Congruences between Derivatives of Geometric L-Functions

David Burns

with an appendix by David Burns, King Fai Lai and Ki-Seng Tan

Abstract. We prove a natural equivariant refinement of a theorem of Lichtenbaum describing the leading terms of Zeta functions of curves over finite fields in terms of Weil-étale cohomology. We then use this result to prove the validity of Chinburg's $\Omega(3)$-Conjecture for all abelian extensions of global function fields, to prove natural refinements and generalisations of the refined Stark conjectures formulated by, amongst others, Gross, Tate, Rubin and Popescu, to prove a variety of explicit restrictions on the Galois module structure of unit groups and divisor class groups and to describe explicitly the Fitting ideals of certain Weil-étale cohomology groups. In an appendix coauthored with K. F. Lai and K-S. Tan we also show that the main conjectures of geometric Iwasawa theory can be proved without using either crystalline cohomology or Drinfeld modules.

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1. Introduction

The main result of the present article is the following

Theorem 1.1. *The central conjecture of* [6] *is valid for all global function fields.*

(For a more explicit statement of this result see Theorem 3.1.)
Theorem 1.1 is a natural equivariant refinement of the leading term formula proved by Lichtenbaum in [27] and also implies an extensive new family of integral congruence relations between the leading terms of L-functions associated to abelian characters of global functions fields (see Remark 3.2). In addition, Theorem 1.1 verifies the conjecture [5, §2.2, Conj. $\mathrm{C}(F/k)$] for all abelian extensions F/k and is also a natural function field analogue of an important special case of the 'equivariant Tamagawa number conjecture' formulated by Flach and the present author in [8] (for more details of this connection see [5, Rem. 2, Rem. 3]). Theorem 1.1 therefore constitutes a strong refinement of all previous results in this area: for example, until now [5, Conj. $\mathrm{C}(F/k)$] was not known to be valid for *any* extension F/k of degree divisible by $\mathrm{char}(k)$. We are also able to deduce from Theorem 1.1 all of the following results.

Corollary 1.2.

(i) *The '$\Omega(3)$-Conjecture' of Chinburg* [13, §4.2] *is valid for all finite abelian extensions of global function fields.*
(ii) *The '$\Omega(1)$-Conjecture' of Chinburg* [13, §4.2] *is valid for all finite abelian tamely ramified extensions of global function fields.*
(iii) *The family of explicit congruence relations between derivatives of abelian L-functions that is described in* [6, (9)] *is valid for all global function fields.*
(iv) *Natural 'higher order' generalisations of all of the following refinements of Stark's conjecture are valid for all global function fields.*
- *The 'guess' formulated by Gross in* [19, top of p. 195]*.*
- *The 'refined class number formula' of Gross* [19, Conj. 4.1]*.*
- *The 'refined class number formula' of Tate* [40] *(see also* [41, $(*)$]*).*
- *The 'refined class number formula' of Aoki et al* [1, Conj. 1.1]*.*
- *The 'refined $\mathfrak{p}$-adic abelian Stark Conjecture' of Gross* [19, Conj. 7.6]*.*

Remarks 1.3.
(i) The 'Ω-Conjectures' of [13, §4.2] are natural function field analogues of the central conjectures of Galois module theory formulated by Chinburg in [14, 15] and, hitherto, have been only very partially verified: for example, the only extensions F/k of degree divisible by $\mathrm{char}(k)$ for which the $\Omega(3)$-Conjecture was previously known to be valid were extensions of degree *equal* to $\mathrm{char}(k)$ (the latter result was first proved by Bae in [3] and later reproved by Popescu in [31]) and we are not aware of any previous results relating to the validity of the $\Omega(1)$-Conjecture in this setting.
(ii) The congruence relations of [6, (9)] constitute a strong refinement of the central conjecture formulated by Rubin in [35] and hence, a fortiori, of the conjecture studied by Popescu in [30].
(iii) The phrase 'higher order' in Corollary 1.2(iv) refers to the fact that our result allows L-functions to vanish to higher order at $s = 0$ than is envisaged in the stated conjectures (see §5.2). In the setting of the original conjectures of Gross, Tate et al

Theorem 1.1 also combines with the final assertion of [6, Cor. 4.3] to give a new proof of the main result of Hayes in [20] (whose original proof relies on the theory of rank one Drinfeld modules) and with the result of [6, Cor. 4.1] to give a more conceptual proof of the main results of, inter alia, Lee [26], Popescu [30, 32], Reid [34] and Tan [36, 37, 38].

In addition to the consequences of Theorem 1.1 that are described in Corollary 1.2 we shall also derive from it a variety of explicit restrictions on the Galois module structures of unit groups and divisor class groups and explicit descriptions of the Fitting ideals of certain natural Weil-étale cohomology groups.

The main contents of this article is as follows. In §2 we recall some necessary background material and fix notation and then in §3 we give a precise statement (Theorem 3.1) of Theorem 1.1 and establish an important reduction step in its proof. In §4 we describe the basic properties of certain natural complexes in geometric Iwasawa theory and then complete the proof of Theorem 3.1 by combining the main result of Crew in [17] with the descent techniques developed by Venjakob and the present author in [12]. (With a view to subsequent applications in non-commutative Iwasawa theory, we do not always assume in §4 that field extensions are abelian.) In §5 we derive a variety of explicit consequences of Theorem 3.1 including Corollary 1.2. Finally, in an appendix coauthored with Lai and Tan, we combine the techniques developed in §4 with a strengthening of the main result of Kueh, Lai and Tan in [25] and Deligne's proof of the Brumer-Stark conjecture to prove the following remarkable fact: all of the main conjectures of geometric Iwasawa theory that are proved by Crew in [17], and hence also both Theorem 1.1 and Corollary 1.2 above, can be deduced from Weil's description of the Zeta functions of curves over a finite field of characteristic p in terms of ℓ-adic homology for any prime $\ell \neq p$, thereby avoiding recourse to the methods of either crystalline cohomology (as used originally by Crew in [17] and also by Bae in [3]) or Drinfeld modules (as used by Hayes in [20]).

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2. Preliminaries

2.1. **Categories.** In this article modules are to be understood, unless explicitly stated otherwise, as left modules. For any left noetherian ring R we write $D(R)$ for the derived category of R-modules and $D^{\mathrm{p}}(R)$ for the full triangulated subcategory of $D(R)$ comprising complexes that are isomorphic to an object of the category $C^{\mathrm{p}}(R)$ of bounded complexes of finitely generated projective R-modules. We also write $D^-(R)$, resp. $D^+(R)$, for the full triangulated subcategory of $D(R)$ comprising complexes C with the property that there exists an integer $m(C)$ such that $H^i(C)$ vanishes for all $i > m(C)$, resp. $i < m(C)$.

2.2. **Determinants.** In this subsection R is a commutative unital noetherian ring. We write $\mathcal{P}(R)$ for the Picard category of graded line bundles on $\mathrm{Spec}(R)$, $\mathbf{1}_R$ for the unit object $(R, 0)$ of $\mathcal{P}(R)$ and $\mathbf{d}_R$ for the determinant functor of Grothendieck-Knudsen-Mumford from [24]. We recall that all morphisms in $\mathcal{P}(R)$ are isomorphisms and that $\mathbf{d}_R$ is well-defined on objects of $D^{\mathrm{p}}(R)$ and takes values in $\mathcal{P}(R)$. If x is any unit of the total quotient ring $Q(R)$ of R, then we write $x\cdot\mathbf{1}_R$ for the element $(xR,0)$ of $\mathcal{P}(R)$. For each object $C^\bullet$ of $D^{\mathrm{p}}(R)$ we shall make frequent use of the following standard (and essentially well-known) facts.

(i) Any homomorphism $\pi : R \to S$ of commutative unital noetherian rings induces a natural functor $\pi_* : \mathcal{P}(R) \to \mathcal{P}(S)$. If one regards S as a right R-module via π, then $S\otimes^{\mathbb{L}}_R C^\bullet$ belongs to $D^{\mathrm{p}}(S)$ and there is a canonical morphism $\pi_*(\mathbf{d}_R(C^\bullet)) \to \mathbf{d}_S(S\otimes^{\mathbb{L}}_R C^\bullet)$ in $\mathcal{P}(S)$. In fact we often regard π as clear from context and simply write $\pi_*(-)$ as $S\otimes_R -$.

