and write $U_{K,S,T}$ for the (finite index) subgroup of $U_{K,S}$ comprising S_K-units that are congruent to 1 modulo all places in T_K. We always assume (as we may) that T is chosen so that $U_{K,S,T}$ is torsion-free.

For each non-archimedean place v of k that is unramified in K/k we write $\mathrm{N}v$ for the cardinality of its residue field and Fr_w for the Frobenius automorphism in G_w and then define

$$\epsilon_v := \mathrm{Nrd}_{\mathbb{Q}[G]}([1 - \mathrm{N}v \cdot \mathrm{Fr}_w^{-1}]_r) \in \mathfrak{Z}(\mathbb{Q}[G]).$$

Here, for any element x of $\mathbb{Q}[G]$ we write $[x]_r$ for the endomorphism $y \mapsto yx$ of $\mathbb{Q}[G]$, regarded as a left $\mathbb{Q}[G]$-module in the obvious way. We then set

$$\theta_{K/k,S,T}(z) := (\prod_{t\in T}\epsilon_t)\theta_{K/k,S}(z). \tag{4}$$

2.2. For any finite non-empty set of places Σ of k that contains S_∞ and any character χ in $\mathrm{Ir}(G)$ we write $r_\Sigma(\chi)$ for the algebraic order at $z = 0$ of the meromorphic function $L_\Sigma(\chi, z)$. We also fix a $\mathbb{C}[G]$-module V_χ of character χ and recall (from, for example, [55, Chap. I, Prop. 3.4]) that

$$r_\Sigma(\chi) = \sum_{v\in\Sigma}\dim_{\mathbb{C}}(V_\chi^{G_w}) - \dim_{\mathbb{C}}(V_\chi^{G}) = \dim_{\mathbb{C}}([V_{\check\chi}, \mathbb{C}\otimes X_{K,\Sigma}]_{\mathbb{C}[G]}). \tag{5}$$

We note that, as $\dim_{\mathbb{C}}(V_\chi^H) = \dim_{\mathbb{C}}(V_{\check\chi}^H)$ for every subgroup H of G, this formula implies $r_\Sigma(\chi) = r_\Sigma(\check\chi)$ (and we shall sometimes use this fact without explicit comment).

Following Rubin [47] we often assume that our fixed set S satisfies the following hypothesis with respect to some chosen integer r with $0 \le r < |S|$.

Hypothesis $\mathbf{H}_r$. Each place in S_r splits completely in K/k.

In the following result we use the r-th order Stickelberger function defined in (2).

Lemma 2.2.1. *If S satisfies Hypothesis H_r then $\theta^{(r)}_{K/k,S,T}(z)$ is holomorphic at $z = 0$.*

Proof. For each character χ in $\mathrm{Ir}(G)$ one has $\epsilon_T^{-1}\theta^{(r)}_{K/k,S,T}(z)e_\chi = z^{-r\chi(1)}L_S(\check\chi, z)e_\chi$ and so it suffices to show that $r_S(\check\chi) \ge r\chi(1)$. But $r_S(\check\chi) = r_S(\chi)$ and, since $V_\chi^G \subseteq V_\chi^{G_0}$, the formula (5) implies that

$$\begin{aligned} r_S(\chi) = \sum_{i=1}^{i=r}\dim_{\mathbb{C}}(V_\chi) + \sum_{i=r+1}^{i=n}\dim_{\mathbb{C}}(V_\chi^{G_i}) + (\dim_{\mathbb{C}}(V_\chi^{G_0}) - \dim_{\mathbb{C}}(V_\chi^{G})) \\ \ge r\dim_{\mathbb{C}}(V_\chi) + (\dim_{\mathbb{C}}(V_\chi^{G_0}) - \dim_{\mathbb{C}}(V_\chi^{G})) \ge r\chi(1), \end{aligned}$$

as required. $\square$

Remark 2.2.2. Hypothesis H_0 is automatically satisfied by S (since S_0 is empty). If K is totally real, then all archimedean places split completely in K/k and so if S contains a non-archimedean place then it can be labeled to satisfy Hypothesis H_r with $r = |S_\infty| = [k : \mathbb{Q}]$ and $S_r = S_\infty$. To discuss the general case we set

$$r_{K,S} := \min\{|S| - 1, |\{v \in S : v \text{ splits completely in } K/k\}|\}.$$

Then $r_{K,S}$ is non-negative (since S is not empty) and for any non-negative integer r a labeling of S exists which satisfies Hypothesis H_r if and only if $r \le r_{K,S}$. The proof of Lemma 2.2.1 also shows that $\theta^{(r)}_{K/k,S,T}(0) = 0$ if $r < r_{K,S}$.

2.3. If S satisfies Hypothesis H_r, then Lemma 2.2.1 allows us to study the value at $z = 0$ of the function $\theta^{(r)}_{K/k,S,T}(z)$. To formulate a precise conjecture in this regard we first introduce some notation.

We fix a Dedekind domain R with field of fractions F and assume that $F[G]$ is semisimple. For each subgroup H of G we write $I(R[H])$ for the two-sided ideal of $R[G]$ generated by the set $\{h - 1 : h \in H\}$. We consider the following set of matrices

$$\mathfrak{M}_S(R[G]) := \{M = (M_{ij}) \in \mathrm{M}_d(R[G]) : d \ge n,\ r < j \le n \Rightarrow M_{ij} \in I(R[G_j])\}.$$

For each M in $\mathfrak{M}_S(R[G])$ one can bound the denominators of $\mathrm{Nrd}_{F[G]}(M)$ in the following way. For every matrix H in $\mathrm{M}_m(R[G])$ there is a unique matrix H^* in $\mathrm{M}_m(F[G])$ with $HH^* = H^*H = \mathrm{Nrd}_{F[G]}(H) \cdot I_m$ and such that for every primitive central idempotent e of $F[G]$ the matrix H^*e is invertible if and only if $\mathrm{Nrd}_{F[G]}(H)e$ is non-zero. We follow Nickel [39] in defining

$$\mathcal{A}(R[G]) := \{x \in \mathfrak{Z}(F[G]) : \text{ if } d > 0 \text{ and } H \in \mathrm{M}_d(R[G]) \text{ then } xH^* \in \mathrm{M}_d(R[G])\}.$$

The interest of this $\mathfrak{Z}(R[G])$-module is explained by the following result.

Lemma 2.3.1. *For any matrix M in $\mathfrak{M}_S(R[G])$ and any element a of $\mathcal{A}(R[G])$ the element $a\mathrm{Nrd}_{F[G]}(M)$ belongs to $R[G]$.*

Proof. Fix M in $\mathrm{M}_d(R[G])$ and a in $\mathcal{A}(R[G])$. Then $\mathrm{Nrd}_{F[G]}(M) \cdot I_d = MM^*$ so $a\mathrm{Nrd}_{F[G]}(M) \cdot I_d = aMM^* = M(aM^*)$. The definition of $\mathcal{A}(R[G])$ ensures that the last term belongs to $\mathrm{M}_d(R[G])$, as required. $\square$

Remark 2.3.2. If $H = I_d$, then $H^* = I_d$ and so $\mathcal{A}(R[G]) \subseteq \mathfrak{Z}(R[G])$. It is easy to see that this inclusion is an equality if G is abelian. In general, for each H in $\mathrm{M}_d(R[G])$ the matrix H^* belongs to $\mathrm{M}_d(\mathcal{M})$ for any maximal order $\mathcal{M}$ in $F[G]$ that contains $R[G]$ (cf. [39, Lem. 4.1]) and so Jacobinski's description in [32] of the central conductor of $\mathcal{M}$ in $R[G]$ implies that for any F-valued character ψ of G the element $\psi(1)^{-1}|G|e_\psi$ belongs to $\mathcal{A}(R[G])$. In certain cases this 'lower bound' on $\mathcal{A}(R[G])$ is sharp (see Proposition 3.6.1(i)) and in general it shows that $|G|\mathfrak{Z}(\mathcal{M}) \subseteq \mathcal{A}(R[G]) \subseteq \mathfrak{Z}(R[G])$.

2.4. We recall that for every homomorphism ϕ in $[U_{K,S}, X_{K,S}]_G$ a $\mathfrak{Z}(\mathbb{R}[G])$-valued regulator $R(\phi)$ is defined in (3).

We also note that any symplectic character ψ in $\mathrm{Ir}(G)$ is real valued so that $\check{\psi} = \psi$ and the leading term $L_S^*(\psi, 0)$ of $L_S(\psi, z)$ at $z = 0$ is a (non-zero) real number. For each v in S and any commutative ring R we write $R[G/G_w]$ for the left $R[G]$-module $R[G] \otimes_{R[G_w]} R$.

We can now state the central conjecture of this article.

Conjecture 2.4.1. *Assume that S satisfies Hypothesis H_r. Fix ϕ in $[U_{K,S}, X_{K,S}]_G$, a in $\mathcal{A}(\mathbb{Z}[G])$ and a non-empty subset S' of S that contains both S_∞ and S_r.*

If $L_S^*(\psi, 0)$ is positive for every symplectic character ψ in $\mathrm{Ir}(G)$, then there exists a matrix $M_{K/k,S,T}(\phi)$ in $\mathfrak{M}_S(\mathbb{Z}[G])$ with

$$\theta^{(r)}_{K/k,S,T}(0)R(\phi) = \mathrm{Nrd}_{\mathbb{Q}[G]}(M_{K/k,S,T}(\phi)). \tag{6}$$

In all cases $\theta^{(r)}_{K/k,S,T}(0)R(\phi)$ is an integral linear combination of elements of the form $\mathrm{Nrd}_{\mathbb{Q}[G]}(M)$ with M in $\mathfrak{M}_S(\mathbb{Z}[G])$ and so $a\theta^{(r)}_{K/k,S,T}(0)R(\phi)$ belongs to $\mathbb{Z}[G]$.

If $r = 0$ or $S' = S$, then $a\theta^{(r)}_{K/k,S,T}(0)R(\phi)$ annihilates $\mathrm{Cl}(\mathcal{O}_{K,S'})$. In all other cases $ba\theta^{(r)}_{K/k,S,T}(0)R(\phi)$ annihilates $\mathrm{Cl}(\mathcal{O}_{K,S'})$ for every b in $\bigcup_{v\in S\setminus S'} \mathrm{Ann}_{\mathbb{Z}[G]}(\mathbb{Z}[G/G_w])$.

Remark 2.4.2.
(i) The existence of symplectic characters in $\mathrm{Ir}(G)$ is a strong restriction on G. For example, there are no such characters if G is abelian, dihedral or of odd order and so in all such cases there should be a matrix as in (6). However, there are many examples in which, for every natural number d, the element $\theta^{(r)}_{K/k,S,T}(0)R(\phi)$ is not equal to $\mathrm{Nrd}_{\mathbb{Q}[G]}(M)$ for any M in $\mathrm{M}_d(\mathbb{Q}[G])$ (see, for example, Remark 4.2.2).
(ii) Our methods will show that in all cases one should predict that $\theta^{(r)}_{K/k,S,T}(0)R(\phi)$ belongs to $\mathfrak{Z}(\mathbb{Q}[G])$ and that for every prime p and every a in $\mathcal{A}(\mathbb{Z}_p[G])$ one has the following: there exists a matrix $M_{K/k,S,T,p}(\phi)$ in $\mathfrak{M}_S(\mathbb{Z}_p[G])$ with

$$\theta^{(r)}_{K/k,S,T}(0)R(\phi) = \mathrm{Nrd}_{\mathbb{Q}_p[G]}(M_{K/k,S,T,p}(\phi)); \tag{7}$$

if $r = 0$ or $S' = S$, then $a\theta^{(r)}_{K/k,S,T}(0)R(\phi) \in \mathbb{Z}_p \otimes \mathrm{Ann}_{\mathbb{Z}[G]}(\mathrm{Cl}(\mathcal{O}_{K,S'}))$; in all other cases $ba\theta^{(r)}_{K/k,S,T}(0)R(\phi) \in \mathbb{Z}_p \otimes \mathrm{Ann}_{\mathbb{Z}[G]}(\mathrm{Cl}(\mathcal{O}_{K,S'}))$ for every $b \in \bigcup_{v\in S\setminus S'} \mathrm{Ann}_{\mathbb{Z}[G]}(\mathbb{Z}[G/G_w])$. (For details of another variant of Conjecture 2.4.1 that should be valid without regard to the sign of $L_S^*(\psi, 0)$ for symplectic characters ψ see Remark 7.4.1.)
(iii) In §3 we show that Conjecture 2.4.1 both refines and generalises the Rubin-Stark Conjecture and Brumer's Conjecture (both of which are formulated only for abelian extensions) and can be reinterpreted in terms of the existence of suitable special elements. In §7.5 (see, in particular, Remark 7.5.2) we also predict that there are matrices $M_{K/k,S,T}(\phi)$ and $M_{K/k,S,T,p}(\phi)$ as in (6) and (7) whose columns satisfy certain

mutual congruence relations and we show that these relations constitute natural non-abelian and 'higher order' generalisations of both the $\mathfrak{p}$-adic abelian Stark conjecture of Gross [30, Conj. 7.6] and of the refined class number formulas conjectured by Gross [30, Conj. 4.1], by Tate [56, $(*)$] and by Aoki, Lee and Tan [1, Conj. 1.1].

2.5. To give an indication of the strength of Conjecture 2.4.1 we discuss the annihilation predictions that it makes in the case that K is totally real.

Proposition 2.5.1. *Let K/k be a finite Galois extension of totally real fields and S a finite set of places of k that contains S_∞, all places which ramify in K/k and at least one non-archimedean place v. If Conjecture 2.4.1 is valid for K/k, then for every ϕ in $[U_{K,S}, X_{K,S}]_G$, every a in $\mathcal{A}(\mathbb{Z}[G])$ and every b in $\mathrm{Ann}_{\mathbb{Z}[G]}(\mathbb{Z}[G/G_w])$ one has*

$$ba\theta^{([k:\mathbb{Q}])}_{K/k,S,T}(0)R(\phi) \in \mathrm{Ann}_{\mathbb{Z}[G]}(\mathrm{Cl}(\mathcal{O}_K)). \tag{8}$$

In particular, if G is abelian and Conjecture 2.4.1 is valid for K/k, then for every ϕ in $[U_{K,S}, X_{K,S}]_G$ and every g in G_w one has

$$(g-1)\theta^{([k:\mathbb{Q}])}_{K/k,S,T}(0)R(\phi) \in \mathrm{Ann}_{\mathbb{Z}[G]}(\mathrm{Cl}(\mathcal{O}_K)). \tag{9}$$

Proof. We set $r := [k : \mathbb{Q}]$ and recall (from Remark 2.2.2) that in this case S can be labeled so that it satisfies Hypothesis H_r with $S_r = S_\infty$. We therefore set $S' := S_\infty$ and note that S' satisfies the conditions of Conjecture 2.4.1 and is also such that $\mathrm{Cl}(\mathcal{O}_{K,S'}) = \mathrm{Cl}(\mathcal{O}_K)$. Thus, since v belongs to $S \setminus S'$, the containment (8) is a direct consequence of Conjecture 2.4.1 in this case.

To deduce the containment (9) from (8) we assume G is abelian. Then for every g in G_w the element $g-1$ annihilates $\mathbb{Z}[G/G_w]$ and also $\mathcal{A}(\mathbb{Z}[G]) = \mathbb{Z}[G]$ (see Remark 2.3.2) and so we can take $b = g-1$ and $a = 1$ in (8). □

Remark 2.5.2. The present author is not aware of any previous use of the values of higher order derivatives of Artin L-series to construct explicit annihilators of the G-module $\mathrm{Cl}(\mathcal{O}_K)$ for *any* non-abelian extension of totally real fields K/k. It also seems far from clear to the present author that in the case of abelian extensions all of the annihilators that are predicted by (9) can be interpreted in terms of a suitable Kolyvagin system of units as in the special cases considered by Rubin in [47, §6.3] and by Büyükboduk in [8].

Example 2.5.3. Upon appropriate specialisation the containment (9) recovers the annihilation result that is proved by Rubin in [46]. To explain this we fix a natural number m and a primitive m-th root of unity ζ_m in $\mathbb{Q}^c$ and write K for the maximal real subfield of $\mathbb{Q}(\zeta_m)$. We take S to be the set comprising the archimedean place ∞ and all prime divisors of m and note that this satisfies Hypothesis H_r with $r = 1$ and $S_1 = \{\infty\}$. Now, since G is abelian, for each prime t with $t \nmid m$ one has $\epsilon_t = 1 - \mathrm{N}t \cdot \mathrm{Fr}_{\mathfrak{t}}^{-1}$ for any place $\mathfrak{t}$ of K above t and so [55, Chap. IV, Lem. 1.1] implies that $2 = w_K$ is an integral linear combination of such elements ϵ_t. The containment

(9) is therefore still valid if one replaces the term $\theta^{(1)}_{K/\mathbb{Q},S,T}(0)$ by $2\theta^{(1)}_{K/\mathbb{Q},S}(0)$. We next recall (from, for example, [55, p. 79]) that for each ψ in $\mathrm{Ir}(G)$ one has

$$2\theta^{(1)}_{K/\mathbb{Q},S}(0)e_\psi = 2L'_S(\check{\psi},0)e_\psi = -\sum_{g\in G}\check{\psi}(g)\log(g(\epsilon_m))e_\psi$$

where we write $L'_S(\chi,z)$ for the first derivative of $L_S(\chi,z)$ and ϵ_m for the cyclotomic element $(1-\zeta_m)(1-\zeta_m^{-1})\in K$. Now, since the cokernel of the inclusion $U_K\to U_{K,S}$ is torsion-free, the restriction map $[U_{K,S},\mathbb{Z}[G]]_G\to[U_K,\mathbb{Z}[G]]_G$ is surjective (by [47, Prop. 1.1(ii)]). For each ϕ_1 in $[U_K,\mathbb{Z}[G]]_G$ we choose an element ϕ'_1 of $[U_{K,S},\mathbb{Z}[G]]_G$ which extends ϕ_1 and write $\tilde{\phi}_1$ for the element of $[U_{K,S},X_{K,S}]_G$ that corresponds to ϕ'_1 as in Proposition 3.3.1. In this case the formula of Proposition 3.3.1 combines with the above displayed formula to imply $2\theta^{(1)}_{K/\mathbb{Q},S}(0)R(\tilde{\phi}_1)=-\phi'_1(\epsilon_m)$. But the group G is generated by the decomposition subgroups G_ℓ of each prime divisor ℓ of m and so for any g in G one has $g-1=\sum_{\ell|m}x_\ell$ for suitable elements x_ℓ of $I(\mathbb{Z}[G_\ell])$. Now since ϵ_m^{g-1} belongs to U_K one has $(g-1)\phi'_1(\epsilon_m)=\phi_1(\epsilon_m^{g-1})$ and so (9) implies that the element $2\phi_1(\epsilon_m^{g-1})=-\sum_{\ell|m}x_\ell 2\theta^{(1)}_{K/\mathbb{Q},S}(0)R(\tilde{\phi}_1)$ annihilates $\mathrm{Cl}(\mathcal{O}_K)$, as proved by Rubin in [46, Th. (2.2) and the following Remark]. In fact these observations combine with Corollary 4.1.3(i) below to give a new proof of Rubin's result (at least if one replaces $\mathrm{Cl}(\mathcal{O}_K)$ by $\mathbb{Z}[\frac{1}{2}]\otimes\mathrm{Cl}(\mathcal{O}_K)$).

2.6. It is natural to ask for an analogue of Conjecture 2.4.1 which does not assume that Hypothesis H_r is satisfied (see, for example, the discussion in the Introduction). Motivated by the approach of [12] we now formulate such a conjecture.

In order to do so we first introduce some notation that will be used throughout this article. For each irreducible complex character ψ of G we fix a subfield E_ψ of $\mathbb{C}$ which is both Galois and of finite degree over $\mathbb{Q}$ and over which ψ can be realised. We fix an indecomposable idempotent f_ψ of $E_\psi[G]e_\psi$. We write $\mathcal{O}_\psi$ for the ring of algebraic integers in E, choose a maximal $\mathcal{O}_\psi$-order $\mathcal{M}_\psi$ in $E[G]$ which contains f_ψ and define an $\mathcal{O}_\psi$-torsion-free right $\mathcal{O}_\psi[G]$-module by setting $T_\psi:=f_\psi\mathcal{M}_\psi$. The associated right $E_\psi[G]$-module $V_\psi:=E_\psi\otimes_{\mathcal{O}_\psi}T_\psi$ has character ψ. For any (left) G-module M we set $M[\psi]:=T_\psi\otimes M$, upon which G acts on the left by $t\otimes m\mapsto tg^{-1}\otimes g(m)$ for each $t\in T_\psi, m\in M$ and $g\in G$.

For any G-module M and subgroup J of G we write $\hat{H}^i(J,M)$ for the Tate cohomology in degree i and M^J, resp. M_J, for the maximal submodule, resp. maximal quotient module, of M upon which J acts trivially. Then we obtain a left, resp. right, exact functor $M\mapsto M^\psi$, resp. $M\mapsto M_\psi$, from the category of left G-modules to the category of $\mathcal{O}_\psi$-modules by setting $M^\psi:=M[\psi]^G$ and $M_\psi:=M[\psi]_G\cong T_\psi\otimes_{\mathbb{Z}[G]}M$.

We write $\mathrm{tr}_{E_\psi/\mathbb{Q}}$ for the field-theoretic trace map $E_\psi\to\mathbb{Q}$ and extend this to a map on group rings $E_\psi[G]\to\mathbb{Q}[G]$ in the obvious way.

We can now state our analogue of Conjecture 2.4.1 in the general case.

Conjecture 2.6.1. *Let K/k be an extension of global fields and S' a finite non-empty set of places of k which contains S_∞. Let S denote the union of S' and the set of places of k which ramify in K/k and fix a finite set of places T of k that is disjoint from S and such that $U_{K,S,T}$ is torsion-free. Fix a non-trivial ψ in $\mathrm{Ir}(G)$ and ρ in $[U^{\psi}_{K,S,T}, X_{K,S,\psi}]_{\mathcal{O}_\psi}$ and set $R(\rho) := \det_{\mathbb{C}}((V_\psi \otimes_{\mathbb{R}[G]} R^{-1}_{K,S}) \circ (\mathbb{C} \otimes_{\mathcal{O}_\psi} \rho))$.*

Then one has

$$|G|^{r_S(\psi)} L^{r_S(\psi)}_{S,T}(\check{\psi}, 0) R(\rho) \in \mathrm{Fit}_{\mathcal{O}_\psi}(\hat{H}^{-1}(G, X_S[\psi])). \tag{10}$$

If ρ belongs to $[U^{\psi}_{K,S'}, X_{K,S',\psi}]_{\mathcal{O}_\psi}$, then also

$$\psi(1)^{-1}|G|^{3+r_S(\psi)} \mathrm{tr}_{E_\psi/\mathbb{Q}}(L^{r_{S'}(\psi)}_{S,T}(\check{\psi}, 0) R(\rho) e_\psi) \in \mathrm{Ann}_{\mathbb{Z}[G]}(\mathrm{Cl}(\mathcal{O}_{K,S'})). \tag{11}$$

We discuss Conjecture 2.6.1 in §12. In particular we show that it refines the conjecture formulated by the present author in [12] and also implies that the algebraic units that are predicted to exist by the Stark-Chinburg Conjecture of [22, Conj. 1] should encode explicit information about the structure of ideal class groups in a way that is strikingly parallel to the cyclotomic results of Rubin that are discussed in Example 2.5.3 above (see Proposition 12.2.1 and Remark 12.2.2).

The connection between Conjectures 2.6.1 and 2.4.1 is explained by the following result (which is proved in §12.3). For any non-empty subset S'' of S which contains S_∞ and any integer r we set

$$\theta^r_{K/k,S'',T}(z) := z^{-r}\theta_{K/k,S'',T}(z).$$

Proposition 2.6.2. *Let K/k, S', S and T be as in Conjecture 2.6.1 and fix an integer r with $r < |S| - 1$ and such that $\theta^r_{K/k,S',T}(z)$ is holomorphic at $z = 0$. Assume that Conjecture 2.6.1 is valid for every non-trivial irreducible complex character of G.*

Then for every ϕ in $[U_{K,S}, X_{K,S}]_G$ one has

$$|G|^r \theta^r_{K/k,S,T}(0) R(\phi) \in \sum_{\psi \in \mathrm{Ir}(G)} \prod_{v \in \Sigma_\psi} \mathrm{Fit}_{\mathcal{O}_\psi}((T_\psi)_{G_w}) e_\psi \tag{12}$$

for any proper subset Σ_ψ of $\{v' \in S : V_\psi^{G_{w'}}$ vanishes$\}$. For every ϕ in $[U_{K,S'}, X_{K,S'}]_G$ one also has

$$|G|^{3+r} \theta^r_{K/k,S,T}(0) R(\phi) \in \mathrm{Ann}_{\mathbb{Z}[G]}(\mathrm{Cl}(\mathcal{O}_{K,S'})). \tag{13}$$

Remark 2.6.3. In any attempt to formulate a closer analogue of Conjecture 2.4.1 in the general case it would be interesting to answer the following questions. Let r be any integer such that $\theta^r_{K/k,S,T}(z) := z^{-r}\theta_{K/k,S,T}(z)$ is holomorphic at $z = 0$.

(i) For which elements x of $\mathfrak{Z}(\mathbb{Q}[G])$ is it true that for all ϕ in $[U_{K,S}, X_{K,S}]_G$ the element $x\theta^r_{K/k,S,T}(0)R(\phi)$ is an integral linear combination of elements of the form $\mathrm{Nrd}_{\mathbb{Q}[G]}(M)$ with M in $\mathrm{M}_d(\mathbb{Z}[G])$?

(ii) If x is any element as in (i), and a is any element of $\mathcal{A}(\mathbb{Z}[G])$, do the elements $xa\theta^r_{K/k,S,T}(0)R(\phi)$ always belong to $\mathrm{Ann}_{\mathbb{Z}[G]}(\mathrm{Cl}(\mathcal{O}_{K,S}))$?

2.7. To end this section we now assume that K/k is an extension of global function fields and show that in this case one can remove all transcendental terms from Conjecture 2.4.1 and 2.6.1.

We write p for the characteristic of k and set $t := p^{-z}$. We then define a $\mathfrak{Z}(\mathbb{C}[G])$-valued function of the variable t by setting $Z_{K/k,S,T}(t) := \theta_{K/k,S,T}(z)$. For each integer r we also define functions $Z^r_{K/k,S,T}(t) = (1-t)^{-r}Z_{K/k,S,T}(t)$ and $Z^{(r)}_{K/k,S,T}(t) := \sum_{\chi\in\mathrm{Ir}(G)}(1-t)^{-r\chi(1)}e_\chi Z_{K/k,S,T}(t)$. For each place w' in S_K we write $\mathrm{val}_{w'}$ for its normalised valuation and $\deg(w')$ for the integer such that the cardinality of the residue field of w' is $p^{\deg(w')}$. We let $D_{K,S}$ denote the element of $[U_{K,S}, X_{K,S}]_G$ which sends each u in $U_{K,S}$ to $\sum_{w'\in S_K}\mathrm{val}_{w'}(u)\deg(w')\cdot w'$ and recall that the induced homomorphism $\mathbb{Q}\otimes D_{K,S}$ is bijective. For each ϕ in $[U_{K,S}, X_{K,S}]_G$ we then set

$$R'(\phi) := \mathrm{Nrd}_{\mathbb{Q}[G]}((\mathbb{Q}\otimes D_{K,S})^{-1}\circ(\mathbb{Q}\otimes\phi)) \in \mathfrak{Z}(\mathbb{Q}[G]).$$

Lemma 2.7.1. *Fix ϕ in $[U_{K,S}, X_{K,S}]_G$.*

(i) *If r is any integer such that $\theta^r_{K/k,S,T}(z)$ is holomorphic at $z=0$, then $Z^r_{K/k,S,T}(t)$ is holomorphic at $t=1$ and $\theta^r_{K/k,S,T}(0)R(\phi) = Z^r_{K/k,S,T}(1)R'(\phi)$.*

(ii) *If S satisfies Hypothesis H_r, then $Z^{(r)}_{K/k,S,T}(t)$ is holomorphic at $t=1$ and $\theta^{(r)}_{K/k,S,T}(0)R(\phi) = Z^{(r)}_{K/k,S,T}(1)R'(\phi)$.*

Proof. For each character ψ in $\mathrm{Ir}(G)$ we define a meromorphic function $Z_{S,T}(\psi,t)$ of the complex variable t by setting $Z_{S,T}(\psi,t) := L_{S,T}(\psi,z)$. For each integer d we also set $Z^d_{S,T}(\psi,t) := (1-t)^{-d}Z_{S,T}(\psi,t)$ and $L^d_{S,T}(\psi,z) := z^{-d}L_{S,T}(\psi,z)$.

It is easy to see that the algebraic order of $Z_{S,T}(\psi,t)$ at $t=1$ is equal to the algebraic order of $L_{S,T}(\psi,z)$ at $z=0$. Since $e_{\check\psi}\theta_{K/k,S,T}(z) = e_{\check\psi}L_{S,T}(\psi,z)$ and $e_{\check\psi}Z_{K/k,S,T}(t) = e_{\check\psi}Z_{S,T}(\psi,t)$ it is therefore enough, when proving both claims, to assume that $L^d_{S,T}(\psi,z)$ is holomorphic at $z=0$ and show that $L^d_{S,T}(\psi,0)R(\phi)e_{\check\psi} = Z^d_{S,T}(\psi,1)R'(\phi)e_{\check\psi}$.

If $L^d_{S,T}(\psi,0)=0$, then $Z^d_{S,T}(\psi,1)=0$ and the required equality is obvious. We thus assume that $L^d_{S,T}(\psi,z)$ is holomorphic at $z=0$ and $L^d_{S,T}(\psi,0)\neq 0$ and hence, by (5), that $\dim_{\mathbb{C}}([V_{\check\psi}, \mathbb{C}\otimes U_{K,S}]_{\mathbb{C}[G]}) = d$. Since $R_{K,S}(u) = \log(p)D_{K,S}(u)$ for each u in $U_{K,S}$ one therefore has

$$\begin{aligned}R(\phi)e_{\check\psi} =& \det_{\mathbb{C}}((\mathbb{C}\otimes R_{K,S})^{-1}\circ(\mathbb{C}\otimes\phi) \mid [V_{\check\psi}, \mathbb{C}\otimes U_{K,S}]_{\mathbb{C}[G]})\\ =&\log(p)^{-d}\det_{\mathbb{C}}((\mathbb{C}\otimes D_{K,S})^{-1}\circ(\mathbb{C}\otimes\phi) \mid [V_{\check\psi}, \mathbb{C}\otimes U_{K,S}]_{\mathbb{C}[G]})\\ =&\log(p)^{-d}R'(\phi)e_{\check\psi}.\end{aligned}$$

On the other hand, since $t = p^{-z}$ an easy change of variable argument shows that

$$\begin{aligned}L^d_{S,T}(\psi,0) =& \lim_{z\to 0} z^{-d}L_{S,T}(\psi,z)\\ =&\log(p)^d\lim_{t\to 1}(1-t)^{-d}Z_{S,T}(\psi,t)\\ =&\log(p)^d Z^d_{S,T}(\psi,1).\end{aligned}$$

The required equality follows by comparing the last two displayed formulas. □

3. The conjectures of Stark, Rubin and Brumer

In this section we explain the link between Conjectures 2.4.1 and 2.6.1 and Stark's Conjecture and show that Conjecture 2.4.1 simultaneously refines and generalises both the Rubin-Stark Conjecture and Brumer's Conjecture. We also reinterpret Conjecture 2.4.1 in terms of the existence of special elements with appropriate properties and discuss an explicit example showing that Conjecture 2.4.1 implies the strongest possible generalisation of Brumer's Conjecture to non-abelian Galois extensions.

3.1. The following result relates Conjecture 2.4.1 and 2.6.1 to Stark's Conjecture.

Lemma 3.1.1. *Let S be any finite non-empty set of places of k which contains S_∞. Then Stark's Conjecture is valid for every complex character of G if and only if for every ϕ in $[U_{K,S},X_{K,S}]_G$ one has $\theta^*_{K/k,S}(0)R(\phi)\in\mathfrak{Z}(\mathbb{Q}[G])$.*

Proof. We fix an injective homomorphism ϕ_0 in $[U_{K,S},X_{K,S}]_G$. Then the induced maps $\mathbb{Q}\otimes\phi_0$ and $\mathbb{R}\otimes\phi_0$ are bijective so $\theta^*_{K/k,S}(0)R(\phi_0)$ belongs to $\mathfrak{Z}(\mathbb{R}[G])^\times$ and for every ϕ in $[U_{K,S},X_{K,S}]_G$ one has

$$(\theta^*_{K/k,S}(0)R(\phi))(\theta^*_{K/k,S}(0)R(\phi_0))^{-1} = \mathrm{Nrd}_{\mathbb{Q}[G]}((\mathbb{Q}\otimes\phi)\circ(\mathbb{Q}\otimes\phi_0)^{-1})\in\mathfrak{Z}(\mathbb{Q}[G]).$$

It thus suffices to prove that Stark's Conjecture is valid if and only if $\theta^*_{K/k,S}(0)R(\phi_0)$ belongs to $\mathfrak{Z}(\mathbb{Q}[G])$. Now in $\mathfrak{Z}(\mathbb{C}[G])$ one has

$$\theta^*_{K/k,S}(0)R(\phi_0) = \sum_{\psi\in\mathrm{Ir}(G)} A(\check{\psi},(\mathbb{Q}\otimes\phi_0)^{-1})^{-1}e_\psi$$

where the notation $A(-,-)$ is as defined by Tate in [55, Chap. I]. But an element $\sum_\psi z_\psi e_\psi$ of $\mathfrak{Z}(\mathbb{C}[G])$ (with $z_\psi\in\mathbb{C}$ for all ψ) belongs to $\mathfrak{Z}(\mathbb{Q}[G])$ if and only if $z_{\psi^\omega} = (z_\psi)^\omega$ for all $\psi\in\mathrm{Ir}(G)$ and all $\omega\in\mathrm{Aut}(\mathbb{C})$. The element $\theta^*_{K/k,S,T}(0)R(\phi_0)$ therefore belongs to $\mathfrak{Z}(\mathbb{Q}[G])$ if and only if $A(\check{\psi},(\mathbb{Q}\otimes\phi_0)^{-1})^\omega = A(\check{\psi}^\omega,(\mathbb{Q}\otimes\phi_0)^{-1})$ for all ψ and all ω. The containment $\theta^*_{K/k,S}(0)R(\phi_0)\in\mathfrak{Z}(\mathbb{Q}[G])$ is thus clearly equivalent to Tate's reformulation [55, Chap. I, Conj. 5.1] of Stark's Conjecture [49] for the (irreducible) complex characters of G. □

3.2. It is convenient at this point to introduce some additional notation.

Let S be a finite non-empty set of places of k that contains S_∞ and all places which ramify in K/k and also satisfies Hypothesis H_r for some non-negative integer r (with $r < |S|$). We write $\pi_{S,r}$, or $\pi_{K,S,r}$ when we need to be more precise, for the natural surjective homomorphism $X_{K,S} \to Y_{K,r}$. Then, as $Y_{K,r}$ is a free G-module, there is a unique homomorphism $\sigma_{S,r}$ in $[Y_{K,r}, X_{K,S}]_G$ that is a section to $\pi_{S,r}$ and satisfies $\sigma_{S,r}(w_i) = w_i - w_0$ for each integer i with $1 \le i \le r$. We usually do not distinguish between $\pi_{S,r}$ and $\mathbb{R}\otimes\pi_{S,r}$ or between $\sigma_{S,r}$ and $\mathbb{R}\otimes\sigma_{S,r}$ and often use the isomorphism $\mathbb{R}\otimes Y_{K,r} \to \mathbb{R}\otimes\sigma_{S,r}(Y_{K,r})$ induced by $\sigma_{S,r}$ to regard $\mathbb{R}\otimes Y_{K,r}$ as a direct summand of $\mathbb{R}\otimes X_{K,S}$.

We set $\mathrm{Ir}_{S,r}(G) := \{\psi \in \mathrm{Ir}(G) : r_S(\psi) = r\psi(1)\}$ and write $e_{S,r}$ for the idempotent $\sum_{\psi\in\mathrm{Ir}_{S,r}(G)} e_\psi$ of $\mathfrak{Z}(\mathbb{C}[G])$. We make frequent use of the following properties of $e_{S,r}$.

Lemma 3.2.1.

(i) *$e_{S,r}$ belongs to $\mathfrak{Z}(\mathbb{Q}[G])$.*

(ii) *The map $e_{S,r}(\mathbb{R}\otimes X_{K,S}) \to e_{S,r}(\mathbb{R}\otimes Y_{K,r})$ induced by $\pi_{S,r}$ is bijective.*

(iii) *$\theta^{(r)}_{K/k,S}(0) = \theta^{(r)}_{K/k,S}(0)e_{S,r}$.*

Proof. The explicit formula (5) implies that

$$\mathrm{Ir}_{S,r}(G) = \{\psi \in \mathrm{Ir}(G) : \dim_{\mathbb{C}}([V_{\check\psi}, \mathbb{C}\otimes X_{K,S}]_{\mathbb{C}[G]}) = r\psi(1)\}. \tag{14}$$

This description implies $\mathrm{Ir}_{S,r}(G)$ is a union of orbits of the action of $\mathrm{Aut}(\mathbb{C})$ on $\mathrm{Ir}(G)$ and so claim (i) is clear. The same description also implies claim (ii) since $Y_{K,r}$ is a free G-module of rank r and so $\dim_{\mathbb{C}}([V_{\check\psi}, \mathbb{C}\otimes Y_{K,r}]_{\mathbb{C}[G]}) = r\psi(1)$ for all $\psi \in \mathrm{Ir}(G)$.

Claim (iii) is valid because $1 - e_{S,r} = \sum_{\psi\in\mathrm{Ir}(G)\setminus\mathrm{Ir}_{S,r}(G)} e_\psi$ and Lemma 2.2.1 implies that for every ψ in $\mathrm{Ir}(G)\setminus\mathrm{Ir}_{S,r}(G)$ one has $r_S(\check\psi) = r_S(\psi) > r\psi(1)$ and hence $\theta^{(r)}_{K/k,S}(0)e_\psi = L_S^{r\psi(1)}(\check\psi, 0)e_\psi = 0$. □

3.3. The next result shows that (even in the non-abelian case) Conjecture 2.4.1 can be interpreted in terms of the existence of special elements of $\mathbb{R}\otimes U_{K,S}$.