(ii) If $Q(R)\otimes^{\mathbb{L}}_R C^\bullet$ is acyclic the morphism $Q(R)\otimes_R \mathbf{d}_R(C^\bullet) \to \mathbf{d}_{Q(R)}(Q(R)\otimes^{\mathbb{L}}_R C^\bullet)$ from (i) combines with the canonical morphism $\mathbf{d}_{Q(R)}(Q(R)\otimes^{\mathbb{L}}_R C^\bullet) \to \mathbf{1}_{Q(R)}$ to give a morphism $\vartheta_{\mathrm{can}} : Q(R)\otimes_R \mathbf{d}_R(C^\bullet) \to \mathbf{1}_{Q(R)}$. In particular, if R is semilocal, $C^\bullet$ is acyclic outside degree i and $H^i(C^\bullet)$ is a (finitely generated) torsion R-module of projective dimension at most one, then $Q(R)\otimes^{\mathbb{L}}_R C^\bullet$ is acyclic and it can be shown that the (initial) Fitting ideal $\mathrm{Fit}_R(H^i(C^\bullet))$ of the R-module $H^i(C^\bullet)$ is an invertible ideal of R which satisfies $\vartheta_{\mathrm{can}}(\mathbf{d}_R(C^\bullet)) = (\mathrm{Fit}_R(H^i(C^\bullet))^{(-1)^{i+1}},0)$.

(iii) If R is a regular ring and $C^\bullet$ is acyclic outside consecutive degrees a and $a+1$ (for any integer a), then any isomorphism of R-modules $\lambda : H^a(C^\bullet)\cong H^{a+1}(C^\bullet)$ naturally induces a morphism in $\mathcal{P}(R)$ of the form $\vartheta_\lambda : \mathbf{d}_R(C^\bullet) \to \mathbf{1}_R$. If R is semisimple (as is the case, for example, if R is a direct factor of the group ring of a finite group over a field of characteristic 0), then an explicit description of ϑ_λ is given in [6, §6.1].

(iv) Assume now that R is isomorphic to a finite product of power series rings in finitely many variables over finite extensions of $\mathbb{Z}_p$ (as is the case if R is the p-adic Iwasawa algebra of a group $H\times\Gamma$ with H finite abelian of order prime to p and Γ topologically isomorphic to $\mathbb{Z}_p^e$ for some natural number e). Then both R and $Q(R)$ are semilocal and regular and if M is a finitely generated torsion R-module that has projective dimension at most one its characteristic ideal $\mathrm{ch}_R(M)$ is equal to $\mathrm{Fit}_R(M)$ (this is easily proved using the structure theory of torsion R-modules and the fact that $R=\bigcap_{\mathfrak{p}} R_{\mathfrak{p}}$ where $\mathfrak{p}$ runs over all prime ideals of R of height one by [4, chap. VII, §1, no. 6, Th. 4]). In particular, if $C^\bullet$ is acyclic outside degree i and such that $H^i(C^\bullet)$ is a torsion R-module of projective dimension at most one, then the observations in (ii) imply that $\vartheta_{\mathrm{can}}(\mathbf{d}_R(C^\bullet)) = (\mathrm{ch}_R(H^i(C^\bullet))^{(-1)^{i+1}},0)$.

2.3. **Further notation.** For each field E we fix a choice of separable closure E^c and for any Galois extension of fields E'/E we set $G_{E'/E} := \mathrm{Gal}(E'/E)$.

For any global function field F we write C_F for the unique geometrically irreducible smooth projective curve with function field F. For any extension of such fields F/k and each finite non-empty set Σ of places of k that contains all places which ramify in F/k we write $\Sigma(F)$ for the set of places of F lying above those in Σ and $\mathcal{O}_{F,\Sigma}$ for the subring of F comprising elements that are integral at all places outside $\Sigma(F)$. We also write $\mathcal{O}^\times_{F,\Sigma}$ for the unit group of $\mathcal{O}_{F,\Sigma}$, set $Y_{F,\Sigma} := \bigoplus_w \mathbb{Z}$ where w runs over $\Sigma(F)$ and write $Y^0_{F,\Sigma}$ for the kernel of the homomorphism $Y_{F,\Sigma}\to\mathbb{Z}$ sending $(n_w)_w$ to $\sum_w n_w$. If F/k is Galois, then each of $\mathcal{O}_{F,\Sigma}, \mathcal{O}^\times_{F,\Sigma}$, $Y_{F,\Sigma}$ and $Y^0_{F,\Sigma}$ has a natural action of $G_{F/k}$. Finally we note that unadorned tensor products are always to be regarded as taken in the category of abelian groups.

3. The leading term formula

3.1. **Statement of the formula.** We fix a finite Galois extension of global function fields F/k and set $G := G_{F/k}$. For each finite non-empty set Σ of places of k that contains all places which ramify in F/k we define an object of $D(\mathbb{Z}[G])$ by setting

$$D^\bullet_{F,\Sigma} := R\operatorname{Hom}_{\mathbb{Z}}(R\Gamma_{\mathrm{W\acute{e}t}}(C_F, j_{F,\Sigma !}\mathbb{Z}), \mathbb{Z}[-2])$$

where the subscript 'Wét' refers to the 'Weil-étale topology' introduced by Lichtenbaum in [27], $j_{F,\Sigma}$ is the natural open immersion $\mathrm{Spec}(\mathcal{O}_{F,\Sigma}) \to C_F$ and $D^\bullet_{F,\Sigma}$ is endowed with the natural contragredient action of G. Then it is known that $D^\bullet_{F,\Sigma}$ belongs to $D^{\mathrm{p}}(\mathbb{Z}[G])$, is acyclic outside degrees 0 and 1 and is such that the spaces $\mathbb{Q}\otimes H^0(D^\bullet_{F,\Sigma}) = H^0(\mathbb{Q}[G]\otimes^{\mathbb{L}}_{\mathbb{Z}[G]} D^\bullet_{F,\Sigma})$ and $\mathbb{Q}\otimes H^1(D^\bullet_{F,\Sigma}) = H^1(\mathbb{Q}[G]\otimes^{\mathbb{L}}_{\mathbb{Z}[G]} D^\bullet_{F,\Sigma})$ identify with $\mathbb{Q}\otimes\mathcal{O}^\times_{F,\Sigma}$ and $\mathbb{Q}\otimes Y^0_{F,\Sigma}$ respectively (cf. [5, Lem. 1]).
For each place w of F we write val_w for associated valuation on F and $\deg_k(w)$ for the degree of the residue field of w over the field of constants of k. We let $\lambda_{F,\Sigma}: \mathcal{O}^\times_{F,\Sigma} \to Y^0_{F,\Sigma}$ denote the homomorphism which sends each element u of $\mathcal{O}^\times_{F,\Sigma}$ to $(\mathrm{val}_w(u)\deg_k(w))_{w\in\Sigma(F)}$. Then the Riemann-Roch Theorem implies that $\mathbb{Q}\otimes\lambda_{F,\Sigma}$ is bijective and we write

$$\vartheta_{F,\Sigma}: \mathbb{Q}[G]\otimes_{\mathbb{Z}[G]}\mathbf{d}_{\mathbb{Z}[G]}(D^\bullet_{F,\Sigma}) \cong \mathbf{d}_{\mathbb{Q}[G]}(\mathbb{Q}[G]\otimes^{\mathbb{L}}_{\mathbb{Z}[G]} D^\bullet_{F,\Sigma}) \to \mathbf{1}_{\mathbb{Q}[G]}$$

for the morphism $\vartheta_{\mathbb{Q}\otimes\lambda_{F,\Sigma}}$ discussed in §2.2(iii).
We write $\mathbb{Q}^c$ for the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$ and set $G^\wedge := \mathrm{Hom}(G, \mathbb{Q}^{c\times}) = \mathrm{Hom}(G,\mathbb{C}^\times)$. For each $\chi\in G^\wedge$ we write $\check{\chi}$ for the contragredient of χ, e_χ for the idempotent $\frac{1}{|G|}\sum_{g\in G}\check{\chi}(g)g$ of $\mathbb{Q}^c[G]$ and $L_\Sigma(\chi,s)$ for the Σ-truncated Dirichlet L-function of χ. We also write q_k for the cardinality of the constant field of k and define a $\mathbb{C}[G]$-valued meromorphic function of the complex variable $t := q_k^{-s}$ by setting

$$Z_{F/k,\Sigma}(t) := \sum_{\chi\in G^\wedge} Z_\Sigma(\chi,t)e_{\check{\chi}}, \tag{1}$$

where for each χ the function $Z_\Sigma(\chi,t)$ is defined by setting $Z_\Sigma(\chi,t) := L_\Sigma(\chi,s)$. We then define a leading term element by setting

$$Z^*_{F/k,\Sigma}(1) := \sum_{\chi\in G^\wedge}\lim_{t\to 1}(1-t)^{-r_\chi}Z_\Sigma(\chi,t)e_{\check{\chi}} \tag{2}$$

where r_χ is the algebraic order of $Z_\Sigma(\chi,t)$ at $t=1$. It is known that $Z^*_{F/k,\Sigma}(1)$ belongs to $\mathbb{Q}[G]^\times$ (see, for example, [5, Lem. 2]).
We can now state a precise version of Theorem 1.1.

Theorem 3.1. *In $\mathcal{P}(\mathbb{Z}[G])$ one has $\vartheta_{F,\Sigma}(\mathbf{d}_{\mathbb{Z}[G]}(D^\bullet_{F,\Sigma})) = Z^*_{F/k,\Sigma}(1)\cdot\mathbf{1}_{\mathbb{Z}[G]}$.*

Remark 3.2. In the 'non-equivariant' case $F=k$ this equality can be shown to be equivalent to the leading term formula proved by Lichtenbaum in [27, Th. 8.2]. The general case was first conjectured by the present author in [5, §2.2, Conj. C(F/k)] and can be interpreted in terms of congruences in the following way. Let Λ be any subring of $\mathbb{Q}$ such that the module $\Lambda\otimes\mathcal{O}^\times_{F,\Sigma}$ is torsion-free. Then the proof of [7, Lem. 3.2.1(ii) with $d_\tau = 1$] shows that there exists an endomorphism d of a finitely generated projective $\Lambda[G]$-module P and an isomorphism κ in $D^{\mathrm{p}}(\Lambda[G])$ from $\Lambda[G]\otimes^{\mathbb{L}}_{\mathbb{Z}[G]} D^\bullet_{F,\Sigma}$

to the complex $P \xrightarrow{d} P$, where the first term is placed in degree 0. We write ι for the composite isomorphism of $\mathbb{Q}[G]$-modules

$$\mathbb{Q}\otimes_\Lambda P \cong (\mathbb{Q}\otimes_\Lambda \ker(d)) \oplus (\mathbb{Q}\otimes_\Lambda \mathrm{im}(d)) \cong (\mathbb{Q}\otimes_\Lambda \mathrm{cok}(d)) \oplus (\mathbb{Q}\otimes_\Lambda \mathrm{im}(d)) \cong \mathbb{Q}\otimes_\Lambda P$$

where the first, resp. third, map is induced by a choice of $\mathbb{Q}[G]$-equivariant section to the tautological map $\mathbb{Q}\otimes_\Lambda P \to \mathbb{Q}\otimes_\Lambda \mathrm{im}(d)$, resp. $\mathbb{Q}\otimes_\Lambda P \to \mathbb{Q}\otimes_\Lambda \mathrm{cok}(d)$, and the second map is $\mathbb{Q}\otimes_\Lambda (H^1(\kappa)\circ(\Lambda\otimes\lambda_{F,\Sigma})\circ H^0(\kappa)^{-1})\oplus \mathrm{id}$. For each $\chi\in G^\wedge$ we then define a normalised leading term by setting

$$A(\chi) := \lim_{t\to 1}(1-t)^{-r_{\check{\chi}}} Z_\Sigma(\check{\chi},t)/\det(\mathbb{Q}^c\otimes_{\mathbb{Q}}\iota \mid e_\chi(\mathbb{Q}^c\otimes_\Lambda P)).$$

It can be shown that, independently of Theorem 3.1, each $A(\chi)$ belongs to the field E generated over $\mathbb{Q}$ by the set $\{\psi(g) : g\in G, \psi\in G^\wedge\}$ and also satisfies $\omega(A(\chi)) = A(\omega\circ\chi)$ for all $\omega\in G_{E/\mathbb{Q}}$. The result of [7, Lem. 3.2.1(ii) with $d_\tau = 1$] shows that Theorem 3.1 implies in addition that each $A(\chi)$ belongs to the integral closure $\mathcal{O}$ of Λ in E, and hence that the sum $\sum_{\chi\in G^\wedge} A(\chi)e_\chi$ belongs to the integral closure $\mathfrak{M}$ of Λ in $\mathbb{Q}[G]$, and moreover that, for differing χ, the elements $A(\chi)$ satisfy a family of mutual congruence relations which are together equivalent to the condition that an element of $\mathfrak{M}$ should belong to the subset $\Lambda[G]$. For example, if $|G|$ is equal to a prime ℓ and $\ell\notin\Lambda^\times$, then the required condition is that for every $\chi\in G^\wedge$ one has $A(\chi)\equiv A(\chi_0)$ (mod $\mathfrak{L}$) where χ_0 is the trivial element of $G^\wedge$ and $\mathfrak{L}$ the unique prime ideal of $\mathcal{O}$ above ℓ. However, explicating the necessary congruences in any very general setting seems to be a difficult algebraic problem.

3.2. **A useful reduction step.** For each prime ℓ we define an object of $D^{\mathrm{p}}(\mathbb{Z}_\ell[G])$ by setting

$$D^\bullet_{F,\Sigma,\ell} := R\,\mathrm{Hom}_{\mathbb{Z}_\ell}(R\Gamma_{\text{ét}}(C_F, j_{F,\Sigma!}\mathbb{Z}_\ell), \mathbb{Z}_\ell[-2]),$$

regarded as endowed with the natural contragredient action of G.

Lemma 3.3. *$D^\bullet_{F,\Sigma,\ell}$ is naturally isomorphic to $\mathbb{Z}_\ell[G]\otimes^{\mathbb{L}}_{\mathbb{Z}[G]} D^\bullet_{F,\Sigma}$ in $D^{\mathrm{p}}(\mathbb{Z}_\ell[G])$.*

Proof. Since the functor $\mathbb{Z}_\ell[G]\otimes_{\mathbb{Z}[G]} -$ identifies with the exact functor $\mathbb{Z}_\ell\otimes -$ it clearly suffices to prove that $\mathbb{Z}_\ell\otimes R\Gamma_{\text{Wét}}(C_F, j_{F,\Sigma!}\mathbb{Z})$ is naturally isomorphic to $C^\bullet_{F/k,\Sigma,\ell} := R\Gamma_{\text{ét}}(C_F, j_{F,\Sigma!}\mathbb{Z}_\ell)$ in $D^{\mathrm{p}}(\mathbb{Z}_\ell[G])$. To do this we fix an object $P^\bullet$ of $C^{\mathrm{p}}(\mathbb{Z}[G])$ that is isomorphic in $D^{\mathrm{p}}(\mathbb{Z}[G])$ to $R\Gamma_{\text{Wét}}(C_F, j_{F,\Sigma!}\mathbb{Z})$ and an object $Q^\bullet$ of $C^{\mathrm{p}}(\mathbb{Z}_\ell[G])$ isomorphic in $D^{\mathrm{p}}(\mathbb{Z}_\ell[G])$ to $C^\bullet_{F/k,\Sigma,\ell}$. For each natural number n we set $\Lambda_n := \mathbb{Z}/\ell^n[G]$. Then the exact sequence $0\to P^\bullet \xrightarrow{\times\ell^n} P^\bullet \to P^\bullet/\ell^n \to 0$ combines with the natural exact triangle

$$R\Gamma_{\text{Wét}}(C_F, j_{F,\Sigma!}\mathbb{Z}) \xrightarrow{\times\ell^n} R\Gamma_{\text{Wét}}(C_F, j_{F,\Sigma!}\mathbb{Z}) \to R\Gamma_{\text{Wét}}(C_F, j_{F,\Sigma!}(\mathbb{Z}/\ell^n)) \to$$

to give an isomorphism $P^\bullet/\ell^n \cong R\Gamma_{\text{Wét}}(C_F, j_{F,\Sigma!}(\mathbb{Z}/\ell^n))$ in $D^{\mathrm{p}}(\Lambda_n)$. In a similar way there is an isomorphism $Q^\bullet/\ell^n \cong R\Gamma_{\text{ét}}(C_F, j_{F,\Sigma!}(\mathbb{Z}/\ell^n))$ in $D^{\mathrm{p}}(\Lambda_n)$. Since the complexes $R\Gamma_{\text{Wét}}(C_F, j_{F,\Sigma!}(\mathbb{Z}/\ell^n))$ and $R\Gamma_{\text{ét}}(C_F, j_{F,\Sigma!}(\mathbb{Z}/\ell^n))$ are canonically isomorphic in $D^{\mathrm{p}}(\Lambda_n)$ (cf. [27, Prop. 2.3(g)]) there is therefore an isomorphism $\alpha_n : Q^\bullet/\ell^n \cong P^\bullet/\ell^n$ in $D^{\mathrm{p}}(\Lambda_n)$. As n varies, the isomorphisms α_n are such that the diagram

$$\begin{array}{ccccccc} \cdots & \longrightarrow & Q^\bullet/\ell^n & \xrightarrow{\pi_n^Q} & Q^\bullet/\ell^{n-1} & \longrightarrow & \cdots \\ & & {\scriptstyle\alpha_n}\Big\downarrow & & {\scriptstyle\alpha_{n-1}}\Big\downarrow & & \\ \cdots & \longrightarrow & P^\bullet/\ell^n & \xrightarrow{\pi_n^P} & P^\bullet/\ell^{n-1} & \longrightarrow & \cdots \end{array}$$

commutes in $D^{\mathrm{p}}(\Lambda_n)$, where π_n^Q and π_n^P denote the natural quotient maps. Since however $Q^\bullet/\ell^n$ consists of projective Λ_n-modules and $P^\bullet/\ell^n$ of Λ_n-modules we can realize each α_n as an actual map of complexes. Moreover, $\alpha_{n-1}\circ\pi_n^Q$ will be homotopic to $\pi_n^P\circ\alpha_n$, i.e.