Proposition 3.3.1. *Let S be a finite non-empty set of places of k that contains S_∞ and all places which ramify in K/k and also satisfies Hypothesis H_r for some non-negative integer r. Assume that $L_S^{r\psi(1)}(\check\psi, 0)$ is non-negative for every symplectic character ψ in $\mathrm{Ir}(G)$. Then there exists a matrix μ in $\mathrm{M}_r(\mathbb{R}[G])$ with*

$$\mathrm{Nrd}_{\mathbb{R}[G]}(\mu) = \theta^{(r)}_{K/k,S,T}(0). \tag{15}$$

Via the $\mathbb{Z}[G]$-basis $\{w_i : 1 \le i \le r\}$ of $Y_{K,r}$ regard μ as an element of $[Y_{K,r}, Y_{K,r}]_G$ and then for each integer j with $1 \le j \le r$ set

$$\xi(\mu)_j := R^{-1}_{K,S}((\sigma_{S,r}\circ\mu)(w_j)) \in \mathbb{R}\otimes U_{K,S}.$$

If for each integer i with $1 \le i \le r$ one chooses an element ϕ_i of $[U_{K,S}, \mathbb{Z}[G]]_G$, then there exists an element $\tilde{\phi}$ of $[U_{K,S}, X_{K,S}]_G$ with

$$\theta^{(r)}_{K/k,S,T}(0)R(\tilde{\phi}) = \mathrm{Nrd}_{\mathbb{R}[G]}((\phi_i(\xi(\mu)_j))_{1\le i,j\le r}).$$

Proof. We set $\theta^{(r)}(0) := \theta^{(r)}_{K/k,S,T}(0)$ and note that Lemma 3.2.1(iii) implies $\theta^{(r)}(0) = \sum_{\psi\in \mathrm{Ir}_{S,r}(G)} L^{r\psi(1)}_{S,T}(\check{\psi},0)e_\psi$. Now $L^{r\psi(1)}_{S,T}(\check{\psi},0) = \epsilon_{T,\psi}L^{r\psi(1)}_{S}(\check{\psi},0)$ where $\epsilon_{T,\psi} \in \mathbb{C}^\times$ is defined by the equality $\epsilon_{T,\psi}e_\psi = e_\psi\prod_{t\in T}\epsilon_t$. In addition, since each ϵ_t belongs to $\mathrm{im}(\mathrm{Nrd}_{\mathbb{R}[G]})$ the factor $\epsilon_{T,\psi}$ is a strictly positive real number whenever ψ is symplectic. Thus, if $L^{r\psi(1)}_{S}(\check{\psi},0)$ is non-negative for every symplectic ψ in $\mathrm{Ir}(G)$, then $L^{r\psi(1)}_{S,T}(\check{\psi},0)$ is strictly positive for every symplectic ψ in $\mathrm{Ir}_{S,0}(G)$ and so $\theta^{(r)}(0)$ belongs to $\mathrm{im}(\mathrm{Nrd}_{A_r})$ where $A_r := \mathbb{R}[G]e_r$ with $e_r := e_{S,r}$. Since the natural map $\mathrm{GL}_r(A_r) \to K_1(A_r)$ is surjective (as a consequence, for example, of [25, (40.31)]) this implies that there exists a matrix μ in $\mathrm{GL}_r(A_r) \subset \mathrm{M}_r(\mathbb{R}[G])$ with $\mathrm{Nrd}_{\mathbb{R}[G]}(\mu) = \mathrm{Nrd}_{A_r}(\mu) = \theta^{(r)}(0)$, as claimed in (15).

For each index i we write $\tilde{\phi}_i$ for the composite $U_{K,S} \xrightarrow{\phi_i} \mathbb{Z}[G] \to Y_{K,r} \xrightarrow{\sigma_{S,r}} X_{K,S}$ where the second arrow maps 1 to w_i. We set $\tilde{\phi} = \sum_{i=1}^{i=r}\tilde{\phi}_i \in [U_{K,S}, X_{K,S}]_G$ and $\lambda := (\mathbb{R}\otimes\tilde{\phi})\circ \mathrm{R}^{-1}_{K,S} \in [\mathbb{R}\otimes X_{K,S}, \mathbb{R}\otimes X_{K,S}]_{\mathbb{R}[G]}$. Then, with respect to the $\mathbb{Z}[G]$-basis $\{w_i : 1\le i\le r\}$ of $Y_{K,r}$, the matrix of $\pi_{S,r}\circ\lambda\circ\sigma_{S,r}\circ\mu$ is $M := (\phi_i(\xi(\mu)_j))_{1\le i,j\le r}$ so

$$\begin{aligned}\mathrm{Nrd}_{\mathbb{R}[G]}(M) =&\mathrm{Nrd}_{\mathbb{R}[G]}(\pi_{S,r}\circ\lambda\circ\sigma_{S,r}\circ\mu)\\ =&\mathrm{Nrd}_{\mathbb{R}[G]}(\mu)\mathrm{Nrd}_{\mathbb{R}[G]}(\pi_{S,r}\circ\lambda\circ\sigma_{S,r})\\ =&\theta^{(r)}(0)\mathrm{Nrd}_{\mathbb{R}[G]}(\pi_{S,r}\circ\lambda\circ\sigma_{S,r}).\end{aligned}$$

Since the map $e_r(\mathbb{R}\otimes X_{K,S}) \to e_r(\mathbb{R}\otimes Y_{K,r})$ induced by $\pi_{S,r}$ is bijective (by Lemma 3.2.1(ii)) the last displayed expression is therefore equal to

$$\begin{aligned}\theta^{(r)}(0)\mathrm{Nrd}_{\mathbb{R}[G]}(\pi_{S,r}\circ\lambda\circ\sigma_{S,r}) =&\theta^{(r)}(0)e_r\mathrm{Nrd}_{\mathbb{R}[G]}(\pi_{S,r}\circ\lambda\circ\sigma_{S,r})\\ =&\theta^{(r)}(0)\mathrm{Nrd}_{\mathbb{R}[G]e_r}(\pi_{S,r}\circ\lambda\circ\sigma_{S,r}\mid e_r(\mathbb{R}\otimes Y_{K,r}))\\ =&\theta^{(r)}(0)\mathrm{Nrd}_{\mathbb{R}[G]e_r}(\lambda\mid e_r(\mathbb{R}\otimes X_{K,S}))\\ =&\theta^{(r)}(0)e_r\mathrm{Nrd}_{\mathbb{R}[G]}(\lambda)\\ =&\theta^{(r)}(0)R(\tilde{\phi}),\end{aligned}$$

as required. □

3.4. We now use Proposition 3.3.1 to prove that Conjecture 2.4.1 specialises to give a refinement of the Rubin-Stark Conjecture [47, Conj. B′].

Proposition 3.4.1. *Conjecture 2.4.1 implies the Rubin-Stark Conjecture.*

Proof. As in [47] we assume that G is abelian and that S satisfies Hypothesis H_r. In this case there are no symplectic characters in $\mathrm{Ir}(G)$ and so a matrix μ satisfying (15) exists. Further, the definition of each element $\xi(\mu)_j$ in Proposition 3.3.1 implies that in $\mathbb{R}\otimes\wedge^r_{\mathbb{Z}[G]}X_{K,S}$ one has

$$\begin{aligned}(\wedge^r_{\mathbb{R}[G]}\mathrm{R}_{K,S})(\wedge_{j=1}^{j=r}\xi(\mu)_j) &= \wedge_{j=1}^{j=r}\mathrm{R}_{K,S}(\xi(\mu)_j) = \wedge_{j=1}^{j=r}(\sigma_{S,r}\circ\mu)(w_j)\\ &= \det(\mu)\cdot\wedge_{j=1}^{j=r}(w_j-w_0) = \theta^r_{K/k,S,T}(0)\cdot\wedge_{j=1}^{j=r}(w_j-w_0)\end{aligned}$$

and so $\wedge_{j=1}^{j=r}\xi(\mu)_j$ is equal to the element $\epsilon_{S,T}$ that occurs in [47, Conj. B′]. The determinant of $(\phi_i(\xi(\mu)_j))_{1\le i,j\le r}$ is therefore equal to $(\wedge_{i=1}^{i=r}\phi_i)(\epsilon_{S,T})$ and so Proposition 3.3.1 implies that Conjecture 2.4.1 can be reinterpreted as assertions about the element $(\wedge_{i=1}^{i=r}\phi_i)(\epsilon_{S,T})$. In particular, the equality (6) implies that $(\wedge_{i=1}^{i=r}\phi_i)(\epsilon_{S,T})$ belongs to $\prod_{j=r+1}^{j=n}I(\mathbb{Z}[G_j])$. Since [47, Conj. B′] asserts only that $(\wedge_{i=1}^{i=r}\phi_i)(\epsilon_{S,T})$ belongs to $\mathbb{Z}[G]$ it is therefore weaker than Conjecture 2.4.1. □

Remark 3.4.2. In [47, Prop. 2.5] Rubin shows that the Rubin-Stark Conjecture specialises to recover the conjecture 'over $\mathbb{Z}$' formulated by Stark in [51]. Results of Rubin [47, Prop. 2.4] and Popescu [40, Th. 5.5.1] also combine to imply that the Rubin-Stark Conjecture for K/k implies the validity for any non-negative integer a of the conjecture formulated by Popescu in [40, Conj. C$(K/k,S,a)$]. From Proposition 3.4.1 it follows that the conjectures of [51] and [40] are also consequences of (the abelian case of) Conjecture 2.4.1.

3.5. The following result shows that Conjecture 2.4.1 implies an explicit non-abelian generalisation of Brumer's Conjecture.

Proposition 3.5.1. *Let S be a finite non-empty set of places of k that contains S_∞ and all places which ramify in K/k and let S' be any non-empty subset of S which contains S_∞. If Conjecture 2.4.1 is valid, then for every a in $\mathcal{A}(\mathbb{Z}[G])$ one has*

$$a\theta_{K/k,S,T}(0)\in\mathrm{Ann}_{\mathbb{Z}[G]}(\mathrm{Cl}(\mathcal{O}_{K,S'})). \tag{16}$$

In particular, if G is abelian then Conjecture 2.4.1 implies Brumer's Conjecture for the extension K/k.

Proof. We set $\theta(z):=\theta_{K/k,S,T}(z)$. Since S automatically satisfies Hypothesis H_0 (with S_0 empty) we may apply Conjecture 2.4.1 with $r=0$ and S' any non-empty subset of S which contains S_∞. In this case Conjecture 2.4.1 implies that for every ϕ in $[U_{K,S},X_{K,S}]_G$ one has $a\theta^{(0)}(0)R(\phi)\in\mathrm{Ann}_{\mathbb{Z}[G]}(\mathrm{Cl}(\mathcal{O}_{K,S'}))$.

To prove the first claim it is thus enough to show that $\theta^{(0)}(0)R(\phi)=\theta(0)$. To do this we write e_0 for the idempotent $e_{S,0}$ in Lemma 3.2.1. Then Lemma 3.2.1(ii) implies that the space $e_0(\mathbb{R}\otimes U_{K,S})$ vanishes and so, as $e_0R(\phi)$ is equal to the reduced norm of the restriction of $R_{K,S}^{-1}\circ(\mathbb{R}\otimes\phi)$ to $e_0(\mathbb{R}\otimes U_{K,S})$, one has $e_0R(\phi)=1$. Since $\theta^{(0)}(0)=\theta(0)=\theta(0)e_0$ one therefore has $\theta^{(0)}(0)R(\phi)=\theta(0)e_0R(\phi)=\theta(0)$, as required.

If G is abelian, then results of Deligne and Ribet imply that $\theta(0)$ belongs to $\mathbb{Z}[G]$. Brumer's Conjecture asserts in addition that $\theta(0)$ annihilates $\mathrm{Cl}(\mathcal{O}_{K,S'})$. To deduce this as a consequence of the first claim one need only note that if G is abelian, then $\mathcal{A}(\mathbb{Z}[G]) = \mathbb{Z}[G]$ (Remark 2.3.2) and so one can take $a = 1$. □

Remark 3.5.2. The Brumer-Stark Conjecture is an explicit refinement of the conjecture of Brumer discussed in Proposition 3.5.1. (For a statement of the Brumer-Stark Conjecture and details of its connection to Brumer's Conjecture see, for example, [41, §3, Conj. $\mathrm{BrSt}(K/k, S_0)$ and Rem. 3]). However the Brumer-Stark Conjecture is itself known to be implied by a special case of the Rubin-Stark Conjecture (cf. [41, Prop. 3.4]) and so Proposition 3.4.1 implies that the Brumer-Stark Conjecture is also a consequence of Conjecture 2.4.1.

3.6. The next result shows that the containment (16) is the strongest possible generalisation of Brumer's Conjecture to non-abelian Galois extensions. This example also shows that the nature of the annihilation predictions in Conjecture 2.4.1 can change considerably if one passes to a subextension.

Proposition 3.6.1. *Let p be a prime that is congruent to 3 modulo 4 and let E be the unique quadratic (imaginary) subfield of $\mathbb{Q}(\zeta_p)$. For any natural number n let K_n be the n-th layer in the anti-cyclotomic $\mathbb{Z}_p$-extension of E. Set $G_n := G_{K_n/\mathbb{Q}}$ and $H_n := G_{K_n/E}$. Since $K_n/\mathbb{Q}$ is unramified outside ∞ and p we set $S := \{\infty, p\}$. We fix a maximal order $\mathcal{M}_n$ in $\mathbb{Q}[G_n]$ with $\mathbb{Z}[G_n] \subseteq \mathcal{M}_n$. For any subfield F of K_n we write h_F for the class number of F and w_F for the number of roots of unity in F. We assume that p does not divide h_E.*

- (i) *Then there are sets T as in §2.1 such that for any element x of $\mathfrak{Z}(\mathbb{Q}[G_n])$ the following conditions are equivalent:-*
  - (a) $x\theta_{K_n/\mathbb{Q},S,T}(0) \in \mathbb{Z}_p \otimes \mathrm{Ann}_{\mathbb{Z}[G_n]}(\mathrm{Cl}(\mathcal{O}_{K_n}))$;
  - (b) $x\theta_{K_n/\mathbb{Q},S,T}(0) \in \mathbb{Z}_p[G_n]$;
  - (c) $x\theta_{K_n/\mathbb{Q},S,T}(0) = y\theta_{K_n/\mathbb{Q},S,T}(0)$ *with* $y \in \mathbb{Z}_p \otimes |G_n|\mathfrak{Z}(\mathcal{M}_n)$.

  In particular, if Conjecture 2.4.1 is valid for $K_n/\mathbb{Q}$, then an element x of $\mathfrak{Z}(\mathbb{Q}[G_n])$ belongs to $\mathbb{Z}_p \otimes \mathcal{A}(\mathbb{Z}[G_n])$ only if $e_{\chi_{\mathrm{sgn}}}(x)$ belongs to $\mathbb{Z}_p \otimes |G_n|\mathfrak{Z}(\mathcal{M}_n)$, where χ_{sgn} is the unique non-trivial linear character of G_n.
- (ii) *If $w_{K_n} = w_E$, then there is a homomorphism ϕ in $[U_{K_n,S}, X_{K_n,S}]_{H_n}$ for which $w_E\theta^{(1)}_{K_n/E,S(E)}(0)R(\phi)$ generates $\mathbb{Z}_p \otimes \mathrm{Ann}_{\mathbb{Z}[H_n]}(\mathrm{Cl}(\mathcal{O}_{K_n,S}))$ as a $\mathbb{Z}_p[H_n]$-module.*

Proof. The group G_n is dihedral and so $\mathrm{Ir}(G_n)$ comprises the trivial character, a non-trivial linear character 'χ_{sgn}' which factors through G_n/H_n and a collection of characters of the form $\mathrm{Ind}_{H_n}^{G_n}\phi$ with $\phi \in \mathrm{Ir}(H_n)$. Since ∞ splits in K_n/E and $|S| > 1$ an analysis of the formula (5) shows that $r_S(\chi_{\mathrm{sgn}}) = 0$ and $r_S(\chi) > 0$ for all $\chi \neq \chi_{\mathrm{sgn}}$ and so $\theta_{K_n/\mathbb{Q},S}(0) = L_S(\chi_{\mathrm{sgn}}, 0)e_{\chi_{\mathrm{sgn}}}$. Now the unique place of E above p is principal and so h_E is equal to the S-class number of E. The analytic class number formula (for both E and $\mathbb{Q}$) therefore implies that $L_S(\chi_{\mathrm{sgn}}, 0) = \zeta'_{E,S}(0)/\zeta'_{\mathbb{Q},S}(0) = w_{\mathbb{Q}}h_E/w_E = 2h_E/w_E$.

Now if $\sigma \in G_n \setminus H_n$, then $e_{\chi_{\rm sgn}} = |G_n|^{-1}(1-\sigma)\Sigma_n$ with $\Sigma_n := \sum_{h\in H_n} h$ and if $T = \{t\}$ for some place t, then $e_{\chi_{\rm sgn}}\epsilon_T = (1-(-1)^{a_t}t)e_{\chi_{\rm sgn}}$ with $a_t = 0$ if t splits in $E/\mathbb{Q}$ and $a_t = 1$ otherwise. For any element $x = \sum_{\chi\in {\rm Ir}(G)} x_\chi e_\chi$ of $\mathfrak{Z}(\mathbb{Q}[G_n])$ one therefore has

$$x\theta_{K_n/\mathbb{Q},S,T}(0) = x_{\chi_{\rm sgn}} p^{-n} w_E^{-1}(1-(-1)^{a_t}t)h_E(1-\sigma)\Sigma_n.$$

We now choose t which is not congruent to either 1 or -1 modulo p. Then since $p \nmid h_E w_E$ the above expression implies that $x\theta_{K_n/\mathbb{Q},S,T}(0)$ belongs to $\mathbb{Z}_p[G_n]$ if and only if the rational number $x_{\chi_{\rm sgn}}$ belongs to $p^n\mathbb{Z}_p$, or equivalently the element $e_{\chi_{\rm sgn}}x$ belongs to $\mathbb{Z}_p\otimes |G_n|\mathfrak{Z}(\mathcal{M}_n)$. Further, if this is true, then $x\theta_{K_n/\mathbb{Q},S,T}(0) = z'\Sigma_n$ for an element z' of $\mathbb{Z}_p[G_n]$ and so, since $\Sigma_n({\rm Cl}(\mathcal{O}_{K_n}))$ belongs to the image of the inflation map ${\rm Cl}(\mathcal{O}_E) \to {\rm Cl}(\mathcal{O}_{K_n})$ and $\mathbb{Z}_p\otimes{\rm Cl}(\mathcal{O}_E)$ vanishes, the element $x\theta_{K_n/\mathbb{Q},S,T}(0)$ belongs to $\mathbb{Z}_p\otimes{\rm Ann}_{\mathbb{Z}[G_n]}({\rm Cl}(\mathcal{O}_{K_n}))$. The equivalence of conditions (a), (b) and (c) is thus clear.

To complete the proof of claim (i) we assume that Conjecture 2.4.1 is valid for $K_n/\mathbb{Q}$. Then for any element x of $\mathbb{Z}_p\otimes\mathcal{A}(\mathbb{Z}[G_n])$ one has $x\theta_{K_n/\mathbb{Q},S,T}(0) \in \mathbb{Z}_p[G_n]$. Thus, by the equivalence of (b) and (c), there exists an element y of $\mathbb{Z}_p\otimes|G_n|\mathfrak{Z}(\mathcal{M}_n)$ with $x\theta_{K_n/\mathbb{Q},S,T}(0) = y\theta_{K_n/\mathbb{Q},S,T}(0)$. Since $\theta_{K_n/\mathbb{Q},S,T}(0) = \theta_{K_n/\mathbb{Q},S,T}(0)e_{\chi_{\rm sgn}} \neq 0$ it follows that $e_{\chi_{\rm sgn}}(x) = e_{\chi_{\rm sgn}}(y)$ belongs to $\mathbb{Z}_p\otimes|G_n|\mathfrak{Z}(\mathcal{M}_n)$. Conversely, if $e_{\chi_{\rm sgn}}(x)$ belongs to $\mathbb{Z}_p\otimes|G_n|\mathfrak{Z}(\mathcal{M}_n)$, then $e_{\chi_{\rm sgn}}(x) = c|G_n|e_{\chi_{\rm sgn}}$ with $c\in\mathbb{Z}_p$ and so Remark 2.3.2 implies that $e_{\chi_{\rm sgn}}(x)$ belongs to $\mathbb{Z}_p\otimes\mathcal{A}(\mathbb{Z}[G_n])$, as required.

To consider claim (ii) we set $\mathcal{O}_n := \mathcal{O}_{K_n,S}$, $X_n := X_{K_n,S}$ and $\theta_n := \theta^{(1)}_{K_n/E,S(E)}(0)$. Since p does not divide h_E and K_n/E is both of p-power degree and ramified at only one place one knows that p does not divide h_{K_n} by [57, Th. 10.4(a)]. One thus has $\mathbb{Z}_p\otimes{\rm Ann}_{\mathbb{Z}[H_n]}({\rm Cl}(\mathcal{O}_n)) = \mathbb{Z}_p[H_n]$ and so it suffices to construct an element ϕ of $[U_{K_n}, X_n]_{H_n}$ with $\theta_n R(\phi) \in \mathbb{Z}_p[H_n]^\times$. Now, since $p\nmid h_{K_n}$ and $w_{K_n} = w_E$, the theory of elliptic units developed by Robert in [45] gives an element y_n of $\mathcal{O}_n^\times := U_{K_n,S}$ with the following properties: the index c of $\mathcal{E}_n := \mathbb{Z}[H_n]\cdot y_n$ in $\mathcal{O}_n^\times$ is prime to p; the homomorphism of H_n-modules $\iota_n : \mathbb{Z}[H_n] \to \mathcal{E}_n$ that sends 1 to y_n is bijective; for each $\chi\in{\rm Ir}(H_n)$ one has $L'_S(\chi,0) = -w_E^{-1}\sum_{h\in H_n}\chi(h)\log|h(y_n)|_v$ with v a choice of archimedean place of K_n (for more details of the deduction of these results from [45] see [20, Rem. 3.21] where y_n is denoted y_N). In addition, if $\mathfrak{p}$ denotes the unique place of K_n above p, then the homomorphism of H_n-modules $a : \mathbb{Z}[H_n] \to X_n$ that sends 1 to $v-\mathfrak{p}$ is bijective. We write ϕ' for the composite $(\iota_n^{-1}\mid_{c\cdot\mathcal{O}_n^\times})\circ(\times c) : \mathcal{O}_n^\times \to \mathbb{Z}[H_n]$ and set $\phi := a\circ\phi'$. Then $w_E\theta^{(1)}_{K_n/E,S}(0)R(\phi) = -\phi'(y_n)$ and since $\mathbb{Z}_p[H_n]\cdot\phi'(y_n) = \mathbb{Z}_p[H_n]$ one has $\phi'(y_n)\in\mathbb{Z}_p[H_n]^\times$, as required. □

4. Statement of the main results

In this section we state our main evidence in support of Conjectures 2.4.1 and 2.6.1 and also discuss several interesting consequences of these results (including Theorems A, B and C).

We recall that a G-module M is said to be 'cohomologically-trivial' if the Tate cohomology group $\hat{H}^i(J, M)$ vanishes in all degrees i and for all subgroups J of G. We note in particular that if ℓ is any prime which does not divide $|G|$, then any $\mathbb{Z}_\ell[G]$-module is automatically cohomologically-trivial.

4.1. We now introduce convenient notation for the relevant leading term conjectures. If K/k is an extension of number fields, then we write 'LTC(K/k)' for the equivariant Tamagawa number conjecture of [17, Conj. 4(iv)] for the pair $(h^0(\mathrm{Spec}(K)), \mathbb{Z}[G])$ where the motive $h^0(\mathrm{Spec}(K))$ is regarded as defined over k and with coefficients $\mathbb{Q}[G]$. If K/k is an extension of global function fields, then we write 'LTC(K/k)' for the conjecture formulated in [10, §2.2] and we recall that this conjecture is an equivariant version of Lichtenbaum's conjecture [34, Conj. 8.1e)] relating special values of Zeta functions of curves over finite fields to Weil étale cohomology. In both cases the reader will find an explicit statement of LTC(K/k) in §6.1.

We write $\mathbb{Z}^*$ for the subring of $\mathbb{Q}$ generated by the inverses of the finitely many primes ℓ for which $\mathbb{Z}_\ell \otimes U_{K,S,\mathrm{tor}}$ is not cohomologically-trivial as a G-module. In the sequel we shall refer to the statement obtained from Conjecture 2.4.1 after replacing both occurrences of $\mathrm{Cl}(\mathcal{O}_{K,S'})$ by $\mathbb{Z}^* \otimes \mathrm{Cl}(\mathcal{O}_{K,S'})$ as 'Conjecture 2.4.1*'.

The following result will be proved in §7 and §8.

Theorem 4.1.1. *If LTC(K/k) is valid, then Conjecture 2.4.1* is valid.*

Remark 4.1.2. In this remark we assume K/k is an extension of fields for which Stark's Conjecture is valid for every complex character of G. Then LTC(K/k) is equivalent to the vanishing of an element of the relative algebraic K-group $K_0(\mathbb{Z}[G], \mathbb{Q}[G])$ (see Remark 6.1.1(ii) below) and so the natural isomorphism $K_0(\mathbb{Z}[G], \mathbb{Q}[G]) \cong \bigoplus_p K_0(\mathbb{Z}_p[G], \mathbb{Q}_p[G])$, where p runs over all primes, decomposes LTC(K/k) into 'p-primary components' for every prime p. In a similar way, for any S and ϕ as in Conjecture 2.4.1, Lemma 3.1.1 implies $\theta^{(r)}_{K/k,S}(0)R(\phi)$ belongs to $\mathfrak{Z}(\mathbb{Q}[G])$ and so one can formulate a natural p-adic analogue of Conjecture 2.4.1 as in Remark 2.4.2(ii). Our proof of Theorem 4.1.1 will show that for any prime p for which $\mathbb{Z}_p \otimes U_{K,S,\mathrm{tor}}$ is a cohomologically-trivial G-module the p-primary component of LTC(K/k) implies the natural p-adic analogue of Conjecture 2.4.1.

We now discuss several interesting consequences of Theorem 4.1.1.

Corollary 4.1.3. *Conjecture 2.4.1* is valid in each of the following cases.*

- (i) *K is an abelian extension of $\mathbb{Q}$ (and k is any subfield of K).*
- (ii) *K/k is a Galois extension of global function fields and either*
    - (a) *$G_{K/k}$ is abelian, or*
    - (b) $\mathrm{char}(k)$ *is coprime to the order of $G_{K/k}$, or*
    - (c) $\mathrm{char}(k) = 2$ *and $G_{K/k}$ is dihedral of order congruent to* 2 *modulo* 4.

Proof. In view of Theorem 4.1.1 it suffices to note that $\mathrm{LTC}(K/k)$ is valid in each of the cases listed above.

In case (i) the validity of $\mathrm{LTC}(K/k)$ follows from results of Greither and the present author [19, Th. 8.1, Rem. 8.1] and Flach [27]. In case (ii)(a) the validity of $\mathrm{LTC}(K/k)$ is proved in [14] and in cases (ii)(b) and (ii)(c) it is proved in [10, Cor. 1 and Rem. 5]. □

Remark 4.1.4.
(i) Aside from the classes of fields considered in Corollary 4.1.3 there are other particular cases in which $\mathrm{LTC}(K/k)$ has been fully verified. To describe such an example we set $\epsilon := \exp(\frac{2\pi i}{7}) + \exp(\frac{-2\pi i}{7})$ and note that $\mathbb{Q}(\epsilon)$ is a cyclic cubic extension of $\mathbb{Q}$. Let σ denote a generator of $G_{\mathbb{Q}(\epsilon)/\mathbb{Q}}$ and write K for the normal closure of $\mathbb{Q}(\epsilon, \sqrt{\epsilon^\sigma/\epsilon^{\sigma^2}})$. Then $G_{K/\mathbb{Q}}$ is isomorphic to the alternating group of order twelve and in [36, Th. 3] Navilarekallu verifies $\mathrm{LTC}(K/k)$ for any subfield k of K. Theorem 4.1.1 therefore implies that Conjecture 2.4.1* is also valid in each of these cases.
(ii) If K is an abelian extension of an imaginary quadratic field F and k any intermediate field of K/F, then the theory of elliptic units implies that Stark's Conjecture is valid for every complex character of $G_{K/k}$ and so one may use the p-adic approach discussed in Remark 4.1.2. In this case Bley has shown that the p-primary component of $\mathrm{LTC}(K/k)$ is valid for every odd prime p which splits in $F/\mathbb{Q}$ and is coprime to the class number of F (cf. [4, Th. 4.2]). Remark 4.1.2 therefore implies that the natural p-adic analogue of Conjecture 2.4.1 is also valid in this case. A similar remark applies in the setting of Corollary 4.3.3 below.

Corollary 4.1.5. *Theorem A is valid.*

Proof. In this argument we use the notation and assumptions of Theorem A. We also set $r := [k : \mathbb{Q}]$ and note that in this case S can be labeled to satisfy Hypothesis H_r with $S_r = S_\infty$. We then set $S' := S_r \cup S_\infty = S_\infty$ so $\mathrm{Cl}(\mathcal{O}_{K,S'}) = \mathrm{Cl}(\mathcal{O}_K)$. Now, since G is abelian, the module $\mathbb{Z}[G/G_w]$ is annihilated by $g - 1$ for any g in G_w and one has $\mathcal{A}(\mathbb{Z}[G]) = \mathbb{Z}[G]$ (by Remark 2.3.2). In particular, since $v \in S \setminus S'$, we can set $b := g - 1$ and $a := 1$ in the statement of Conjecture 2.4.1. Further, since K is real one has $U_{K,S,\mathrm{tor}} = \{\pm 1\}$ and so $\mathbb{Z}_\ell \otimes U_{K,S,\mathrm{tor}}$ is a cohomologically-trivial G-module for every odd prime ℓ. This implies that $\mathbb{Z}^*$ is equal to either $\mathbb{Z}$ or $\mathbb{Z}[\frac{1}{2}]$ and it is now easy to check that Theorem A follows by applying Corollary 4.1.3(i) in this case. □

The following consequence of Theorem 4.1.1 both refines and generalises the results of Büyükboduk in [8] (for more details see Remark 4.1.7 below). This result will be proved in §9.1.

Corollary 4.1.6. *Let K/k be an odd degree tamely ramified Galois extension of totally real fields. Assume that both*

(i) *for all primes p the p-adic Stark Conjecture at $s = 1$ of Serre and Tate is valid for all p-adic characters of G, and*

(ii) *for all prime divisors p of $|G|$ the p-adic Iwasawa μ-invariant for the cyclotomic $\mathbb{Z}_p$-extension of K vanishes.*

Then Conjecture 2.4.1, and hence also the containments of Proposition 2.5.1, are valid for K/k.

Remark 4.1.7. The p-adic Stark Conjecture at $s = 1$ is discussed by Tate in [55, Chap. VI, §5] where it is attributed to Serre [48]. In fact there are imprecisions in the discussion of [55, Chap. VI, §5] but a precise statement of the conjecture can be found in [21, Rem. 7.2]. If K is totally real, then this conjecture predicts an explicit formula for the leading terms at $s = 1$ of p-adic Artin L-functions of p-adic characters ψ of G and it is known that Leopoldt's Conjecture for K at p is equivalent to the validity of the p-adic Stark Conjecture at $s = 1$ for all p-adic permutation characters ψ of G. Now in this case $U_{K,S,\mathrm{tor}} = \{\pm 1\}$ and so if $|G|$ is odd, then Conjectures 2.4.1 and 2.4.1* are equivalent. In particular, if one sets $S' = S_{[k:\mathbb{Q}]} = S_\infty$ and fixes a prime p that is at most tamely ramified in K/k and is such that both the p-adic Stark Conjecture at $s = 1$ is valid for all p-adic characters of G and if p divides $|G|$ a suitable μ-invariant vanishes, then (the proof of) Corollary 4.1.6 combines with the final paragraph of Conjecture 2.4.1* to construct explicit annihilators of $\mathbb{Z}_p \otimes \mathrm{Cl}(\mathcal{O}_K)$ from the higher Stickelberger element $\theta^{(r)}_{K/k,S,T}(0)$. In the special case that K/k is abelian and of order dividing $p-1$ for some prime p that is both coprime to the discriminant of K/k and such that Leopoldt's Conjecture is valid for K at p, such annihilators have already been constructed by Büyükboduk in [8].

4.2. It seems reasonable to believe that LTC(K/k) implies Conjecture 2.4.1. However, at this stage the only unconditional evidence that we have in favour of Conjecture 2.4.1 (rather than Conjecture 2.4.1*) is the next result. To state this result we recall that if ℓ is any rational prime that is congruent to 1 modulo 12 and also such that the Legendre symbols $(\frac{\ell}{7})$ and $-(\frac{\ell}{5})$ are equal to 1, then there is a unique field $K_{3,5,7,\ell}$ which contains $\mathbb{Q}(\sqrt{21}, \sqrt{5})$, is Galois over $\mathbb{Q}$ with group isomorphic to the quaternion group of order eight and is such that the extension $K_{3,5,7,\ell}/\mathbb{Q}$ is ramified precisely at $3, 5, 7, \ell$ and the archimedean place ∞ (cf. [24, Prop. 4.1.3]).

The following result will be proved in §10.

Theorem 4.2.1. *Conjecture 2.4.1 is valid in both of the following cases.*

(i) *K/k is quadratic.*

(ii) *$K = K_{3,5,7,\ell}$ for any prime ℓ as above and $k = \mathbb{Q}$. Further, if in each such case one sets $S := \{3, 5, 7, \ell, \infty\}$, then $\theta_{K/\mathbb{Q},S,T}(0) \neq 0$ and $L^*_S(\psi, 0) = L_S(\psi, 0)$ is a strictly negative real number for the unique irreducible complex symplectic character ψ of $G_{K/\mathbb{Q}}$.*

Remark 4.2.2. Assume the notation of Theorem 4.2.1(ii). Then for every ϕ in $[U_{K,S}, X_{K,S}]_G$ one has $\theta^{(0)}_{K/k,S,T}(0)R(\phi) = \theta_{K/\mathbb{Q},S,T}(0)$ (by the argument of Proposition

3.5.1). Now $\psi = \check{\psi}$ and so $\theta_{K/\mathbb{Q},S,T}(0)e_\psi = \epsilon_{T,\psi}L_S(\psi,0)e_\psi$ where $\epsilon_{T,\psi}$ is as defined in the proof of Proposition 3.3.1. We recall that, since ψ is symplectic, $\epsilon_{T,\psi}$ is a strictly positive real number. The product $\epsilon_{T,\psi}L_S(\psi,0)$ is thus a strictly negative real number and so $\theta_{K/\mathbb{Q},S,T}(0) \notin \mathrm{im}(\mathrm{Nrd}_{\mathbb{R}[G]})$. In this case therefore the element $\theta^{(0)}_{K/k,S,T}(0)R(\phi)$ is not equal to $\mathrm{Nrd}_{\mathbb{Q}[G]}(M)$ for any M in $\mathrm{M}_d(\mathbb{Q}[G])$ and any natural number d.

4.3. We now fix a character ψ in $\mathrm{Ir}(G)$ and a field E_ψ as in §2.6 and use the trace map $\mathrm{tr}_{E_\psi/\mathbb{Q}} : E_\psi[G] \to \mathbb{Q}[G]$. We write $\mathcal{D}_{E_\psi/\mathbb{Q}}$ for the different of the extension $E_\psi/\mathbb{Q}$.

In the following result we refer to the Strong Stark Conjecture that is formulated by Chinburg in [23, Conj. 2.2]. This result will be proved in §11 and §12.

Theorem 4.3.1. *Let S be a finite non-empty set of places of k which contains S_∞ and all places which ramify in K/k. Assume that either*

(a) *K/k is an extension of global function fields, or*
(b) *K/k is an extension of number fields and ψ validates the Strong Stark Conjecture.*

Then both of the following assertions are valid.

(i) *Assume that S satisfies Hypothesis H_r and fix a non-empty subset S' of S which contains both S_∞ and S_r and a homomorphism ϕ in $[U_{K,S}, X_{K,S}]_G$. Set $x := 1$ if $r = 0$ or $S = S'$ and in all other cases let x be any element of $\bigcup_{v\in S\setminus S'} \mathrm{Ann}_{\mathbb{Z}[G]}(\mathbb{Z}[G/G_w])$. Then for any element y of the fractional ideal $\psi(1)^{-1}|G|^2\mathcal{D}^{-1}_{E_\psi/\mathbb{Q}}$ one has $\mathrm{tr}_{E_\psi/\mathbb{Q}}(yx\theta^{(r)}_{K/k,S,T}(0)R(\phi)e_\psi) \in \mathrm{Ann}_{\mathbb{Z}[G]}(\mathrm{Cl}(\mathcal{O}_{K,S'}))$.*
(ii) *Conjecture 2.6.1 is valid for ψ.*

Remark 4.3.2. In Proposition 9.2.1 we will prove that if K is totally real and the p-adic Stark Conjecture at $s = 1$ is valid for all p-adic characters of G, then the Strong Stark Conjecture is valid for all complex characters of G.

In the remainder of this section we shall derive some interesting consequences of Theorem 4.3.1.

Corollary 4.3.3. *If K/k is an extension of number fields, then the assertions of Theorem 4.3.1 are valid in each of the following cases:-*

(i) *The character ψ is rational valued;*
(ii) *$k = \mathbb{Q}$ and ψ has degree one;*
(iii) *$r = 0$ and in the statement of Theorem 4.3.1 one replaces $\mathrm{Ann}_{\mathbb{Z}[G]}(\mathrm{Cl}(\mathcal{O}_{K,S'}))$ by $\mathbb{Z}[\frac{1}{2}] \otimes \mathrm{Ann}_{\mathbb{Z}[G]}(\mathrm{Cl}(\mathcal{O}_{K,S'}))$.*

Proof. In cases (i) and (ii) it is enough to note that the Strong Stark Conjecture is valid in the indicated cases. In case (i) this conjecture is proved by Tate in [55, Ch. II, Th. 6.8]. In case (ii) the conjecture is proved by Ritter and Weiss in [43] if the conductor of ψ is odd and the remaining 2-primary difficulties are resolved by Flach in [27].