$$\alpha_{n-1}\circ\pi_n^Q-\pi_n^P\circ\alpha_n=d\circ h+h\circ d$$

for some map $h: Q^\bullet/\ell^n\to P^\bullet/\ell^{n-1}[-1]$. But, for each i, the projection $P^i/\ell^n\to P^i/\ell^{n-1}$ is surjective and Q^i/ℓ^n is a projective Λ_n-module and so we can lift h to a map $h': Q^\bullet/\ell^n\to P^\bullet/\ell^n[-1]$. If we then replace α_n by $\alpha_n+d\circ h'+h'\circ d$, the above diagram will actually be a commutative diagram of maps of complexes. So by induction we may assume that, taken together, the maps α_n constitute a map of inverse systems of complexes. The inverse limit of such a compatible system then gives an isomorphism in $D^{\mathrm{p}}(\mathbb{Z}_\ell[G])$ of the form

$$C^\bullet_{F/k,\Sigma,\ell}\cong Q^\bullet\cong\varprojlim_n Q^\bullet/\ell^n\cong\varprojlim_n P^\bullet/\ell^n\cong\mathbb{Z}_\ell\otimes P^\bullet\cong\mathbb{Z}_\ell\otimes R\Gamma_{\mathrm{W\acute{e}t}}(C_F,j_{F,\Sigma!}\mathbb{Z}),$$

as required. □

In the following result we write

$$\vartheta_{F,\Sigma,\ell}:\mathbf{d}_{\mathbb{Q}_\ell[G]}(\mathbb{Q}_\ell[G]\otimes^{\mathbb{L}}_{\mathbb{Z}_\ell[G]}D^\bullet_{F,\Sigma,\ell})\to\mathbf{1}_{\mathbb{Q}_\ell[G]}$$

for the morphism in $\mathcal{P}(\mathbb{Q}_\ell[G])$ that is constructed in the same way as the morphism $\vartheta_{F,\Sigma}$ in Theorem 3.1 but with the role of $\mathbb{Q}\otimes\lambda_{F,\Sigma}$ played by the composite of $\mathbb{Q}_\ell\otimes_{\mathbb{Q}}(\mathbb{Q}\otimes\lambda_{F,\Sigma})$ and the isomorphisms $H^a(\mathbb{Q}_\ell[G]\otimes^{\mathbb{L}}_{\mathbb{Z}_\ell[G]}D^\bullet_{F,\Sigma,\ell})\cong\mathbb{Q}_\ell\otimes_{\mathbb{Q}}(\mathbb{Q}\otimes H^a(D^\bullet_{F,\Sigma}))$ induced by Lemma 3.3.
In the sequel we set $p:=\mathrm{char}(k)$.

Proposition 3.4. *Theorem 3.1 is valid if and only if in $\mathcal{P}(\mathbb{Z}_p[G])$ one has*

$$\vartheta_{F,\Sigma,p}(\mathbf{d}_{\mathbb{Z}_p[G]}(D^\bullet_{F,\Sigma,p}))=Z^*_{F/k,\Sigma}(1)\cdot\mathbf{1}_{\mathbb{Z}_p[G]}. \tag{3}$$

Proof. The natural diagonal functor $\mathcal{P}(\mathbb{Z}[G])\to\prod_\ell\mathcal{P}(\mathbb{Z}_\ell[G])$, where ℓ runs over all primes, is faithful and so the equality of Theorem 3.1 is valid if at each prime ℓ it is valid after applying $\mathbb{Z}_\ell[G]\otimes_{\mathbb{Z}[G]}-$. But from Lemma 3.3 and the definition of $\vartheta_{F,\Sigma,\ell}$ one has $\mathbb{Z}_\ell[G]\otimes_{\mathbb{Z}[G]}\vartheta_{F,\Sigma}(\mathbf{d}_{\mathbb{Z}[G]}(D^\bullet_{F,\Sigma}))=\vartheta_{F,\Sigma,\ell}(\mathbf{d}_{\mathbb{Z}_\ell[G]}(D^\bullet_{F,\Sigma,\ell}))$ and so it is enough to prove that $\vartheta_{F,\Sigma,\ell}(\mathbf{d}_{\mathbb{Z}_\ell[G]}(D^\bullet_{F,\Sigma,\ell}))=Z^*_{F/k,\Sigma}(1)\cdot\mathbf{1}_{\mathbb{Z}_\ell[G]}$ for every prime ℓ. To prove the claim it therefore suffices to show that if $\ell\neq p$, then the latter equality follows from results proved in [5]. To do this we note first that, because of the different normalisations used here and in [5], $Z^*_{F/k,\Sigma}(1)$ is equal to the element $Z^*_{F/k,\Sigma}(1)^\#$ which occurs in the (conjectural) equality of [5, (3)]. We also note that [5, Lem. 1(i)] induces a canonical isomorphism in $D^{\mathrm{p}}(\mathbb{Z}[G])$ between $D^\bullet_{F,\Sigma}$ and the Weil-étale cohomology complex $R\Gamma_{\mathcal{W}}(U_{F,\Sigma},\mathbb{G}_m)$ which occurs in [5, (3)]. After taking this into account, and using [5, Rem. A1] to translate the image under the natural localisation map $K_0(\mathbb{Z}[G],\mathbb{Q}) \to K_0(\mathbb{Z}_\ell[G],\mathbb{Q}_\ell)$ of the equality [5, (3)] from the language of relative algebraic K-theory to that of graded invertible $\mathbb{Z}_\ell[G]$-modules, one obtains the required equality $\vartheta_{F,\Sigma,\ell}(\mathbf{d}_{\mathbb{Z}_\ell[G]}(D^\bullet_{F,\Sigma,\ell})) = Z^*_{F/k,\Sigma}(1)\cdot \mathbf{1}_{\mathbb{Z}_\ell[G]}$. It is therefore clear that if $\ell \neq p$, then the validity of this equality follows directly by combining the results of [5, Th. 3.1(i) and Lem. 2]. □

4. Iwasawa theory

In this section we complete the proof of Theorem 3.1 by using the descent formalism developed by Venjakob and the present author in [12] to deduce the required equality (3) from Crew's proof in [17] of the relevant main conjecture in geometric Iwasawa theory. (Later, in Appendix A, we shall also present an alternative, and conceptually simpler, proof of all of the main conjectures that are proved by Crew in [17].)
For any profinite group $\mathcal{G}$ we write $\Lambda(\mathcal{G})$ for the Iwasawa algebra $\varprojlim_U \mathbb{Z}_p[\mathcal{G}/U]$, where U runs over all open normal subgroups of $\mathcal{G}$. We recall that if $\mathcal{G}$ is a compact p-adic Lie group, then $\Lambda(\mathcal{G})$ is left noetherian and its total quotient ring $Q(\Lambda(\mathcal{G}))$ is left regular. For any Galois extension of fields F/E we often abbreviate $\Lambda(G_{F/E})$ to $\Lambda(F/E)$.

4.1. **The complexes.** In this subsection we fix a compact p-adic Lie extension K/k (of global function fields of characteristic p) and, with a view to subsequent applications in non-commutative Iwasawa theory, we do not assume that K/k is abelian. We set $\mathcal{G} := G_{K/k}$ and for any closed normal subgroup V of $\mathcal{G}$ we write $\Lambda(\mathcal{G}/V)^\#$ for $\Lambda(\mathcal{G}/V)$, regarded as a (left) $\Lambda(\mathcal{G}/V)$-module via multiplication on the left and also endowed with the following commuting (left) action of $G_{k^c/k}$: each σ in $G_{k^c/k}$ acts on $\Lambda(\mathcal{G}/V)^\#$ as right multiplication by σ_V^{-1} where σ_V is the image of σ in $\mathcal{G}/V$. We regard $\Lambda(\mathcal{G}/V)^\#$ as an étale sheaf of $\Lambda(\mathcal{G}/V)$-modules on $U_{k,\Sigma} := \mathrm{Spec}(\mathcal{O}_{k,\Sigma})$ in the natural way and, following the approach of Nekovář [29], we then obtain an object of $D^-(\Lambda(\mathcal{G}/V))$ by setting

$$C^\bullet_{K^V/k,\Sigma} := R\Gamma_{\text{ét}}(C_k, j_{k,\Sigma!}(\Lambda(\mathcal{G}/V)^\#)),$$

where $j_{k,\Sigma} : U_{k,\Sigma} \to C_k$ is the natural open immersion.
We write $x \mapsto x^\sim$ for the $\mathbb{Z}_p$-linear anti-involution of $\Lambda(\mathcal{G}/V)$ that inverts elements of $\mathcal{G}/V$. For each $\Lambda(\mathcal{G}/V)$-module M we regard $\mathrm{Hom}_{\Lambda(\mathcal{G}/V)}(M,\Lambda(\mathcal{G}/V))$ as a $\Lambda(\mathcal{G}/V)$-module by setting $x(\theta)(m) := \theta(m)x^\sim$ for all x in $\Lambda(\mathcal{G}/V)$, θ in $\mathrm{Hom}_{\Lambda(\mathcal{G}/V)}(M,\Lambda(\mathcal{G}/V))$ and m in M. We note in particular that if M is a finitely generated projective $\Lambda(\mathcal{G}/V)$-module, then so is $\mathrm{Hom}_{\Lambda(\mathcal{G}/V)}(M,\Lambda(\mathcal{G}/V))$. This therefore induces an exact functor $R\,\mathrm{Hom}_{\Lambda(\mathcal{G}/V)}(-,\Lambda(\mathcal{G}/V)[-2])$ from $D^-(\Lambda(\mathcal{G}/V))$ to $D^+(\Lambda(\mathcal{G}/V))$ which restricts to give an exact functor from $D^{\mathrm{p}}(\Lambda(\mathcal{G}/V))$ to $D^{\mathrm{p}}(\Lambda(\mathcal{G}/V))$. We set

$$D^\bullet_{K^V/k,\Sigma} := R\,\mathrm{Hom}_{\Lambda(\mathcal{G}/V)}(C^\bullet_{K^V/k,\Sigma},\Lambda(\mathcal{G}/V)[-2])$$

and in the next result record the basic properties of such complexes.
We recall that a $\Lambda(\mathcal{G})$-module M is said to be 'torsion' if the scalar extension $Q(\Lambda(\mathcal{G}))\otimes_{\Lambda(\mathcal{G})} M$ vanishes. For each place v in Σ we write $\mathcal{G}_v$ for the decomposition subgroup in $\mathcal{G}$ of a choice of place of K lying above v. We also write k^∞ for the constant $\mathbb{Z}_p$-extension of k.

Proposition 4.1.

(i) *For each closed normal subgroup V of $\mathcal{G}$ there is a natural isomorphism in $D^{\mathrm{p}}(\Lambda(\mathcal{G}/V))$ of the form* $\Lambda(\mathcal{G}/V)\otimes^{\mathbb{L}}_{\Lambda(\mathcal{G})} D^\bullet_{K/k,\Sigma}\cong D^\bullet_{K^V/k,\Sigma}$.

(ii) *If for each v in Σ the decomposition subgroup $\mathcal{G}_v$ is both non-trivial and normal, then $D^\bullet_{K/k,\Sigma}$ is acyclic outside degree 1 and the $\Lambda(\mathcal{G})$-module $H^1(D^\bullet_{K/k,\Sigma})$ is both torsion and of projective dimension one.*

(iii) *If K contains k^∞, then $D^\bullet_{K/k,\Sigma}$ is acyclic outside degree 1 and $H^1(D^\bullet_{K/k,\Sigma})$ is finitely generated over $\Lambda(K/k^\infty)$ and both torsion and of projective dimension one over $\Lambda(\mathcal{G})$.*

Proof. Regarding Σ as fixed, for any closed normal subgroup V of $\mathcal{G}$ we set $C^\bullet_{K^V/k}:=C^\bullet_{K^V/k,\Sigma}$ and $D^\bullet_{K^V/k}:=D^\bullet_{K^V/k,\Sigma}$. Then it is well known that $C^\bullet_{K^V/k}$, and hence also its dual $D^\bullet_{K^V/k}$, belongs to $D^{\mathrm{p}}(\Lambda(\mathcal{G}/V))$ (see, for example, [18, Prop. 1.6.5(2)]). The rest of claim (i) is true because the standard 'projection formula' isomorphism $\Lambda(\mathcal{G}/V)\otimes^{\mathbb{L}}_{\Lambda(\mathcal{G})}C^\bullet_{K/k}\cong C^\bullet_{K^V/k}$ (cf. [18, Prop. 1.6.5(3)]) induces an isomorphism in $D^{\mathrm{p}}(\Lambda(\mathcal{G}/V))$ of the form

$$\begin{aligned}\Lambda(\mathcal{G}/V)\otimes^{\mathbb{L}}_{\Lambda(\mathcal{G})}D^\bullet_{K/k}&=\Lambda(\mathcal{G}/V)\otimes^{\mathbb{L}}_{\Lambda(\mathcal{G})}R\,\mathrm{Hom}_{\Lambda(\mathcal{G})}(C^\bullet_{K/k},\Lambda(\mathcal{G})[-2])\\&\cong R\,\mathrm{Hom}_{\Lambda(\mathcal{G}/V)}(\Lambda(\mathcal{G}/V)\otimes^{\mathbb{L}}_{\Lambda(\mathcal{G})}C^\bullet_{K/k},\Lambda(\mathcal{G}/V)[-2])\\&\cong R\,\mathrm{Hom}_{\Lambda(\mathcal{G}/V)}(C^\bullet_{K^V/k},\Lambda(\mathcal{G}/V)[-2])=D^\bullet_{K^V/k}.\end{aligned}$$

To compute the groups $H^i(D^\bullet_{K/k})$ we write $O(\mathcal{G})$ for the set of open normal subgroups of $\mathcal{G}$ and for U in $O(\mathcal{G})$ set $D^\bullet_U:=R\,\mathrm{Hom}_{\mathbb{Z}_p}(C^\bullet_{K^U/k},\mathbb{Z}_p[-2])$, regarded as an element of $D^{\mathrm{p}}(\mathbb{Z}_p[\mathcal{G}/U])$ via the natural contragredient action of $\mathcal{G}/U$. For each finitely generated projective $\Lambda(\mathcal{G})$-module P there is a natural isomorphism of (left) $\Lambda(\mathcal{G})$-modules $\mathrm{Hom}_{\Lambda(\mathcal{G})}(P,\Lambda(\mathcal{G}))\cong\varprojlim_{U\in O(\mathcal{G})}\mathrm{Hom}_{\mathbb{Z}_p}(P_U,\mathbb{Z}_p)$ where P_U denotes the module of U-coinvariants of P and $\mathrm{Hom}_{\mathbb{Z}_p}(P_U,\mathbb{Z}_p)$ is endowed with the natural contragredient action of $\Lambda(\mathcal{G})$. This fact combines with the argument of [18, Prop. 1.6.5] to imply that $H^i(D^\bullet_{K/k})$ identifies with $\varprojlim_{U\in O(\mathcal{G})}H^i(D^\bullet_U)$ where the limit is taken with respect to the natural transition morphisms. Now from Lemma 3.3 and [5, Lem. 1] one knows that $H^i(D^\bullet_U)=0$ if $i\notin\{0,1\}$, that $H^0(D^\bullet_U)$ is canonically isomorphic to $\mathbb{Z}_p\otimes\mathcal{O}^\times_{K^U,\Sigma}$ and that there is a canonical exact sequence of $\mathbb{Z}_p[\mathcal{G}/U]$-modules of the form

$$0\to\mathbb{Z}_p\otimes\mathrm{Cl}(\mathcal{O}_{K^U,\Sigma})\to H^1(D^\bullet_U)\to\mathbb{Z}_p\otimes Y_{K^U,\Sigma}\to\mathbb{Z}_p\to 0.$$

If $U\subseteq U'$ with $U'\in O(\mathcal{G})$ then, with respect to the above identifications, the transition morphisms $H^0(D^\bullet_U)\to H^0(D^\bullet_{U'})$ and $H^1(D^\bullet_U)_{\mathrm{tor}}\to H^1(D^\bullet_{U'})_{\mathrm{tor}}$ are induced by the field theoretic norm $N_{U,U'}:K^U\to K^{U'}$, whilst the transition morphism $H^1(D^\bullet_U)_{\mathrm{tf}}\to H^1(D^\bullet_{U'})_{\mathrm{tf}}$ is induced by the direct sum over each place v in Σ of the maps $N'_{U,U',v}:\mathbb{Z}_p\otimes Y_{K^U,\{v\}}\to\mathbb{Z}_p\otimes Y_{K^{U'},\{v\}}$ defined by setting $N'_{U,U',v}(\epsilon)(v)=\sum_{w\mid v}\epsilon(w)$ for all $\epsilon:\Sigma(K^U)\to\mathbb{Z}_p$ (for details of the calculation of such transition morphisms see, for example, [9, Prop. 5.1]). Passing to the limit over U in $O(\mathcal{G})$ of the above displayed exact sequences preserves exactness since all involved modules are compact and therefore gives rise to an exact sequence of the form

$$0\to\varprojlim_{N_{U,U'}}\mathbb{Z}_p\otimes\mathrm{Cl}(\mathcal{O}_{K^U,\Sigma})\to H^1(D^\bullet_{K/k})\to\bigoplus_{v\in\Sigma}\Lambda(\mathcal{G})\otimes_{\Lambda(\mathcal{G}_v)}\mathbb{Z}_p\to\mathbb{Z}_p\to 0. \tag{4}$$