In case (iii) it is enough to verify the equality of Strong Stark Conjecture after inverting 2 for all characters ψ in $\mathrm{Ir}(G)$ with $r_S(\psi) = 0$. It is straightforward to deduce this case of the conjecture from Wiles' proof of the main conjecture of Iwasawa theory for totally real fields (for details see, for example, Nickel [38, Cor. 2, p.24]). $\square$

Example 4.3.4. Let K be a Galois extension of $\mathbb{Q}$ such that $G := G_{K/\mathbb{Q}}$ is isomorphic to the alternating group of order twelve. Then $\mathrm{Ir}(G)$ comprises the trivial character, two characters of degree one (and order three) and a character of degree three that is rational valued. Corollary 4.3.3(i) and (ii) therefore combines with Theorem 4.3.1(b)(ii) to imply that Conjecture 2.6.1 is valid for every non-trivial character of G. Thus, if r is any integer with $r < |S| - 1$ and such that the function $\theta^r_{K/k,S,T}(z) := z^{-r}\theta_{K/k,S,T}(z)$ is holomorphic at $z = 0$, then Proposition 2.6.2 implies that for every ϕ in $[U_{K,S}, X_{K,S}]_G$ the element $|G|^{3+r}\theta^r_{K/k,S,T}(0)R(\phi)$ belongs to $\mathbb{Z}[G]$ and annihilates $\mathrm{Cl}(\mathcal{O}_{K,S})$. For example, if K is the field described in Remark 4.1.4, then one can take $S = \{\infty, 2, 7, 43\}$ and for this set the group $\mathrm{Cl}(\mathcal{O}_{K,S})$ is non-trivial, the function $\theta^2_{K/\mathbb{Q},S,T}(z)$ is holomorphic at $z = 0$ and $\theta^2_{K/\mathbb{Q},S,T}(0)e_\psi$ is non-zero for every non-trivial character ψ of G. It follows that for every injective homomorphism ϕ in $[U_{K,S}, X_{K,S}]_G$ the element $2^{10}3^5\theta^2_{K/\mathbb{Q},S,T}(0)R(\phi)$ is non-zero, belongs to $\mathbb{Z}[G]$ and annihilates the (non-trivial) module $\mathrm{Cl}(\mathcal{O}_{K,S})$.

In the sequel we write 1_Γ for the trivial complex character of any finite group Γ.

Corollary 4.3.5. *Theorem B is valid.*

Proof. We assume the notation of Theorem B and set $r := [k : \mathbb{Q}]$. We fix a finite set S of places of k containing S_∞, all places which ramify in K/k and at least one non-archimedean place v. Since K is totally real this set S satisfies Hypothesis H_r with $S_r := S_\infty$ and so Theorem B(i) follows immediately from Lemma 2.2.1.

To prove Theorem B(ii) we may apply Theorem 4.3.1(b)(i) with $S' = S_\infty$ (so $\mathrm{Cl}(\mathcal{O}_{K,S'}) = \mathrm{Cl}(\mathcal{O}_K)$) and $y = \psi(1)^{-1}|G|^2$ to deduce that if $\psi \in \mathrm{Ir}(G)$ validates the Strong Stark Conjecture, then

$$\psi(1)^{-1}|G|^2 x\,\mathrm{tr}_{E_\psi/\mathbb{Q}}(\theta^{(r)}_{K/k,S,T}(0)R(\phi)e_\psi) \in \mathrm{Ann}_{\mathbb{Z}[G]}(\mathrm{Cl}(\mathcal{O}_K)) \tag{17}$$

for any x in $\mathrm{Ann}_{\mathbb{Z}[G]}(\mathbb{Z}[G/G_w])$, where w is any fixed place of K above v. If now ψ is also rational valued, then we take $E_\psi = \mathbb{Q}$ and note Corollary 4.3.3(i) implies that (17) is valid unconditionally. Since the assertion of Theorem B(ii) is obviously valid for any ψ such that $\theta^{(r)}_{K/k,S,T}(0)e_\psi = 0$, it is thus enough to show that if $\psi \neq 1_G$ and $\theta^{(r)}_{K/k,S,T}(0)e_\psi \neq 0$ (or equivalently $r_S(\psi) = r\psi(1)$), then $\psi(1)^{-1}|G|e_\psi$ belongs to $\mathrm{Ann}_{\mathbb{Z}[G]}(\mathbb{Z}[G/G_w])$ and so is a valid choice for x in (17). But $\psi(1)^{-1}|G|e_\psi = \sum_{g\in G}\psi(g^{-1})g$ clearly belongs to $\mathbb{Z}[G]$ and so it suffices to show that if $\psi \neq 1_G$ and $r_S(\psi) = r\psi(1)$, then $e_\psi\mathbb{C}[G/G_w]$ vanishes. Now in this case (5) implies $r_S(\psi) = r\psi(1) = r_{S_\infty}(\psi)$ and hence that $\dim_\mathbb{C}(V_\psi^{G_{w'}}) = 0$ for all $v' \in S \setminus S_\infty$. In particular therefore, the space $V_\psi^{G_w}$ vanishes. Since $e_\psi\mathbb{C}[G/G_w]$ is isomorphic to a direct sum of copies of $V_\psi^{G_w}$ it must therefore also vanish, as required.

Finally we observe that Theorem B(iii) is an obvious consequence of Theorem B(ii) and the following fact: if S contains at least two non-archimedean places, then $|S| > r+1$ and so (5) implies that $\theta^{(r)}_{K/k,S,T}(0)e_{1_G} = 0$. □

Example 4.3.6. Let K/k be an extension of totally real fields for which G is isomorphic to the quaternion group of order eight. Write τ for the unique element of G of order two and ψ for the unique irreducible complex character of G of degree two. Fix a finite set S of places of k which contains S_∞, all places which ramify in K/k and a non-archimedean place v. Then for every ϕ in $[U_{K,S}, X_{K,S}]_G$ one has

$$2^5\theta^{([k:\mathbb{Q}])}_{K/k,S,T}(0)R(\phi)(1-\tau) \in \mathrm{Ann}_{\mathbb{Z}[G]}(\mathrm{Cl}(\mathcal{O}_K)).$$

To show this we note that if G_w is trivial, then (5) implies $\theta^{([k:\mathbb{Q}])}_{K/k,S,T}(0)$ vanishes and so the containment is obvious. We may therefore assume that G_w is not trivial and so contains τ. Since τ is central in G this implies that the element $1-\tau$ annihilates $\mathbb{Z}[G/G_w]$. Then, as $(1-\tau)e_\psi = (1-\tau)$, the displayed containment follows directly from (17) with $x = 1-\tau$ and $E_\psi = \mathbb{Q}$.

Corollary 4.3.7. *Theorem C is valid.*

Proof. We assume the notation of Theorem C. Then Theorem 4.3.1(a)(ii) combines with Proposition 2.6.2 (with $S' = S$) to imply that the element $|G|^{3+r}\theta^r_{K/k,S,T}(0)R(\phi)$ belongs to $\mathrm{Ann}_{\mathbb{Z}[G]}(\mathrm{Cl}(\mathcal{O}_{K,S}))$. To deduce Theorem C from this we need only apply Lemma 2.7.1(i) with $\phi = D_{K,S}$ (so $R'(\phi) = 1$) and then note that $Z^r_{K/k,S,T}(1)$ cannot be zero since, by assumption, $Z^{r+1}_{K/k,S,T}(t)$ is not holomorphic at $t = 1$. □

5. Reduction steps

With the exception of §12.2 (in which we discuss a particular case of Conjecture 2.6.1) the rest of this article is dedicated to proving results announced in §2 and §4.

We will use the following convenient notation: for any natural number d we set $\langle d\rangle := \{i \in \mathbb{Z} : 1 \le i \le d\}$.

5.1. As the first step in our proof of Theorem 4.1.1 we now prove an important reduction step concerning Conjecture 2.4.1. The basis for this reduction is provided by the following result.

We fix sets S and T as in Conjecture 2.4.1.

Lemma 5.1.1. *There exists a finite set S'' of places of k which do not belong to $S \cup T$, are each totally split in K/k and are such that the following conditions are both satisfied by the set $S' := S \cup S''$.*

(i) $\mathrm{Cl}(\mathcal{O}_{K,S'}) = 0$.

(ii) *There exists an exact sequence of G-modules of the form*

$$0 \to U_{K,S',T} \xrightarrow{\subseteq} U_{K,S'} \to \mathbb{F}_T^\times \to 0$$

where $\mathbb{F}_T^\times$ denotes the direct sum of the multiplicative groups of the residue fields of all places in T_K.

Proof. With S' as in the statement of the lemma, there exists an exact sequence of G-modules of the form

$$0 \to U_{K,S',T} \xrightarrow{\subseteq} U_{K,S'} \to \mathbb{F}_T^\times \to \mathrm{Cl}(\mathcal{O}_{K,S'})_T \to \mathrm{Cl}(\mathcal{O}_{K,S'}) \to 0,$$

where $\mathrm{Cl}(\mathcal{O}_{K,S'})_T$ is the quotient of the group of fractional ideals of $\mathcal{O}_{K,S'}$ that are coprime to all places in T_K by the subgroup of principal ideals with a generator congruent to 1 modulo all places in T_K (cf. [47, (1)]). To verify claims (i) and (ii) it is therefore enough to show that we may choose S'' so that $\mathrm{Cl}(\mathcal{O}_{K,S'})_T$ vanishes. But class field theory identifies $\mathrm{Cl}(\mathcal{O}_{K,S'})_T$ with $G_{H_{S',T}/K}$ where $H_{S',T}$ is the maximal abelian extension of K which is unramified outside T_K and is such that all places in S'_K, resp. in T_K, are totally split, resp. at most tamely ramified, and so the existence of a suitable set S'' follows as a consequence of Tchebotarev's Density Theorem. □

In the sequel we fix a set $S' = S \cup S''$ as in Lemma 5.1.1. We set $m := |S''|$ and $n' := |S'| - 1 = n + m$, label the elements of S'' as $\{\mu_j : j \in \langle m \rangle\}$ and then label the elements of S' as $\{v'_i : 0 \le i \le n'\}$ where

$$v'_i := \begin{cases} v_0, & i = 0 \\ \mu_i, & 1 \le i \le m \\ v_{i-m}, & m < i \le n'. \end{cases}$$

We set $r' := m + r$ and note that, with the above labeling, the set S' satisfies Hypothesis $\mathrm{H}_{r'}$ with $S_{r'} = \{v'_i : i \in \langle r' \rangle\}$.

Proposition 5.1.2. *Assume that the second paragraph of Conjecture 2.4.1 is valid with S and r replaced by S' and r'.*

(i) *Then the second paragraph of Conjecture 2.4.1 is valid as stated.*
(ii) *Fix ϕ in $[U_{K,S}, X_{K,S}]_G$ and a in $\mathcal{A}(\mathbb{Z}[G])$. Then $a^2\theta^{(r)}_{K/k,S,T}(0)R(\phi)$ belongs to $\mathbb{Z}[G]$ and annihilates any quotient of $\mathrm{Cl}(\mathcal{O}_{K,S})$ that is cohomologically-trivial as a G-module.*

Our proof of Proposition 5.1.2 starts with the following technical result. Before stating this result we note that, since $\mathrm{Cl}(\mathcal{O}_{K,S'})$ vanishes, there is a natural exact sequence of G-modules

$$0 \to U_{K,S} \to U_{K,S'} \xrightarrow{\epsilon} I_{K,S''} \xrightarrow{\pi} \mathrm{Cl}(\mathcal{O}_{K,S}) \to 0 \tag{18}$$

in which $I_{K,S''}$ denotes the free abelian group on ideals of K that correspond to places in S''_K. We note that since every place in S'' is non-archimedean (as S'' is disjoint from S) and splits completely in K/k the G-module $I_{K,S''}$ is free of rank m.

Lemma 5.1.3. *Fix a free G-submodule M of $I_{K,S''}$ of rank m and write A for the transition matrix from $I_{K,S''}$ to M with respect to any choices of G-bases (so A belongs to $\mathrm{M}_m(\mathbb{Z}[G])$). Fix an integer t with $t\cdot\ker(\pi)\subseteq M$. Then for every ϕ in $[U_{K,S},X_{K,S}]_G$ there exists $\phi'_{M,t}$ in $[U_{K,S'},X_{K,S'}]_G$ with $\theta^{(r')}_{K/k,S'}(0)R(\phi'_{M,t})\mathrm{Nrd}_{\mathbb{Q}[G]}(A)=\theta^{(r)}_{K/k,S}(0)R(\phi)t^m$.*

Proof. For each integer j with $0\le j\le n+m$ we fix a place w'_j of K above v'_j. If j belongs to $\langle m\rangle$, then w'_j is non-archimedean and we write val_j for the associated normalised valuation on $K^\times$. We consider the composite homomorphism of G-modules

$$\epsilon' : U_{K,S'}\xrightarrow{\epsilon} I_{K,S''}\xrightarrow{\beta} Y_{K,S''}$$

where β sends each ideal $\mathfrak{a}$ to $\sum_{j=1}^{j=m}(\sum_{g\in G}\mathrm{val}_j(g^{-1}(\mathfrak{a}))g)w'_j$. Then, since $\mathrm{im}(\epsilon)=\ker(\pi)$ (because (18) is exact), one has $\mathrm{im}(t\cdot\epsilon)=t\cdot\mathrm{im}(\epsilon)\subseteq M$ and so the homomorphism $\epsilon'_{M,t}:=A^{-1}t\cdot\epsilon'$ belongs to $[U_{K,S'},Y_{K,S''}]_G\subseteq[U_{K,S'},Y_{K,S'_{r'}}]_G$.

Since $\mathrm{im}(\epsilon)$ is torsion-free, the exact sequence (18) implies that for any finitely generated free G-module F the restriction map $[U_{K,S'},F]_G\to[U_{K,S},F]_G$ is surjective (cf. [47, Prop. 1.1(ii)]). In particular, by applying this observation with $F=Y_{K,S'_{r'}}$ we may choose ϕ' in $[U_{K,S'},Y_{K,S'_{r'}}]_G$ which restricts to give $\pi_{S,r}\circ\phi$ on $U_{K,S}$ where ϕ is the given element of $[U_{K,S},X_{K,S}]_G$. We then set

$$\phi'_{M,t}:=\sigma_{S',r'}\circ(\phi'+\epsilon'_{M,t})\in[U_{K,S'},X_{K,S'}]_G$$

where $\sigma_{S',r'}\in[Y_{K,S'_{r'}},X_{K,S'}]_G$ sends each w'_j to $w'_j-w'_0$. Taken with respect to the $\mathbb{R}[G]$-basis $\{w'_j : i\in\langle r'\rangle\}$ of $\mathbb{R}\otimes Y_{K,S'_{r'}}$ the matrix of $\pi_{S',r'}\circ(\mathbb{R}\otimes\phi'_{M,t})\circ\mathrm{R}^{-1}_{K,S'}\circ\sigma_{S',r'}$ is equal to

$$M_{S'}:=\begin{pmatrix} M_S & 0\\ * & t\cdot\Delta_{S,S'}\cdot A^{-1}\end{pmatrix}$$

where M_S is the matrix of $\pi_{S,r}\circ(\mathbb{R}\otimes\phi)\circ\mathrm{R}^{-1}_{K,S}\circ\sigma_{S,r}$ with respect to the basis $\{w'_j : j\in\langle r\rangle\}$ of $\mathbb{R}\otimes Y_{K,r}$ and $\Delta_{S,S'}$ is the diagonal $m\times m$ matrix with jj-entry equal to $(\log\mathrm{N}w_j)^{-1}$.

We now use the idempotents $e_r:=e_{S,r}$ and $e_{r'}:=e_{S',r'}$ from Lemma 3.2.1. Since every place in $S''=S'\setminus S$ splits completely in K/k it is straightforward to show, by using the description (14), that $\mathrm{Ir}_{S,r}(G)=\mathrm{Ir}_{S',r'}(G)$ and hence that $e_r=e_{r'}$. Since the homomorphisms $e_{r'}(\mathbb{R}\otimes\pi_{S',r'})$ and $e_r(\mathbb{R}\otimes\pi_{S,r})$ are both bijective (by Lemma

3.2.1(ii)) one therefore has

$$\begin{aligned} e_{r'}R(\phi'_{M,t}) &= e_{r'}\mathrm{Nrd}_{\mathbb{R}[G]}((\mathbb{R}\otimes\phi'_{M,t})\circ \mathrm{R}_{K,S'}^{-1}) \\ &= e_{r'}\mathrm{Nrd}_{\mathbb{R}[G]}(\pi_{S',r'}\circ(\mathbb{R}\otimes\phi'_{M,t})\circ \mathrm{R}_{K,S'}^{-1}\circ\sigma_{S',r'}) \\ &= e_{r'}\mathrm{Nrd}_{\mathbb{R}[G]}(M_{S'}) \\ &= e_{r'}\mathrm{Nrd}_{\mathbb{R}[G]}(M_S)t^m\mathrm{Nrd}_{\mathbb{Q}[G]}(A)^{-1}\mathrm{Nrd}_{\mathbb{R}[G]}(\Delta_{S,S'}) \\ &= e_rR(\phi)t^m\mathrm{Nrd}_{\mathbb{Q}[G]}(A)^{-1}\mathrm{Nrd}_{\mathbb{R}[G]}(\Delta_{S,S'}). \end{aligned}$$

Now $\theta^{(r')}_{K/k,S'}(0) = \theta^{(r')}_{K/k,S'}(0)e_{r'}$ and $\theta^{(r)}_{K/k,S}(0) = \theta^{(r)}_{K/k,S}(0)e_r$ by Lemma 3.2.1(iii). In addition, as $e_\psi\mathrm{Nrd}_{\mathbb{R}[G]}(\Delta_{S,S'}) = e_\psi\prod_{j=1}^{j=m}(\log \mathrm{N}w_j)^{-\psi(1)}$ for every ψ in $\mathrm{Ir}(G)$, one has $\theta^{(r')}_{K/k,S'}(0) = \theta^{(r)}_{K/k,S}(0)\mathrm{Nrd}_{\mathbb{R}[G]}(\Delta_{S,S'})^{-1}$. Using these observations, the claimed equality is obtained by simply multiplying the last displayed formula by $\mathrm{Nrd}_{\mathbb{Q}[G]}(A)\theta^{(r')}_{K/k,S'}(0)$. □

Returning to the proof of Proposition 5.1.2(i) we take $M = I_{K,S''}$, $A = \mathrm{id}$ and $t = 1$ in Lemma 5.1.3. Then $\mathrm{Nrd}_{\mathbb{Q}[G]}(A) = 1$ and so Lemma 5.1.3 implies that $\theta^{(r)}_{K/k,S,T}(0)R(\phi) = \theta^{(r')}_{K/k,S',T}(0)R(\phi'_{M,t})$. To prove Proposition 5.1.2(i) it is thus enough to show that for every matrix M' in $\mathfrak{M}_{S'}(\mathbb{Z}[G])$ there is a matrix M in $\mathfrak{M}_S(\mathbb{Z}[G])$ with $\mathrm{Nrd}_{\mathbb{Q}[G]}(M) = \mathrm{Nrd}_{\mathbb{Q}[G]}(M')$. To do this we recall that such a matrix M' belongs to $\mathrm{M}_{d'}(\mathbb{Z}[G])$ for some $d' \geq n'$ and we fix a permutation Π of the set $\langle d'\rangle$ with $\Pi(a) = a - m$ for all a with $r' < a \leq n'$. We then obtain a suitable matrix M in $\mathfrak{M}_S(\mathbb{Z}[G]) \cap \mathrm{M}_{d'}(\mathbb{Z}[G])$ by first using Π to permute the columns of M' and then multiplying every element in the first column of the new matrix by $(-1)^{\mathrm{sgn}(\Pi)}$.

To prove Proposition 5.1.2(ii) we now fix a cohomologically-trivial quotient Q of the G-module $\mathrm{Cl}(\mathcal{O}_{K,S})$. Then the kernel P of the composite surjective homomorphism $I_{K,S''} \xrightarrow{\pi} \mathrm{Cl}(\mathcal{O}_{K,S}) \to Q$ is finitely generated, torsion-free and cohomologically-trivial. This implies that P is a projective $\mathbb{Z}[G]$-module (by [2, Th. 8]) and hence a locally-free $\mathbb{Z}[G]$-module (by Swan's Theorem [25, (32.1)]). By Roiter's Lemma [25, (31.6)] it follows that, for any given integer D', there exists a free G-submodule P' of P for which the quotient P/P' is finite and of cardinality D coprime to D'. Since $\ker(\pi) \subseteq P$ we may apply Lemma 5.1.3 with $M = P'$ and $t = D$ to deduce that

$$D^m\theta^{(r)}_{K/k,S,T}(0)R(\phi) = \theta^{(r')}_{K/k,S',T}(0)R(\phi'_{M,t})f_A \tag{19}$$

with $f_A := \mathrm{Nrd}_{\mathbb{Q}[G]}(A)$. Now for each element a of $\mathcal{A}(\mathbb{Z}[G])$ Lemma 5.1.4 below implies that af_A belongs to $\mathbb{Z}[G]$ and annihilates Q. Hence, since the hypothesis of Proposition 5.1.2 implies that $a\theta^{(r')}_{K/k,S',T}(0)R(\phi'_{M,t})$ belongs to $\mathbb{Z}[G]$, we can multiply (19) by a^2 to deduce that $D^ma^2\theta^{(r)}_{K/k,S,T}(0)R(\phi) = (a\theta^{(r')}_{K/k,S',T}(0)R(\phi'_{M,t}))(af_A)$ belongs to $\mathbb{Z}[G]$ and annihilates Q. Since D can be chosen to be coprime to any given integer this implies that $a^2\theta^{(r)}_{K/k,S,T}(0)R(\phi)$ belongs to $\mathbb{Z}[G]$ and annihilates Q, as claimed by Proposition 5.1.2(ii).

This completes our proof of Proposition 5.1.2.

Lemma 5.1.4. *Let R be a Dedekind domain of characteristic zero with field of fractions F. Let A be an endomorphism of a finitely generated free $R[G]$-module. Then for any a in $\mathcal{A}(R[G])$ the element* $a\mathrm{Nrd}_{F[G]}(A)$ *belongs to* $\mathfrak{Z}(R[G])$ *and annihilates the module* $\mathrm{cok}(A)$.

Proof. We assume that A is an endomorphism of a free $R[G]$-module P of rank d and, having fixed an $R[G]$-basis of P, write H for the corresponding matrix in $\mathrm{M}_d(R[G])$. We also write A^* for the endomorphism of $F\otimes_R P$ that corresponds, with respect to the induced $F[G]$-basis of $F\otimes_R P$, to the matrix H^* defined in §2.3.

Lemma 2.3.1 implies that for each element a of $\mathcal{A}(R[G])$ the product $a\mathrm{Nrd}_{E[G]}(A) = a\mathrm{Nrd}_{E[G]}(H)$ belongs to $\mathfrak{Z}(R[G])$ and so acts on $\mathrm{cok}(A)$. In addition, for every element x of P one has $a\mathrm{Nrd}_{E[G]}(A)(x) = aAA^*(x) = A(aA^*(x))$. But the definition of $\mathcal{A}(R[G])$ implies that aA^* restricts to give an endomorphism of P and so the element $a\mathrm{Nrd}_{E[G]}(A)(x) = A(aA^*(x))$ belongs to $\mathrm{im}(A)$. It is therefore clear that $a\mathrm{Nrd}_{E[G]}(A)$ annihilates $\mathrm{cok}(A)$, as required. □

6. The leading term conjecture

In this section we discuss the leading term conjecture that occurs in Theorem 4.1.1.

6.1. We must first introduce some additional notation.

Let R be an integral domain of characteristic zero with field of fractions F and $\mathfrak{A}$ an R-order in a finite dimensional semisimple F-algebra A. For any extension field E of F we write A_E for the (semisimple) E-algebra $E\otimes_F A$ and $K_0(\mathfrak{A}, A_E)$ for the relative algebraic K-group of the inclusion $\mathfrak{A}\subset A_E$. This group is functorial in the pair $(\mathfrak{A}, A_E)$ and also sits in a long exact sequence of relative K-theory

$$K_1(\mathfrak{A}) \xrightarrow{\partial^2_{\mathfrak{A},A_E}} K_1(A_E) \xrightarrow{\partial^1_{\mathfrak{A},A_E}} K_0(\mathfrak{A}, A_E) \xrightarrow{\partial^0_{\mathfrak{A},A_E}} K_0(\mathfrak{A}).$$

In the case $R\subset\mathbb{Q}$ and $E\subseteq\mathbb{R}$ we write

$$\delta_{\mathfrak{A},A_E} : \mathfrak{Z}(A_E)^\times \to K_0(\mathfrak{A}, A_E)$$

for the 'extended boundary homomorphism' defined in [17, Lem. 9] and we recall that $\delta_{\mathfrak{A},A_E}\circ\mathrm{Nrd}_{A_E} = \partial^1_{\mathfrak{A},A_E}$ where Nrd_{A_E} denotes the homomorphism $K_1(A_E)\to\mathfrak{Z}(A_E)^\times$ induced by taking reduced norms. If $\mathfrak{A} = \mathbb{Z}[G]$ and $A_E = \mathbb{R}[G]$, then we often abbreviate $\partial^i_{\mathfrak{A},A_E}$ and $\delta_{\mathfrak{A},A_E}$ to ∂^i_G and δ_G respectively.

Let G be a finite group. For $R[G]$-modules M and N an element of $\mathrm{Ext}^2_{R[G]}(N, M)$ is said to be 'perfect' if it is represented by a sequence $0\to M\to C^0 \xrightarrow{d} C^1\to N\to 0$ in which C^0 and C^1 are both finitely generated and cohomologically-trivial. In this case we regard the complex $C^0 \xrightarrow{d} C^1$, with C^0 placed in degree 0, as an object $C^\bullet$ of $D^{\mathrm{p}}(R[G])$ for which $H^0(C^\bullet) = M$ and $H^1(C^\bullet) = N$. We recall that to each such complex $C^\bullet$ and each isomorphism of $E[G]$-modules $\lambda : E \otimes_R M \cong E \otimes_R N$ one can associate a canonical Euler characteristic $\chi(C^\bullet, \lambda)$ in $K_0(R[G], E[G])$. (For more details about this construction see Appendix A.)

We further recall that if Σ is any finite non-empty set of places of k which contains all archimedean places (if any) and all places that ramify in K/k and is also such that $\mathrm{Cl}(\mathcal{O}_{K,\Sigma})$ vanishes, then the canonical element $\tau_{K/k,\Sigma}$ of $\mathrm{Ext}^2_G(X_{K,\Sigma}, U_{K,\Sigma})$ that is defined by Tate in [54] is perfect.

We can now explicitly recall the relevant equivariant leading term conjecture.

Conjecture LTC(K/k): *Let Σ be any finite non-empty set of places of k that contains all archimedean places (if any) and all places that ramify in K/k and is also such that $\mathrm{Cl}(\mathcal{O}_{K,\Sigma})$ vanishes. If $C^\bullet_{K/k,\Sigma}$ is any representative in $D^{\mathrm{p}}(\mathbb{Z}[G])$ of the canonical class $\tau_{K/k,\Sigma}$ then in $K_0(\mathbb{Z}[G], \mathbb{R}[G])$ one has $\delta_G(\theta^*_{K/k,\Sigma}(0)) = \chi(C^\bullet_{K/k,\Sigma}, \mathrm{R}_{K,\Sigma})$.*

Remark 6.1.1.
(i) If K/k is an extension of number fields, then [18, (29)] implies that LTC(K/k) coincides with the equivariant Tamagawa number conjecture of [17, Conj. 4] for the pair $(h^0(\mathrm{Spec}(K)), \mathbb{Z}[G])$. If K/k is an extension of global function fields, then [10, §2.2, Remarks 1, 2 and 3] combine with the proof of [16, Th. 3.2] to imply that LTC(K/k) coincides with the conjecture formulated in [10, §2.2] and hence refines Lichtenbaum's conjecture [34, Conj. 8.1e)].
(ii) With Σ as in LTC(K/k) we set

$$T\Omega(\mathbb{Q}(0)_K, \mathbb{Z}[G]) := \delta_G(\theta^*_{K/k,\Sigma}(0)) - \chi(C^\bullet_{K/k,\Sigma}, \mathrm{R}_{K,\Sigma}) \in K_0(\mathbb{Z}[G], \mathbb{R}[G])$$

(so LTC(K/k) simply asserts that $T\Omega(\mathbb{Q}(0)_K, \mathbb{Z}[G])$ vanishes). We recall from, for example, [7, p. 1445] that $T\Omega(\mathbb{Q}(0)_K, \mathbb{Z}[G])$ is independent of the choices of both Σ and $C^\bullet_{K/k,\Sigma}$ and has the following properties: for each subgroup H of G the image of $T\Omega(\mathbb{Q}(0)_K, \mathbb{Z}[G])$ under the restriction homomorphism $K_0(\mathbb{Z}[G], \mathbb{R}[G]) \to K_0(\mathbb{Z}[H], \mathbb{R}[H])$ is $T\Omega(\mathbb{Q}(0)_K, \mathbb{Z}[H])$; for each normal subgroup H of G the image of $T\Omega(\mathbb{Q}(0)_K, \mathbb{Z}[G])$ under the coinflation homomorphism $K_0(\mathbb{Z}[G], \mathbb{R}[G]) \to K_0(\mathbb{Z}[G/H], \mathbb{R}[G/H])$ is $T\Omega(\mathbb{Q}(0)_{K^H}, \mathbb{Z}[G/H])$; Stark's Conjecture is valid for every complex character of G if and only if $T\Omega(\mathbb{Q}(0)_K, \mathbb{Z}[G])$ belongs to the subgroup $K_0(\mathbb{Z}[G], \mathbb{Q}[G])$ of $K_0(\mathbb{Z}[G], \mathbb{R}[G])$.
(iii) We will often use the fact (proved in [18, §3, Cor. 1]) that the validity of the Strong Stark Conjecture for all complex characters of G is equivalent to asserting that $T\Omega(\mathbb{Q}(0)_K, \mathbb{Z}[G])$ belongs to the subgroup $K_0(\mathbb{Z}[G], \mathbb{Q}[G])_{\mathrm{tor}}$ of $K_0(\mathbb{Z}[G], \mathbb{R}[G])$.

6.2. In the remainder of §6 we prove an algebraic result that is key to computing explicitly the Euler characteristic $\chi(C^\bullet_{K/k,\Sigma}, \mathrm{R}_{K,\Sigma})$ that occurs in LTC(K/k).

To do this we must first describe a convenient resolution of certain types of G-modules. We therefore fix a finite non-empty set of places $S = \{v_i : 0 \le i \le n\}$ of k which contains S_∞ and all places that ramify in K/k and also satisfies Hypothesis H_r for some non-negative integer r. For any subgroup H of G we set $T_H := \sum_{h \in H} h \in \mathbb{Z}[G]$ and for any subset Σ of S we identify $Y_{K^H,\Sigma}$ with $T_H(Y_{K,\Sigma})$ by means of the homomorphism induced by sending each place v in Σ_{K^H} to $T_H(w)$ where, as before, w is a choice of place of K above v.

For any integer $m \in \langle n \rangle$ we set $S_m := \{v_i : i \in \langle m \rangle\}$ and abbreviate Y_{K,S_m} to $Y_{K,m}$. We also fix a finitely generated subring R of $\mathbb{Q}$.

Lemma 6.2.1. *We suppose given data of the following kind:-*

- *an integer m with $r \le m \le n$;*
- *a surjection of finitely generated $R[G]$-modules $\rho : M \to R \otimes Y_{K,m}$.*

Then there exists a finitely generated free $R[G]$-module F of rank d with $d \ge n+2$, a surjective homomorphism of $R[G]$-modules $\pi : F \to M$ and an ordered $R[G]$-basis $\{b_i : i \in \langle d \rangle\}$ of F with all of the following properties.

We write ξ for the composite homomorphism $\rho \circ \pi : F \to R \otimes Y_{K,m}$.

(i) *One has $\xi(b_i) = w_{i,K}$ for each $i \in \langle m \rangle$ and $\xi(b_i) = 0$ for each $i \in \langle d \rangle \setminus \langle m \rangle$. In particular, if $\mathcal{Z}$ is the set of integers j in $\langle m \rangle$ such that v_j splits completely in K/k, then* $\ker(\pi) \subseteq \sum_{i \in \langle d \rangle \setminus \mathcal{Z}} R[G] \cdot b_i$.

(ii) *Assume $m = n$. For each normal subgroup A of G we identify the restriction ξ^A of ξ with the composite homomorphism $F^A = T_A(F) \xrightarrow{\xi} R \otimes T_A(Y_{K,m}) = R \otimes Y_{K^A,m}$ and for each index i set $b_i^A := T_A(b_i)$. Then $\xi^A(b_i^A) = w_{i,K^A}$ if $i \in \langle n \rangle$ and $\xi^A(b_i^A) = 0$ if $i \in \langle d \rangle \setminus \langle n \rangle$. In particular, if $\mathcal{Z}_A$ is the set of integers j in $\langle n \rangle$ such that v_j splits completely in K^A/k, then* $\ker(\pi^A) \subseteq \sum_{i \in \langle d \rangle \setminus \mathcal{Z}_A} R[G/A] \cdot b_i^A$.

Proof. We first fix a free $R[G]$-module F_1 of rank m, with basis $\{b'_i : i \in \langle m \rangle\}$ say, and write $\pi_1 : F_1 \to R \otimes Y_{K,m}$ for the homomorphism of $R[G]$-modules which sends b'_i to w_i for each index $i \in \langle m \rangle$. We set $N := \ker(\pi_1)$ and fix a surjective homomorphism of $R[G]$-modules $\pi_2 : F_2 \to N$ in which F_2 is a finitely generated free $R[G]$-module, of rank c such that $c \ge n + 2 - m$. We also fix a $R[G]$-basis $\{b''_j : j \in \langle c \rangle\}$ of F_2.

We now set $F := F_1 \oplus F_2$ and let $\pi : F \to M$ denote the homomorphism of $R[G]$-modules which restricts on F_1 to give π_1 and on F_2 to give the composite of π_2 and the inclusion $N \subseteq M$. It is clear that π is surjective. Further, if we set $d := m + c$ (so $d \ge n+2$), $b_i := b'_i$ for each i with $i \in \langle m \rangle$ and $b_i := b''_{i-m}$ for each $i \in \langle d \rangle \setminus \langle m \rangle$, then it is straightforward to check that the set $\{b_i : i \in \langle d \rangle\}$ is an ordered $R[G]$-basis of F which satisfies the first claims in both (i) and (ii).

We now prove the assertion concerning $\ker(\pi^A)$ in claim (ii) (leaving to the reader the easier proof of the assertion in claim (i) regarding $\ker(\pi)$). We therefore assume that $m = n$ and set $\Sigma_A := \{v_i : i \in \mathcal{Z}_A\}$. Now the definition of $\mathcal{Z}_A$ ensures that the elements $\{w_{i,K^A} : i \in \mathcal{Z}_A\}$ of Y_{K^A,Σ_A} are linearly independent over $R[G/A]$. Hence there is an isomorphism of $\mathbb{R}[G/A]$-modules $\mathbb{R}[G/A]\cdot w_{i,K^A} \cong \mathbb{R}[G/A]$ for each $i \in \mathcal{Z}_A$ and also a direct sum decomposition of $R[G/A]$-modules

$$R \otimes Y_{K^A,m} = R \otimes Y_{K^A,S_m\setminus\Sigma_A} \oplus \bigoplus_{i\in\mathcal{Z}_A} R[G/A]\cdot w_{i,K^A}.$$

From the properties of the elements $\{\xi^A(b_i^A) : i \in \langle d\rangle\}$ described in claim (ii) it is thus clear that $\ker(\pi^A) \subseteq \ker(\xi^A) \subseteq \sum_{i\in\langle d\rangle\setminus\mathcal{Z}_A} R[G/A]\cdot b_i^A$, as required. □

Before stating the main result of this section we introduce further notation. We define the Euler characteristic of an object $C^\bullet$ of $D^{\mathrm{p}}(R[G])$ by setting

$$\chi(C^\bullet) := \sum_{i\in\mathbb{Z}}(-1)^i[P^i] \in K_0(R[G]) \tag{20}$$

where $P^\bullet$ is a bounded complex of finitely generated projective $R[G]$-modules that is isomorphic in $D^{\mathrm{p}}(R[G])$ to $C^\bullet$ and $[P^i]$ denotes the element of $K_0(R[G])$ corresponding to P^i. (This definition is easily checked to be independent of the choice of $P^\bullet$.) We also recall for each ψ in $\mathrm{Ir}(G)$ the functor $M \mapsto M_\psi$ from G-modules to $\mathcal{O}_\psi$-modules that is defined in §2.6.

Proposition 6.2.2. *We suppose given data of the following kind:-*

- *an object $C^\bullet$ of $D^{\mathrm{p}}(R[G])$ which is acyclic outside degrees 0 and 1 and is such that $H^0(C^\bullet)$ is R-torsion-free and there exists a surjection of $R[G]$-modules $\rho : H^1(C^\bullet) \to R\otimes Y_{K,m}$ where m is an integer with $r \le m \le n$;*
- *an isomorphism of $\mathbb{R}[G]$-modules $\lambda : \mathbb{R}\otimes_R H^0(C^\bullet) \cong \mathbb{R}\otimes_R H^1(C^\bullet)$;*
- *a natural number $\hbar$.*

Let π, F, d and $\{b_i : i \in \langle d\rangle\}$ be constructed as in Lemma 6.2.1 with respect to the given surjection $\rho : H^1(C^\bullet) \to R\otimes Y_{K,m}$. Then there exists an endomorphism ϑ of the $R[G]$-module F which satisfies all of the following conditions.

Let $C^\bullet_\vartheta$ denote the complex $F \xrightarrow{\vartheta} F$ where the first term is placed in degree 0.

(i) *There exists an exact triangle in $D^{\mathrm{p}}(R[G])$ of the form*

$$C^\bullet_\vartheta \xrightarrow{\alpha} C^\bullet \to Q[0] \to C^\bullet_\vartheta[1] \tag{21}$$

where $H^1(\alpha)$ is induced by π and the module $Q := H^0(C^\bullet)/\operatorname{im}(H^0(\alpha))$ is finite and of cardinality coprime to $|G|\hbar$.

(ii) *One can take Q to be the zero module in claim (i) if and only if $\chi(C^\bullet)$ vanishes.*

(iii) *Let the sets $\mathcal{Z}$ and $\mathcal{Z}_A$ be defined as in Lemma 6.2.1.*
   (a) *Then $\operatorname{im}(\vartheta) \subseteq \sum_{i\in\langle d\rangle\setminus\mathcal{Z}} R[G]\cdot b_i$.*
   (b) *If $m = n$, then $\operatorname{im}(\vartheta^A) \subseteq \sum_{i\in\langle d\rangle\setminus\mathcal{Z}_A} R[G/A]\cdot b_i^A$ for all normal subgroups A of G.*

(iv) *For each integer t in $\langle d\rangle$ we write $F(t)$ for the $R[G]$-submodule of F that is generated by $\{b_i : t < i \le d\}$. Then for every ψ in* $\mathrm{Ir}(G)$ *for which there exists an integer t_ψ with $\mathbb{C}\otimes_{\mathcal{O}_\psi}\mathrm{im}(\vartheta)_\psi = \mathbb{C}\otimes_{\mathcal{O}_\psi} F(t_\psi)_\psi$ there is a direct sum decomposition $\mathbb{C}\otimes_{\mathcal{O}_\psi} F_\psi = (\mathbb{C}\otimes_{\mathcal{O}_\psi}\ker(\vartheta)_\psi)\oplus(\mathbb{C}\otimes_{\mathcal{O}_\psi}\mathrm{im}(\vartheta)_\psi)$.*

(v) *The algebra $\mathbb{Q}[G]$ is semisimple and so there are $\mathbb{Q}[G]$-equivariant sections ι_1 and ι_2 to the surjections $\mathbb{Q}\otimes_R F\to\mathbb{Q}\otimes_R\mathrm{im}(\vartheta)$ and $\mathbb{Q}\otimes_R F\to\mathbb{Q}\otimes_R\mathrm{cok}(\vartheta)\cong\mathbb{Q}\otimes_R H^1(C^\bullet)$ that are induced by ϑ and π respectively. For any subfield E of $\mathbb{R}$ this induces a direct sum decomposition of $E[G]$-modules*

$$E\otimes_R F = E\otimes_R\ker(\vartheta)\oplus(E\otimes_{\mathbb{Q}}\iota_1)(E\otimes_R\mathrm{im}(\vartheta))$$

and so for λ' in $[E\otimes_R H^0(C^\bullet), E\otimes_R H^1(C^\bullet)]_{E[G]}$ there is a unique $\langle\lambda',\vartheta,\alpha,\iota_1,\iota_2\rangle$ in $[E\otimes_R F, E\otimes_R F]_{E[G]}$ that is equal to $(E\otimes_{\mathbb{Q}}\iota_2)\circ\lambda'\circ(E\otimes_R H^0(\alpha))$ on $E\otimes_R\ker(\vartheta)$ and to $E\otimes_R\vartheta$ on $(E\otimes_{\mathbb{Q}}\iota_1)(E\otimes_R\mathrm{im}(\vartheta))$. If λ' is invertible, then $\langle\lambda',\vartheta,\alpha,\iota_1,\iota_2\rangle$ is invertible. In particular, for the given isomorphism of $\mathbb{R}[G]$-modules λ the element $\mathrm{Nrd}_{\mathbb{R}[G]}(\langle\lambda,\vartheta,\alpha,\iota_1,\iota_2\rangle)$ belongs to $\mathfrak{Z}(\mathbb{R}[G])^\times$ and

$$\chi(C^\bullet,\lambda)=\delta_{R[G],\mathbb{R}[G]}(\mathrm{Nrd}_{\mathbb{R}[G]}(\langle\lambda,\vartheta,\alpha,\iota_1,\iota_2\rangle))+\chi(Q[0],0)$$

where Q is the finite module in claim (i).

The proof of this result will occupy the rest of §6.

6.3. Since $C^\bullet$ belongs to $D^{\mathrm{p}}(R[G])$ and is acyclic outside degrees 0 and 1 a standard argument shows that there exists an exact sequence of $R[G]$-modules

$$0\to H^0(C^\bullet)\xrightarrow{\nu}P\xrightarrow{\mu}F\xrightarrow{\pi}H^1(C^\bullet)\to 0 \tag{22}$$

for which the following property is satisfied: if we let $\tilde{C}^\bullet$ denote the complex $P\xrightarrow{\mu}F$ where P occurs in degree 0 and the cohomology groups are identified with $H^0(C^\bullet)$ and $H^1(C^\bullet)$ by means of (22), then there exists an isomorphism ς in $D(R[G])$ between $\tilde{C}^\bullet$ and $C^\bullet$ such that $H^i(\varsigma)$ is the identity map on $H^i(C^\bullet)$ for $i=0,1$. Since F is a free $R[G]$-module and $C^\bullet$ belongs to $D^{\mathrm{p}}(R[G])$, the isomorphism ς implies that the module P has a finite projective resolution and so is cohomologically-trivial. But the exactness of (22) implies that P is also both finitely generated and torsion-free as an R-module and so P is a projective $R[G]$-module.

The existence of λ implies that the $\mathbb{Q}[G]$-modules $\mathbb{Q}\otimes_R H^0(C^\bullet)$ and $\mathbb{Q}\otimes_R H^1(C^\bullet)$ are isomorphic (by [25, Vol. I, Ex. 6, p.139]). The Krull-Schmidt Theorem [25, (6.12)] then combines with the exact sequence (22) to imply that the $\mathbb{Q}[G]$-module $\mathbb{Q}\otimes_R P$ is isomorphic to $\mathbb{Q}\otimes_R F$ and hence is free of rank greater than one. In addition, for each prime p in $\mathrm{Spec}(R)$ the $\mathbb{Z}_p[G]$-module $\mathbb{Z}_p\otimes_R P$ is isomorphic to $\mathbb{Z}_p\otimes_R F$ (by [25, (32.1)]). Then Roiter's Lemma [25, (31.6)] implies that there exists an $R[G]$-submodule P' of P for which the quotient P/P' is finite and of cardinality coprime to $|G|\hbar$ and one has an isomorphism of G-modules $\iota : F\to P'$.

We write ϑ for the composite of ι and the restriction to P' of the homomorphism μ in (22) and note that this gives an exact commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F & \xrightarrow{\iota} & P & \longrightarrow & P/P' & \longrightarrow & 0 \\ & & {\scriptstyle\vartheta}\big\downarrow & & {\scriptstyle\mu}\big\downarrow & & \big\downarrow & & \\ 0 & \longrightarrow & F & \xrightarrow{\mathrm{id}} & F & \longrightarrow & 0 & \longrightarrow & 0. \end{array}$$

When combined with the isomorphism ς used above the first square of this diagram gives a morphism α in $D(R[G])$ from the complex $C^\bullet_\vartheta$ (as described in the statement of the proposition) to $C^\bullet$ and hence the diagram gives rise to an exact triangle of the form (21) in which $H^1(\alpha)$ is the map $\mathrm{cok}(\vartheta) \to H^1(C^\bullet)$ induced by π and the module $Q := P/P'$ is finite and of cardinality coprime to $|G|\hbar$.

To prove claim (ii) we must show that one can take Q to be the zero module in (21) if and only if $\chi(C^\bullet)$ vanishes. But if Q is the zero module, then the fact that (21) is an exact triangle implies $C^\bullet$ is isomorphic to $C^\bullet_\vartheta$ in $D^{\mathrm{p}}(R[G])$ and hence that $\chi(C^\bullet) = \chi(C^\bullet_\vartheta) = [F] - [F] = 0$. To prove the converse we assume $\chi(C^\bullet)$ vanishes. Since $\tilde{C}^\bullet$ is isomorphic to $C^\bullet$ in $D^{\mathrm{p}}(R[G])$, one therefore has $[P] - [F] = \chi(\tilde{C}^\bullet) = \chi(C^\bullet) = 0$. Then, as the $R[G]$-rank of P is greater than one, the Bass Cancellation Theorem [25, (41.20)] combines with the equality $[P] = [F]$ to imply that P is isomorphic to F as an $R[G]$-module. In the construction of (21) made above we can therefore take $P' = P$ so that Q vanishes, as required.

Since $\mathrm{im}(\vartheta) \subseteq \mathrm{im}(\mu) = \ker(\pi)$ the inclusion of claim (iii)(a) follows from Lemma 6.2.1(i). In a similar way, for each normal subgroup A one has $\mathrm{im}(\vartheta^A) \subseteq \mathrm{im}(\mu^A) \subseteq \ker(\pi^A)$ and so the inclusion of claim (iii)(b) follows from Lemma 6.2.1(ii).

This completes the proof of claims (i), (ii) and (iii) of Proposition 6.2.2.

6.4. We write $\mathrm{Ir}_{\mathbb{Q}}(G)$ for the set of irreducible $\mathbb{Q}$-valued characters of G. For any finitely generated $\mathbb{Q}[G]$-module W we write $g(W)$ for the minimal number of generators of W as a $\mathbb{Q}[G]$-module.

For each pair of natural numbers m and m' we set $\langle m, m'\rangle := \langle m\rangle \times \langle m'\rangle$ and we order elements of this set lexicographically.

In the next result we also use the notation $F(t)$ introduced in Proposition 6.2.2(iv).

Lemma 6.4.1. *For each μ in $[F,F]_{R[G]}$ there exists an element ϵ_μ of $[F,F]_{R[G]}$ that is bijective and has the following property: if ψ is any character in $\mathrm{Ir}(G)$ for which there exists an integer t_ψ with $\mathbb{C}\otimes_{\mathcal{O}_\psi} \mathrm{im}(\mu)_\psi = \mathbb{C}\otimes_{\mathcal{O}_\psi} F(t_\psi)_\psi$, then there is a direct sum decomposition $\mathbb{C}\otimes_{\mathcal{O}_\psi} F_\psi = (\mathbb{C}\otimes_{\mathcal{O}_\psi} \ker(\mu\circ\epsilon_\mu)_\psi) \oplus (\mathbb{C}\otimes_{\mathcal{O}_\psi} \mathrm{im}(\mu\circ\epsilon_\mu)_\psi)$.*

Proof. For any $R[G]$-module U and any character χ in $\mathrm{Ir}_{\mathbb{Q}}(G)$ we write $U_{[\chi]}$ for the $\mathbb{Q}[G]$-module $e_\chi(\mathbb{Q}[G]\otimes_{R[G]} U)$. Then, since μ is defined over $R[G]$ (and $R \subset \mathbb{Q}$), one has $\mathbb{C}\otimes_{\mathcal{O}_\psi} \mathrm{im}(\mu)_\psi = \mathbb{C}\otimes_{\mathcal{O}_\psi} F(t_\psi)_\psi$ for some integer t_ψ, respectively $\mathbb{C}\otimes_{\mathcal{O}_\psi} F_\psi = (\mathbb{C}\otimes_{\mathcal{O}_\psi} \ker(\mu\circ\epsilon)_\psi) \oplus (\mathbb{C}\otimes_{\mathcal{O}_\psi} \mathrm{im}(\mu\circ\epsilon)_\psi)$ for a given automorphism ϵ of F, if and only if $\mathrm{im}(\mu)_{[\psi_{\mathbb{Q}}]} = F(t_\psi)_{[\psi_{\mathbb{Q}}]}$, respectively $F_{[\psi_{\mathbb{Q}}]} = \ker(\mu\circ\epsilon)_{[\psi_{\mathbb{Q}}]} \oplus \mathrm{im}(\mu\circ\epsilon)_{[\psi_{\mathbb{Q}}]}$, where

$\psi_{\mathbb{Q}}$ is the unique element of $\mathrm{Ir}_{\mathbb{Q}}(G)$ which contains ψ as a constituent over $\mathbb{C}$. It is therefore enough to verify that there exists an automorphism of the $R[G]$-module F which verifies the stated conditions with each module of the form $\mathbb{C}\otimes_{\mathcal{O}_\psi} M_\psi$ replaced by $M_{[\psi_{\mathbb{Q}}]}$.

We set $I := \ker(\mu)\cap\mathrm{im}(\mu)$. We also write Υ for the subset of $\mathrm{Ir}_{\mathbb{Q}}(G)$ comprising characters χ which satisfy the following condition: $\mathrm{im}(\mu)_{[\chi]} = F(t_\chi)_{[\chi]}$ for an integer t_χ but $F_{[\chi]} \neq \ker(\mu)_{[\chi]}\oplus\mathrm{im}(\mu)_{[\chi]}$ (or equivalently $I_{[\chi]}\neq\{0\}$).

For the moment we fix a character χ in Υ and a decomposition $e_\chi = \sum_{i=1}^{i=d_\chi} f_{i,\chi}$ of e_χ as a sum of primitive (mutually orthogonal) idempotents of the algebra $e_\chi\mathbb{Q}[G]$.

We recall the $R[G]$-basis $\{b_i : i\in\langle d\rangle\}$ of F and for each $\underline{x}\in\langle d,d_\chi\rangle$ we set $b_{\underline{x},\chi} := f_{x_2,\chi}b_{x_1}$ and $V_{\underline{x},\chi} := \mathbb{Q}[G]\cdot b_{\underline{x},\chi} = \mathbb{Q}[G]f_{x_2,\chi}\cdot b_{x_1}$. We note that $F_{[\chi]}$ is the direct sum of $V_{\underline{x},\chi}$ as $\underline{x}$ ranges over $\langle d,d_\chi\rangle$ and that all such $\mathbb{Q}[G]$-modules $V_{\underline{x},\chi}$ are isomorphic since each $\mathbb{Q}[G]f_{x_2,\chi}$ is a minimal left ideal of the simple algebra $e_\chi\mathbb{Q}[G]$.

We set $\Pi_\chi := \oplus_{\underline{x}\in\langle t_\chi,d_\chi\rangle}V_{\underline{x},\chi}$ and write $\pi_\chi : F_{[\chi]}\to\Pi_\chi$ for the natural projection homomorphism. We set $g_\chi := g(I_{[\chi]})$ and note in passing that, since $I_{[\chi]}\subseteq F(t_\chi)_{[\chi]}$, one has $g_\chi \le g(F(t_\chi)_{[\chi]}) = d_\chi(d-t_\chi)$. We also set $Z_\chi := \pi_\chi(\ker(\mu)_{[\chi]})$ and note that $g(Z_\chi) = g(\ker(\mu)_{[\chi]}) - g_\chi = (g(F_{[\chi]}) - g(\mathrm{im}(\mu)_{[\chi]})) - g_\chi = (d_\chi d - d_\chi(d-t_\chi)) - g_\chi = g(\Pi_\chi) - g_\chi$. By applying Lemma 6.4.2 below (with $W=\Pi_\chi$ and $V = V_{\underline{x},\chi}$ for any given $\underline{x}\in\langle t_\chi,d_\chi\rangle$) we can therefore choose a subset Σ_χ of $\langle t_\chi,d_\chi\rangle$ with $|\Sigma_\chi| = g_\chi$ and such that the $\mathbb{Q}[G]$-module $X_\chi := \oplus_{\underline{x}\in\Sigma_\chi}V_{\underline{x},\chi}$ is a direct complement of Z_χ in Π_χ.

We now choose a minimal set of generators $\{w_{i,\chi} : i\in\langle g_\chi\rangle\}$ of the $\mathbb{Q}[G]$-module $I_{[\chi]}$. Then for each i in $\langle g_\chi\rangle$ and each $\underline{x}$ in $\Sigma^*_\chi := \langle d,d_\chi\rangle\setminus\langle t_\chi,d_\chi\rangle$ we can choose an element $\mu_{i,\underline{x},\chi}$ of $\mathbb{Q}[G]$ such that $w_{i,\chi} = \sum_{\underline{x}\in\Sigma^*_\chi}\mu_{i,\underline{x},\chi}b_{\underline{x},\chi}$. We then consider the associated $g_\chi\times d_\chi(d-t_\chi)$-matrix $M = (\mu_{i,\underline{x},\chi})$ where the rows of M are indexed by the set $\langle g_\chi\rangle$ and the columns by the set Σ^*_χ. By changing the elements $w_{i,\chi}$ we can assume that M contains g_χ columns which together form an identity matrix (to do this one applies 'elementary row operations' to M: this can be justified because each element $b_{\underline{x},\chi}$ generates the simple $\mathbb{Q}[G]$-module $V_{\underline{x},\chi}$ and so for any $z\in\mathbb{Q}[G]$ with $zb_{\underline{x},\chi}\neq 0$ there exists $z'\in\mathbb{Q}[G]$ with $z'zb_{\underline{x},\chi} = b_{\underline{x},\chi}$). We then fix a subset Σ'_χ of Σ^*_χ that corresponds to a set of columns of M which together constitute a $g_\chi\times g_\chi$ identity matrix. We then fix a bijection of sets $\iota_\chi : \Sigma'_\chi\to\Sigma_\chi$ and for each $\underline{x}\in\Sigma'_\chi$ an isomorphism of $\mathbb{Q}[G]$-modules $\theta_{\underline{x},\chi} : V_{\underline{x},\chi}\cong V_{\iota_\chi(\underline{x}),\chi}$. We also fix a natural number $m_{\underline{x},\chi}$ that is large enough to ensure that $m_{\underline{x},\chi}\theta_{\underline{x},\chi}(b_{\underline{x},\chi})\in F$.

For each integer i with $t_\chi < i\le d$ we set $\Sigma'_{\chi,i} := \{\underline{x}\in\Sigma'_\chi : x_1 = i\}$. We then define $\epsilon_{\mu,\chi}$ to be the automorphism of the $R[G]$-module F which for each $j\in\langle d\rangle$ satisfies

$$\epsilon_{\mu,\chi}(b_j) = \begin{cases} b_j - \sum_{\underline{x}\in\Sigma'_{\chi,j}} m_{\underline{x},\chi}\theta_{\underline{x},\chi}(b_{\underline{x},\chi}), & \text{if } t_\chi < j\le d \text{ and } \Sigma'_{\chi,j}\neq\emptyset\\ b_j, & \text{otherwise.}\end{cases}$$

We now define ϵ_μ to be the composite (in any order) of the automorphisms $\epsilon_{\mu,\chi}$ as χ varies over Υ and in the remainder of the proof we verify that this automorphism has the property stated in the lemma. To do this we fix κ in $\mathrm{Ir}_{\mathbb{Q}}(G)$ for which $\mathrm{im}(\mu)_{[\kappa]} = F(t)_{[\kappa]}$ for some integer t.

If $\kappa \notin \Upsilon$, then the definition of Υ implies that $F_{[\kappa]} = \ker(\mu)_{[\kappa]} \oplus \mathrm{im}(\mu)_{[\kappa]}$. Since in this case one also has $(\epsilon_\mu)_{[\kappa]} = \mathrm{Id}_{F_{[\kappa]}}$ (because $e_\kappa V_{\underline{x},\chi}$ vanishes for all χ in Υ) it is therefore clear that $F_{[\kappa]} = \ker(\mu \circ \epsilon_\mu)_{[\kappa]} \oplus \mathrm{im}(\mu \circ \epsilon_\mu)_{[\kappa]}$.

We henceforth assume that $\kappa \in \Upsilon$. We define an automorphism $\xi := e_\kappa(\epsilon_\mu) = e_\kappa(\epsilon_{\mu,\kappa}) \in [F_{[\kappa]}, F_{[\kappa]}]_{\mathbb{Q}[G]}$ and note that $\ker(\mu \circ \epsilon_\mu)_{[\kappa]} = \xi^{-1}(\ker(\mu)_{[\kappa]})$. For each $\underline{x}$ in Σ_χ we write $\underline{x}' = (x_1', x_2')$ for the element $\iota_\chi^{-1}(\underline{x})$ of $\Sigma'_{\chi,x_1'}$ and we recall that (by our assumptions on the matrix M) there exists a unique integer i in $\langle g_\kappa\rangle$ such that $w_{i,\kappa} = b_{\underline{x}',\kappa}$. Since $b_{\underline{x}',\kappa} = f_{x_2',\kappa} b_{x_1'}$ and $f_{x_2',\kappa} b_{\underline{x},\kappa} = 0$ for all $\underline{x} \in \Sigma'_{\kappa,x_1'} \setminus \{\underline{x}'\}$ one therefore has

$$\begin{aligned}\xi^{-1}(w_{i,\kappa}) = f_{x_2',\kappa}\epsilon_{\mu,\kappa}^{-1}(b_{x_1'}) &= f_{x_2',\kappa} b_{x_1'} + \sum_{\underline{x}\in\Sigma'_{\kappa,x_1'}} m_{\underline{x},\kappa}\theta_{\underline{x},\kappa}(f_{x_2',\kappa} b_{\underline{x},\kappa}) \\ &= f_{x_2',\kappa} b_{x_1'} + m_{\underline{x}',\kappa}\theta_{\underline{x}',\kappa}(b_{\underline{x}',\kappa})\end{aligned}$$

and so $m_{\underline{x}',\kappa}\theta_{\underline{x}',\kappa}(b_{\underline{x}',\kappa}) = \pi_\kappa(\xi^{-1}(w_{i,\kappa})) \in \pi_\kappa(\ker(\mu\circ\epsilon_\mu)_{[\kappa]})$. Since the $\mathbb{Q}[G]$-module X_κ is generated by $\{m_{\underline{x}',\kappa}\theta_{\underline{x}',\kappa}(b_{\underline{x}',\kappa}) : \underline{x}' \in \Sigma'_\kappa\}$ it follows that $X_\kappa \subseteq \pi_\kappa(\ker(\mu\circ\epsilon_\mu)_{[\kappa]})$. But for every $w \in \ker(\mu)_{[\kappa]}$ the definition of $\epsilon_{\mu,\kappa}$ implies that $\pi_\kappa(\xi^{-1}(w)) - \pi_\kappa(w) \in X_\kappa$ and so $\pi_\kappa(\ker(\mu \circ \epsilon_\mu)_{[\kappa]}) = \pi_\kappa(\ker(\mu)_{[\kappa]}) + X_\kappa = Z_\kappa + X_\kappa = \Pi_\kappa$. Since $g(\Pi_\kappa) = g(\ker(\mu \circ \epsilon_\mu)_{[\kappa]})$ this implies that $\ker(\mu \circ \epsilon_\mu)_{[\kappa]}$ is disjoint from $\ker(\pi_\kappa)_{[\kappa]} = F(t_\kappa)_{[\kappa]} = \mathrm{im}(\mu)_{[\kappa]} = \mathrm{im}(\mu\circ\epsilon_\mu)_{[\kappa]}$ and hence that $F_{[\kappa]} = \ker(\mu\circ\epsilon_\mu)_{[\kappa]}\oplus\mathrm{im}(\mu\circ\epsilon_\mu)_{[\kappa]}$, as required. □

Lemma 6.4.2. *Let W be a finitely generated left $\mathbb{Q}[G]$-module that is isomorphic to a direct sum of m copies of a simple (left) $\mathbb{Q}[G]$-module V. Then any $\mathbb{Q}[G]$-submodule U of W is isomorphic to a direct sum of m' copies of V for some $m' \le m$. If B is any minimal generating set of W, then there exists a subset B' of B with $|B'| = |B| - g(U)$ and such that the $\mathbb{Q}[G]$-submodule of W that is generated by B' is a direct complement of U in W.*

Proof. The first assertion follows immediately from the Krull-Schimdt theorem for the (semisimple) algebra $\mathbb{Q}[G]$. For any subset Σ of W we write $\langle\Sigma\rangle$ for the $\mathbb{Q}[G]$-submodule of W that is generated by Σ.

The claimed result is clear if U vanishes or $U = W$ and hence if $g(W) = 1$. We therefore assume that $g(W) > 1$ and $U \neq \{0\}$. We note also that, as $|B'| = g(W) - g(U)$, one has $\langle B'\rangle \oplus U = W$ if and only if $\langle B'\rangle \cap U = \{0\}$. To prove the existence of such a set B' we argue by induction on $g(W)$ and so choose an element b of B and set $B' := B \setminus \{b\}$ and $W' := \langle B'\rangle$.

We first assume that $U \subseteq W'$. We replace W by W' and B by B' and deduce from the inductive hypothesis that there exists a subset B_1' of B' with $|B_1'| = g(W') - g(U) = g(W) - g(U) - 1$ and $U \cap \langle B_1'\rangle = \{0\}$. Since $U \subseteq W'$ the set $B_1 := B_1' \cup \{b\}$ is easily checked to satisfy the required conditions.

We now assume that $U \not\subseteq W'$. Then $g(U\cap W') = g(U)-1$. We replace W by W', B by B' and U by $U' := U\cap W'$. By the inductive hypothesis there exists a subset B_1' of B' with $|B_1'| = g(W') - g(U') = g(W) - g(U)$ and $U \cap \langle B_1'\rangle = U'\cap\langle B_1'\rangle = \{0\}$. In this case the set $B_1 := B_1'$ therefore satisfies the required conditions. $\square$

Returning to the proof of Proposition 6.2.2, we now apply Lemma 6.4.1 with μ equal to the endomorphism ϑ of F that was constructed in §6.3 to satisfy claims (i), (ii) and (iii) of the proposition. We then replace ϑ and α (as constructed in §6.3) by $\vartheta' := \vartheta\circ\epsilon_\vartheta$ and $\alpha' := \alpha\circ\beta$ respectively, where ϵ_ϑ is the automorphism constructed in Lemma 6.4.1 (with $\mu = \vartheta$) and β is the morphism of complexes $C^\bullet_{\vartheta\circ\epsilon_\vartheta} \to C^\bullet_\vartheta$ that is equal to ϵ_ϑ in degree 0 and to the identity map in degree 1. With these choices the validity of Proposition 6.2.2(iv) follows directly from Lemma 6.4.1 whilst, since $\mathrm{im}(H^0(\alpha)) = \mathrm{im}(H^0(\alpha'))$ and $\mathrm{im}(\vartheta^A) = \mathrm{im}((\vartheta')^A)$ for all normal subgroups A of G, the validity of Proposition 6.2.2(i), (ii) and (iii) is also preserved. We therefore assume henceforth that ϑ and α are chosen in this way.

6.5. To complete the proof of Proposition 6.2.2 we write $\Theta_{\lambda'}$ for the endomorphism $\langle\lambda',\vartheta,\alpha,\iota_1,\iota_2\rangle$ of $E\otimes_R F$ that occurs in Proposition 6.2.2(v). If λ' is invertible, then the definition of $\Theta_{\lambda'}$ implies that $\Theta_{\lambda'}((E\otimes_\mathbb{Q}\iota_1)(E\otimes_R \mathrm{im}(\vartheta))) = E\otimes_R\mathrm{im}(\vartheta)$ and $\Theta_{\lambda'}(E\otimes_R\ker(\vartheta)) = (E\otimes_\mathbb{Q}\iota_2)(E\otimes X_{K,S})$. Since $E\otimes_R F$ is equal to the direct sum of $E\otimes_R\mathrm{im}(\vartheta)$ and $(E\otimes_\mathbb{Q}\iota_2)(E\otimes X_{K,S})$ one therefore has $\mathrm{im}(\Theta_{\lambda'}) = E\otimes_R F$ and so $\Theta_{\lambda'}$ is invertible, as claimed.

To prove the second claim in Proposition 6.2.2(v) we write λ^α for the composite isomorphism of $\mathbb{R}[G]$-modules

$$(\mathbb{R}\otimes_R H^1(\alpha))^{-1}\circ\lambda\circ(\mathbb{R}\otimes_R H^0(\alpha)) : \mathbb{R}\otimes_R H^0(C^\bullet_\vartheta)\to\mathbb{R}\otimes_R H^1(C^\bullet_\vartheta).$$

Then, since $\mathbb{R}\otimes_R Q[0]$ is acyclic, the additivity criterion of Lemma A.1.2 applies to the exact triangle (21) (with $C_1^\bullet = C^\bullet_\vartheta, C_2^\bullet = C^\bullet, C_3^\bullet = Q[0], \lambda_1 = \lambda, \lambda_2 = \lambda^\alpha$ and λ_3 the identity map on the zero space) to prove that $\chi(C^\bullet,\lambda) = \chi(C^\bullet_\vartheta,\lambda^\alpha)+\chi(Q[0],0)$ in $K_0(R[G],\mathbb{R}[G])$. To complete the proof of Proposition 6.2.2(v) (and hence of Proposition 6.2.2 itself) it therefore suffices to note that $\chi(C^\bullet_\vartheta,\lambda^\alpha) = \delta_{R[G],\mathbb{R}[G]}(\mathrm{Nrd}_{\mathbb{R}[G]}(\Theta_\lambda))$ as a consequence of Lemma A.1.1(iii).

7. From LTC(K/k) to integral matrices

In this section we deduce the claims in the second paragraph of Conjecture 2.4.1 from the assumed validity of LTC(K/k). Following Proposition 5.1.2(i) we will assume, unless stated otherwise, that S is a finite non-empty set of places of k that contains S_∞ and all places which ramify in K/k and is also such that $\mathrm{Cl}(\mathcal{O}_{K,S})$ vanishes and the natural sequence of G-modules $0\to U_{K,S,T}\to U_{K,S}\to\mathbb{F}_T^\times\to 0$ is exact, where $\mathbb{F}_T^\times$ is as defined in Lemma 5.1.1(ii). We assume that S is labeled so it satisfies Hypothesis H_r with r equal to the integer $r_{K,S}$ defined in Remark 2.2.2.

7.1. In the following result we use the complex $C^\bullet_{K/k,S}$ that occurs in the statement of LTC(K/k) (with $\Sigma = S$).

Proposition 7.1.1. *There exists data $\pi, F, d, \{b_i : i \in \langle d\rangle\}, \vartheta, \alpha, Q, \iota_1$ and ι_2 as in Proposition 6.2.2 (with $R = \mathbb{Z}$, $m = n$, $\lambda = R_{K,S}$, $\hbar$ any given natural number and both $C^\bullet$ and $\rho : H^1(C^\bullet) \to Y_{K,n}$ as specified in the proof below) such that*

$$\chi(C^\bullet_{K/k,S}, R_{K,S}) = \delta_G(\epsilon_T^{-1}\mathrm{Nrd}_{\mathbb{R}[G]}(\langle R_{K,S}, \vartheta, \alpha, \iota_1, \iota_2\rangle)) + \chi(Q[0], 0).$$

One can take $Q = 0$ in this equality if and only if $\chi(C^\bullet_{K/k,S})$ vanishes.

Proof. We first note that, since each place in T is unramified in K/k, there exists an exact sequence of G-modules

$$0 \to \bigoplus_{t\in T}\mathbb{Z}[G] \xrightarrow{(1-\mathrm{N}\,t\cdot\mathrm{Fr}_t^{-1})_t} \bigoplus_{t\in T}\mathbb{Z}[G] \to \mathbb{F}_T^\times \to 0. \tag{23}$$

This sequence implies that the G-module $\mathbb{F}_T^\times$ is cohomologically-trivial. Now, since $X_{K,S}$ is torsion-free, the universal coefficient spectral sequence $H^p(G, \mathrm{Ext}^q(X_{K,S}, \mathbb{F}_T^\times)) \Rightarrow \mathrm{Ext}^{p+q}_G(X_{K,S}, \mathbb{F}_T^\times)$ degenerates to give an isomorphism of abelian groups of the form $\mathrm{Ext}^i_G(X_{K,S}, \mathbb{F}_T^\times) \cong H^i(G, [X_{K,S}, \mathbb{F}_T^\times]_\mathbb{Z})$ and the latter group vanishes for each $i \geq 1$ since the G-module $[X_{K,S}, \mathbb{F}_T^\times]_\mathbb{Z} \cong [X_{K,S}, \mathbb{Z}]_\mathbb{Z} \otimes \mathbb{F}_T^\times$ is cohomologically-trivial (because $\mathbb{F}_T^\times$ is). It follows that the group $\mathrm{Ext}^i_G(X_{K,S}, \mathbb{F}_T^\times)$ vanishes for each $i \geq 1$. Upon applying $\mathrm{Ext}^*_G(X_{K,S}, -)$ to the exact sequence of Lemma 5.1.1(ii), one therefore finds that the natural map

$$\iota_T : \mathrm{Ext}^2_G(X_{K,S}, U_{K,S,T}) \to \mathrm{Ext}^2_G(X_{K,S}, U_{K,S})$$

is bijective.

We now consider the following commutative diagram of exact sequences of G-modules

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & & \\
0 \arrow[r] & U_{K,S,T} \arrow[r, "\nu"] \arrow[d, "\subseteq"] & P \arrow[r, "\mu"] \arrow[d, "\kappa"] & F \arrow[r, "\pi"] \arrow[d, equal] & X_{K,S} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & U_{K,S} \arrow[r, "\nu'"] \arrow[d] & \Psi^0 \arrow[r, "\mu'"] \arrow[d] & F \arrow[r, "\pi"] & X_{K,S} \arrow[r] & 0 \\
 & \mathbb{F}_T^\times \arrow[r, equal] \arrow[d] & \mathbb{F}_T^\times \arrow[d] & & & \\
 & 0 & 0. & & &
\end{tikzcd} \tag{24}$$

The first row of this diagram is a choice of representative of the extension class $\iota_T^{-1}(\tau_{K/k,S})$ (it is easy to see that a representative of the above form exists). In the second row we write Ψ^0 for the push-out of ν via the inclusion $U_{K,S,T} \subseteq U_{K,S}$ and κ, ν' and μ' for the maps that are induced by (μ and) the definition of this push-out. This construction ensures that the second row of (24) is both exact and represents $\tau_{K/k,S}$. The first column of (24) is assumed to be exact (by our choice of S) and the exactness of the second column in (24) then follows from the commutativity of the diagram.

We let $C^\bullet_{K/k,S}$ denote the complex $\Psi^0 \to F$ where Ψ^0 occurs in degree 0, the differential is the map μ' in (24) and the cohomology groups are identified with $U_{K,S}$ and $X_{K,S}$ by means of the second row in (24). (Since the second row of (24) represents $\tau_{K/k,S}$ this is a valid choice for the complex $C^\bullet_{K/k,S}$ in $\mathrm{LTC}(K/k)$.) Since F is a free G-module and $C^\bullet_{K/k,S}$ belongs to $D^{\mathrm{p}}(\mathbb{Z}[G])$, the module Ψ^0 has a finite projective resolution and so is cohomologically-trivial. Then, since the G-modules Ψ^0 and $\mathbb{F}_T^\times$ are both cohomologically-trivial, the exactness of the second column of (24) implies that P is cohomologically-trivial. If we now let $C^\bullet$ denote the complex $P \xrightarrow{\mu} F$ where P is placed in degree 0, then the diagram (24) gives an exact triangle in $D^{\mathrm{p}}(\mathbb{Z}[G])$

$$C^\bullet \to C^\bullet_{K/k,S} \to \mathbb{F}_T^\times[0] \to C^\bullet[1]. \tag{25}$$

This triangle combines with Lemma A.1.2 (with $C_1^\bullet = C^\bullet, C_2^\bullet = C^\bullet_{K/k,S}, C_3^\bullet = \mathbb{F}_T^\times[0], \lambda_1 = \lambda_2 = R_{K,S}$ and λ_3 the identity map on the zero space) to give an equality $\chi(C^\bullet_{K/k,S}, R_{K,S}) = \chi(C^\bullet, R_{K,S}) + \chi(\mathbb{F}_T^\times[0], 0)$ in $K_0(\mathbb{Z}[G], \mathbb{R}[G])$. The equality of the proposition is thus valid because

$$\begin{aligned}
&\chi(C^\bullet, R_{K,S}) + \chi(\mathbb{F}_T^\times[0], 0)\\
=&\delta_G(\mathrm{Nrd}_{\mathbb{R}[G]}(\langle R_{K,S}, \vartheta, \alpha, \iota_1, \iota_2\rangle)) + \chi(Q[0], 0) + \chi(\mathbb{F}_T^\times[0], 0)\\
=&\delta_G(\epsilon_T^{-1}\mathrm{Nrd}_{\mathbb{R}[G]}(\langle R_{K,S}, \vartheta, \alpha, \iota_1, \iota_2\rangle)) + \chi(Q[0], 0)
\end{aligned}$$

where the first equality is obtained by applying Proposition 6.2.2(v) (with $C^\bullet$ as above, $R = \mathbb{Z}$, $\lambda = R_{K,S}$, $m = n$ and ρ equal to the projection $\pi_{S,n} : X_{K,S} \to Y_{K,n}$) and the second is valid because the exact sequence (23) combines with Lemma A.1.1(iii) to imply that $\chi(\mathbb{F}_T^\times[0], 0) = \delta_G(\epsilon_T^{-1})$.

Now (23) also implies that $\chi(\mathbb{F}_T^\times[0])$ vanishes in $K_0(\mathbb{Z}[G])$ and so (25) implies that $\chi(C^\bullet) = \chi(C^\bullet_{K/k,S})$. Given this equality Proposition 6.2.2(ii) implies that one can take $Q = 0$ in the last displayed formula if and only if $\chi(C^\bullet_{K/k,S})$ vanishes, as claimed. □

7.2. For each intermediate field F of K/k we write r_F for the integer $r_{F,S}$ defined in Remark 2.2.2. We also fix a normal subgroup H of G and set $L := K^H$ and assume, as we may, that S is numbered so that (it satisfies Hypothesis H_{r_K} and) each place in S_{r_L} splits completely in L/k. If E denotes either K or L, and $\Delta := G_{E/k}$, then we write Υ_E for the set of characters ψ in $\mathrm{Ir}(\Delta) \subseteq \mathrm{Ir}(G)$ with $\dim_{\mathbb{C}}(\ker(\vartheta)_\psi) = r_E\psi(1)$.

Just as in the proof of Lemma 3.2.1 it is easily seen that Υ_E, and hence also its complement Υ'_E in $\mathrm{Ir}(\Delta)$, is closed under the action of $\mathrm{Aut}(\mathbb{C})$ on $\mathrm{Ir}(G)$ and so the idempotent $e_E := \sum_{\chi\in\Upsilon_E} e_\chi$ belongs to $\mathbb{Q}[G]$.

Proposition 7.2.1. *We use the notation of Proposition 7.1.1 and also fix a homomorphism ϕ in $[U_{K,S}, X_{K,S}]_G$. We assume the validity of LTC(K/k).*

(i) *If $L^*_S(\psi, 0)$ is strictly positive for all symplectic characters ψ in $\mathrm{Ir}(G)$, then there exists a matrix U in $\mathrm{GL}_d(\mathbb{Z}[G])$ with*

$$\theta^{(r_E)}_{E/k,S,T}(0)R(\phi) = \mathrm{Nrd}_{\mathbb{Q}[G]}(U)e_E\mathrm{Nrd}_{\mathbb{Q}[G]}(\langle \mathbb{Q}\otimes\phi, \vartheta, \alpha, \iota_1, \iota_2\rangle).$$

(ii) *Let p be a prime divisor of the natural number $\hbar$ in Proposition 7.1.1. Then $\theta^{(r_E)}_{E/k,S,T}(0)R(\phi)$ belongs to $\mathbb{Q}[\Delta]$ and there is a matrix U_p in $\mathrm{GL}_d(\mathbb{Z}_p[G])$ with*

$$\theta^{(r_E)}_{E/k,S,T}(0)R(\phi) = \mathrm{Nrd}_{\mathbb{Q}_p[G]}(U_p)e_E\mathrm{Nrd}_{\mathbb{Q}[G]}(\langle \mathbb{Q}\otimes\phi, \vartheta, \alpha, \iota_1, \iota_2\rangle).$$

Proof. At the outset we note that the $\mathbb{C}[G]$-module $\mathbb{C}\otimes\ker(\vartheta) = \mathbb{C}\otimes U_{K,S}$ is isomorphic to $\mathbb{C}\otimes X_{K,S}$. The formula (5) therefore implies that for each $\psi\in\Upsilon_E$, resp. $\psi\in\Upsilon'_E$, the value at $z=0$ of $z^{-r_E\psi(1)}L_S(\check{\psi}, z)$ is equal to $L^*_S(\check{\psi}, 0)$, resp. 0. From the definition of $\theta^{(r_E)}_{E/k,S,T}(z)$ in (2) it follows that $\theta^{(r_E)}_{E/k,S,T}(0) = e_E\epsilon_T\theta^*_{K/k,S}(0)$.