For each U there is also a natural exact sequence of $\mathbb{Z}_p[\mathcal{G}/U]$-modules

$$0 \to \mathbb{Z}_p \otimes \mathcal{O}^\times_{K^U,\Sigma} \to \mathbb{Z}_p \otimes Y^0_{K^U,\Sigma} \to \mathbb{Z}_p \otimes \mathrm{Cl}^0_{K^U} \to \mathbb{Z}_p \otimes \mathrm{Cl}(\mathcal{O}_{K^U,\Sigma}) \to \mathbb{Z}_p/p^{m(U)} \to 0$$

where we write Cl^0_F for the degree 0 divisor class group of a function field F, $m(U)$ is a non-negative integer and $\mathcal{G}/U$-acts trivially on $\mathbb{Z}_p/p^{m(U)}$. This sequence is compatible with the maps induced by the norms $N_{U,U'}$ as $U \le U'$ run over $O(\mathcal{G})$ and if $v \in \Sigma$ then the limit $\varprojlim_{N_{U,U'}} \mathbb{Z}_p \otimes Y_{K^U,v}$ vanishes if v has infinite residue degree in K/k and is otherwise canonically isomorphic to $\Lambda(\mathcal{G}) \otimes_{\Lambda(\mathcal{G}_v)} \mathbb{Z}_p$. Hence, upon passing to the limit over U in $O(\mathcal{G})$, the last displayed sequence induces an exact sequence

$$(5)\quad 0 \to H^0(D^\bullet_{K/k}) \to \bigoplus_{v\in\Sigma^K_{\mathrm{fin}}} \Lambda(\mathcal{G}) \otimes_{\Lambda(\mathcal{G}_v)} \mathbb{Z}_p \to X_{K/k} \to \varprojlim_{N_{U,U'}} \mathbb{Z}_p \otimes \mathrm{Cl}(\mathcal{O}_{K^U,\Sigma}) \to 0$$

where Σ^K_{fin} denotes the subset of Σ comprising those places which have finite residue degree in K/k and we set $X_{K/k} := \varprojlim_{N_{U,U'}} \mathbb{Z}_p \otimes \mathrm{Cl}^0_{K^U}$. If now the subgroup $\mathcal{G}_v$ is both non-trivial and normal, then $\Lambda(\mathcal{G}) \otimes_{\Lambda(\mathcal{G}_v)} \mathbb{Z}_p$ is annihilated by (left) multiplication by $g_v - 1 \in \Lambda(\mathcal{G})$ for any $g_v \in \mathcal{G}_v$ and so is a torsion $\Lambda(\mathcal{G})$-module. From (4) and (5) it thus follows that, under the assumptions of claim (ii), the $\Lambda(\mathcal{G})$-modules $H^0(D^\bullet_{K/k})$ and $H^1(D^\bullet_{K/k})$ are both torsion. To complete the proof of claim (ii) it is thus enough to show $D^\bullet_{K/k}$ is represented by a complex in $C^{\mathrm{p}}(\Lambda(\mathcal{G}))$ of the form $P^0 \to P^1$ (since then $H^0(D^\bullet_{K/k})$ is isomorphic to a torsion submodule of the projective module P^0, and hence vanishes, and so $P^0 \to P^1$ is a projective resolution of $H^1(D^\bullet_{K/k})$). To do this it suffices to fix a complex $Q^\bullet$ in $C^{\mathrm{p}}(\Lambda(\mathcal{G}))$ of the form $\cdots \to Q^{-1} \xrightarrow{d^{-1}} Q^0 \xrightarrow{d^0} Q^1$ that is isomorphic in $D(\Lambda(\mathcal{G}))$ to $D^\bullet_{K/k}$ and prove $\mathrm{cok}(d^{-1})$ is a projective $\Lambda(\mathcal{G})$-module (since then we will obtain a complex of the required form by setting $P^0 := \mathrm{cok}(d^{-1})$ and $P^1 := Q^1$ and using the map induced by d^0 as the differential). But for each U in $O(\mathcal{G})$ the complex $\tau_{\ge 0}Q^\bullet_U$ given by the truncation $\mathrm{cok}(d^{-1}_U) \xrightarrow{d^0_U} Q^1_U$ to degrees 0 and 1 of $Q^\bullet_U := \mathbb{Z}_p[\mathcal{G}/U] \otimes_{\Lambda(\mathcal{G})} Q^\bullet$ is isomorphic in $D^{\mathrm{p}}(\mathbb{Z}_p[\mathcal{G}/U])$ to $\mathbb{Z}_p[\mathcal{G}/U] \otimes^{\mathbb{L}}_{\Lambda(\mathcal{G})} D^\bullet_{K/k} \cong D^\bullet_U$ and so the $\mathbb{Z}_p[\mathcal{G}/U]$-module $\mathrm{cok}(d^{-1}_U)$ is both $\mathbb{Z}_p$-free (because $\ker(d^0_U) \cong H^0(D^\bullet_U) \cong \mathbb{Z}_p \otimes \mathcal{O}^\times_{K^U,\Sigma}$ and Q^1_U are $\mathbb{Z}_p$-free) and of finite projective dimension (because Q^1_U is projective and $\tau_{\ge 0}Q^\bullet_U \cong D^\bullet_U$ belongs to $D^{\mathrm{p}}(\mathbb{Z}_p[\mathcal{G}/U])$). From [2, Th. 8] we deduce that $\mathrm{cok}(d^{-1}_U)$ is a projective $\mathbb{Z}_p[\mathcal{G}/U]$-module. Upon passing to the limit over U (and noting that all involved modules are compact) we then deduce that $\mathrm{cok}(d^{-1}) \cong \varprojlim_U \mathrm{cok}(d^{-1})_U = \varprojlim_U \mathrm{cok}(d^{-1}_U)$ is a projective $\Lambda(\mathcal{G})$-module, as required. This proves claim (ii).

If now $k^\infty \subseteq K$, then each place has infinite residue degree in K/k so Σ^K_{fin} is empty and (5) implies that the group $H^0(D^\bullet_{K/k})$ vanishes. Since $D^\bullet_{K/k}$ is acyclic outside degree 1 the same argument as in (ii) then shows that the finitely generated $\Lambda(\mathcal{G})$-module $H^1(D^\bullet_{K/k})$ has projective dimension one. We next observe that any $\Lambda(\mathcal{G})$-module that is finitely generated over $\Lambda(K/k^\infty)$ is automatically a torsion module. Hence, to complete the proof of claim (iii) we choose an open subgroup J of G_{K/k^∞} that is pro-p and normal in $\mathcal{G}$ and note that, by Nakayama's Lemma, $H^1(D^\bullet_{K/k})$ is finitely generated over $\Lambda(K/k^\infty)$ if the module of J-coinvariants $H^1(D^\bullet_{K/k})_J$ is finitely generated over $\mathbb{Z}_p$. Now, since both $D^\bullet_{K/k}$ and $D^\bullet_{K^J/k}$ are acyclic in degrees greater than 1, the isomorphism in claim (i) with $V = J$ induces an isomorphism $H^1(D^\bullet_{K/k})_J \cong H^1(D^\bullet_{K^J/k})$ and so in the rest of the argument we assume K/k^∞ is finite. In particular, we may choose a finite extension k' of k with $K = (k')^\infty$. We write κ' for the constant field of k' and C for the complete smooth curve over κ' for which $k' = \kappa'(C)$. We let A be the Jacobian variety of C over κ' and fix a power N of p with $A[p] \subseteq A(\mathbb{F}_N)$ if p is odd and $A[4] \subseteq A(\mathbb{F}_N)$ is $p = 2$, where $\mathbb{F}_N$ is the field with N elements. We set $\tilde{k} := k'\mathbb{F}_N$ and $\tilde{K} := K\mathbb{F}_N$ and write $\tilde{\kappa}$ for the constant field of $\tilde{k}$. Then $\tilde{K} = \tilde{k}^\infty$, $A[p] \subseteq A(\tilde{\kappa})$ and the isomorphism $H^1(D^\bullet_{\tilde{K}/k})_{G_{\tilde{K}/K}} \cong H^1(D^\bullet_{K/k})$ (induced by claim (i)) means that it is enough to prove that $H^1(D^\bullet_{\tilde{K}/k})$ is a finitely generated $\mathbb{Z}_p$-module. To do this we set $\tilde{\mathcal{G}} := G_{\tilde{K}/k}$ and for each place v of k write $\tilde{\mathcal{G}}_v$ for the decomposition subgroup of v in $\tilde{\mathcal{G}}$. Now, since no place splits completely in k^∞/k, each group $\tilde{\mathcal{G}}_v$ is open in $\tilde{\mathcal{G}}$ and so the module $\Lambda(\tilde{\mathcal{G}}) \otimes_{\Lambda(\tilde{\mathcal{G}}_v)} \mathbb{Z}_p$ is finitely generated over $\mathbb{Z}_p$. In view of (4) and (5) (with K and $\mathcal{G}$ replaced by $\tilde{K}$ and $\tilde{\mathcal{G}}$) it is therefore enough for us to prove that the $\mathbb{Z}_p$-module $X_{\tilde{K}/k}$ is finitely generated. Our proof of claim (iii) is thus completed by Lemma 4.2 below. □