Under the hypothesis of claim (i) the element $\theta^*_{K/k,S}(0)$ belongs to $\mathrm{im}(\mathrm{Nrd}_{\mathbb{R}[G]})$ and so $\delta_G(\theta^*_{K/k,S}(0)) = \partial^1_G(\mathrm{Nrd}^{-1}_{\mathbb{R}[G]}(\theta^*_{K/k,S}(0)))$ belongs to $\mathrm{im}(\partial^1_G) = \ker(\partial^0_G)$. The equality of LTC$(K/k)$ therefore implies that

$$\chi(C^\bullet_{K/k,S}) = \partial^0_G(\chi(C^\bullet_{K/k,S}, R_{K,S})) = \partial^0_G(\delta_G(\theta^*_{K/k,S}(0))) = 0.$$

We may thus choose $Q=0$ in the formula of Proposition 7.1.1 and so the latter result implies that

$$\delta_G(\theta^*_{K/k,S}(0)) = \chi(C^\bullet_{K/k,S}, R_{K,S}) = \delta_G(\epsilon_T^{-1}\mathrm{Nrd}_{\mathbb{R}[G]}(\Theta_{R_{K,S}}))$$

where for any homomorphism λ in $[\mathbb{R}\otimes U_{K,S}, \mathbb{R}\otimes X_{K,S}]_{\mathbb{R}[G]}$ we set $\Theta_\lambda := \langle\lambda, \vartheta, \alpha, \iota_1, \iota_2\rangle \in [\mathbb{R}\otimes F, \mathbb{R}\otimes F]_{\mathbb{R}[G]}$. Now $\epsilon_T \in \mathrm{im}(\mathrm{Nrd}_{\mathbb{R}[G]})$ so the last displayed equality implies $\epsilon_T\theta^*_{K/k,S}(0)\mathrm{Nrd}_{\mathbb{R}[G]}(\Theta_{R_{K,S}})^{-1} \in \ker(\delta_G)\cap\mathrm{im}(\mathrm{Nrd}_{\mathbb{R}[G]})$. But $\ker(\delta_G)\cap\mathrm{im}(\mathrm{Nrd}_{\mathbb{R}[G]}) = \mathrm{Nrd}_{\mathbb{Q}[G]}(\mathrm{im}(\partial^2_G))$ and, since $d\geq 2$ (by Lemma 6.2.1), the Surjective Stability Theorem of Bass [25, (40.41), (40.42)] implies that the natural map $\mathrm{GL}_d(\mathbb{Z}[G]) \to K_1(\mathbb{Z}[G])$ is surjective. We may therefore choose a matrix U in $\mathrm{GL}_d(\mathbb{Z}[G])$ such that

$$\epsilon_T\theta^*_{K/k,S}(0) = \mathrm{Nrd}_{\mathbb{Q}[G]}(U)\mathrm{Nrd}_{\mathbb{R}[G]}(\Theta_{R_{K,S}}). \tag{26}$$

After multiplying this equality by $e_ER(\phi)$, and recalling $\theta^{(r_E)}_{E/k,S,T}(0) = e_E\epsilon_T\theta^*_{K/k,S}(0)$, the equality of claim (i) follows by applying Lemma 7.2.2 below (with $\Sigma = S$).

Regarding claim (ii) we first recall, from Remark 6.1.1(ii), that LTC(K/k) implies LTC(E/k) and hence that $\theta^{(r_E)}_{E/k,S,T}(0)R(\phi)$ belongs to $\mathbb{Q}[\Delta]$. With p as in claim (ii) we fix an embedding $j:\mathbb{R}\to\mathbb{C}_p$. We write $j_* : \mathfrak{Z}(\mathbb{R}[G])^\times \to \mathfrak{Z}(\mathbb{C}_p[G])^\times$ and

$j'_* : K_0(\mathbb{Z}[G], \mathbb{R}[G]) \to K_0(\mathbb{Z}_p[G], \mathbb{C}_p[G])$ for the induced homomorphisms and recall from [7, Lem. 2.2] that there is a commutative diagram

$$\begin{array}{ccc} \mathfrak{Z}(\mathbb{R}[G])^\times & \xrightarrow{\delta_G} & K_0(\mathbb{Z}[G], \mathbb{R}[G]) \\ {\scriptstyle j_*}\downarrow & & \downarrow{\scriptstyle j'_*} \\ \mathfrak{Z}(\mathbb{C}_p[G])^\times & \xrightarrow{\partial^1_{\mathbb{Z}_p[G],\mathbb{C}_p[G]} \circ \mathrm{Nrd}^{-1}_{\mathbb{C}_p[G]}} & K_0(\mathbb{Z}_p[G], \mathbb{C}_p[G]). \end{array} \tag{27}$$

Now, since p divides $\hbar$, the module $\mathbb{Z}_p \otimes Q$ vanishes (by Proposition 6.2.2(i)) and so $j'_*(\chi(Q[0], 0)) = 0$. The above diagram therefore combines with LTC(K/k) and Proposition 7.1.1 to imply that the element $j_*(\epsilon_T \theta^*_{K/k,S}(0)\mathrm{Nrd}_{\mathbb{R}[G]}(\Theta_{R_{K,S}})^{-1})$ belongs to $\ker(\partial^1_{\mathbb{Z}_p[G],\mathbb{C}_p[G]} \circ \mathrm{Nrd}^{-1}_{\mathbb{C}_p[G]}) = \mathrm{Nrd}_{\mathbb{Q}_p[G]}(\mathrm{im}(\partial^2_{\mathbb{Z}_p[G],\mathbb{C}_p[G]}))$. Since the natural map $\mathrm{GL}_d(\mathbb{Z}_p[G]) \to K_1(\mathbb{Z}_p[G])$ is surjective (by [25, (40.41), (40.42)]) there is therefore a matrix U_p in $\mathrm{GL}_d(\mathbb{Z}_p[G])$ with $j_*(\epsilon_T \theta^*_{K/k,S}(0)) = \mathrm{Nrd}_{\mathbb{Q}_p[G]}(U_p) j_*(\mathrm{Nrd}_{\mathbb{R}[G]}(\Theta_{R_{K,S}}))$. From here the proof of claim (ii) proceeds in just the same way that claim (i) is deduced from (26). □

Lemma 7.2.2. *Let Σ be any finite non-empty set of places of k that contains S_∞ and S_r and fix ϕ in $[U_{K,\Sigma}, X_{K,\Sigma}]_G$. Let $C^\bullet$ be any complex as in Proposition 6.2.2 for which $\mathbb{Q} \otimes_R H^0(C^\bullet) = \mathbb{Q} \otimes U_{K,\Sigma}$. Then, in terms of the notation of Proposition 6.2.2(v), one has* $\mathrm{Nrd}_{\mathbb{R}[G]}(\langle R_{K,\Sigma}, \vartheta, \alpha, \iota_1, \iota_2\rangle)R(\phi) = \mathrm{Nrd}_{\mathbb{Q}[G]}(\langle \mathbb{Q} \otimes \phi, \vartheta, \alpha, \iota_1, \iota_2\rangle)$.

Proof. We set $\phi_\mathbb{Q} = \mathbb{Q} \otimes \phi$ and $\phi_\mathbb{R} = \mathbb{R} \otimes \phi$. With λ denoting $R_{K,\Sigma}, \phi_\mathbb{R}$ or $\phi_\mathbb{Q}$ we also set $\Theta_\lambda := \langle \lambda, \vartheta, \alpha, \iota_1, \iota_2\rangle$. Then, by comparing the explicit definitions of these endomorphisms, one finds that $\Theta^{-1}_{R_{K,\Sigma}} \circ \Theta_{\phi_\mathbb{R}}$ is the identity on $(\mathbb{R} \otimes_\mathbb{Q} \iota_1)(\mathbb{R} \otimes_R \mathrm{im}(\vartheta))$ and equal to $(R^{-1}_{K,\Sigma} \circ \phi_\mathbb{R})^\alpha := (\mathbb{R} \otimes_R H^0(\alpha))^{-1} \circ (R^{-1}_{K,\Sigma} \circ \phi_\mathbb{R}) \circ (\mathbb{R} \otimes_R H^0(\alpha))$ on $\mathbb{R} \otimes_R \ker(\vartheta)$. From the commutative diagram

$$\begin{array}{ccc} \mathbb{R} \otimes_R \ker(\vartheta) & \xrightarrow{(R^{-1}_{K,\Sigma} \circ \phi_\mathbb{R})^\alpha} & \mathbb{R} \otimes_R \ker(\vartheta) \\ {\scriptstyle \mathbb{R} \otimes_R H^0(\alpha)}\downarrow & & \downarrow{\scriptstyle \mathbb{R} \otimes_R H^0(\alpha)} \\ \mathbb{R} \otimes U_{K,\Sigma} & \xrightarrow{R^{-1}_{K,\Sigma} \circ \phi_\mathbb{R}} & \mathbb{R} \otimes U_{K,\Sigma} \end{array}$$

it therefore follows that $\mathrm{Nrd}_{\mathbb{R}[G]}(\Theta_{R_{K,\Sigma}})^{-1}\mathrm{Nrd}_{\mathbb{Q}[G]}(\Theta_{\phi_\mathbb{Q}}) = \mathrm{Nrd}_{\mathbb{R}[G]}(\Theta^{-1}_{R_{K,\Sigma}} \circ \Theta_{\phi_\mathbb{R}}) = \mathrm{Nrd}_{\mathbb{R}[G]}((R^{-1}_{K,\Sigma} \circ \phi_\mathbb{R})^\alpha)$ is equal to $R(\phi)$, as claimed. □

7.3. In this subsection we fix any finite non-empty set of places S of k that contains S_∞ and all places which ramify in K/k and is such that $\mathrm{Cl}(\mathcal{O}_{K,S})$ vanishes. We label S so that it satisfies Hypothesis H_r with $r := r_{K,S}$ and also continue to use the notation of Proposition 6.2.2 and Lemma 6.2.1. For typographical convenience we often do not distinguish between the morphism α and its scalar extensions $\mathbb{Q} \otimes_R \alpha$ and $\mathbb{R} \otimes_R \alpha$. We will similarly regard $[U_{K,S}, X_{K,S}]_G$ as a sublattice of $[\mathbb{Q} \otimes U_{K,S}, \mathbb{Q} \otimes X_{K,S}]_{\mathbb{Q}[G]}$.

If $m = n$, then we let E denote either K or L and set $J := G_{K/E}$, $\Delta := G_{E/k}$ and $r_E = r_{E,S}$. In this case we also assume, as we may, that S is numbered so that all places in S_{r_L} split completely in L/k. If $m \neq n$, then we set $E := K$, $\Delta := G$, $r_E := r_{K,S} = r$ and write J for the identity subgroup of G. In both cases we use the idempotent e_E of $\mathbb{Q}[\Delta]$ that is defined at the beginning of §7.2.

For any homomorphism κ in $[F^J, \mathbb{Q} \otimes Y_{E,r_E}]_{R[\Delta]}$ and any pair of integers $(s,t) \in \langle d, r_E\rangle$ we define a unique element $M(\kappa)_{st}$ of $\mathbb{Q}[\Delta]$ by setting

$$\kappa(b_s^J) = \sum_{t=1}^{t=r_E} M(\kappa)_{st} \cdot w_{i,E}. \tag{28}$$

Using Proposition 6.2.2(iii) we also define for each pair of integers $(s,t) \in \langle d, d\rangle \setminus \langle d, r_E\rangle$ a unique element $M(\vartheta^J)_{st}$ of $R[\Delta]$ by setting

$$\vartheta^J(b_s^J) = \sum_{t=r_E+1}^{t=d} M(\vartheta^J)_{st} \cdot b_t^J. \tag{29}$$

We then define a matrix $M(\kappa, \vartheta^J)$ in $\mathrm{M}_d(\mathbb{Q}[\Delta])$ by setting

$$M(\kappa, \vartheta^J)_{st} := \begin{cases} M(\kappa)_{st}, & \text{if } (s,t) \in \langle d, r_E\rangle, \\ M(\vartheta^J)_{st}, & \text{if } (s,t) \in \langle d, d\rangle \setminus \langle d, r_E\rangle. \end{cases} \tag{30}$$

In the next result we abbreviate the homomorphism $\pi_{E,S,r_E} : X_{E,S} \to Y_{E,r_E}$ to π_E and write ρ_E for the (surjective) restriction homomorphism $[F^J, R \otimes Y_{E,r_E}]_{R[\Delta]} \to [\ker(\vartheta^J), R \otimes Y_{E,r_E}]_{R[\Delta]}$. We do not distinguish between π_E and ρ_E and the induced maps $\mathbb{Q} \otimes \pi_E$ and $\mathbb{Q} \otimes_R \rho_E$.

Lemma 7.3.1. *We suppose given data of the following kind:-*

- $\phi \in [U_{K,S}, X_{K,S}]_G$;
- $\phi^K \in [F, R \otimes Y_{K,r_K}]_{R[G]}$ *with* $\rho_K(\phi^K) = \pi_K \circ \phi \circ H^0(\alpha)$;
- $\phi^L \in \mathbb{Q} \otimes_R [F^H, R \otimes Y_{L,r_L}]_{R[\Gamma]}$ *with* $\rho_L(\phi^L) = \pi_L \circ \phi^H \circ H^0(\alpha)^H$.

Then $e_E \mathrm{Nrd}_{\mathbb{Q}[G]}(\langle \phi, \vartheta, \alpha, \iota_1, \iota_2\rangle) = \mathrm{Nrd}_{\mathbb{Q}[\Delta]}(M(\phi^E, \vartheta^J))$.

Proof. Since $\mathrm{Nrd}_{\mathbb{Q}[G]}(\langle \phi, \vartheta, \alpha, \iota_1, \iota_2\rangle)$ is independent of the choice of sections ι_1 and ι_2 we assume henceforth that for each integer $i \in \langle r_E\rangle$ one has $\iota_2(w_{i,E}) = b_i^J$. (The existence of such a section is a consequence of Lemma 6.2.1.) This implies in particular that $M(\phi^E, \vartheta^J)$ is the matrix, with respect to the basis $\{b_i^J : i \in \langle d\rangle\}$, of

$$\xi := (\iota_2)^J \circ \phi^E + \mathbb{Q} \otimes_R \vartheta^J \in [\mathbb{Q} \otimes_R F^J, \mathbb{Q} \otimes_R F^J]_{\mathbb{Q}[\Delta]}.$$

It therefore suffices to prove that $e_\psi e_E \mathrm{Nrd}_{\mathbb{Q}[G]}(\langle \phi, \vartheta, \alpha, \iota_1, \iota_2\rangle) = e_\psi \mathrm{Nrd}_{\mathbb{Q}[\Delta]}(\xi)$ for all ψ in $\mathrm{Ir}(\Delta)$. To do this we use the functor $M \mapsto M_\psi$ defined in §2.6 and, for convenience, do not distinguish between an $\mathcal{O}_\psi$-module M_ψ and the complex vector space that it spans. In particular, for each $\mu \in [\mathbb{Q} \otimes_R F^J, \mathbb{Q} \otimes_R F^J]_{\mathbb{Q}[\Delta]}$ and $\psi \in \mathrm{Ir}(\Delta)$ one has $e_\psi \mathrm{Nrd}_{\mathbb{Q}[\Delta]}(\mu) = \det_{\mathbb{C}}(\mu_\psi)$ for the endomorphism μ_ψ of $F_\psi^J = F_\psi$ that is induced by μ.

Now if $\psi \in \mathrm{Ir}(\Delta)$ is such that $\dim_{\mathbb{C}}(\ker(\vartheta^J)_\psi) > r_E\psi(1)$, then the definition of e_E implies $e_\psi e_E = 0$ and so we must show ξ_ψ is singular. But in this case $\dim_{\mathbb{C}}(\mathrm{im}(\vartheta^J)_\psi) < \dim_{\mathbb{C}}(F_\psi) - r_E\psi(1)$ and so $\dim_{\mathbb{C}}(\mathrm{im}(\xi_\psi)) \le \dim_{\mathbb{C}}(\mathrm{im}(\phi^E)_\psi) + \dim_{\mathbb{C}}(\mathrm{im}(\vartheta^J)_\psi) < r_E\psi(1) + (\dim_{\mathbb{C}}(F_\psi) - r_E\psi(1)) = \dim_{\mathbb{C}}(F_\psi)$. This shows that ξ_ψ is not surjective, as required.

In the rest of the argument we fix $\psi \in \mathrm{Ir}(\Delta)$ with $\dim_{\mathbb{C}}(\ker(\vartheta^J)_\psi) = r_E\psi(1)$. In this case $e_\psi e_E = e_\psi$, the projection map $\pi_{E,\psi}$ is bijective and, as $\rho_E(\phi^E) = \pi_E \circ \phi \circ H^0(\alpha)^J$, the endomorphism $\langle \phi, \vartheta, \alpha, \iota_1, \iota_2\rangle_\psi$ is equal to ξ'_ψ with

$$\xi' := (\iota_2)^J \circ \phi^E \circ (\hat{\iota}_1)^J + \mathbb{Q} \otimes \vartheta^J \in [\mathbb{Q} \otimes_R F^J, \mathbb{Q} \otimes_R F^J]_{\mathbb{Q}[\Delta]}$$

where we write $\hat{\iota}_1$ for the homomorphism $\mathbb{Q} \otimes_R F \to \mathbb{Q} \otimes_R \ker(\vartheta)$ induced by the decomposition $\mathbb{R} \otimes_R F = \mathbb{R} \otimes_R \ker(\vartheta) \oplus \iota_1(\mathbb{R} \otimes_R \mathrm{im}(\vartheta))$. It therefore suffices to prove that matrix representatives of ξ'_ψ and ξ_ψ have the same determinant.

Now, since each place in S_{r_E} splits completely in E/k, Proposition 6.2.2(iii) implies that $\mathrm{im}(\vartheta^J)_\psi \subseteq F(r_E)^J_\psi = F(r_E)_\psi$. Since $\dim_{\mathbb{C}}(\mathrm{im}(\vartheta^J)_\psi) = \dim_{\mathbb{C}}(F_\psi) - \dim_{\mathbb{C}}(\ker(\vartheta^J)_\psi) = \dim_{\mathbb{C}}(F_\psi) - r_E\psi(1) = \dim_{\mathbb{C}}(F(r_E)_\psi)$ one therefore has $\mathrm{im}(\vartheta^J)_\psi = F(r_E)_\psi$. From Proposition 6.2.2(iv) it follows that $F^J_\psi = \ker(\vartheta^J)_\psi \oplus \mathrm{im}(\vartheta^J)_\psi$ and so we obtain an ordered $\mathbb{C}$-basis $\mathcal{B}$ of F_ψ by concatenating ordered $\mathbb{C}$-bases $\mathcal{B}_1$ of $\ker(\vartheta^J)_\psi$ and $\mathcal{B}_2$ of $\mathrm{im}(\vartheta^J)_\psi$ (in this order). This also implies that the restriction $\vartheta'_\psi : \mathrm{im}(\vartheta)_\psi \to \mathrm{im}(\vartheta)_\psi$ of ϑ_ψ is bijective and so we may assume that the section ι_1 is chosen so that $\iota_{1,\psi} = (\vartheta'_\psi)^{-1}$. The direct sum decomposition $F_\psi = \iota_{2,\psi}(\mathrm{cok}(\vartheta^J)_\psi) \oplus \mathrm{im}(\vartheta^J)_\psi$ also shows that we obtain an ordered $\mathbb{C}$-basis $\mathcal{B}'$ of F_ψ by concatenating an ordered $\mathbb{C}$-basis $\mathcal{B}'_1$ of $\iota_{2,\psi}(\mathrm{cok}(\vartheta^J)_\psi)$ with the basis $\mathcal{B}_2$ of $\mathrm{im}(\vartheta^J)_\psi$. Now, using the above choice of ι_1, the matrices with respect to the bases $\mathcal{B}$ and $\mathcal{B}'$ of the endomorphisms ξ_ψ and ξ'_ψ are block matrices of the form

$$\begin{pmatrix} A_\psi & 0 \\ * & B_\psi \end{pmatrix}, \quad \text{resp.} \quad \begin{pmatrix} A_\psi & 0 \\ 0 & B_\psi \end{pmatrix},$$

where A_ψ is the matrix of $\iota^J_{2,\psi} \circ \rho_E(\phi^E)_\psi \in [\ker(\vartheta^J)_\psi, \iota_{2,\psi}(\mathrm{cok}(\vartheta^J)_\psi)]_{\mathbb{C}}$ with respect to the bases $\mathcal{B}_1$ and $\mathcal{B}'_1$ and B_ψ is the matrix of $\vartheta'_\psi \in [\mathrm{im}(\vartheta^J)_\psi, \mathrm{im}(\vartheta^J)_\psi]_{\mathbb{C}}$ with respect to the basis $\mathcal{B}_2$. It is therefore clear that, computing with respect to the bases $\mathcal{B}$ and $\mathcal{B}'$, the determinants of ξ_ψ and ξ'_ψ are both equal to $\det(A_\psi)\det(B_\psi)$. □

7.4. We now complete the deduction of the second paragraph of Conjecture 2.4.1 from $\mathrm{LTC}(K/k)$ and also justify Remark 2.4.2(ii). We therefore assume throughout this subsection that $\mathrm{LTC}(K/k)$ is valid. We also assume that S is as described at the beginning of §7 and that (following Remark 2.2.2) the integer r in Conjecture 2.4.1 is equal to $r_{K,S}$. We set $m := n$ and fix ϕ in $[U_{K,S}, X_{K,S}]_G$ and then ϕ^K as in Lemma 7.3.1. Then the matrix $M(\phi^K, \vartheta)$ defined in (30) (with $R = \mathbb{Z}$) belongs to $\mathrm{M}_d(\mathbb{Z}[G])$ and is also such that $\mathrm{Nrd}_{\mathbb{Q}[G]}(M(\phi^K, \vartheta)) = e_K\mathrm{Nrd}_{\mathbb{Q}[G]}(\langle \phi, \vartheta, \alpha, \iota_1, \iota_2\rangle)$ by Lemma 7.3.1.

We assume first that $L_S^*(\psi,0)$ is positive for every symplectic character ψ in $\mathrm{Ir}(G)$. Then Proposition 7.2.1(i) implies that

$$\theta^{(r_K)}_{K/k,S,T}(0)R(\phi) = \mathrm{Nrd}_{\mathbb{Q}[G]}(M_{K/k,S,T}(\phi))$$

with $M_{K/k,S,T}(\phi) := U\cdot M(\phi^K,\vartheta)$. For any integer j with $r_K < j \le n$ we write N_j for the normal closure of G_j in G. Then, upon multiplying the equality of (29) with J equal to the identity subgroup (and $E = K$) by the element $T_{N_j} := \sum_{n\in N_j} n \in \mathbb{Z}[G]$ and then using Proposition 6.2.2(iii)(b) with $A = N_j$, one finds that $T_{N_j}\cdot M(\phi^K,\vartheta)_{sj} = 0$ and hence that $M(\phi^K,\vartheta)_{sj} \in \ker(T_{N_j})$ for all integers $s \in \langle d\rangle$. Since $\ker(T_{N_j})$ is equal to the two-sided ideal $I(\mathbb{Z}[G_j])$ of $\mathbb{Z}[G]$ generated by $\{g-1 : g \in G_j\}$ this implies that every element in the j-th column of $M(\phi^K,\vartheta)$, and hence also of $M_{K/k,S,T}(\phi)$, belongs to $I(\mathbb{Z}[G_j])$. The matrix $M_{K/k,S,T}(\phi)$ therefore belongs to $\mathfrak{M}_S(\mathbb{Z}[G])$ and satisfies the required equality (6) with $r = r_{K,S}$.

We now consider the general case. If R denotes $\mathbb{Z}$ or $\mathbb{Z}_\ell$ for some prime ℓ and F is the field of fractions of R, then we write $\mathfrak{I}(R[G])$ for the R-submodule of $\mathfrak{Z}(F[G])$ that is generated by the set $\{\mathrm{Nrd}_{F[G]}(M) : M \in \mathfrak{M}_S(R[G])\}$. Then $\mathfrak{I}(\mathbb{Z}[G])$ is a finitely generated sublattice of $\mathfrak{Z}(\mathbb{Q}[G])$ and for every prime ℓ the pro-ℓ completion $\mathbb{Z}_\ell\otimes\mathfrak{I}(\mathbb{Z}[G])$ of $\mathfrak{I}(\mathbb{Z}[G])$ is equal to $\mathfrak{I}(\mathbb{Z}_\ell[G])$. Now the same type of argument as used in the preceding paragraph combines with Proposition 7.2.1(ii) to imply that for each prime divisor p of $\hbar$ the matrix $M_{K/k,S,T,p}(\phi) := U_p\cdot M(\phi^K,\vartheta)$ belongs to $\mathfrak{M}_S(\mathbb{Z}_p[G])$ and satisfies $\theta^{(r_K)}_{K/k,S,T}(0)R(\phi) = \mathrm{Nrd}_{\mathbb{Q}_p[G]}(M_{K/k,S,T,p}(\phi))$, as predicted by Remark 2.4.2(ii). This shows in particular that the element $\theta^{(r_K)}_{K/k,S,T}(0)R(\phi)$ of $\mathfrak{Z}(\mathbb{Q}[G])$ belongs to $\mathfrak{I}(\mathbb{Z}_p[G]) = \mathbb{Z}_p\otimes\mathfrak{I}(\mathbb{Z}[G])$ for each prime divisor p of $\hbar$. By varying the integer $\hbar$ in Proposition 7.1.1 we deduce that $\theta^{(r_K)}_{K/k,S,T}(0)R(\phi)$ belongs to $\mathbb{Z}_p\otimes\mathfrak{I}(\mathbb{Z}[G])$ for every prime p, and hence that $\theta^{(r_K)}_{K/k,S,T}(0)R(\phi)$ belongs to $\mathfrak{I}(\mathbb{Z}[G])$, as required.

This completes our deduction of the second paragraph of Conjecture 2.4.1 from $\mathrm{LTC}(K/k)$.

Remark 7.4.1. In all cases $\theta^*_{K/k,S}(0)^2 \in \mathrm{im}(\mathrm{Nrd}_{\mathbb{R}[G]})$ and so $\mathrm{LTC}(K/k)$ implies $2\chi(C^\bullet_{K/k,S}) = 0$ in $K_0(\mathbb{Z}[G])$. But if P and F are as in diagram (24), then $2\chi(C^\bullet_{K/k,S}) = [P\oplus P]-[F\oplus F]$ and so the Bass Cancellation Theorem implies $P\oplus P$ is isomorphic as a G-module to $F\oplus F$. Thus, if $C^\bullet$ is the complex in (25), then $C^\bullet\oplus C^\bullet$ is isomorphic to a complex of the form $F\oplus F \xrightarrow{\vartheta'} F\oplus F$ for a homomorphism ϑ' for which there is a surjective homomorphism of G-modules $\mathrm{cok}(\vartheta') \to X_{K,S}\oplus X_{K,S}$. By mimicking the argument used above to deduce from $\mathrm{LTC}(K/k)$ the existence of a matrix $M_{K/k,S,T}(\phi)$ as in (6) one finds that in all cases $\mathrm{LTC}(K/k)$ implies the existence of a matrix M in $\mathfrak{M}_S(\mathbb{Z}[G])$ with $(\theta^{(r)}_{K/k,S,T}(0)R(\phi))^2 = \mathrm{Nrd}_{\mathbb{Q}[G]}(M)$.

7.5. We now justify Remark 2.4.2(iii). Following the proof of Proposition 5.1.2(i) we can and will assume that S is as described at the beginning of §7 and is labeled as in §7.2. We also set $\Gamma := G_{L/k} \cong G/H$, $r = r_K$ and $r' = r_L$. As in §3.3 we identify $Y_{K,r}$ with the direct summand $\sigma_{S,r}(Y_{K,r})$ of $X_{K,S}$ and so regard the projection map $\pi_{S,r} : X_{K,S} \to Y_{K,r}$ as an element of $[X_{K,S}, X_{K,S}]_G$. We fix ϕ in $[U_{K,S}, X_{K,S}]_G$ and set $\hat{\phi} := \pi_{S,r} \circ \phi \in [U_{K,S}, X_{K,S}]_G$. Since $U^H_{K,S} = U_{L,S}$ and $\mathrm{im}(\pi_{S,r})^H = Y^H_{K,r} = T_H(Y_{K,r}) = Y_{L,r} \subset X_{L,S}$ we will regard $\hat{\phi}^H$ as an element of $[U_{L,S}, X_{L,S}]_\Gamma$.

For any subgroup J of G we set $I(J) := I(\mathbb{Z}[J])$. We also write $I'(J)$ for the augmentation ideal of $\mathbb{Z}[J]$ and note that $I'(J) \subseteq I(J)$. We recall that if j is an integer with $r < j \le r'$, then v_j splits completely in L/k and so the decomposition subgroup G_j of w_j is contained in H and hence $I(G_j) \subseteq I(H)$. For a natural number m and a quotient Q of G we write $\{e_{Q,i} : i \in \langle m\rangle\}$ for the standard basis of the direct sum $\mathbb{Z}[Q]^m$ of m copies of $\mathbb{Z}[Q]$. We also write ϖ_m for the natural projection map $\mathrm{M}_m(\mathbb{Z}[G]) \to \mathrm{M}_m(\mathbb{Z}[\Gamma])$.

Theorem 7.5.1. *Assume that LTC(K/k) is valid and that for every symplectic character ψ in $\mathrm{Ir}(G)$ the leading term $L^*_S(\psi, 0)$ is positive. Then there exist matrices $M(\phi) := M_{K/k,S,T}(\phi)$ in $\mathrm{M}_d(\mathbb{Z}[G]) \cap \mathfrak{M}_S(\mathbb{Z}[G])$ and $M(\hat{\phi}^H) := M_{L/k,S,T}(\hat{\phi}^H)$ in $\mathrm{M}_d(\mathbb{Z}[\Gamma]) \cap \mathfrak{M}_S(\mathbb{Z}[\Gamma])$ which together satisfy all of the following conditions.*

(i) $\theta^{(r)}_{K/k,S,T}(0)R(\phi) = \mathrm{Nrd}_{\mathbb{Q}[G]}(M(\phi))$ *and* $\theta^{(r')}_{L/k,S,T}(0)R(\hat{\phi}^H) = \mathrm{Nrd}_{\mathbb{Q}[\Gamma]}(M(\hat{\phi}^H))$.

(ii) $\varpi_d(M(\phi)) = M(\hat{\phi}^H)$.

(iii) *Let $M(\hat{\phi}^H)_0 \in \mathrm{M}_d(\mathbb{Z}[\Gamma])$ be the matrix obtained from $M(\hat{\phi}^H)$ by replacing the first r' columns by 0 and regard this matrix as an element of $[\mathbb{Z}[\Gamma]^d, \mathbb{Z}[\Gamma]^d]_\Gamma$ in the natural way. For each integer j with $r < j \le r'$ let ξ_j denote the element of $[\mathbb{Z}[\Gamma]^d, I(G_j)/I(G_j)I(H)]_\Gamma$ which sends each element $e_{\Gamma,i}$ to the class in $I(G_j)/I(G_j)I(H)$ of $M(\phi)_{ij} \in I(G_j)$. Then there is an injective homomorphism of Γ-modules $\beta : \ker(M(\hat{\phi}^H)_0) \to U_{L,S}$ such that the following diagram commutes*

$$\begin{array}{ccccc}
\ker(M(\hat{\phi}^H)_0) & \xrightarrow{\subseteq} & \mathbb{Z}[\Gamma]^d & \xrightarrow{\xi_j} & I(G_j)/I(G_j)I(H) \\
{\scriptstyle\beta}\downarrow & & & & \parallel \\
U_{L,S} & \xrightarrow{\mathrm{rec}_j} & G_j^{\mathrm{ab}} & \xrightarrow{\eta} & I(G_j)/I(G_j)I(H).
\end{array}$$

Here rec_j is the composite of the localisation map $U_{L,S} \to L^\times_{w_j}$ with the local reciprocity map $L^\times_{w_j} \to G^{\mathrm{ab}}_j$ and η is the natural composite homomorphism $G^{\mathrm{ab}}_j \cong I'(G_j)/I'(G_j)^2 \to I(G_j)/I(G_j)I(H)$.

Proof. Throughout this proof we use the notation of §7.3 and §7.4.

We fix $\hat{\phi}^K$ in $[F, Y_{K,r}]_G$ with $\rho_K(\hat{\phi}^K) = \pi_K \circ \hat{\phi} \circ H^0(\alpha)$. Then $\rho_K(\hat{\phi}^K) = \pi_K \circ \phi \circ H^0(\alpha)$ because $\phi - \hat{\phi} \in \ker(\pi_K)$ and so the argument of §7.4 implies $\theta^{(r)}_{K/k,S,T}(0)R(\phi) =$

$\mathrm{Nrd}_{\mathbb{Q}[G]}(M(\phi))$ with $M(\phi) := U \cdot M(\hat{\phi}^K, \vartheta) \in \mathrm{M}_d(\mathbb{Z}[G]) \cap \mathfrak{M}_S(\mathbb{Z}[G])$. We set $\hat{\phi}^L := (\hat{\phi}^K)^H \in [F^H, Y_{L,r'}]_\Gamma$. Then $\rho_L(\hat{\phi}^L) = \pi_L \circ \hat{\phi}^H \circ H^0(\alpha)^H$ and so by a similar argument to the above one deduces that $\theta^{(r')}_{L/k,S,T}(0)R(\hat{\phi}^H) = \mathrm{Nrd}_{\mathbb{Q}[G]}(M(\hat{\phi}^H))$ with $M(\hat{\phi}^H) := \varpi_d(U) \cdot M(\hat{\phi}^L, \vartheta^J) \in \mathrm{M}_d(\mathbb{Z}[\Gamma]) \cap \mathfrak{M}_S(\mathbb{Z}[\Gamma])$. Claim (i) is therefore satisfied by these choices of $M(\phi)$ and $M(\hat{\phi}^H)$.

To prove claim (ii) it suffices to show that $\varpi_1(M(\hat{\phi}^K, \vartheta)_{ij}) = M(\hat{\phi}^L, \vartheta^J)_{ij}$ for all i and j. After recalling the explicit definition (30) of the matrices $M(\hat{\phi}^K, \vartheta)$ and $M(\hat{\phi}^L, \vartheta^J)$ it is clear that if j satisfies $1 \le j \le r$ or $n < j \le d$ then $\varpi_1(M(\hat{\phi}^K, \vartheta)_{ij}) = M(\hat{\phi}^L, \vartheta^J)_{ij}$. On the other hand, if $r < j \le r'$, then $M(\hat{\phi}^L, \vartheta^J)_{ij} = 0$ since $\mathrm{im}(\hat{\phi}^L) \subseteq Y_{L,r}$, whilst $M(\hat{\phi}^K, \vartheta)_{ij}$ belongs to $I(G_j) \subseteq I(H) = \ker(\varpi_1)$ since $M(\hat{\phi}^K, \vartheta)$ belongs to $\mathfrak{M}_S(\mathbb{Z}[G])$.

We recall the basis $\{b_i : i \in \langle d \rangle\}$ of F that is fixed in Lemma 6.2.1. To prove claim (iii) we identify $\mathbb{Z}[G]^d$ with F by the map sending each $e_{G,i}$ to b_i. This induces an identification of $\mathbb{Z}[\Gamma]^d$ with F^H in such a way that $M(\hat{\phi}^H)_0$ is the matrix, with respect to the basis $\{b_i^H : i \in \langle d \rangle\}$, of the endomorphism ϑ^H of F^H. In particular the morphism α induces an injective homomorphism $\beta := H^0(\alpha)^H$ from $\ker(M(\hat{\phi}^H)_0) = \ker(\vartheta^H) = \ker(\vartheta)^H$ to $U^H_{K,S} = U_{L,S}$. It is thus enough to show that if $x = \sum_{i=1}^{i=d} \mu_i b_i^H$ belongs to $\ker(\vartheta)^H$, with each μ_i in $\mathbb{Z}[\Gamma]$, then the elements $\mathrm{rec}_j(H^0(\alpha)^H(x)) \in G_j^{\mathrm{ab}} \cong I'(G_j)/I'(G_j)^2$ and $\sum_{i=1}^{i=d} \mu_i M(\phi)_{ij} \in I(G_j)$ coincide after projection to $I(G_j)/I(G_j)I(H)$. This fact is proved by mimicking the argument of [11, §10]. (Indeed, in terms of the notation used in [11, §10], one need only replace ϕ, I_j and G_j by $\vartheta, I(G_j)$ and the normal closure of G_j in G respectively.) $\square$

Remark 7.5.2.
(i) Let p be any prime. Without any condition on the leading terms $L_S^*(\psi, 0)$ the proof of Theorem 7.5.1 can be repeated with the matrix U replaced by the matrix $U_p \in \mathrm{GL}_d(\mathbb{Z}_p[G])$ described in §7.4. This shows that, if $\mathrm{LTC}(K/k)$ is valid, then in all cases there exist matrices $M_{K/k,S,T,p}(\phi)$ which satisfy (7) as well as natural analogues of all of the conditions described in Theorem 7.5.1.
(ii) Let G be abelian. In this case the properties that are described in the statement and proof of Theorem 7.5.1 can be reinterpreted by using exterior power operations in place of reduced norms (cf. the proof of Proposition 3.4.1). In this way it is straightforward to show that these properties imply the claims that are made in [11, Th. 3.1(i), (ii)]. We recall (from [11, §4]) that, upon appropriate specialisation, the claims of [11, Th. 3.1(i), (ii)] in turn imply natural 'higher order' generalisations of the $\mathfrak{p}$-adic abelian Stark conjecture of Gross [30, Conj. 7.6], of the refined class number formulas conjectured by Gross [30, Conj. 4.1], by Tate [56, (∗)] and by Aoki, Lee and Tan [1, Conj. 1.1] and of the 'guess' that is formulated by Gross in [30, top of p. 195].

(iii) In [31] Gross raises the problem of generalising [30, Conj. 7.6] to abelian L-series which vanish to arbitrary order at $s = 0$. The observations in (ii) show that the predicted existence of matrices as in Theorem 7.5.1 simultaneously generalises [30, Conj. 7.6] to L-series which can vanish to arbitrary order at $s = 0$ and also need not be abelian.

8. LTC(K/k) implies Conjecture 2.4.1*

In this section we complete the proof of Theorem 4.1.1. To do this we fix a finite non-empty set of places S of k which contains S_∞ and all places which ramify in K/k and is also such that $\mathrm{Cl}(\mathcal{O}_{K,S})$ vanishes.

8.1. The key to our proof is provided by the following result.

Proposition 8.1.1. *Let Σ be any non-empty subset of S which contains S_∞. Then there exists a complex $C^\bullet_{K/k,\Sigma}$ in $D^{\mathrm{p}}(\mathbb{Z}[G])$ and an element $\theta^*_{K/k,S,\Sigma}$ of $\mathfrak{Z}(\mathbb{R}[G])^\times$ such that all of the following conditions are satisfied.*

(i) *$C^\bullet_{K/k,\Sigma}$ is acyclic outside degrees 0 and 1.*
(ii) *$H^0(C^\bullet_{K/k,\Sigma}) = U_{K,\Sigma}$ and there exists a canonical isomorphism of $\mathbb{Q}[G]$-modules $\mathbb{Q}\otimes H^1(C^\bullet_{K/k,\Sigma}) \cong \mathbb{Q}\otimes X_{K,\Sigma}$.*
(iii) *There exists an injective homomorphism of G-modules $\mathrm{Cl}(\mathcal{O}_{K,\Sigma}) \to H^1(C^\bullet_{K/k,\Sigma})$.*
(iv) *For any proper subset Σ' of Σ there exists a surjective homomorphism of G-modules $\pi_{\Sigma,\Sigma'} : H^1(C^\bullet_{K/k,\Sigma}) \to Y_{K,\Sigma'}$.*
(v) *If $\Sigma = S$, then $C^\bullet_{K/k,\Sigma}$ is equal to the complex that occurs in LTC(K/k).*
(vi) *Fix ψ in $\mathrm{Ir}(G)$. If $\Sigma = S$ or both $\psi \neq 1_G$ and $r_\Sigma(\psi) = r_S(\psi)$, then $e_\psi\theta^*_{K/k,S,\Sigma} = e_\psi\theta^*_{K/k,S}(0) = L^*_S(\check{\psi},0)e_\psi$.*
(vii) *Using the identifications of claim (ii) we regard $R_{K,\Sigma}$ as an isomorphism of $\mathbb{R}[G]$-modules $\mathbb{R}\otimes H^0(C^\bullet_{K/k,\Sigma}) \cong \mathbb{R}\otimes H^1(C^\bullet_{K/k,\Sigma})$. Then in $K_0(\mathbb{Z}[G],\mathbb{R}[G])$ one has $\delta_G(\theta^*_{K/k,S}(0)) - \chi(C^\bullet_{K/k,S},\mathrm{R}_{K,S}) = \delta_G(\theta^*_{K/k,S,\Sigma}) - \chi(C^\bullet_{K/k,\Sigma},\mathrm{R}_{K,\Sigma})$. Thus LTC($K/k$) is valid if and only if $\delta_G(\theta^*_{K/k,S,\Sigma}) = \chi(C^\bullet_{K/k,\Sigma},\mathrm{R}_{K,\Sigma})$.*

The proof of Proposition 8.1.1 is given in §8.3. In the next subsection we use it to complete the deduction of Conjecture 2.4.1* from LTC(K/k).

8.2. In this subsection we assume that S is as in Conjecture 2.4.1 (and so satisfies Hypothesis H_r for a non-negative integer r with $r < |S|$). We write μ_K for the torsion subgroup of $U_{K,S}$.

We start with a useful reduction step.

Lemma 8.2.1. *Assume that LTC(K/k) is valid.*

(i) *If $|S| > r+1$, then Conjecture 2.4.1* is valid if for every a in $\mathcal{A}(\mathbb{Z}[G])$, every $\Sigma \subseteq S$ with both $S_\infty \cup S_r \subseteq \Sigma$ and $|\Sigma| > r$, every ϕ in $[U_{K,\Sigma}, X_{K,\Sigma}]_G$ and every* *prime p for which the G-module $\mathbb{Z}_p \otimes \mu_K$ is cohomologically-trivial one has*

$$a\theta^{(r)}_{K/k,S,T}(0)R(\phi) \in \mathbb{Z}_p \otimes \mathrm{Ann}_{\mathbb{Z}[G]}(\mathrm{Cl}(\mathcal{O}_{K,\Sigma})). \tag{31}$$

(ii) *If $|S| = r+1$, then Conjecture 2.4.1* is valid if for every a in $\mathcal{A}(\mathbb{Z}[G])$, every ϕ in $[U_{K,S}, X_{K,S}]_G$ and every prime p for which the G-module $\mathbb{Z}_p \otimes \mu_K$ is cohomologically-trivial one has*

$$a\theta^{(r)}_{K/k,S,T}(0)R(\phi) \in \mathbb{Z}_p \otimes \mathrm{Ann}_{\mathbb{Z}[G]}(\mathrm{Cl}(\mathcal{O}_{K,S})). \tag{32}$$

Proof. As in Conjecture 2.4.1 we fix a non-empty subset S' of S with $S_\infty \cup S_r \subseteq S'$, an element a of $\mathcal{A}(\mathbb{Z}[G])$ and a homomorphism ϕ' in $[U_{K,S}, X_{K,S}]_G$. Since $\mathrm{LTC}(K/k)$ is assumed to be valid the argument of §7 shows that the second paragraph of Conjecture 2.4.1 is valid and hence that the element $a\theta^{(r)}_{K/k,S,T}(0)R(\phi')$ belongs to $\mathbb{Z}[G]$. To prove Conjecture 2.4.1* it is thus enough to show that for each prime p for which the G-module $\mathbb{Z}_p \otimes \mu_K$ is cohomologically-trivial one has

$$xa\theta^{(r)}_{K/k,S,T}(0)R(\phi') \in \mathbb{Z}_p \otimes \mathrm{Ann}_{\mathbb{Z}[G]}(\mathrm{Cl}(\mathcal{O}_{K,S'})) \tag{33}$$

with $x = 1$ if $r = 0$ or $S' = S$ and in all other cases $x \in \mathrm{Ann}_{\mathbb{Z}[G]}(\mathbb{Z}[G/G_w])$ for any $v \in S \setminus S'$. We must therefore deduce (33) from the validity of either (31) or (32) for all suitable a, Σ and ϕ.

If $S' = S$, then $x = 1$ and so (33) is equivalent to either (31) or (32) with $\Sigma = S$ and $\phi' = \phi$.

If $r = 0$ and $|S| = r + 1$, then $S' = S$ so this case is covered by the preceding paragraph. If $r = 0$ and $|S| > r+1$, then $e_\psi(\mathbb{C} \otimes U_{K,S})$ vanishes for all ψ in $\mathrm{Ir}(G)$ with $r_S(\psi) = r$ so $\theta^{(r)}_{K/k,S,T}(0)R(\phi') = \theta^{(r)}_{K/k,S,T}(0)$ for each ϕ' in either $[U_{K,\Sigma}, X_{K,\Sigma}]_G$ or $[U_{K,S}, X_{K,S}]_G$. Since in this case $x = 1$ the containment (33) is therefore equivalent to (31) with $\Sigma = S'$.

We therefore assume that $S' \neq S$ and $r \neq 0$. We choose any place v in $S \setminus S'$ and set $\Sigma = S' \cup \{v\}$. Then $|\Sigma| > r$ and so $\Sigma = S$ if $|S| = r+1$. The surjective homomorphism $X_{K,\Sigma} \to Y_{K,r}$ splits (since $Y_{K,r}$ is a free G-module) and so the induced homomorphism $\varrho : [U_{K,\Sigma}, X_{K,\Sigma}]_G \to [U_{K,\Sigma}, Y_{K,r}]_G$ is surjective. Given ϕ' in $[U_{K,S}, X_{K,S}]_G$ we may thus choose ϕ in $[U_{K,\Sigma}, X_{K,\Sigma}]_G$ for which $\varrho(\phi)$ is equal to the restriction to $U_{K,\Sigma}$ of $\pi_{S,r} \circ \phi'$. But, since $S_r \subseteq \Sigma$, Lemma 2.2.1 implies that the inclusion $e_\psi(\mathbb{C} \otimes U_{K,\Sigma}) \subseteq e_\psi(\mathbb{C} \otimes U_{K,S})$ is bijective for every ψ in $\mathrm{Ir}(G)$ with $r_S(\psi) = r\psi(1)$. One therefore has $\theta^{(r)}_{K/k,S,T}(0)R(\phi') = \theta^{(r)}_{K/k,S,T}(0)R(\phi)$ and so from the assumed validity of either (31) or (32) we deduce that $a\theta^{(r)}_{K/k,S,T}(0)R(\phi')$ belongs to $\mathbb{Z}_p \otimes \mathrm{Ann}_{\mathbb{Z}[G]}(\mathrm{Cl}(\mathcal{O}_{K,\Sigma})) = \mathrm{Ann}_{\mathbb{Z}_p[G]}(\mathbb{Z}_p \otimes \mathrm{Cl}(\mathcal{O}_{K,\Sigma}))$.

Now the group of fractional ideals of K that lie above v is isomorphic as a G-module to $\mathbb{Z}[G/G_w]$ and so there is an exact sequence $\mathbb{Z}_p[G/G_w] \to \mathbb{Z}_p \otimes \mathrm{Cl}(\mathcal{O}_{K,S'}) \to \mathbb{Z}_p \otimes \mathrm{Cl}(\mathcal{O}_{K,\Sigma})$. This implies that $xa\theta^{(r)}_{K/k,S,T}(0)R(\phi') \in \mathbb{Z}_p \otimes \mathrm{Ann}_{\mathbb{Z}[G]}(\mathrm{Cl}(\mathcal{O}_{K,S'}))$ for every $x \in \mathrm{Ann}_{\mathbb{Z}[G]}(\mathbb{Z}[G/G_w])$, as required to prove (33). $\square$

If $|S| > r+1$ we let Σ be any subset of S as in Lemma 8.2.1(i). If $|S| = r+1$ we set $\Sigma := S$. (In all cases therefore S_r is a proper subset of Σ.)

We recall that $\mathbb{Z}^*$ denotes the subring of $\mathbb{Q}$ generated by the inverses of the (finitely many) primes ℓ for which the G-module $\mathbb{Z}_\ell \otimes \mu_K$ is not cohomologically-trivial. The G-module $\mu^* := \mathbb{Z}^* \otimes \mu_K$ is therefore cohomologically-trivial and so the inclusion $\mu_K = (U_{K,\Sigma})_{\rm tor} \subseteq U_{K,\Sigma} = H^0(C^\bullet_{K/k,\Sigma})$ gives rise to an exact triangle in $D^{\rm p}(\mathbb{Z}^*[G])$ of the form $\mu^*_K[0] \to \mathbb{Z}^* \otimes C^\bullet_{K/k,\Sigma} \to C^\bullet \to \mu^*_K[1]$, and hence by Lemma A.1.2 to an equality

$$\chi(\mathbb{Z}^* \otimes C^\bullet_{K/k,\Sigma}, R_{K,\Sigma}) = \chi(C^\bullet, R_{K,\Sigma}) + \chi(\mu^*_K[0], 0)$$

in $K_0(\mathbb{Z}^*[G], \mathbb{R}[G])$, where $C^\bullet$ is acyclic outside degrees 0 and 1 and is such that $H^0(C^\bullet) = \mathbb{Z}^* \otimes (H^0(C^\bullet_{K/k,\Sigma})/H^0(C^\bullet_{K/k,\Sigma})_{\rm tor})$ and $H^1(C^\bullet) = \mathbb{Z}^* \otimes H^1(C^\bullet_{K/k,\Sigma})$. In view of Proposition 8.1.1(i), (ii) and (iv) (with $\Sigma' = S_r$) we may therefore apply Proposition 6.2.2 to $C^\bullet$ and with $R = \mathbb{Z}^*$, $m = r$, $\rho = \mathbb{Z}^* \otimes \pi_{\Sigma,S_r}$ (where π_{Σ,S_r} is the homomorphism in Proposition 8.1.1(iv)) and $\lambda = R_{K,\Sigma}$ to deduce that

$$\begin{aligned}(34)\quad \chi(\mathbb{Z}^* \otimes C^\bullet_{K/k,\Sigma}, R_{K,\Sigma}) = &\\ &\delta_{\mathbb{Z}^*[G],\mathbb{R}[G]}(\mathrm{Nrd}_{\mathbb{R}[G]}(\langle R_{K,\Sigma}, \vartheta, \alpha, \iota_1, \iota_2\rangle)) + \chi(Q[0], 0) + \chi(\mu^*_K[0], 0)\end{aligned}$$

for a finite $\mathbb{Z}^*[G]$-module Q that is cohomologically-trivial and can be chosen to have order coprime to any given natural number $\hbar$.

Now whenever there exists an injective homomorphism of cyclic groups $M \to M'$ there is also a surjective homomorphism of the form $M' \to M$. In particular, as μ_K is cyclic and $\mathbb{F}_T^\times$ is a G-equivariant direct sum of cyclic groups, the inclusion $\mu_K \to \mathbb{F}_T^\times$ (which exists by our choice of T) implies there is a surjective homomorphism of G-modules from $\mathbb{F}_T^\times$ to μ_K. This homomorphism extends to give a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & \bigoplus_{t\in T}\mathbb{Z}^*[G] & \xrightarrow{(1-\mathrm{N}\,t\cdot \mathrm{Fr}_{\mathfrak{t}}^{-1})_t} & \bigoplus_{t\in T}\mathbb{Z}^*[G] & \to & \mathbb{Z}^*\otimes\mathbb{F}_T^\times & \to & 0\\ & & {\scriptstyle\epsilon}\downarrow & & \| & & \downarrow & & \\ 0 & \to & \bigoplus_{t\in T}\mathbb{Z}^*[G] & \xrightarrow{\varphi} & \bigoplus_{t\in T}\mathbb{Z}^*[G] & \to & \mu^*_K & \to & 0\end{array}$$

where the upper row is the scalar extension of (23). This diagram combines with Lemma A.1.1(iii) to imply that

$$\begin{aligned}\chi(\mu^*_K[0], 0) &= -\,\delta_{\mathbb{Z}^*[G],\mathbb{R}[G]}(\mathrm{Nrd}_{\mathbb{Q}[G]}(\varphi))\\ &= \delta_{\mathbb{Z}^*[G],\mathbb{R}[G]}(\prod_{t\in T}\mathrm{Nrd}_{\mathbb{Q}[G]}(1-\mathrm{N}\,t\cdot\mathrm{Fr}_{\mathfrak{t}}^{-1})^{-1}) + \delta_{\mathbb{Z}^*[G],\mathbb{R}[G]}(\mathrm{Nrd}_{\mathbb{Q}[G]}(\epsilon))\\ &= \delta_{\mathbb{Z}^*[G],\mathbb{R}[G]}(\epsilon_T^{-1}\mathrm{Nrd}_{\mathbb{Q}[G]}(\epsilon)).\end{aligned}$$

By substituting this formula into the equality (34), and then using the final assertion of Proposition 8.1.1(vii), we deduce that if $\mathrm{LTC}(K/k)$ is valid, then

$$\delta_{\mathbb{Z}^*[G],\mathbb{R}[G]}(\epsilon_T\theta^*_{K/k,S,\Sigma}) = \delta_{\mathbb{Z}^*[G],\mathbb{R}[G]}(\mathrm{Nrd}_{\mathbb{Q}[G]}(\epsilon)\mathrm{Nrd}_{\mathbb{R}[G]}(\langle R_{K,\Sigma}, \vartheta, \alpha, \iota_1, \iota_2\rangle)) + \chi(Q[0], 0).$$

We now fix a prime p for the G-module $\mathbb{Z}_p \otimes \mu_K$ is cohomologically-trivial and we assume, as we may, that the order of Q is coprime to p. We note that there is a natural analogue of the commutative diagram (27) in which one replaces $\mathbb{Z}$ by $\mathbb{Z}^*$. In fact $j'_*(\chi(Q[0],0)) = 0$ since $\mathbb{Z}_p \otimes_{\mathbb{Z}^*} Q$ vanishes and so this commutative diagram combines with the last displayed formula to imply that in $\mathfrak{Z}(\mathbb{C}_p[G])$ one has

$$j_*(\epsilon_T \theta^*_{K/k,S,\Sigma}) = \mathrm{Nrd}_{\mathbb{Q}_p[G]}(A_p) j_*(\mathrm{Nrd}_{\mathbb{R}[G]}(\langle R_{K,\Sigma}, \vartheta, \alpha, \iota_1, \iota_2\rangle))$$

for a suitable matrix A_p in $\mathrm{M}_d(\mathbb{Z}_p[G])$.

We now fix ϕ in $[U_{K,\Sigma}, X_{K,\Sigma}]_G$. Then, after multiplying the last displayed equality by $e_K R(\phi)$, and recalling both Lemmas 7.2.2 and 7.3.1, one finds that

$$e_K \epsilon_T \theta^*_{K/k,S,\Sigma} R(\phi) = \mathrm{Nrd}_{\mathbb{Q}_p[G]}(A_p \cdot M(\phi^K, \vartheta))$$

for any suitable homomorphism ϕ^K in $[F, \mathbb{Z}^* \otimes Y_{K,r}]_{\mathbb{Z}^*[G]}$. In addition, since both $S_r \subseteq \Sigma$, and $\Sigma = S$ if $|S| = r+1$, Proposition 8.1.1(vi) implies that $e_K \epsilon_T \theta^*_{K/k,S,\Sigma} = \theta^{(r)}_{K/k,S,T}(0)$. Upon substituting this fact into the last displayed equality one obtains the required containment (31) or (32) from the following result.

Lemma 8.2.2. *For every a in $\mathcal{A}(\mathbb{Z}[G])$ the element $a\mathrm{Nrd}_{\mathbb{Q}_p[G]}(A_p \cdot M(\phi^K, \vartheta))$ belongs to $\mathbb{Z}_p \otimes \mathrm{Ann}_{\mathbb{Z}[G]}(\mathrm{Cl}(\mathcal{O}_{K,\Sigma}))$.*

Proof. We write $\alpha(\phi^K, \vartheta)'_p$ and $\alpha(\phi^K, \vartheta)_p$ for the endomorphisms of the $\mathbb{Z}_p[G]$-module $F_p := \mathbb{Z}_p \otimes_{\mathbb{Z}^*} F$ which correspond, with respect to the $\mathbb{Z}_p[G]$-basis $\{b_i : i \in \langle d\rangle\}$, to the matrices $A_p \cdot M(\phi^K, \vartheta)$ and $M(\phi^K, \vartheta)$ respectively. We also fix a in $\mathcal{A}(\mathbb{Z}[G])$ and note that, as $\mathcal{A}(\mathbb{Z}[G]) \subset \mathcal{A}(\mathbb{Z}_p[G])$, Lemma 5.1.4 implies $a\mathrm{Nrd}_{\mathbb{Q}_p[G]}(A_p \cdot M(\phi^K, \vartheta))$ belongs to $\mathbb{Z}_p[G]$ and annihilates $\mathrm{cok}(\alpha(\phi^K, \theta)'_p)$. To prove the required containment it is therefore enough to show that $\mathbb{Z}_p \otimes \mathrm{Cl}(\mathcal{O}_{K,\Sigma})$ is isomorphic as a $\mathbb{Z}_p[G]$-module to a subquotient of $\mathrm{cok}(\alpha(\phi, \theta)'_p)$.

To do this we note first that $\mathrm{im}(\alpha(\phi^K, \vartheta)'_p) \subseteq \mathrm{im}(\alpha(\phi^K, \vartheta)_p)$ and hence that $\mathrm{cok}(\alpha(\phi^K, \vartheta)_p)$ is a quotient of $\mathrm{cok}(\alpha(\phi^K, \vartheta)'_p)$. We consider next the following exact commutative diagram

$$\begin{array}{ccccccc}
F_p & \xrightarrow{\alpha(\phi^K,\vartheta)_p} & F_p & \longrightarrow & \mathrm{cok}(\alpha(\phi^K, \vartheta)_p) & \longrightarrow & 0 \\
\Big\| & & \Big\downarrow{\scriptstyle \theta_1} & & \Big\downarrow{\scriptstyle \theta_2} & & \\
F_p & \xrightarrow{\mathbb{Z}_p \otimes_{\mathbb{Z}^*} \vartheta} & F_p & \longrightarrow & \mathbb{Z}_p \otimes_{\mathbb{Z}^*} \mathrm{cok}(\vartheta) & \longrightarrow & 0 \\
& & \Big\downarrow{\scriptstyle \theta_4} & & \Big\downarrow{\scriptstyle \theta_3} & & \\
& & \mathbb{Z}_p \otimes Y_{K,r} & = & \mathbb{Z}_p \otimes Y_{K,r}. & &
\end{array}$$

In this diagram we use the following notation: θ_1 is the homomorphism which, for each subset $\{x_i : i \in \langle d\rangle\}$ of $\mathbb{Z}_p[G]$, sends $\sum_{i=1}^{i=d} x_i b_i$ to $\sum_{i=r+1}^{i=d} x_i b_i$; a comparison of the definitions (29) and (30) (with J the identity group, $E = K$ and $\kappa = \phi^K$)

then shows that the first upper square commutes and we define θ_2 to be the unique homomorphism which makes the second upper square commute; θ_3 is the composite $\mathbb{Z}_p \otimes_{\mathbb{Z}^*} \mathrm{cok}(\vartheta) = \mathbb{Z}_p \otimes_{\mathbb{Z}^*} H^1(C^\bullet_\vartheta) \to \mathbb{Z}_p \otimes_{\mathbb{Z}^*} H^1(C^\bullet) = \mathbb{Z}_p \otimes H^1(C^\bullet_{K/k,\Sigma}) \to \mathbb{Z}_p \otimes Y_{K,r}$ where the first arrow is the surjection induced by the triangle (21) in the context of our present application of Proposition 6.2.2 (as described just prior to (34)) and the final arrow is $\mathbb{Z}_p \otimes_{\mathbb{Z}^*} \pi_{\Sigma,S_r}$ where π_{Σ,S_r} is the homomorphism in Proposition 8.1.1(iv); θ_4 is defined to make the lower square commute. The rows in the diagram are obviously exact and central column is exact as a consequence of Lemma 6.2.1(i) (with $R = \mathbb{Z}^*$ and $m = r$). The commutativity of the diagram then implies that the final column is exact and hence that there is a surjection from $\mathrm{cok}(\alpha(\phi^K, \vartheta)_p)$ to $\mathbb{Z}_p \otimes_{\mathbb{Z}^*} \ker(\pi_{\Sigma,S_r})$. Since Proposition 8.1.1(iii) implies there is an injective homomorphism of $\mathbb{Z}_p[G]$-modules from $\mathbb{Z}_p \otimes \mathrm{Cl}(\mathcal{O}_{K,\Sigma})$ to $\mathbb{Z}_p \otimes_{\mathbb{Z}^*} \ker(\pi_{\Sigma,S_r})$ it follows that $\mathbb{Z}_p \otimes \mathrm{Cl}(\mathcal{O}_{K,\Sigma})$ is isomorphic to a subquotient of $\mathrm{cok}(\alpha(\phi^K, \theta)_p)$, and hence also of $\mathrm{cok}(\alpha(\phi^K, \theta)'_p)$, as required. □

8.3. It remains to prove Proposition 8.1.1 and to do this we adapt an approach used by Nickel (which is itself dependent upon earlier work of Ritter and Weiss [42] and of Greither [29]). We write $F_{S\setminus\Sigma}$ for the free G-module on the set $\{x_v : v \in S \setminus \Sigma\}$ and for each $v \in S \setminus \Sigma$ we use the G_w-module W_w^* that is described in [37, (2.1)]. We recall that there is an exact sequence of finitely generated G-modules

$$0 \to U_{K,\Sigma} \oplus F_{S\setminus\Sigma} \to A \oplus F_{S\setminus\Sigma} \xrightarrow{d} B \to \nabla \to 0. \tag{35}$$

Here A and B are cohomologically-trivial and ∇ is such that $\mathrm{Cl}(\mathcal{O}_{K,\Sigma}) \subseteq \nabla_{\mathrm{tor}}$ and there is a canonical exact sequence of G-modules

$$0 \to \overline{\nabla} \to Y_{K,\Sigma} \oplus W^*_{S\setminus\Sigma} \xrightarrow{\epsilon} \mathbb{Z} \to 0 \tag{36}$$

where we set $W^*_{S\setminus\Sigma} := \bigoplus_{v\in S\setminus\Sigma} \mathrm{ind}^G_{G_w}(W^*_w)$ and for any module M we write $\overline{M}$ for the quotient of M by M_{tor} (for more details of these constructions see [37, (2.3) and the end of the proof of Lem. 2.4]). We write $C^\bullet_{S,\Sigma}$ for the complex $A \oplus F_{S\setminus\Sigma} \xrightarrow{d} B$ where the first term is placed in degree 0 and we use (35) to identify $H^0(C^\bullet_{S,\Sigma})$ and $H^1(C^\bullet_{S,\Sigma})$ with $U_{K,\Sigma} \oplus F_{S\setminus\Sigma}$ and ∇ respectively.

For each v in S we write I_w for the inertia subgroup of w in G and set $e_w := e_{G_w}$ and $T_w := |G_w|e_{G_w} = \sum_{g\in G_w} g \in \mathbb{Z}[G_w]$. We recall that for each v in $S \setminus \Sigma$ there is a canonical surjective homomorphism of G_w-modules $\kappa_w : \mathbb{Z}[G_w] \oplus \mathbb{Z}[G_w] \to W^*_w$ (cf. [37, Prop. 4.4]) and we set $d_w := \kappa_w((1, e_w)) \in \mathbb{Q} \otimes W^*_w$. We write h_K for the class number of K and for each v in $S \setminus \Sigma$ we choose a generator u_w of the h_K-th power of the prime ideal of $\mathcal{O}_K$ that corresponds to w. We then write

$$\lambda_{S,\Sigma} : \mathbb{R} \otimes (U_{K,\Sigma} \oplus F_{S\setminus\Sigma}) \to \mathbb{R} \otimes (Y_{K,\Sigma} \oplus W^*_{S\setminus\Sigma})$$

for the homomorphism of $\mathbb{R}[G]$-modules that is equal to $R_{K,\Sigma}$ on $\mathbb{R}\otimes U_{K,\Sigma}$ and for each v in $S\setminus\Sigma$ satisfies

$$\lambda_{S,\Sigma}(x_v) = (h_K\log(\mathrm{N}w)e_w + 1 - e_w)d_w - e_w\sum_{s\in S_{\infty,K}}\log|u_w|_s s. \tag{37}$$

It is easy to check that for each place v in $S\setminus\Sigma$ the projection map ϵ in (36) sends d_w to 1 (cf. [37, just before (4.1)]) and hence that $\epsilon(\lambda_{S,\Sigma}(x_v))$ is equal to $h_K\log(\mathrm{N}w) - \sum_{s\in S_{\infty,K}}\log|u_w|_s = -\sum_s\log|u_w|_s = 0$ where in the last sum s runs over all places of K. This shows that $\mathrm{im}(\lambda_{S,\Sigma})\subset\mathbb{R}\otimes\ker(\epsilon) = \mathbb{R}\otimes\nabla$ and so allows us to regard $\lambda_{S,\Sigma}$ as a homomorphism of $\mathbb{R}[G]$-modules from $\mathbb{R}\otimes H^0(C^\bullet_{S,\Sigma}) = \mathbb{R}\otimes(U_{K,\Sigma}\oplus F_{S\setminus\Sigma})$ to $\mathbb{R}\otimes\nabla = \mathbb{R}\otimes H^1(C^\bullet_{S,\Sigma})$.

Lemma 8.3.1. *The map $\lambda_{S,\Sigma}:\mathbb{R}\otimes H^0(C^\bullet_{S,\Sigma})\to\mathbb{R}\otimes H^1(C^\bullet_{S,\Sigma})$ is bijective. Further, LTC(K/k) is valid if and only if in $K_0(\mathbb{Z}[G],\mathbb{R}[G])$ one has $\chi(C^\bullet_{S,\Sigma},\lambda_{S,\Sigma}) = \delta_G(\theta_{S,\Sigma})$ where $\theta_{S,\Sigma}$ is the unique element of $\mathfrak{Z}(\mathbb{R}[G])^\times$ which for each ψ in $\mathrm{Ir}(G)$ satisfies $e_\psi\theta_{S,\Sigma} = e_\psi L_S^*(\check{\psi},0)\prod_{v\in S\setminus\Sigma}(-h_K|G_w|)^{\dim_{\mathbb{C}}(V_\psi^{G_w})}$.*

Proof. The map $\lambda_{S,\Sigma}$ differs from the bijective map $\lambda_\Sigma^{\mathrm{mod}}:\mathbb{R}\otimes(U_{K,\Sigma}\oplus F_{S\setminus\Sigma})\to\mathbb{R}\otimes\nabla$ that is defined in [37, (4.1)] (with $h = h_K$) in the following way: $\lambda_{S,\Sigma}$ and $\lambda_\Sigma^{\mathrm{mod}}$ agree upon restriction to $\mathbb{R}\otimes U_{K,\Sigma}$ but for each place $v\in S\setminus\Sigma$ one has $\lambda_{S,\Sigma}(x_v) = \lambda_\Sigma^{\mathrm{mod}}(x_v) + (1-e_w)\sum_{s\in S_{\infty,K}}\log|u_w|_s s$.

For each ψ in $\mathrm{Ir}(G)$ we fix a $\mathbb{C}$-basis of $\mathbb{C}\otimes_{\mathcal{O}_\psi}H^0(C^\bullet_{S,\Sigma})_\psi$, resp. of $\mathbb{C}\otimes_{\mathcal{O}_\psi}H^1(C^\bullet_{S,\Sigma})_\psi$, by concatenating $\mathbb{C}$-bases of $\mathbb{C}\otimes_{\mathcal{O}_\psi}U_{K,\Sigma,\psi}$ and $\mathbb{C}\otimes_{\mathcal{O}_\psi}F_{S\setminus\Sigma,\psi}$, resp. of $\mathbb{C}\otimes_{\mathcal{O}_\psi}X_{K,\Sigma,\psi}$ and $\mathbb{C}\otimes_{\mathcal{O}_\psi}W^*_{S\setminus\Sigma,\psi}$. It is easily checked that, with respect to these bases, the homomorphism $\mathbb{C}\otimes_{\mathcal{O}_\psi}H^0(C^\bullet_{S,\Sigma})_\psi\to\mathbb{C}\otimes_{\mathcal{O}_\psi}H^1(C^\bullet_{S,\Sigma})_\psi$ induced by $\lambda_{S,\Sigma}$, resp. $\lambda_\Sigma^{\mathrm{mod}}$, is represented by a block matrix of the form

$$M'_\psi := \begin{pmatrix} A_\psi & 0\\ * & B_\psi\end{pmatrix},\quad\text{resp. } M_\psi := \begin{pmatrix} A_\psi & 0\\ ** & B_\psi\end{pmatrix},$$

where A_ψ is the matrix of the map $\mathbb{C}\otimes_{\mathcal{O}_\psi}U_{K,\Sigma,\psi}\to\mathbb{C}\otimes_{\mathcal{O}_\psi}X_{K,\Sigma,\psi}$ induced by $R_{K,\Sigma}$ and B_ψ the matrix of the map $\mathbb{C}\otimes_{\mathcal{O}_\psi}F_{S\setminus\Sigma,\psi}\to\mathbb{C}\otimes_{\mathcal{O}_\psi}W^*_{S\setminus\Sigma,\psi}$ that is induced by the map $\mathbb{R}\otimes F_{S\setminus\Sigma}\to\mathbb{R}\otimes W^*_{S\setminus\Sigma}$ sending each x_v to $(h_K\log(\mathrm{N}w)e_w + 1 - e_w)d_w$. It follows that $\det(M'_\psi) = \det(A_\psi)\det(B_\psi) = \det(M_\psi)$. Now M_ψ is invertible (as $\lambda_\Sigma^{\mathrm{mod}}$ is bijective) and so $e_\psi\mathrm{Nrd}_{\mathbb{R}[G]}(\lambda_{S,\Sigma}\circ(\lambda_\Sigma^{\mathrm{mod}})^{-1}) = e_\psi\det(M'_\psi M_\psi^{-1}) = e_\psi\det(M'_\psi)\det(M_\psi)^{-1} = e_\psi$. Since this is true for all ψ it shows that $\lambda_{S,\Sigma}$ is bijective (as claimed) and also that $\mathrm{Nrd}_{\mathbb{R}[G]}(\lambda_{S,\Sigma}\circ(\lambda_\Sigma^{\mathrm{mod}})^{-1}) = 1$. This last equality combines with Lemma A.1.1(ii) to imply $\chi(C^\bullet_{S,\Sigma},\lambda_\Sigma^{\mathrm{mod}}) = \chi(C^\bullet_{S,\Sigma},\lambda_{S,\Sigma})$. Given this equality the claimed interpretation of LTC(K/k) follows immediately from [37, §4, Rem. 1]. □

We now fix a place w_σ of K that lies above a place in Σ and for each $v\in S\setminus\Sigma$ we set $b_v = -T_w(w_\sigma) + \kappa_w((1,T_w))\in Y_{K,\Sigma}\oplus W^*_{S\setminus\Sigma}$. We note that $\epsilon(b_v) = 0$ and hence that b_v belongs to $\overline{\nabla}$. We then write δ for the homomorphism of G-modules from $F_{S\setminus\Sigma}$ to ∇ which sends each x_v to any pre-image of b_v under the natural map $\nabla \to \overline{\nabla}$. An obvious adaptation of the proof of [37, Lem. 4.5] shows that for each v in $S\setminus\Sigma$ the element $\kappa_w((1,T_w))$ is a $\mathbb{Q}[G_w]$-generator of the module $\mathbb{Q}\otimes W_w^*$ and this fact implies that the homomorphism δ is injective. We choose a lift δ' of δ through the surjection $B\to\nabla$ which occurs in (35) and then consider the following exact commutative diagram

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & U_{K,\Sigma} & \longrightarrow & A & \xrightarrow{d'} & B/\operatorname{im}(\delta') & \longrightarrow & \nabla/\operatorname{im}(\delta) & \longrightarrow & 0\\
 & & \uparrow{\scriptstyle \pi_1} & & {\scriptstyle \pi_2}\uparrow & & {\scriptstyle \pi_3}\uparrow & & {\scriptstyle \pi_4}\uparrow & & \\
0 & \longrightarrow & U_{K,\Sigma}\oplus F_{S\setminus\Sigma} & \longrightarrow & A\oplus F_{S\setminus\Sigma} & \xrightarrow{d} & B & \longrightarrow & \nabla & \longrightarrow & 0\\
 & & \uparrow{\scriptstyle \iota_1} & & \uparrow{\scriptstyle \iota_2} & & \uparrow{\scriptstyle \delta'} & & \uparrow{\scriptstyle \delta} & & \\
0 & \longrightarrow & F_{S\setminus\Sigma} & = & F_{S\setminus\Sigma} & \xrightarrow{0} & F_{S\setminus\Sigma} & = & F_{S\setminus\Sigma} & \longrightarrow & 0.
\end{array} \tag{38}$$

All maps in the central row of this diagram are as in (35), the maps ι_i and π_j are the natural injections and projections and the maps in the upper row are induced by those in the middle row. All rows in the diagram are exact and all columns are short exact sequences.

We write $C^\bullet_{K/k,\Sigma}$ and $C^\bullet_{S\setminus\Sigma}$ for the complexes $A \xrightarrow{d'} B/\operatorname{im}(\delta')$ and $F_{S\setminus\Sigma} \xrightarrow{0} F_{S\setminus\Sigma}$ with the first module placed in degree 0 in both cases. It is then clear that $C^\bullet_{K/k,\Sigma}$ is acyclic outside degrees 0 and 1. Further, since $\mathrm{Cl}(\mathcal{O}_{K,\Sigma})\subseteq\nabla$ and $\operatorname{im}(\delta)$ is torsion-free there is an induced injective homomorphism $\mathrm{Cl}(\mathcal{O}_{K,\Sigma})\to\nabla/\operatorname{im}(\delta) = H^1(C^\bullet_{K/k,\Sigma})$. Using the description of $\overline{\nabla}$ given in (36) it is also straightforward to see that the inclusion $X_{K,\Sigma}\subseteq\nabla$ induced by (36) extends to give an identification $\mathbb{Q}\otimes X_{K,\Sigma} = \mathbb{Q}\otimes H^1(C^\bullet_{K/k,\Sigma})$ and that if Σ' is any proper subset of Σ, then the natural projection $Y_{K,\Sigma}\to Y_{K,\Sigma'}$ induces a surjective homomorphism from $\overline{\nabla}$ to $Y_{K,\Sigma'}$.

The second and third columns of (38) give rise to an exact triangle in $D^{\mathrm{p}}(\mathbb{Z}[G])$ of the form $C^\bullet_{S\setminus\Sigma}\xrightarrow{\iota} C^\bullet_{S,\Sigma}\xrightarrow{\pi} C^\bullet_{K/k,\Sigma}\to C^\bullet_{S\setminus\Sigma}[1]$. This triangle has the property that, with respect to the identification $\mathbb{R}\otimes X_{K,\Sigma} = \mathbb{R}\otimes H^1(C^\bullet_{K/k,\Sigma})$ described in the preceding paragraph, there is an exact commutative diagram of $\mathbb{R}[G]$-modules

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{R}\otimes H^0(C^\bullet_{S\setminus\Sigma}) & \xrightarrow{\mathbb{R}\otimes H^0(\iota)} & \mathbb{R}\otimes H^0(C^\bullet_{S,\Sigma}) & \xrightarrow{\mathbb{R}\otimes H^0(\pi)} & \mathbb{R}\otimes H^0(C^\bullet_{K/k,\Sigma}) & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle \mathrm{id}} & & {\scriptstyle \lambda}\downarrow & & {\scriptstyle R_{K,\Sigma}}\downarrow & & \\
0 & \longrightarrow & \mathbb{R}\otimes H^1(C^\bullet_{S\setminus\Sigma}) & \xrightarrow{\mathbb{R}\otimes H^1(\iota)} & \mathbb{R}\otimes H^1(C^\bullet_{S,\Sigma}) & \xrightarrow{\mathbb{R}\otimes H^1(\pi)} & \mathbb{R}\otimes H^1(C^\bullet_{K/k,\Sigma}) & \longrightarrow & 0
\end{array}$$

with λ the isomorphism sending (u, f) in $H^0(C^\bullet_{S,\Sigma}) = U_{K,\Sigma} \oplus F_{S\setminus\Sigma}$ to $R_{K,\Sigma}(u) + \delta(f)$. Lemma A.1.2 therefore applies in this context to imply $\chi(C^\bullet_{S,\Sigma}, \lambda) = \chi(C^\bullet_{K/k,\Sigma}, R_{K,\Sigma}) + \chi(C^\bullet_{S\setminus\Sigma}, \mathbb{R}\otimes \mathrm{id}_{F_{S\setminus\Sigma}})$ and so, since $\chi(C^\bullet_{S\setminus\Sigma}, \mathbb{R}\otimes\mathrm{id}_{F_{S\setminus\Sigma}}) = 0$ (as a consequence of Lemma A.1.1(iii)), one has $\chi(C^\bullet_{K/k,\Sigma}, R_{K,\Sigma}) = \chi(C^\bullet_{S,\Sigma}, \lambda)$. But $\chi(C^\bullet_{S,\Sigma}, \lambda) = \chi(C^\bullet_{S,\Sigma}, \lambda_{S,\Sigma}) + \delta_G(\mathrm{Nrd}_{\mathbb{R}[G]}(\lambda \circ \lambda^{-1}_{S,\Sigma}))$ (by Lemma A.1.1(ii)) and so Lemma 8.3.1 implies $\mathrm{LTC}(K/k)$ is valid if and only if $\chi(C^\bullet_{K/k,\Sigma}, R_{K,\Sigma}) = \delta_G(\theta_{S,\Sigma}\mathrm{Nrd}_{\mathbb{R}[G]}(\lambda \circ \lambda^{-1}_{S,\Sigma}))$. Thus, if we set $\theta^*_{K/k,S,\Sigma} := \theta_{S,\Sigma}\mathrm{Nrd}_{\mathbb{R}[G]}(\lambda\circ\lambda^{-1}_{S,\Sigma})$, then the proof of Proposition 8.1.1 is completed by the following result.