Lemma 4.2. *The $\mathbb{Z}_p$-module $X_{\tilde{K}/k}$ is generated by $\dim_{\mathbb{F}_p}(A[p])$ elements.*

Proof. (This proof was shown to us by K-S. Tan.) For each natural number n let $\tilde{k}_n$ and $\tilde{\kappa}_n$ be the unique subfield of $\tilde{K}$ of degree p^n over $\tilde{k}$ and the unique extension of $\tilde{\kappa}$ of degree p^n respectively. Then $\mathbb{Z}_p \otimes \mathrm{Cl}^0_{\tilde{k}_n}$ identifies with the subgroup $A(\tilde{\kappa}_n)[p^\infty]$ of p-power torsion points in $A(\tilde{\kappa}_n)$. The assumption $A[p] \subseteq A(\tilde{\kappa})$ thus implies that $(\mathbb{Z}_p \otimes \mathrm{Cl}^0_{\tilde{k}_n})[p] = A[p]$ for all n. To prove that $X_{\tilde{K}/k} = \varprojlim_n(\mathbb{Z}_p \otimes \mathrm{Cl}^0_{\tilde{k}_n})$ can be generated by $\dim_{\mathbb{F}_p}(A[p])$ elements it is therefore enough to show that each norm map $\mathrm{N}_{n+1,n} : A(\tilde{\kappa}_{n+1})[p^\infty] \to A(\tilde{\kappa}_n)[p^\infty]$ is surjective. To do this we fix P in $A(\tilde{\kappa}_n)[p^\infty]$, a p-division point Q of P and a generator σ of $G_{\tilde{\kappa}^c/\tilde{\kappa}_n}$. Then $R := \sigma(Q) - Q$ belongs to $A[p] \subseteq A(\tilde{\kappa})$. This implies $\sigma(R) = R$ and hence $\sigma^i(Q) = Q + iR$ for each non-negative integer i. This in turn implies that σ^p fixes Q, so Q belongs to $A(\tilde{\kappa}_{n+1})$, and also that $\mathrm{N}_{n+1,n}(Q) = \sum_{i=0}^{i=p-1} \sigma^i(Q) = pQ + \sum_{i=0}^{i=p-1} iR$. If $p \neq 2$, then this sum is equal to $pQ + (\frac{p-1}{2})pR = pQ = P$ and so $P = \mathrm{N}_{n+1,n}(Q)$, as required. If $p = 2$, then the sum is equal to $pQ + R = P + R$. But $R \in A[2]$ and, since $A[4] \subseteq A(\tilde{\kappa})$, there exists an element R' of $A(\tilde{\kappa})$ with $R = 2R' = \mathrm{N}_{n+1,n}(R')$. Hence one has $P = (P + R) - R = \mathrm{N}_{n+1,n}(Q - R')$, as required. □

Remark 4.3. Assume that $\mathcal{G} := G_{K/k}$ is abelian. If the quotient $\mathcal{G}/V$ has rank at least one and no place in Σ splits completely in K^V/k, then the results of Proposition 4.1(i) and (ii) combine with the functorial properties of $\mathbf{d}_{\Lambda(\mathcal{G})}$ described in §2.2(i) to imply that $\vartheta_{\mathrm{can}}(\mathbf{d}_{\Lambda(\mathcal{G}/V)}(D^\bullet_{K^V/k,\Sigma})) = \pi_{V,*}(\vartheta_{\mathrm{can}}(\mathbf{d}_{\Lambda(\mathcal{G})}(D^\bullet_{K/k,\Sigma})))$ where π_V is the projection homomorphism $\Lambda(\mathcal{G}) \to \Lambda(\mathcal{G}/V)$ and in both cases the morphism ϑ_{can} is as described in §2.2(ii). If $\mathcal{G}$ is isomorphic to $H \times \mathbb{Z}_p^d$ for some finite abelian group H of order prime to p and natural number d and no place in Σ splits completely in K/k, then Proposition 4.1(ii) also combines with the final assertion of §2.2(iv) and the fact that characteristic ideals are multiplicative in the exact sequences of torsion modules (4) and (5) to imply that

$$\vartheta_{\mathrm{can}}(\mathbf{d}_{\Lambda(\mathcal{G})}(D^\bullet_{K/k,\Sigma})) = (\mathrm{ch}_{\Lambda(\mathcal{G})}(X_{K/k})^{-1}\mathrm{ch}_{\Lambda(\mathcal{G})}(\mathbb{Z}_p)\prod_{v\in\Sigma\setminus\Sigma^K_{\mathrm{fin}}}\mathrm{ch}_{\Lambda(\mathcal{G})}(\Lambda(\mathcal{G}/\mathcal{G}_v))^{-1},0).$$

4.2. **The main conjecture.** For any finite abelian extension F/k that is unramified outside a finite non-empty set of places Σ the value $\theta_{F/k,\Sigma}$ of $Z_{F/k,\Sigma}(t)$ at $t=1$ is both well-defined and belongs to $\mathbb{Z}_p[G_{F/k}]$ (cf. [39, Chap. V, Th. 1.2] and note $\theta_{F/k,\Sigma}$ is equal to $\Theta_\Sigma(1)$ in the notation of loc. cit. and that, since $p=\mathrm{char}(k)$, the integer e which occurs in [39, Chap. V, Th. 1.2] is coprime to p). Further, if $\tilde{F}$ is an intermediate field of F/k and $\tilde{F}/k$ is unramified outside a non-empty subset $\tilde{\Sigma}$ of Σ, then

$$\pi_{F,\tilde{F}}(\theta_{F/k,\Sigma}) = \theta_{\tilde{F}/k,\tilde{\Sigma}}\prod_{v\in\Sigma\setminus\tilde{\Sigma}}(1-\mathrm{Fr}_{\tilde{F}}(v)^{-1}) \tag{6}$$

where $\pi_{F,\tilde{F}}$ is the natural projection $\mathbb{Z}_p[G_{F/k}]\to\mathbb{Z}_p[G_{\tilde{F}/k}]$ and for any abelian extension L/k and any place v that does not ramify in L/k we write $\mathrm{Fr}_L(v)$ for the (arithmetic) Frobenius of v in $G_{L/k}$. In particular, for any abelian p-adic Lie extension K/k that is unramified outside Σ we may set

$$\theta_{K/k,\Sigma} := \varprojlim_F \theta_{F/k,\Sigma}\in\Lambda(K/k)$$

where F runs over all finite extensions of k inside K and the limit is taken with respect to the maps $\pi_{F,\tilde{F}}$. We shall deduce the equality in the next result from the validity of a natural main conjecture in geometric Iwasawa theory.