Lemma 8.3.2. *Fix ψ in $\mathrm{Ir}(G)$. If either $\Sigma = S$ or both $\psi \neq 1_G$ and $r_\Sigma(\psi) = r_S(\psi)$, then $e_\psi\theta^*_{K/k,S,\Sigma} = e_\psi L^*_S(\check{\psi}, 0)$.*

Proof. If $\Sigma = S$, then both $F_{S\setminus\Sigma}$ and $W^*_{S\setminus\Sigma}$ vanish and $\lambda = \lambda_{S,\Sigma} = R_{K,\Sigma}$ and so the claimed equality is an immediate consequence of the explicit formula for $\theta_{S,\Sigma}$ given in Lemma 8.3.1. We therefore assume that $\psi \neq 1_G$ and $r_\Sigma(\psi) = r_S(\psi)$. In this case it suffices to prove that both $e_\psi\theta_{S,\Sigma} = e_\psi L^*_S(\check{\psi}, 0)$ and $e_\psi\mathrm{Nrd}_{\mathbb{R}[G]}(\lambda\circ\lambda^{-1}_{S,\Sigma}) = e_\psi$.

Since V^G_ψ vanishes and $r_\Sigma(\psi) = r_S(\psi)$ the formula (5) implies that $V^{G_w}_\psi$ vanishes for each v in $S\setminus\Sigma$ and so the formula for $\theta_{S,\Sigma}$ in Lemma 8.3.1 implies that $e_\psi\theta_{S,\Sigma} = e_\psi L^*_S(\check{\psi}, 0)$. Also $e_\psi e_w = e_\psi T_w = 0$ since $V^{G_w}_\psi$ vanishes and so

$$e_\psi\lambda(x_v) = e_\psi\delta(x_v) = e_\psi(\kappa_w(1, T_w)) - e_\psi T_w(w_\sigma) = e_\psi(\kappa_w(1, 0)) = e_\psi(d_w) = e_\psi\lambda_{S,\Sigma}(x_v),$$

where the penultimate equality follows because $d_w = \kappa_w((1, e_w))$ and the last is obtained by multiplying (37) by e_ψ. It follows that $e_\psi(\mathbb{C}\otimes_\mathbb{R}\lambda) = e_\psi(\mathbb{C}\otimes_\mathbb{R}\lambda_{S,\Sigma})$ and hence that $e_\psi\mathrm{Nrd}_{\mathbb{R}[G]}(\lambda\circ\lambda^{-1}_{S,\Sigma}) = e_\psi\mathrm{Nrd}_{\mathbb{R}[G]}(\mathrm{id}) = e_\psi$, as required. □

9. The proof of Corollary 4.1.6

In this section we prove Corollary 4.1.6 and the result discussed in Remark 4.3.2.

9.1. As already observed in Remark 4.1.7, the conditions of Corollary 4.1.6 ensure that Conjectures 2.4.1 and 2.4.1* coincide. Given Theorem 4.1.1 it is thus enough to prove that if the hypotheses of Corollary 4.1.6 are satisfied, then $\mathrm{LTC}(K/k)$ is valid. To do this we recall that $\mathrm{LTC}(K/k)$ is valid if and only if the element $T\Omega(\mathbb{Q}(0)_K, \mathbb{Z}[G])$ defined in Remark 6.1.1(ii) vanishes.

For any finite group Γ, prime p and isomorphism of fields $j : \mathbb{C} \cong \mathbb{C}_p$ we write $j_{\Gamma,*}$ for the induced homomorphism of K-groups $K_0(\mathbb{Z}[\Gamma], \mathbb{R}[\Gamma]) \to K_0(\mathbb{Z}_p[\Gamma], \mathbb{C}_p[\Gamma])$. We recall from [7, Lem. 2.1] that the induced homomorphism

$$K_0(\mathbb{Z}[\Gamma], \mathbb{R}[\Gamma]) \xrightarrow{(j_{\Gamma,*})_{p,j}} \prod_{p,j} K_0(\mathbb{Z}_p[\Gamma], \mathbb{C}_p[\Gamma])$$

is injective, where the product runs over all primes p and all isomorphisms $j : \mathbb{C} \cong \mathbb{C}_p$. To prove Corollary 4.1.6 it is thus enough to show that if p is any prime that is tamely ramified in K/k and such that the hypotheses (i) and (ii) of Corollary 4.1.6 are satisfied, then $T\Omega(\mathbb{Q}(0)_K, \mathbb{Z}[G])$ belongs to $\ker(j_{G,*})$ for every isomorphism $j : \mathbb{C} \cong \mathbb{C}_p$.

To do this we recall from [17, Th. 5.3] that

$$j_{G,*}(T\Omega(\mathbb{Q}(0)_K, \mathbb{Z}[G]) + \psi_p^*(j_{G,*}(T\Omega(\mathbb{Q}(1)_K, \mathbb{Z}[G]))) = j_{G,*}(T\Omega^{\mathrm{loc}}(\mathbb{Q}(0)_K, \mathbb{Z}[G]))$$

for canonical elements $T\Omega(\mathbb{Q}(1)_K, \mathbb{Z}[G])$ and $T\Omega^{\mathrm{loc}}(\mathbb{Q}(1)_K, \mathbb{Z}[G])$ of $K_0(\mathbb{Z}[G], \mathbb{R}[G])$ and a canonical involution ψ_p^* of $K_0(\mathbb{Z}_p[G], \mathbb{C}_p[G])$. We further recall that, since p is assumed to be at most tamely ramified in K/k, one knows (from [5, §7]) that the element $j_{G,*}(T\Omega^{\mathrm{loc}}(\mathbb{Q}(0)_K, \mathbb{Z}[G]))$ vanishes. Given the last displayed equality it is therefore enough for us to show that if both the p-adic Stark Conjecture at $s = 1$ is valid for all p-adic characters of G and if p divides $|G|$ the p-adic μ-invariant of the cyclotomic $\mathbb{Z}_p$-extension of K vanishes, then $T\Omega(\mathbb{Q}(1)_K, \mathbb{Z}[G])$ belongs to $\ker(j_{G,*})$. But by the very definition of $T\Omega(\mathbb{Q}(1)_K, \mathbb{Z}[G])$ in [17, Conj. 4.1(iii)] one knows that $j_{G,*}(T\Omega(\mathbb{Q}(1)_K, \mathbb{Z}[G]))$ vanishes if and only if the image under $j_{G,*}$ of the equivariant Tamagawa number conjecture for the pair $(\mathbb{Q}(1)_K, \mathbb{Z}[G])$ is valid. Hence, under our stated hypotheses on p, the required vanishing of $j_{G,*}(T\Omega(\mathbb{Q}(1)_K, \mathbb{Z}[G]))$ can be deduced from the recent proof by Ritter and Weiss [44] of the main conjecture of non-commutative Iwasawa theory for totally real fields. For full details of this deduction see the proof of [15, Cor. 2.9].

This completes the proof of Corollary 4.1.6.

9.2. The following result justifies Remark 4.3.2.

Proposition 9.2.1. *Assume that K is totally real. If the p-adic Stark Conjecture at $s = 1$ is valid for all primes p and all p-adic characters of G, then the Strong Stark Conjecture is valid for all complex characters of G.*

Proof. For each subgroup H of G and each quotient Γ of H we write ρ_H^G and q_Γ^H for the natural homomorphisms $K_0(\mathbb{Z}[G], \mathbb{R}[G]) \to K_0(\mathbb{Z}[H], \mathbb{R}[H])$ and $K_0(\mathbb{Z}[H], \mathbb{R}[H]) \to K_0(\mathbb{Z}[\Gamma], \mathbb{R}[\Gamma])$. We also continue to use the notation introduced in §9.1.

Remark 6.1.1(iii) combines with the description of $K_0(\mathbb{Z}[G], \mathbb{Q}[G])_{\mathrm{tor}}$ given in [9, Prop. 4.1] to imply that the Strong Stark Conjecture is valid for all complex characters of G if and only if $T\Omega(\mathbb{Q}(0)_K, \mathbb{Z}[G])$ belongs to $\ker(j_{\Gamma,*} \circ q_\Gamma^H \circ \rho_H^G)$ for every prime p, every isomorphism $j : \mathbb{C} \cong \mathbb{C}_p$, every cyclic subgroup H of G and every quotient $\Gamma = H/J$ that is of order prime to p. Now Remark 6.1.1(ii) also implies that the image of $T\Omega(\mathbb{Q}(0)_K, \mathbb{Z}[G])$ under $q_\Gamma^H \circ \rho_H^G$ is equal to $T\Omega(\mathbb{Q}(0)_{K^J}, \mathbb{Z}[\Gamma])$ and so it is enough to prove that for any (cyclic) Galois extension F/E, with $\Delta = G_{F/E}$, every prime p with $p \nmid |\Delta|$ and every isomorphism $j : \mathbb{C} \cong \mathbb{C}_p$ the validity of the p-adic Stark Conjecture at $s = 1$ for all p-adic characters of Δ implies that $T\Omega(\mathbb{Q}(0)_F, \mathbb{Z}[\Delta])$ belongs to $\ker(j_{\Delta,*})$. But since p does not divide $|\Delta|$ it is tamely ramified in F/E and so the required vanishing of $j_{\Delta,*}(T\Omega(\mathbb{Q}(0)_F, \mathbb{Z}[\Delta]))$ follows by the same argument as in §9.1. $\square$

10. The proof of Theorem 4.2.1

10.1. In this subsection we fix a quadratic extension of fields K/k. We set $G := G_{K/k}$ and write g for the non-trivial element of G.

In this case $\mathrm{LTC}(K/k)$ has been proved by Kim [33, §2.4, Rem. i)] and so Theorem 4.1.1 implies that Conjecture 2.4.1* is valid for K/k. One also has $\mathcal{A}(\mathbb{Z}[G]) = \mathbb{Z}[G]$ and $\mathrm{Ann}_{\mathbb{Z}[G]}(\mathbb{Z}) = (g-1)\mathbb{Z}[G]$. To complete the proof of Conjecture 2.4.1 for K/k, and hence of Theorem 4.2.1(i), it is thus enough to prove the following result.

Proposition 10.1.1. *Fix ϕ in $[U_{K,S}, X_{K,S}]_G$. If $r = 0$ or $S' = S$, then $\theta^{(r)}_{K/k,S,T}(0)R(\phi)$ belongs to $\mathrm{Ann}_{\mathbb{Z}[G]}(\mathrm{Cl}(\mathcal{O}_{K,S'}))$. In all other cases $(g-1)\theta^{(r)}_{K/k,S,T}(0)R(\phi)$ belongs to $\mathrm{Ann}_{\mathbb{Z}[G]}(\mathrm{Cl}(\mathcal{O}_{K,S'})$.*

Proof. If $r = 0$, then the containment $\theta^{(r)}_{K/k,S,T}(0)R(\phi) \in \mathrm{Ann}_{\mathbb{Z}[G]}(\mathrm{Cl}(\mathcal{O}_{K,S'}))$ is equivalent to Brumer's Conjecture (by the argument of Proposition 3.5.1) and so is proved by Tate in [55, Chap. IV, Th. 5.4 and the discussion of §6]. If $S' = S$, then the required containment follows directly from Theorem 10.1.2 below. We may therefore assume in the sequel that $r > 0$ and $S' \neq S$.

Now if there exists a place in $S \setminus S'$ which splits in K/k, then S contains more than r places which split in K/k and also satisfies $|S| > r + 1$ and so Remark 2.2.2 implies $\theta^{(r)}_{K/k,S,T}(0) = 0$. It is therefore obvious that $\theta^{(r)}_{K/k,S,T}(0)R(\phi)$ belongs to $\mathbb{Z}[G]$ and annihilates $\mathrm{Cl}(\mathcal{O}_{K,S'})$.

We thus assume that no place in $S\setminus S'$ splits in K/k. We write $\mathrm{Id}_{S\setminus S'}$ for the group of fractional ideals of K generated by ideals corresponding to the (non-archimedean) places in $S_K \setminus S'_K$ and note that there exists an exact sequence of G-modules of the form $\mathrm{Id}_{S\setminus S'} \to \mathrm{Cl}(\mathcal{O}_{K,S'}) \to \mathrm{Cl}(\mathcal{O}_{K,S})$.

Now the decomposition subgroup of every place in $S\setminus S'$ is equal to G and so $\mathrm{Id}_{S\setminus S'}$ is annihilated by $g-1$. Thus, since Theorem 10.1.2 implies $\theta^{(r)}_{K/k,S,T}(0)R(\phi)$ belongs to $\mathrm{Ann}_{\mathbb{Z}[G]}(\mathrm{Cl}(\mathcal{O}_{K,S}))$, the above sequence implies $(g-1)\theta^{(r)}_{K/k,S,T}(0)R(\phi)$ belongs to $\mathrm{Ann}_{\mathbb{Z}[G]}(\mathrm{Cl}(\mathcal{O}_{K,S'}))$, as required. □

Theorem 10.1.2. (Macias Castillo [35]) *If K/k is a quadratic extension, then for every ϕ in $[U_{K,S}, X_{K,S}]_G$ the element $\theta^{(r)}_{K/k,S,T}(0)R(\phi)$ belongs to $\mathrm{Ann}_{\mathbb{Z}[G]}(\mathrm{Cl}(\mathcal{O}_{K,S}))$.*

10.2. In this subsection we fix a field $K := K_{3,5,7,\ell}$ as in Theorem 4.2.1(ii).

In this case $\mathrm{LTC}(K/\mathbb{Q})$ is proved in [18, Th. 4.1] and so Theorem 4.1.1 implies the validity of Conjecture 2.4.1* for $K/\mathbb{Q}$. To complete the proof of Conjecture 2.4.1 for $K/\mathbb{Q}$ we will show that for any a in $\mathcal{A}(\mathbb{Z}[G])$ the element $a\theta^{(r)}_{K/\mathbb{Q},S,T}(0)R(\phi)$ belongs to $\mathbb{Z}[G]$ and annihilates $\mathrm{Cl}(\mathcal{O}_K)$. Our proof of this result depends heavily on results of Chinburg in [22].

We write ψ_2 for the unique element of $\mathrm{Ir}(G)$ of degree two, J for the unique subgroup of G of order two and τ for the unique non-trivial element of J. Then the decomposition subgroups of $3, 5, 7, \ell$ and ∞ are of orders $8, 4, 8, 4$ and 2 respectively (cf. [22, p. 53 and p.73]) and so all of these subgroups contain J. Since J is central and ψ_2 is both irreducible and faithful it follows that if V is a representation of character ψ_2, then the space V^{G_w} vanishes for all v in S. By combining these facts with the formula (5) one finds that $r_S(\psi_2) = 0$ and $r_S(\psi) > 0$ for all $\psi \neq \psi_2$. Since $\check{\psi}_2 = \psi_2$ this implies both that $\theta_{K/\mathbb{Q},S,T}(0) = \epsilon_T L_S(\psi_2, 0)e_{\psi_2}$ and $L_S(\psi_2, 0) = L(\psi_2, 0)$.

Next we recall from [22, Prop. 4.3.7] that $L(\psi_2, 0) = -2^4 m$ for an odd rational integer m. We also note that if $\mathcal{M}$ is any maximal order in $\mathbb{Q}[G]$ that contains $\mathbb{Z}[G]$, then both ae_{ψ_2} and $\epsilon_T e_{\psi_2}$ belong to $\mathfrak{Z}(\mathcal{M})e_{\psi_2} = \mathbb{Z}e_{\psi_2}$, the former by Remark 2.3.2 and the latter because ϵ_T belongs to $\mathrm{Nrd}_{\mathbb{Q}[G]}(\mathbb{Z}[G]) \subset \mathrm{Nrd}_{\mathbb{Q}[G]}(\mathcal{M}) = \mathfrak{Z}(\mathcal{M})$. But $\theta^{(0)}_{K/\mathbb{Q},S,T}(0)R(\phi) = \theta_{K/\mathbb{Q},S,T}(0)$ (by the argument of Proposition 3.5.1) and $e_{\psi_2} = \frac{1}{2}(1-\tau)$ and so $a\theta^{(0)}_{K/\mathbb{Q},S,T}(0)R(\phi) = a\epsilon_T L(\psi_2, 0)e_{\psi_2} = -2^4 m \cdot ae_{\psi_2}\epsilon_T e_{\psi_2}$ belongs to $2^3 \cdot \mathbb{Z}[G]$. Since the 2-primary part of $\mathrm{Cl}(\mathcal{O}_K)$ has exponent 2 (by [22, Prop. 4.2.5]) it is therefore clearly annihilated by $a\theta^{(0)}_{K/\mathbb{Q},S,T}(0)R(\phi)$. On the other hand, the element $a\theta^{(0)}_{K/\mathbb{Q},S,T}(0)R(\phi) = a\theta^{(0)}_{K/\mathbb{Q},S,T}(0)R(\phi)e_{\psi_2}$ annihilates every odd primary component of $\mathrm{Cl}(\mathcal{O}_K)$ as a consequence of Theorem 4.3.1(b)(i) (with $\psi = \psi_2$, $E_\psi = \mathbb{Q}$, $r = 0$, $S' = S_\infty$ and $x = 1$) and Corollary 4.3.3(i).

At this stage we have proved Conjecture 2.4.1 for the extension $K/\mathbb{Q}$ and also observed that $L^*_S(\psi_2, 0) = L_S(\psi_2, 0)$ is strictly negative. This therefore completes the proof of Theorem 4.2.1.

11. The proof of Theorem 4.3.1(i)

In this section we use the notation introduced in §2.6.

11.1. We fix a non-empty subset S' of S which contains $S_\infty \cup S_r$ and define an auxiliary set of places

$$\Sigma := \begin{cases} S' & \text{if } r = 0 \text{ or } S' = S, \\ S' \cup \{v\} & \text{if } r \neq 0, S \neq S' \text{ and } v \text{ is any choice of place in } S \setminus S'. \end{cases}$$

We write $C^\bullet_\Sigma$ for the complex $C^\bullet_{K/k,\Sigma}$ in Proposition 8.1.1. Then, since S_r is a proper subset of Σ, Proposition 8.1.1(iv) implies that there exists a surjective homomorphism of G-modules $\pi_\Sigma : H^1(C^\bullet_\Sigma) \to Y_{K,r}$.

We also fix a character ψ in $\mathrm{Ir}(G)$ and set $E := E_\psi, \mathcal{O} := \mathcal{O}_\psi$ and $\mathrm{pr}_\psi := \psi(1)^{-1}|G|e_\psi = \sum_{g\in G} \psi(g^{-1})g \in \mathcal{O}[G]$.

Proposition 11.1.1. *Theorem 4.3.1(i) is valid if for all ψ in $\mathrm{Ir}(G)$ with $r_S(\psi) = r\psi(1)$ and all ϕ in $[U_{K,S}, X_{K,S}]_G$ one has*

$$\theta^{(r)}_{K/k,S,T}(0)R(\phi)e_\psi \in \mathrm{Ann}_{\mathcal{O}}(\ker(\pi_\Sigma)_\psi)e_\psi. \tag{39}$$

Proof. If $r_S(\psi) \neq r\psi(1)$, then Lemma 2.2.1 implies that $r_S(\check{\psi}) = r_S(\psi) > r\psi(1)$ and so $\theta^{(r)}_{K/k,S,T}(0)e_\psi = L^{r\psi(1)}_{S,T}(\check{\psi},0)e_\psi = 0$ and the containment of Theorem 4.3.1(i) is obvious. We therefore assume that $r_S(\psi) = r\psi(1)$ and that the containment (39) is valid and, by using these assumptions, we must prove the containment of Theorem 4.3.1(i).

As a first step we apply Lemma 11.1.2(i) below to deduce that the element $\Omega_\psi := |G|\theta^{(r)}_{K/k,S,T}(0)R(\phi)\mathrm{pr}_\psi$ belongs to $\mathcal{O} \otimes \mathrm{Ann}_{\mathbb{Z}[G]}(\ker(\pi_\Sigma))$. Thus, since Proposition 8.1.1(iii) implies there is an injective homomorphism of G-modules from $\mathrm{Cl}(\mathcal{O}_{K,\Sigma})$ to $\ker(\pi_\Sigma)$, it follows that Ω_ψ belongs to $\mathcal{O} \otimes \mathrm{Ann}_{\mathbb{Z}[G]}(\mathrm{Cl}(\mathcal{O}_{K,\Sigma}))$.

Now if $r = 0$ or $S' = S$, then $\Sigma = S'$ and so Lemma 11.1.2(ii) below implies that for every y' in $\mathcal{D}^{-1}_{E/\mathbb{Q}}$ the element $\mathrm{tr}_{E/\mathbb{Q}}(y'\Omega_\psi)$ belongs to $\mathrm{Ann}_{\mathbb{Z}[G]}(\mathrm{Cl}(\mathcal{O}_{K,S'}))$. Since $y'\Omega_\psi = yL^{r\psi(1)}_{S,T}(\check{\psi},0)R(\phi)e_\psi$ with $y := \psi(1)^{-1}|G|^2y' \in \psi(1)^{-1}|G|^2\mathcal{D}^{-1}_{E/\mathbb{Q}}$ this implies the containment of Theorem 4.3.1(i) with $x = 1$.

If $r > 0$ and $S' \neq S$, then $\Sigma = S' \cup \{v\}$ for a chosen place v in $S \setminus S'$. Since the group of fractional ideals of K above v is isomorphic as a G-module to $\mathbb{Z}[G/G_w]$ there is an exact sequence of G-modules $\mathbb{Z}[G/G_w] \to \mathrm{Cl}(\mathcal{O}_{K,S'}) \to \mathrm{Cl}(\mathcal{O}_{K,\Sigma})$. This sequence combines with the containment $\Omega_\psi \in \mathcal{O} \otimes \mathrm{Ann}_{\mathbb{Z}[G]}(\mathrm{Cl}(\mathcal{O}_{K,\Sigma}))$ to imply that $x\Omega_\psi \in \mathcal{O} \otimes \mathrm{Ann}_{\mathbb{Z}[G]}(\mathrm{Cl}(\mathcal{O}_{K,S'}))$ for any $x \in \mathrm{Ann}_{\mathbb{Z}[G]}(\mathbb{Z}[G/G_w])$. Then, by applying Lemma 11.1.2(ii) just as above, this implies the containment of Theorem 4.3.1(i). $\square$

Lemma 11.1.2. *Let M be a finitely generated left G-module.*

(i) *For any y_ψ in $\mathrm{Ann}_{\mathcal{O}}(M_\psi)e_\psi$ the element $\psi(1)^{-1}|G|^2y_\psi = |G|y_\psi\mathrm{pr}_\psi$ belongs to $\mathcal{O} \otimes \mathrm{Ann}_{\mathbb{Z}[G]}(M)$.*

(ii) *If x is any element of E with $x\mathrm{pr}_\psi \in \mathcal{O} \otimes_{\mathbb{Z}} \mathrm{Ann}_{\mathbb{Z}[G]}(M)$, then for every y in $\mathcal{D}^{-1}_{E/\mathbb{Q}}$ one has $\mathrm{tr}_{E/\mathbb{Q}}(yx\mathrm{pr}_\psi) \in \mathrm{Ann}_{\mathbb{Z}[G]}(M)$.*

Proof. The action of $\sum_{g\in G} g$ on $M[\psi]$ induces a homomorphism of $\mathcal{O}$-modules $M_\psi \to M^\psi$ whose cokernel is equal to $\hat{H}^0(G, M[\psi])$. Since the abelian group $\hat{H}^0(G, M[\psi])$ is annihilated by $|G|$ (by [2, §6, Cor. 1]) this implies that if y_ψ belongs to $\mathrm{Ann}_{\mathcal{O}}(M_\psi)e_\psi$, then $|G|y_\psi$ belongs to $\mathrm{Ann}_{\mathcal{O}}(M^\psi)e_\psi$. To prove claim (i) it is thus enough to show that if x is any element of $\mathrm{Ann}_{\mathcal{O}}(M^\psi)e_\psi$, then the element $x\mathrm{pr}_\psi$ belongs to $\mathcal{O}\otimes\mathrm{Ann}_{\mathbb{Z}[G]}(M)$.

To do this we set $n := \psi(1)$. By enlarging E (and $\mathcal{O}$) if necessary, we can assume that T_ψ is a free $\mathcal{O}$-module of rank n. We then fix an $\mathcal{O}$-basis $\{t_i : i \in \langle n\rangle\}$ of T_ψ and write $\rho_\psi : G \to \mathrm{GL}_n(E)$ for the associated representation. For each m in M and each index i the element $T_i(m) := \sum_{g\in G} g(t_i \otimes m)$ belongs to M^ψ. Now in $\mathcal{O} \otimes M$ we have

$$T_i(m) = \sum_{g\in G} t_i g^{-1} \otimes g(m) = \sum_{g\in G}\sum_{j=1}^{j=n} \rho_\psi(g^{-1})_{ij} t_j \otimes g(m) = \sum_{j=1}^{j=n} t_j \otimes \left(\sum_{g\in G} \rho_\psi(g^{-1})_{ij} g(m)\right).$$

But, if we write x' for the element of $\mathrm{Ann}_{\mathcal{O}}(M^\psi)$ that is defined by the equality $x = x'e_\psi$, then x' annihilates $T_i(m) \in M^\psi$ and so, since $\{t_i : i \in \langle n\rangle\}$ is an $\mathcal{O}$-basis of T_ψ, the above equation implies that $x'\sum_{g\in G}\rho_\psi(g^{-1})_{ij}g(m)=0$ for all i and j. Each element $c(x')_{ij}:=x'\sum_{g\in G}\rho_\psi(g^{-1})_{ij}g$ therefore belongs to $\mathrm{Ann}_{\mathcal{O}[G]}(\mathcal{O}\otimes M)$. In particular, the element

$$\sum_{i=1}^{i=n}c(x')_{ii}=x'\sum_{g\in G}\left(\sum_{i=1}^{i=n}\rho_\psi(g^{-1})_{ii}\right)g=x'\sum_{g\in G}\psi(g^{-1})g=x'\mathrm{pr}_\psi=x\mathrm{pr}_\psi$$

belongs to $\mathrm{Ann}_{\mathcal{O}[G]}(\mathcal{O}\otimes M)=\mathcal{O}\otimes\mathrm{Ann}_{\mathbb{Z}[G]}(M)$, as required to prove claim (i).

The hypotheses of claim (ii) imply $yx\mathrm{pr}_\psi$ belongs to $\mathcal{D}_{E/\mathbb{Q}}^{-1}\otimes\mathrm{Ann}_{\mathbb{Z}[G]}(M)$. The element $\mathrm{tr}_{E/\mathbb{Q}}(yx\mathrm{pr}_\psi)$ therefore belongs to $\mathrm{tr}_{E/\mathbb{Q}}(\mathcal{D}_{E/\mathbb{Q}}^{-1})\otimes\mathrm{Ann}_{\mathbb{Z}[G]}(M)\subseteq\mathrm{Ann}_{\mathbb{Z}[G]}(M)$, as required. □

11.2. To prove Theorem 4.3.1(i) we are reduced to proving the containment (39). To do this we first choose a free G-submodule F of $H^1(C^\bullet_\Sigma)$ for which the given surjection $\pi_\Sigma:H^1(C^\bullet_\Sigma)\to Y_{K,r}$ restricts to give an isomorphism $F\cong Y_{K,r}$. We then choose a free G-submodule $\mathcal{E}$ of $U_{K,\Sigma}$ that has rank r and satisfies $R_{K,\Sigma}(\mathcal{E})\subset\mathbb{R}\otimes F$ (the existence of such a module $\mathcal{E}$ is guaranteed by the isomorphism of Proposition 8.1.1(ii)) and consider the following exact commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & U_{K,\Sigma}/\mathcal{E} \arrow[r] & \mathrm{cok}(\iota) \arrow[r, "d''"] & \mathrm{cok}(\epsilon) \arrow[r] & \ker(\pi_\Sigma) \arrow[r] & 0 \\
0 \arrow[r] & U_{K,\Sigma} \arrow[r, "c'"] \arrow[u] & A \arrow[r, "d'"] \arrow[u] & B/\operatorname{im}(\delta') \arrow[r, "e'"] \arrow[u] & H^1(C^\bullet_\Sigma) \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & \mathcal{E} \arrow[r, equal] \arrow[u, "\subseteq"'] & \mathcal{E} \arrow[r, "0"] \arrow[u, "\iota"'] & F \arrow[r, equal] \arrow[u, "\epsilon"'] & F \arrow[r] \arrow[u, "\subseteq"'] & 0.
\end{tikzcd}$$

In this diagram the second row is the first row of (38), ι is the restriction of c' to $\mathcal{E}$, ϵ is a lift of the inclusion $F\to H^1(C^\bullet_\Sigma)$ through the surjection e', the first three columns are the tautological short exact sequences and the fourth column is the obvious short exact sequence obtained by noting that our choice of F gives a direct sum decomposition of G-modules $H^1(C^\bullet_\Sigma)=F\oplus\ker(\pi_\Sigma)$. We then take all homomorphisms in the first row of the diagram to be those that are induced by the fact that all lower squares in the diagram commute.

Writing $C^\bullet$ and $C^\bullet_{\mathcal{E},F}$ for the complexes $\mathrm{cok}(\iota)\xrightarrow{d''}\mathrm{cok}(\epsilon)$ and $\mathcal{E}\xrightarrow{0}F$, where in both cases the first term is placed in degree 0, the above diagram combines with Lemma A.1.2 to imply that in $K_0(\mathbb{Z}[G],\mathbb{R}[G])$ one has an equality

$$\chi(C^\bullet_\Sigma,R_{K,\Sigma})=\chi(C^\bullet_{\mathcal{E},F},R'_{K,\Sigma})+\chi(C^\bullet,R''_{K,\Sigma})$$

where $R'_{K,\Sigma}$ is the isomorphism $\mathbb{R}\otimes H^0(C^\bullet_{\mathcal{E},F})=\mathbb{R}\otimes\mathcal{E}\to\mathbb{R}\otimes F=\mathbb{R}\otimes H^1(C^\bullet_{\mathcal{E},F})$ obtained by restricting $R_{K,\Sigma}$ and $R''_{K,\Sigma}$ the isomorphism $\mathbb{R}\otimes H^0(C^\bullet)=\mathbb{R}\otimes U_{K,\Sigma}/\mathcal{E}\cong$

$\mathbb{R}\otimes\ker(\pi_\Sigma)=\mathbb{R}\otimes H^1(C^\bullet)$ induced by $R_{K,\Sigma}$. We now fix an isomorphism of (free) G-modules $\kappa: F\to\mathcal{E}$. Then

$$\chi(C^\bullet_{\mathcal{E},F},R'_{K,\Sigma})=\chi(C^\bullet_{\mathcal{E},F},R'_{K,\Sigma})-\chi(C^\bullet_{\mathcal{E},F},(\mathbb{R}\otimes\kappa)^{-1})=\delta_G(\mathrm{Nrd}_{\mathbb{R}[G]}(R'_{K,\Sigma}\circ(\mathbb{R}\otimes\kappa)))$$

where the first equality is because $\chi(C^\bullet_{\mathcal{E},F},(\mathbb{R}\otimes\kappa)^{-1})$ vanishes as a consequence of Lemma A.1.1(iii) and the second follows from Lemma A.1.1(ii). By using the equality of Proposition 8.1.1(vii) one therefore has

$$\begin{aligned}\chi(C^\bullet,R''_{K,\Sigma})&=\chi(C^\bullet_\Sigma,R_{K,\Sigma})-\delta_G(\mathrm{Nrd}_{\mathbb{R}[G]}(R'_{K,\Sigma}\circ(\mathbb{R}\otimes\kappa)))\\ &=\delta_G(\theta^*_{K/k,S,\Sigma}\mathrm{Nrd}_{\mathbb{R}[G]}(R'_{K,\Sigma}\circ(\mathbb{R}\otimes\kappa))^{-1})-T\Omega(\mathbb{Q}(0)_K,\mathbb{Z}[G])\end{aligned}\tag{40}$$

where $T\Omega(\mathbb{Q}(0)_K,\mathbb{Z}[G])$ is the element defined in Remark 6.1.1(ii).

Following Proposition 11.1.1 we assume in the sequel that $r_S(\psi)=r\psi(1)$ and hence, by (5), that the inclusions $\mathbb{C}\otimes_{\mathcal{O}_\psi}\mathcal{E}^\psi\subseteq\mathbb{C}\otimes_{\mathcal{O}_\psi}U^\psi_{K,\Sigma}\subseteq\mathbb{C}\otimes_{\mathcal{O}_\psi}U^\psi_{K,S}$ are bijective. This implies $U^\psi_{K,\Sigma}=U^\psi_{K,S}$ and thus $H^0(C^\bullet)^\psi=(U_{K,\Sigma}/\mathcal{E})^\psi=U^\psi_{K,\Sigma}/\mathcal{E}^\psi=U^\psi_{K,S}/\mathcal{E}^\psi$ where the second equality is valid because $\mathcal{E}$ is a free G-module. It also implies that $H^0(C^\bullet)^\psi$, and hence also $H^1(C^\bullet)_\psi=\ker(\pi_\Sigma)_\psi$, is finite.

We also assume that the homomorphism $\phi^\psi: U^\psi_{K,S}\to X^\psi_{K,S}$ induced by ϕ has finite kernel since if this is not true, then $R(\phi)e_\psi=0$ and so the containment (39) is obvious. We further assume that K/k is an extension of function fields or ψ validates the Strong Stark Conjecture. In this case the element $T\Omega(\mathbb{Q}(0)_K,\mathbb{Z}[G])$ belongs to the kernel of the homomorphism ι_ψ in Lemma A.1.3 (this follows from Remark 6.1.1(iii) and the fact that the Strong Stark Conjecture is proved for function fields by Bae in [3]). By combining Lemma A.1.3, with $d(C^\bullet,\psi)=0$, together with (40) and the explicit descriptions of $H^0(C^\bullet)^\psi$ and $H^1(C^\bullet)_\psi$ given above we thus obtain an equality of invertible $\mathcal{O}$-modules

$$\mathcal{O}\cdot e_\psi\theta^*_{K/k,S,\Sigma}\mathrm{Nrd}_{\mathbb{R}[G]}(R'_{K,\Sigma}\circ(\mathbb{R}\otimes\kappa))^{-1}=\mathrm{Fit}_{\mathcal{O}}(U^\psi_{K,S}/\mathcal{E}^\psi)^{-1}\mathrm{Fit}_{\mathcal{O}}(\ker(\pi_\Sigma)_\psi)e_\psi.$$

Next we note that our choice of Σ ensures that if $|S|=r+1$, then $\Sigma=S$ and so Proposition 8.1.1(vi) implies that

$$e_\psi\theta^*_{K/k,S,\Sigma}=L_S^{r\psi(1)}(\check{\psi},0)e_\psi=\epsilon_T^{-1}\theta^{(r)}_{K/k,S,T}(0)e_\psi.$$

In addition, one has

$$\begin{aligned}\mathrm{Nrd}_{\mathbb{R}[G]}(R'_{K,\Sigma}\circ(\mathbb{R}\otimes\kappa))R(\phi)e_\psi=&e_\psi\mathrm{Nrd}_{\mathbb{R}[G]}(R'_{K,\Sigma}\circ(\mathbb{R}\otimes\kappa))\mathrm{Nrd}_{\mathbb{R}[G]}(R^{-1}_{K,S}\circ(\mathbb{R}\otimes\phi))\\ =&e_\psi\mathrm{Nrd}_{\mathbb{R}[G]}(\mathbb{R}\otimes(\phi\circ\kappa))\end{aligned}$$

where in the last expression we regard κ as a map from F to $U_{K,S}$. In addition, our assumptions imply that the composite $(\phi\circ\kappa)^\psi=\phi^\psi\circ\kappa^\psi$ is injective and so

$$\mathcal{O}\cdot e_\psi\mathrm{Nrd}_{\mathbb{R}[G]}(\mathbb{R}\otimes(\phi\circ\kappa))=\mathcal{O}\cdot\det_{\mathbb{C}}((\phi\circ\kappa)^\psi)e_\psi=\mathrm{Fit}_{\mathcal{O}}(\mathrm{cok}((\phi\circ\kappa)^\psi))e_\psi.$$

Now $\mathrm{cok}(\kappa^\psi)=U^\psi_{K,S}/\mathcal{E}^\psi$ and, as $\ker(\phi^\psi)$ is finite and $X^\psi_{K,S}$ is torsion free, also $\ker(\phi^\psi)=(U^\psi_{K,S})_{\mathrm{tor}}=\mu^\psi_K$. The kernel-cokernel sequence of $\phi^\psi\circ\kappa^\psi$ thus gives an exact sequence of finite $\mathcal{O}$-modules $0 \to \mu_K^\psi \to U_{K,S}^\psi/\mathcal{E}^\psi \to \mathrm{cok}((\phi\circ\kappa)^\psi) \to \mathrm{cok}(\phi^\psi) \to 0$ and hence an equality

$$\mathrm{Fit}_{\mathcal{O}}(\mathrm{cok}((\phi\circ\kappa)^\psi)) = \mathrm{Fit}_{\mathcal{O}}(\mu_K^\psi)^{-1}\mathrm{Fit}_{\mathcal{O}}(\mathrm{cok}(\phi^\psi))\mathrm{Fit}_{\mathcal{O}}(U_{K,S}^\psi/\mathcal{E}^\psi).$$

By combining the last five displayed formulas one finds that

$$\begin{aligned}
&\mathrm{Fit}_{\mathcal{O}}(\mu_K^\psi)\cdot \epsilon_T^{-1}\theta_{K/k,S,T}^{(r)}(0)R(\phi)e_\psi\\
=&\mathrm{Fit}_{\mathcal{O}}(\mathrm{cok}((\theta\circ\kappa)^\psi))\cdot e_\psi\theta^*_{K/k,S,\Sigma}\mathrm{Nrd}_{\mathbb{R}[G]}(R'_{K,\Sigma}\circ(\mathbb{R}\otimes\kappa))^{-1}\\
=&\mathrm{Fit}_{\mathcal{O}}(\ker(\pi_\Sigma)_\psi)\mathrm{Fit}_{\mathcal{O}}(\mathrm{cok}(\phi^\psi))e_\psi\\
\subseteq&\mathrm{Ann}_{\mathcal{O}}(\ker(\pi_\Sigma)_\psi)e_\psi.
\end{aligned}$$

The required containment (39) is therefore a consequence of the following result.

Lemma 11.2.1. *The complex number* $\epsilon_{T,\psi}$ *that is defined by the equality* $\epsilon_T e_\psi = \epsilon_{T,\psi}e_\psi$ *belongs to* $\mathrm{Fit}_{\mathcal{O}}(\mu_K^\psi)$.

Proof. The sequence (23) induces an exact sequence of $\mathcal{O}$-modules

$$0 \to \bigoplus_{t\in T}\mathbb{Z}[G]^\psi \xrightarrow{(1-\mathrm{N}\,t\cdot\mathrm{Fr}_{\mathfrak{t}}^{-1})_t^\psi} \bigoplus_{t\in T}\mathbb{Z}[G]^\psi \to \mathbb{F}_T^{\times,\psi} \to 0.$$

This implies that $\epsilon_{T,\psi} = \prod_{t\in T}\det_{\mathbb{C}}((1-\mathrm{N}\,t\cdot\mathrm{Fr}_{\mathfrak{t}}^{-1})_t^\psi)$ belongs to $\mathrm{Fit}_{\mathcal{O}}(\mathbb{F}_T^{\times,\psi})$. Now, since $U_{K,S,T}$ is assumed to be torsion-free, there is an injective homomorphism of G-modules $\mu_K \to \mathbb{F}_T^\times$. This induces an injective homomorphism of $\mathcal{O}$-modules $\mu_K^\psi \to \mathbb{F}_T^{\times,\psi}$ which in turn implies $\mathrm{Fit}_{\mathcal{O}}(\mathbb{F}_T^{\times,\psi}) \subseteq \mathrm{Fit}_{\mathcal{O}}(\mu_K^\psi)$ and hence that $\epsilon_{T,\psi}$ belongs to $\mathrm{Fit}_{\mathcal{O}}(\mu_K^\psi)$, as required. $\square$

12. The proof of Theorem 4.3.1(ii)

In this section we prove Theorem 4.3.1(ii) and Proposition 2.6.2 and also show that Conjecture 2.6.1 predicts that there are explicit links between Stark units and the structure of ideal class groups.

To do this we fix a finite extension of global fields K/k, a finite non-empty set of places S of k that contains S_∞ and all places which ramify in K/k and a finite set of places T of k that is disjoint from S and such that $U_{K,S,T}$ is torsion-free. We often suppress explicit reference to K, preferring to abbreviate notation such as $U_{K,\Sigma,T}$ or $\mathrm{R}_{K,\Sigma}$ to $U_{\Sigma,T}$ and R_Σ respectively. For any module M we write $\overline{M}$ for the quotient of M by its ($\mathbb{Z}$-)torsion submodule M_{tor}.

We fix a non-trivial irreducible complex character ψ of G and set $\mathcal{O} := \mathcal{O}_\psi$, $E := E_\psi$ and $r := r_S(\psi)$.

12.1. In this subsection we prove Theorem 4.3.1(ii). To do this we must show that the containments (10) and (11) are both valid if either K/k is an extension of function fields or the Strong Stark Conjecture is valid for ψ.

As a first step we interpret the conjectural containment (10) in terms of the central conjecture of [12]. To do this we write $\mathrm{R}_S^{(\psi)} : \mathbb{C}\otimes_{\mathcal{O}} \wedge^r_{\mathcal{O}} U^{\psi}_{S,T} \xrightarrow{\sim} \mathbb{C}\otimes_{\mathcal{O}} \wedge^r_{\mathcal{O}} X^{\psi}_S$ for the isomorphism induced by R_S.

Lemma 12.1.1. *The containment (10) is valid for all ρ in $[U^{\psi}_{S,T}, X_{S,\psi}]_{\mathcal{O}_\psi}$ if and only if ψ validates* [12, Conj. 2.1].

Proof. For each element x of $\mathbb{C}\otimes_{\mathcal{O}} \wedge^r_{\mathcal{O}} X^{\psi}_S$ we set

$$u_x := |G|^r L^r_{S,T}(\check{\psi}, 0)\cdot \wedge^r_{\mathbb{C}}(\mathrm{R}_S^{(\psi)})^{-1}(x) \in \mathbb{C}\otimes_{\mathcal{O}} \wedge^r_{\mathcal{O}} U^{\psi}_{S,T}.$$

Then

$$\begin{aligned}(41)\quad & |G|^r L^r_{S,T}(\check{\psi},0)R(\rho)x \\ &= |G|^r L^r_{S,T}(\check{\psi},0)((\mathbb{C}\otimes_{\mathcal{O}} \wedge^r_{\mathcal{O}}\rho)\circ \wedge^r_{\mathbb{C}}(\mathrm{R}_S^{(\psi)})^{-1})(x) = (\mathbb{C}\otimes_{\mathcal{O}} \wedge^r_{\mathcal{O}}\rho)(u_x).\end{aligned}$$

Now [12, Conj. 2.1] asserts that for each x in $\wedge^r_{\mathcal{O}}\overline{X_{S,\psi}} \subset \mathbb{C}\otimes_{\mathcal{O}} \wedge^r_{\mathcal{O}} X^{\psi}_S$ the element u_x belongs to $\mathrm{Fit}_{\mathcal{O}}(\hat{H}^{-1}(G, X_S[\psi]))\cdot \wedge^r_{\mathcal{O}} U^{\psi}_{S,T}$. If this is true, then $(\wedge^r_{\mathcal{O}}\rho)(u_x)$ belongs to $\mathrm{Fit}_{\mathcal{O}}(\hat{H}^{-1}(G, X_S[\psi]))\cdot \wedge^r_{\mathcal{O}}\overline{X_{S,\psi}}$ and so (41) implies that $|G|^r L^r_{S,T}(\check{\psi},0)R(\rho)x$ belongs to $\mathrm{Fit}_{\mathcal{O}}(\hat{H}^{-1}(G, X_S[\psi]))\cdot \wedge^r_{\mathcal{O}}\overline{X_{S,\psi}}$. Since this should be true for any element x of the projective $\mathcal{O}$-module $\wedge^r_{\mathcal{O}}\overline{X_{S,\psi}}$ this implies that $|G|^r L^r_{S,T}(\check{\psi},0)R(\rho)$ belongs to $\mathrm{Fit}_{\mathcal{O}}(\hat{H}^{-1}(G, X_S[\psi]))$, as claimed by (10).

Conversely, if (10) is valid, then (41) implies that for every ρ in $[U^{\psi}_{S,T}, X_{S,\psi}]_{\mathcal{O}_\psi}$ and every x in $\wedge^r_{\mathcal{O}}\overline{X_{S,\psi}}$ one has

$$(\wedge^r_{\mathcal{O}}\rho)(u_x) \in \mathrm{Fit}_{\mathcal{O}}(\hat{H}^{-1}(G, X_S[\psi]))\cdot \wedge^r_{\mathcal{O}}\overline{X_{S,\psi}}.$$

Now since $U^{\psi}_{S,T}$ and $\overline{X_{S,\psi}}$ are both projective $\mathcal{O}$-modules of rank r we can choose ρ in such a way that the cokernel of the induced map $U^{\psi}_{S,T} \to \overline{X_{S,\psi}}$ is coprime to any given integer. The last displayed containment therefore implies that u_x belongs to $\mathrm{Fit}_{\mathcal{O}}(\hat{H}^{-1}(G, X_S[\psi]))\cdot \wedge^r_{\mathcal{O}} U^{\psi}_{S,T}$, as predicted by [12, Conj. 2.1]. □

If K/k is an extension of function fields or the Strong Stark Conjecture is valid for ψ, then [12, Th. 4.1] proves that [12, Conj. 2.1] is valid. Lemma 12.1.1 therefore implies that (10) is also valid in these cases.

We now consider the second containment (11) of Conjecture 2.6.1. As observed in §11.2, if K/k is an extension of function fields or ψ validates the Strong Stark Conjecture, then the element $T\Omega(\mathbb{Q}(0)_K, \mathbb{Z}[G])$ defined in Remark 6.1.1(ii) belongs to the kernel of the homomorphism ι_ψ in Lemma A.1.3. From Proposition 8.1.1(vii) with $\Sigma = S'$ one therefore has

$$\iota_\psi(\delta_G(\theta^*_{K/k,S,S'})) = \iota_\psi(\chi(C^\bullet_{K/k,S'}, \mathrm{R}_{S'})). \tag{42}$$

To study this equality we set $C^\bullet := C^\bullet_{K/k,S'}$ and $r' := r_{S'}(\psi)$. We note that if $r' < r$, then $L^{r'}_S(\check{\psi}, 0) = 0$ and so the required containment (11) is obvious. We shall therefore assume in the rest of this section that $r' = r$.

Lemma 12.1.2.

(i) $H^0(C^\bullet)^\psi = U^\psi_{S'}$ *and* $\overline{X_{S',\psi}} \subseteq \overline{H^1(C^\bullet)_\psi}$.
(ii) $|G|\mathrm{Ann}_\mathcal{O}(H^1(C^\bullet)_{\psi,\mathrm{tor}}) \subseteq \mathrm{Ann}_\mathcal{O}(\mathrm{Cl}(\mathcal{O}_{K,S'})_\psi)$.
(iii) $e_\psi\theta^*_{K/k,S,S'} = e_\psi L^{r'}_S(\check{\psi}, 0)$.

Proof. The equality $H^0(C^\bullet)^\psi = U^\psi_{S'}$ follows immediately from Proposition 8.1.1(ii). Next we note that the sequence (36) (with $\Sigma = S'$) gives an inclusion $X_{S'} \subseteq \overline{\nabla}$. This in turn induces an inclusion $\overline{X_{S',\psi}} \subseteq \overline{H^1(C^\bullet)_\psi}$ because $H^1(C^\bullet) = \nabla/\operatorname{im}(\delta)$ and $\operatorname{im}(\delta)$ is disjoint from $X_{S'}$ in $\overline{\nabla}$. This proves claim (i).

Proposition 8.1.1(iii) gives rise to an exact sequence of G-modules of the form $0 \to \mathrm{Cl}(\mathcal{O}_{K,S'}) \to H^1(C^\bullet) \to M \to 0$ and hence to an exact sequence of $\mathcal{O}$-modules $\mathrm{Tor}^1_\mathcal{O}(T_\psi, M) \to \mathrm{Cl}(\mathcal{O}_{K,S'})_\psi \to H^1(C^\bullet)_{\psi,\mathrm{tor}}$. The inclusion of claim (ii) therefore follows because the $\mathcal{O}$-module $\mathrm{Tor}^1_\mathcal{O}(T_\psi, M)$ is isomorphic to the Tate cohomology group $\hat{H}^{-2}(G, M[\psi])$ and so is annihilated by multiplication by $|G|$.

Since $\psi \neq 1_G$ and (by assumption) $r' = r$ the formula of claim (iii) follows immediately from Lemma 8.3.2 with $\Sigma = S'$. □

Now Lemma 11.2.1 implies $L^{r'}_{S,T}(\check{\psi}, 0) = \epsilon_{T,\psi}L^{r'}_S(\check{\psi}, 0)$ belongs to $L^{r'}_S(\check{\psi}, 0)\cdot\mathrm{Fit}_\mathcal{O}(\mu^\psi_K)$. By combining the observations of Lemma 12.1.2 with the formulas of Lemma A.1.3 and the equality (42) one therefore obtains the following inclusions

$$\begin{aligned}
|G|^{1+r'}L^{r'}_{S,T}(\check{\psi}, 0)\cdot\wedge^{r'}_\mathcal{O}(\overline{X_{S',\psi}}) &\subseteq |G|^{1+r'}\mathrm{Fit}_\mathcal{O}(\mu^\psi_K)L^{r'}_S(\check{\psi}, 0)\cdot\wedge^{r'}_\mathcal{O}(\overline{X_{S',\psi}})\\
&\subseteq |G|^{1+r'}\mathrm{Fit}_\mathcal{O}(\mu^\psi_K)L^{r'}_S(\check{\psi}, 0)\cdot\wedge^{r'}_\mathcal{O}(\overline{H^1(C^\bullet)_\psi})\\
&= |G|\mathrm{Fit}_\mathcal{O}(H^1(C^\bullet)_{\psi,\mathrm{tor}})\cdot\wedge^{r'}_\mathbb{C}((\mathbb{C}\otimes_\mathbb{R}\mathrm{R}_{S'})(\overline{U^\psi_{S'}}))\\
&\subseteq \mathrm{Ann}_\mathcal{O}(\mathrm{Cl}(\mathcal{O}_{K,S'})_\psi)\cdot\wedge^{r'}_\mathbb{C}((\mathbb{C}\otimes_\mathbb{R}\mathrm{R}_{S'})(\overline{U^\psi_{S'}})).
\end{aligned} \tag{43}$$

To prove (11) we fix ρ in $[U^\psi_{S'}, X_{S',\psi}]_{\mathcal{O}_\psi}$. Then the argument of Lemma 12.1.1 (with S replaced by S') combines with (43) to imply $\mathcal{L} := |G|^{1+r'}L^{r'}_{S,T}(\check{\psi}, 0)R(\rho)$ belongs to $\mathrm{Ann}_\mathcal{O}(\mathrm{Cl}(\mathcal{O}_{K,S'})_\psi)$. Lemma 11.1.2 then implies $\mathrm{tr}_{E/\mathbb{Q}}(|G|\mathcal{L}\mathrm{pr}_\psi)$ belongs to $\mathrm{Ann}_{\mathbb{Z}[G]}(\mathrm{Cl}(\mathcal{O}_{K,S'}))$ and this implies (11) as $|G|\mathcal{L}\mathrm{pr}_\psi = \psi(1)^{-1}|G|^{3+r}L^{r'}_{S,T}(\check{\psi}, 0)R(\rho)e_\psi$. This completes the proof of Theorem 4.3.1(ii).

12.2. In the next result we explain the connection between Conjecture 2.6.1 and the 'Stark units' that are predicted to exist by Chinburg in [22]. To do this we note that if $\psi \neq 1_G$ and satisfies $r_S(\psi) = 1$, then the formula (5) implies that there is a unique place v_1 in S for which the space $V_\psi^{G_{w_1}}$ does not vanish, where G_{w_1} is the decomposition subgroup in G of any fixed place w_1 of K above v_1.

We continue to abbreviate $\mathcal{O}_\psi$ and E_ψ to $\mathcal{O}$ and E respectively.

Proposition 12.2.1. *Assume that k is a number field, ψ has degree two, $r_S(\psi) = 1$, $|S| > 1$ and the unique place v_1 in S for which $V_\psi^{G_{w_1}}$ does not vanish is archimedean. Then the group G_{w_1} has order two.*

Fix an embedding of K in $\mathbb{C}$ that corresponds to w_1 and use this to regard K as a subfield of $\mathbb{C}$. If ψ validates Conjecture 2.6.1, then for every element d of the fractional $\mathcal{O}$-ideal $\mathrm{Fit}_\mathcal{O}(\hat{H}^{-1}(G, X_S[\psi]))^{-1}\mathcal{D}_{E/\mathbb{Q}}^{-1}$ the element

$$\epsilon(d) := \exp(-\sum_{\gamma \in G_{E/\mathbb{Q}}} \gamma(d) L'_{S,T}(\psi^\gamma, 0))$$

is a real unit in K which has all of the following properties:-

(i) *either $\epsilon(d)$ or $-\epsilon(d)$ is congruent to 1 modulo all of the places in T_K;*
(ii) *$|\epsilon(d)|_w = 1$ if w is any place of K that does not lie above v_1;*
(iii) *for each $g \in G$ one has*

$$-\log|g^{-1}(\epsilon(d))|_{w_1} = \sum_{\gamma \in G_{E/\mathbb{Q}}} \gamma(d)(\psi^\gamma(g) + \psi^\gamma(g\tau))L'_{S,T}(\psi^\gamma, 0)$$

where τ is the unique non-trivial element of G_{w_1};
(iv) *if d belongs to $\mathcal{O}$, then for every homomorphism ϕ in $[U_K, \mathbb{Z}[G]]_G$ the element $2^{-2}|G|^4\phi(\epsilon(d))$ belongs to $\mathbb{Z}[G]$ and annihilates $\mathrm{Cl}(\mathcal{O}_K)$.*

Proof. Since ψ has degree two and $r_S(\psi) = 1$ the formula (5) implies $\dim_\mathbb{C}(V_\psi^{G_{w_1}}) = 1 < \dim_\mathbb{C}(V_\psi)$ and so the group G_{w_1} is non-trivial. Hence, since w_1 is archimedean, the order of G_{w_1} is equal to two.

We assume henceforth that ψ validates Conjecture 2.6.1. Then [12, Conj. 2.1] is valid (by Lemma 12.1.1) and so [12, Prop. 3.3] implies that the element $\epsilon(d)$ defined above satisfies the conditions (i), (ii) and (iii).

To prove claim (iv) we first decompose e_ψ as a sum of (non-zero) indecomposable idempotents $\sum_{j=1}^{j=n} f_j$ of $E[G]$ with f_1 equal to the idempotent f_ψ fixed in §2.6 when defining the lattice $T_\psi = f_\psi\mathcal{M}_\psi$. We set $S' := S_\infty$ so $U_{K,S'} = U_K$ and $\mathcal{O}_{K,S'} = \mathcal{O}_K$. Then, as $\psi \neq 1_G$, $\mathrm{pr}_\psi(w_1)$ belongs to $\mathcal{O} \otimes X_{S'}$ and so, as d belongs to $\mathcal{O}$, the element $x_1 := 2^{-1}df_1(w_1)$ satisfies $|G|x_1 = df_1(2^{-1}|G|e_\psi(w_1)) = df_1(\mathrm{pr}_\psi(w_1)) \in \overline{X_{S',\psi}}$. The inclusion (43) therefore implies that the element u_{x_1} defined just before (41) is such that $|G|^2u_{x_1} = |G|^2L'_{S,T}(\check{\psi}, 0)\mathrm{R}_S^{-1}(2^{-1}|G|df_1(w_1))$ belongs to $\mathrm{Ann}_\mathcal{O}(\mathrm{Cl}(\mathcal{O}_K)_\psi) \cdot \overline{U_K^\psi}$.

But if one regards ϕ as an element of $[E\otimes U_K, E[G]]_{E[G]}$, then $\phi(\overline{U_K^\psi})\subseteq \mathbb{Z}[G]^\psi$ and so

$$|G|^2\phi(u_{x_1})\in \mathrm{Ann}_{\mathcal{O}}(\mathrm{Cl}(\mathcal{O}_K)_\psi)\cdot\mathbb{Z}[G]^\psi.$$

Now $\mathbb{Z}[G]^\psi = |G|T_\psi$ whilst $T_\psi\subseteq e_\psi\mathcal{M}_\psi, 2^{-1}|G|e_\psi = \mathrm{pr}_\psi$ and $2^{-1}|G|\mathcal{M}_\psi\subseteq\mathcal{O}[G]$ (the latter by [32]) and so $2^{-2}|G|\mathbb{Z}[G]^\psi\subseteq 2^{-1}|G|\mathcal{M}_\psi\cdot 2^{-1}|G|e_\psi\subseteq\mathcal{O}[G]\cdot\mathrm{pr}_\psi$. The last displayed containment therefore implies that $2^{-2}|G|^3\phi(u_{x_1})$ belongs to the lattice $\mathrm{Ann}_{\mathcal{O}}(\mathrm{Cl}(\mathcal{O}_K)_\psi)\cdot\mathcal{O}[G]\mathrm{pr}_\psi$. From Lemma 11.1.2(i) it then follows that $2^{-2}|G|^4\phi(u_{x_1})$ belongs to $\mathcal{O}\otimes\mathrm{Ann}_{\mathbb{Z}[G]}(\mathrm{Cl}(\mathcal{O}_K))$.

We next set $\Gamma := G_{E/\mathbb{Q}}$ and for each integer j with $1\le j\le n$ and each $\gamma\in\Gamma$ also $x_{j,\gamma} := 2^{-1}\gamma(df_j)(w_1)$. Then by using $\gamma(f_j)$ as the idempotent that occurs in the definition of T_{ψ^γ} in §2.6 the same argument as above shows that the element $2^{-2}|G|^4\phi(u_{x_{j,\gamma}})$ belongs to $\mathcal{O}\otimes\mathrm{Ann}_{\mathbb{Z}[G]}(\mathrm{Cl}(\mathcal{O}_K))$. Note also that $u_{x_{j,\gamma}}$ coincides with the element $\eta(\gamma(d),\check{\psi},j)$ defined in [12] (at least provided one sets $f_{\psi,j} := \gamma(f_j)$ in the notation of loc. cit.). The proof of [12, Prop. 3.2 and Prop. 3.3] thus shows that $\pm\epsilon(d)$ is equal to the sum $\sum_{\gamma\in\Gamma}\sum_{1\le j\le n}u_{x_{j,\gamma}}$ and hence that $2^{-2}|G|^4\phi(\epsilon(d)) = \pm\sum_{\gamma\in\Gamma}\sum_{1\le j\le n}2^{-2}|G|^4\phi(u_{x_{j,\gamma}})\in\mathcal{O}\otimes\mathrm{Ann}_{\mathbb{Z}[G]}(\mathrm{Cl}(\mathcal{O}_K))$. Since $2^{-2}|G|^4\phi(\epsilon(d))$ also belongs to $\mathbb{Q}[G]$ and $\mathbb{Q}[G]\cap(\mathcal{O}\otimes\mathrm{Ann}_{\mathbb{Z}[G]}(\mathrm{Cl}(\mathcal{O}_K))) = \mathrm{Ann}_{\mathbb{Z}[G]}(\mathrm{Cl}(\mathcal{O}_K))$ it follows that $2^{-2}|G|^4\phi(\epsilon(d))$ belongs to $\mathrm{Ann}_{\mathbb{Z}[G]}(\mathrm{Cl}(\mathcal{O}_K))$, as required. □

Remark 12.2.2. The hypotheses in the first sentence of Proposition 12.2.1 coincide with those of the 'Stark-Chinburg Conjecture' of [22, Conj. 1]. Indeed, the discussion which follows [12, Prop. 3.3] shows that properties (i), (ii) and (iii) in Proposition 12.2.1 imply that $\epsilon(d)$ is an algebraic unit of the sort first discussed by Stark [52] and later conjectured to exist by Chinburg in [22, Conj. 1]. However the prediction of a precise link between such Stark units and the structure of ideal class groups is new. The prediction of Proposition 12.2.1(iv) is also remarkably similar to the results of Rubin (in the abelian case) that are discussed in Example 2.5.3. In particular, it would be interesting to know if this prediction is best possible and whether the computational techniques of the articles surveyed in [26, §14] could be used to obtain supporting evidence for it.

12.3. We now prove Proposition 2.6.2 and so assume the notation and hypotheses of that result.

Since $\theta^r_{K/k,S',T}(z)$ is holomorphic at $z=0$ one has $r_S(\psi)\ge r_{S'}(\psi)\ge r$ for every ψ in $\mathrm{Ir}(G)$. In particular, if $r_S(\psi)\ne r$ (as is the case for the trivial character because $|S|-1>r$), then $r_S(\psi)>r$ and so $\theta^r_{K/k,S,T}(0)e_\psi = L^r_{S,T}(\check{\psi},0)e_\psi = 0$. For each ϕ in $[U_S, X_S]_G$, resp. $[U_{S'}, X_{S'}]_G$, one therefore has

$$\theta^r_{K/k,S,T}(0)R(\phi) = \sum_\psi \theta^r_{K/k,S,T}(0)R(\phi)e_\psi = \sum_\psi L^r_{S,T}(\check{\psi},0)R(\phi(\psi))e_\psi \tag{44}$$

where ψ runs over all (non-trivial) characters in $\mathrm{Ir}(G)$ with $r_S(\psi) = r_{S'}(\psi) = r$ and $\phi(\psi)$ is the element of $[U_S^\psi, X_{S,\psi}]_{\mathcal{O}}$, resp. $[U_{S'}^\psi, X_{S',\psi}]_{\mathcal{O}}$, induced by ϕ.

To deduce (12) from (10) it is thus enough to recall (from [12, Rem. 2.3]) that for any non-trivial ψ in $\mathrm{Ir}(G)$ one has $\mathrm{Fit}_{\mathcal{O}}(\hat{H}^{-1}(G, X_S[\psi])) \subseteq \prod_v \mathrm{Fit}_{\mathcal{O}_\psi}((T_\psi)_{G_v})$ where v runs over any proper subset of $\{v' \in S : V_\psi^{G_{w'}} \text{ vanishes}\}$.

We now assume that ϕ belongs to $[U_{S'}, X_{S'}]_G$. We recall from §12.1 that for each non-trivial ψ with $r_{S'}(\psi) = r_S(\psi) = r$ the element $|G|^{3+r} L^r_{S,T}(\check{\psi}, 0) R(\phi(\psi)) e_\psi$ belongs to $\mathcal{O}_\psi \otimes \mathrm{Ann}_{\mathbb{Z}[G]}(\mathrm{Cl}(\mathcal{O}_{K,S'}))$. From (44) it follows that $|G|^{3+r}\theta^r_{K/k,S,T}(0)R(\phi)$ belongs to $\mathcal{O}' \otimes \mathrm{Ann}_{\mathbb{Z}[G]}(\mathrm{Cl}(\mathcal{O}_{K,S'}))$ where $\mathcal{O}'$ is the ring of integers of any number field that contains E_ψ for all ψ which occur in (44). But Stark's Conjecture implies that $|G|^{3+r}\theta^r_{K/k,S,T}(0)R(\phi)$ belongs to $\mathbb{Q}[G]$ (by the argument of Lemma 3.1.1) and so the required containment (13) follows because $\mathbb{Q}[G] \cap (\mathcal{O}' \otimes \mathrm{Ann}_{\mathbb{Z}[G]}(\mathrm{Cl}(\mathcal{O}_{K,S'}))) = \mathrm{Ann}_{\mathbb{Z}[G]}(\mathrm{Cl}(\mathcal{O}_{K,S'}))$. This completes the proof of Proposition 2.6.2.

Appendix A. Euler characteristics

A.1. We quickly review the relevant properties of the Euler characteristics that are used in this article. For more details of this general approach see [9].

We fix a Dedekind domain R with field of fractions F, an R-order $\mathfrak{A}$ in a finite dimensional semisimple F-algebra A and for any extension field E of F we write A_E for the (semisimple) E-algebra $E \otimes_F A$. We use the description of the relative algebraic K-group $K_0(\mathfrak{A}, A_E)$ in terms of explicit generators and relations that is given by Swan in [53, p. 215]. We recall in particular that in this description each element of $K_0(\mathfrak{A}, A_E)$ is represented by a triple $[P, \phi, Q]$ where P and Q are finitely generated projective left $\mathfrak{A}$-modules and ϕ is an isomorphism of (left) A_E-modules $E \otimes_R P \cong E \otimes_R Q$.

We find it convenient to use a different normalisation to that specified in [9]. Thus, for any object $C^\bullet$ of $D^{\mathrm{p}}(\mathfrak{A})$ that is acyclic outside degrees 0 and 1 and any isomorphism of A_E-modules $\lambda : E \otimes_R H^0(C^\bullet) \cong E \otimes_R H^1(C^\bullet)$ we write $\chi(C^\bullet, \lambda)$ for the element $-\chi_{\mathfrak{A}}(C^\bullet, \lambda^{-1}(C^\bullet))$ of $K_0(\mathfrak{A}, A_E)$ that is defined in [9, Def. 2.6].

Lemma A.1.1. *Let $C^\bullet$ and λ be as above.*

(i) *In $K_0(\mathfrak{A})$ one has $\partial^0_{\mathfrak{A},A_E}(\chi(C^\bullet, \lambda)) = \chi(C^\bullet)$ where $\chi(C^\bullet)$ is as defined in (20).*

(ii) *For any other isomorphism of A_E-modules $\lambda' : E \otimes_R H^0(C^\bullet) \cong E \otimes_R H^1(C^\bullet)$ one has $\chi(C^\bullet, \lambda') - \chi(C^\bullet, \lambda) = \delta_{\mathfrak{A},A_E}(\mathrm{Nrd}_{A_E}(\lambda' \circ \lambda^{-1}))$.*

(iii) *Let q be an isomorphism in $D^{\mathrm{p}}(\mathfrak{A})$ from $C^\bullet$ to a complex $C^\bullet_\phi$ of the form $P \xrightarrow{\phi} P$ for a finitely generated projective $\mathfrak{A}$-module P. Let λ^q be the isomorphism $(E \otimes_R H^1(q)) \circ \lambda \circ (E \otimes_R H^0(q))^{-1} : E \otimes_R H^0(C^\bullet_\phi) \cong E \otimes_R H^1(C^\bullet_\phi)$. Let ι_1 and ι_2 be A_E-equivariant sections to the tautological surjections $E \otimes_R P \to E \otimes_R \mathrm{im}(\phi)$ and $E \otimes_R P \to E \otimes_R \mathrm{cok}(\phi) = E \otimes_R H^1(C^\bullet)$ and write $\langle \lambda, \phi, q, \iota_1, \iota_2 \rangle$ for the automorphism of $E \otimes_R P$ that is equal to $\iota_2 \circ \lambda^q$ on $E \otimes_R H^0(C^\bullet) = E \otimes_R \ker(\phi)$ and to $E \otimes_R \phi$ on $\iota_1(E \otimes_R \mathrm{im}(\phi))$. Then $\chi(C^\bullet, \lambda) = \chi(C^\bullet_\phi, \lambda^q) = \delta_{\mathfrak{A},A_E}(\mathrm{Nrd}_{A_E}(\langle \lambda, \phi, q, \iota_1, \iota_2 \rangle))$.*

Proof. Claims (i) and (ii) follow from [9, Th. 2.1(i) and (ii)]. Regarding claim (iii) we note first that [9, Th. 2.1(iii)] implies $\chi(C^\bullet, \lambda) = \chi(C^\bullet_\phi, \lambda^q)$. Now, in terms of Swan's description of $K_0(\mathfrak{A}, A_E)$, the class $\chi(C^\bullet_\phi, \lambda^q)$ is defined in [9, §2] to be the element $[P, \Theta, P]$ with $\Theta := \langle \lambda, \phi, q, \iota_1, \iota_2 \rangle$. The claimed equality is therefore true because the definition of $\partial^1_{\mathfrak{A},A_E}$ implies that $[P, \Theta, P] = \partial^1_{\mathfrak{A},A_E}(\Theta) = \delta_{\mathfrak{A},A_E}(\mathrm{Nrd}_{A_E}(\Theta))$. □

Lemma A.1.2. *Let $C^\bullet_1 \xrightarrow{\alpha} C^\bullet_2 \xrightarrow{\beta} C^\bullet_3 \xrightarrow{\gamma} C^\bullet_1[1]$ be an exact triangle in $D^{\mathrm{p}}(\mathfrak{A})$ in which each complex C_i is acyclic outside degrees* 0 *and* 1 *and $E \otimes_R H^0(\gamma)$ is the zero map. For each $i = 1, 2, 3$ let $\lambda_i : E \otimes_R H^0(C^\bullet) \cong E \otimes_R H^1(C^\bullet)$ be an isomorphism of A_E-modules such that the following diagram commutes*

$$\begin{array}{ccccc} E \otimes_R H^0(C^\bullet_1) & \xrightarrow{E\otimes_R H^0(\alpha)} & E \otimes_R H^0(C^\bullet_2) & \xrightarrow{E\otimes_R H^0(\beta)} & E \otimes_R H^0(C^\bullet_3) \\ \lambda_1 \big\downarrow & & \lambda_2 \big\downarrow & & \lambda_3 \big\downarrow \\ E \otimes_R H^1(C^\bullet_1) & \xrightarrow{E\otimes_R H^1(\alpha)} & E \otimes_R H^1(C^\bullet_2) & \xrightarrow{E\otimes_R H^1(\beta)} & E \otimes_R H^1(C^\bullet_3). \end{array}$$

Then $\chi(C^\bullet_2, \lambda_2) = \chi(C^\bullet_1, \lambda_1) + \chi(C^\bullet_3, \lambda_3)$ in $K_0(\mathfrak{A}, A_E)$.

Proof. This result is a special case of the additivity criterion of [9, Th. 2.8]. □

Let G be a finite group. For each ψ in $\mathrm{Ir}(G)$ we write $I_{\mathcal{O}_\psi}$ for the multiplicative group of invertible $\mathcal{O}_\psi$-modules in $\mathbb{C}$. For any module M we also write $\overline{M}$ for the quotient M/M_{tor}.

Lemma A.1.3. *For each ψ in $\mathrm{Ir}(G)$ there is a homomorphism of abelian groups $\iota_\psi : K_0(\mathbb{Z}[G], \mathbb{R}[G]) \to I_{\mathcal{O}_\psi}$ with the following two properties.*

(i) *Let $C^\bullet$ be an object of $D^{\mathrm{p}}(\mathbb{Z}[G])$ that is acyclic outside degrees* 0 *and* 1*. Then for any isomorphism of $\mathbb{R}[G]$-modules $\lambda : \mathbb{R} \otimes H^0(C^\bullet) \cong \mathbb{R} \otimes H^1(C^\bullet)$ one has*

$$|G|^{d(C^\bullet,\psi)}\mathrm{Fit}_{\mathcal{O}_\psi}(H^0(C^\bullet)^\psi_{\mathrm{tor}})\iota_\psi(\chi(C^\bullet, \lambda)) \cdot \wedge^{d(C^\bullet,\psi)}_{\mathcal{O}_\psi}\overline{H^1(C^\bullet)_\psi}$$
$$= \mathrm{Fit}_{\mathcal{O}_\psi}(H^1(C^\bullet)_{\psi,\mathrm{tor}}) \cdot \wedge^{d(C^\bullet,\psi)}_{\mathcal{O}_\psi}(\lambda(\overline{H^0(C^\bullet)})^\psi)$$

with $d(C^\bullet, \psi) = \dim_{\mathbb{C}}(\mathbb{C} \otimes_{\mathcal{O}_\psi} H^1(C^\bullet)_\psi)$.

(ii) *For every x in $\mathfrak{Z}(\mathbb{R}[G])^\times$ one has $\iota_\psi(\delta_G(x))e_\psi = e_\psi x\mathcal{O}_\psi$.*

Proof. This is essentially well known but for the reader's convenience we sketch a proof. To do this we fix ψ in $\mathrm{Ir}(G)$ and set $\mathcal{O} := \mathcal{O}_\psi$ and $\iota := \iota_\psi$.

One obtains well-defined homomorphisms $\varrho : K_0(\mathbb{Z}[G], \mathbb{R}[G]) \to K_0(\mathcal{O}, \mathbb{C})$ and $\varpi : K_0(\mathcal{O}, \mathbb{C}) \to I_{\mathcal{O}}$ by setting $\varrho([P, \phi, P']) = [P^\psi, \phi^\psi, P'^{,\psi}]$ and defining $\varpi([B, \mu, B'])$ to be the fractional $\mathcal{O}$-ideal $\mathfrak{b}$ that satisfies $(\wedge^d_{\mathbb{C}}\mu)(\wedge^d_{\mathcal{O}}B) = \mathfrak{b} \cdot \wedge^d_{\mathcal{O}}B'$ where d is the (common) rank of the finitely generated projective $\mathcal{O}$-modules B and B'. We set $\iota := \varpi \circ \varrho$.

Let now $C^\bullet$ and λ be as in claim (i). We choose a representative of $C^\bullet$ of the form $C^0 \xrightarrow{d} C^1$ where C^0 and C^1 are finitely generated cohomologically-trivial G-modules and C^0 occurs in degree 0 and we write $C^{\bullet,\psi}$ for the object $C^{0,\psi} \xrightarrow{d^\psi} C^{1,\psi}$ of $D^{\rm p}(\mathcal{O})$, where $C^{0,\psi}$ occurs in degree 0. Then the argument of [13, §6.2] shows that there are natural identifications $H^0(C^{\bullet,\psi}) = H^0(C^\bullet)^\psi$ and $H^1(C^{\bullet,\psi}) = H^1(C^\bullet)_\psi$ and that, with respect to these identifications, one has $\varrho(\chi(C^\bullet,\lambda)) = \chi(C^{\bullet,\psi},|G|^{-1}\cdot\lambda^\psi)$ and also

$$\iota(\chi(C^\bullet,\lambda)) = \varpi(\chi(C^{\bullet,\psi},|G|^{-1}\cdot\lambda^\psi)) = \varpi([\overline{H^0},|G|^{-1}\cdot\lambda^\psi,\overline{H_1}])\mathrm{Fit}_\mathcal{O}(H^0_{\rm tor})^{-1}\mathrm{Fit}_\mathcal{O}(H_{1,\rm tor})$$

where we set $H^0 := H^0(C^\bullet)^\psi$ and $H_1 := H^1(C^\bullet)_\psi$. On the other hand the definition of ϖ implies that

$$|G|^{-d}\cdot\wedge^d_\mathcal{O}\lambda^\psi(\overline{H^0}) = \wedge^d_\mathbb{C}(|G|^{-1}\cdot\lambda^\psi)(\wedge^d_\mathcal{O}\overline{H^0}) = \varpi([\overline{H^0},|G|^{-1}\cdot\lambda^\psi,\overline{H_1}])\cdot\wedge^d_\mathcal{O}\overline{H_1}.$$

By combining the last two displayed formulas one obtains the formula of claim (i).

To prove claim (ii) it suffices to prove that for each prime p and each isomorphism of fields $j:\mathbb{C}\cong\mathbb{C}_p$ one has $j(\iota(\delta_G(x)))\mathcal{O}_j e_{\psi^j} = j_*(e_\psi x)\mathcal{O}_j$ where $\mathcal{O}_j$ is the completion of $\mathcal{O}$ at the place corresponding to j and we write ψ^j for the irreducible $\mathbb{C}_p$-valued character of G obtained by composing ψ with j and j_* for the homomorphism $\mathfrak{Z}(\mathbb{C}[G])^\times\to\mathfrak{Z}(\mathbb{C}_p[G])^\times$ sending each element $\sum_{g\in G}c_g g$ to $\sum_{g\in G}j(c_g)g$. But, by using the diagram (27), one computes that $j(\iota(\delta_G(x))) = \det_{\mathbb{C}_p}(\kappa^{\psi^j})\mathcal{O}_j$ for any automorphism κ of $\mathbb{C}_p[G]$ with $\mathrm{Nrd}_{\mathbb{C}_p[G]}(\kappa) = j_*(x)$. The required equality therefore follows because by the definition of $\mathrm{Nrd}_{\mathbb{C}_p[G]}$ one has $\det_{\mathbb{C}_p}(\kappa^{\psi^j})e_{\psi^j} = e_{\psi^j}\mathrm{Nrd}_{\mathbb{C}_p[G]}(\kappa) = e_{\psi^j}j_*(x) = j_*(e_\psi x)$. $\square$

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