Proposition 4.4. *Let K/k be an abelian p-adic Lie extension of finite rank that is unramified outside a finite non-empty set of places Σ and set $\mathcal{G}:=G_{K/k}$. If no place of Σ splits completely in K/k, then one has $\vartheta_{\mathrm{can}}(\mathbf{d}_{\Lambda(\mathcal{G})}(D^\bullet_{K/k,\Sigma}))^{-1}=\theta_{K/k,\Sigma}\cdot\mathbf{1}_{\Lambda(\mathcal{G})}$.*

Proof. We first choose a Galois extension K'/k that is ramified precisely at the places in Σ and such that both $K\subseteq K'$ and $\mathcal{G}':=G_{K'/k}$ is (topologically) isomorphic to a group of the form $H\times\Gamma$ with H a finite abelian group of order prime to p and Γ isomorphic to $\mathbb{Z}_p^d$ for some natural number d (the existence of such an extension K'/k is guaranteed by the results of Kisilevsky in [23]). Then the functorial behaviour described in Remark 4.3 and (6) imply that it is enough to prove the claimed equality after replacing K/k by K'/k and so we shall assume henceforth that K/k is ramified at every place in Σ and that $\mathcal{G}=H\times\Gamma$ with H and Γ as above.
We set $C^\bullet:=C^\bullet_{K/k,\Sigma}$ and $D^\bullet:=D^\bullet_{K/k,\Sigma}$. Then, since $\Lambda(\mathcal{G})$ is semi-local, Proposition 4.1(ii) implies that $D^\bullet$ is isomorphic in $D^p(\Lambda(\mathcal{G}))$ to a complex of the form

$$\tilde{D}^\bullet:\Lambda(\mathcal{G})^m\xrightarrow{\delta}\Lambda(\mathcal{G})^m$$

where the first term is placed in degree 0 and the differential δ is injective. Now if one endows $\mathrm{Hom}_{\Lambda(\mathcal{G})}(\Lambda(\mathcal{G})^m,\Lambda(\mathcal{G}))$ with the $\Lambda(\mathcal{G})$-action described just prior to Proposition 4.1, then the map $\phi\mapsto\phi(1)^\sim$ induces an isomorphism of $\Lambda(\mathcal{G})$-modules $\mathrm{Hom}_{\Lambda(\mathcal{G})}(\Lambda(\mathcal{G})^m,\Lambda(\mathcal{G}))\cong\Lambda(\mathcal{G})^m$. Thus, since there are natural isomorphisms $C^\bullet\cong R\,\mathrm{Hom}_{\Lambda(\mathcal{G})}(D^\bullet,\Lambda(\mathcal{G})[-2])\cong\mathrm{Hom}_{\Lambda(\mathcal{G})}(\tilde{D}^\bullet,\Lambda(\mathcal{G})[-2])$, one has $\vartheta_{\mathrm{can}}(\mathbf{d}_{\Lambda(\mathcal{G})}(D^\bullet))^{-1}=$

$\det_{\Lambda(\mathcal{G})}(\delta) \cdot \mathbf{1}_{\Lambda(\mathcal{G})}$ and $\vartheta_{\mathrm{can}}(\mathbf{d}_{\Lambda(G)}(C^\bullet))^{-1} = \det_{\Lambda(\mathcal{G})}(\delta)^{\sim} \cdot \mathbf{1}_{\Lambda(\mathcal{G})}$. This implies, in particular, that $\vartheta_{\mathrm{can}}(\mathbf{d}_{\Lambda(G)}(D^\bullet)) = \tau_*(\vartheta_{\mathrm{can}}(\mathbf{d}_{\Lambda(G)}(C^\bullet)))$ where τ is the endomorphism of $\Lambda(\mathcal{G})$ sending each element x to $x^{\sim}$. Now, since $C^\bullet \cong \mathrm{Hom}_{\Lambda(\mathcal{G})}(\tilde{D}^\bullet, \Lambda(\mathcal{G})[-2])$ is acyclic outside degree 2 and $H^2(C^\bullet)$ is a torsion $\Lambda(\mathcal{G})$-module of projective dimension one, the final assertion of §2.2(iv) implies that $\vartheta_{\mathrm{can}}(\mathbf{d}_{\Lambda(\mathcal{G})}(C^\bullet))^{-1} = (\mathrm{ch}_{\Lambda(\mathcal{G})}(H^2(C^\bullet)), 0)$ and hence that $\vartheta_{\mathrm{can}}(\mathbf{d}_{\Lambda(G)}(D^\bullet))^{-1} = \tau_*(\vartheta_{\mathrm{can}}(\mathbf{d}_{\Lambda(G)}(C^\bullet))^{-1}) = (\tau(\mathrm{ch}_{\Lambda(\mathcal{G})}(H^2(C^\bullet))), 0) = (\mathrm{ch}_{\Lambda(\mathcal{G})}(H^2(C^\bullet)^{\sim}), 0)$ where for each $\Lambda(\mathcal{G})$-module M we set $M^{\sim} := \Lambda(\mathcal{G}) \otimes_{\Lambda(\mathcal{G}),\tau} M$. The claimed equality is therefore equivalent to the following equality of $\Lambda(\mathcal{G})$-ideals

$$\mathrm{ch}_{\Lambda(\mathcal{G})}(H^2(C^\bullet)^{\sim}) = \theta_{K/k,\Sigma} \cdot \Lambda(\mathcal{G}). \tag{7}$$

To prove this equality we note that $\mathbb{Z}_p[H]$ decomposes as a product $\prod_{i\in I} \mathcal{O}_i$ of unramified extensions of $\mathbb{Z}_p$ and so $\Lambda(\mathcal{G})$ is the corresponding product of power series rings in d-variables $\Lambda_i := \mathcal{O}_i[[\Gamma]]$. For any finitely generated torsion $\Lambda(\mathcal{G})$-module M there is an induced decomposition $M = \bigoplus_{i\in I} M_i$ with $M_i := \mathcal{O}_i \otimes_{\mathbb{Z}_p[H]} M$ and so it is enough to prove for each index i that $\mathrm{ch}_{\Lambda_i}(H^2(C^\bullet)_i^{\sim}) = \theta_{K/k,\Sigma} \cdot \Lambda_i$. To deduce this equality from the results of Crew in [17] one fixes the p-adic characters ρ_0 and χ in [17, p. 396] in the following way. After identifying the fundamental group $\pi_1(U_{k,\Sigma})$ with $G_{M_\Sigma(K)/k}$, where $M_\Sigma(K)$ is the maximal unramified outside Σ extension of K inside k^c, one fixes ρ_0 and χ to be the natural restriction homomorphisms $G_{M_\Sigma(K)/k} \twoheadrightarrow G_{K^H/k} \cong \Gamma \subset \Lambda_i^\times$ and $G_{M_\Sigma(K)/k} \twoheadrightarrow G_{K^\Gamma/k} \cong H \to \Lambda_i^\times$ where the last arrow is induced by the given projection $\mathbb{Z}_p[H] \to \mathcal{O}_i \subset \Lambda_i$. For these characters one has isomorphisms $H^2(C^\bullet)_i^{\sim} \cong H^2_c(U_{k,\Sigma}, \mathcal{F}) \cong M(\rho)$ where we write $\mathcal{F}$ for the sheaf $\mathcal{G}$ defined in [17, §3], the first isomorphism is because $\mathcal{F}$ is isomorphic to $\Lambda(\mathcal{G})_i$ rather than $\Lambda(\mathcal{G})_i^\#$, $\rho = \rho_0 \times \chi$, the module $M(\rho)$ is as defined in [17, §3] and the second isomorphism is the 'curious isomorphism' derived in [17, Rem. (3.2)] (the latter isomorphism is valid in this case because, by assumption, all places in Σ ramify in K/k). The required equality is thus equivalent to the equality $\mathrm{ch}_{\Lambda_i}(M(\rho)) = \theta_{K/k,\Sigma} \cdot \Lambda_i$ that is proved in [17, (3.1.1) and the discussion on p. 396]. □

Remark 4.5. Assume $\mathcal{G}$ is isomorphic to $H \times \mathbb{Z}_p^d$ with H finite abelian of order prime to p and $d > 0$. Then, since $\mathrm{ch}_{\Lambda(\mathcal{G})}(H^2(C^\bullet_{K/k,\Sigma})^{\sim}) = \tau(\mathrm{ch}_{\Lambda(\mathcal{G})}(H^2(C^\bullet_{K/k,\Sigma}))$ and $\theta_{K/k,\Sigma} \cdot \Lambda(\mathcal{G}) = \tau(\theta^{\sim}_{K/k,\Sigma} \cdot \Lambda(\mathcal{G}))$, the equality (7) shows that in this case the equality of Proposition 4.4 is also equivalent to $\mathrm{ch}_{\Lambda(\mathcal{G})}(H^2(C^\bullet_{K/k,\Sigma})) = \theta^{\sim}_{K/k,\Sigma} \cdot \Lambda(\mathcal{G})$.

4.3. **The descent theory.** In this subsection we shall prove the required equality (3) by combining Proposition 4.4 for the extension Fk^∞/k with the descent techniques developed by Venjakob and the present author in [11, 12]. To do this we set $K := Fk^\infty$ and (regarding K, F and Σ as fixed) also $D^\bullet := D^\bullet_{K/k,\Sigma}$ and $D^\bullet_0 := D^\bullet_{F,\Sigma,p}$. We write γ for the topological generator of $\Gamma := G_{k^\infty/k}$ given by $x \mapsto x^{q_k}$. For any finite group Δ we set $\Delta^* := \mathrm{Hom}(\Delta, \mathbb{Q}_p^{c\times})$. For each ρ in G^* we write $\mathbb{Z}_p[\rho]$ for the ring generated over $\mathbb{Z}_p$ by the values of ρ and β^0_ρ for the image under $\mathbb{Q}_p^c \otimes_{\mathbb{Z}_p[\rho]} -$ of the Bockstein homomorphism in degree 0 of the triple $(D^\bullet_\rho, \mathbb{Z}_p, \gamma)$, as defined in [11, §3.1, §3.3].
We first record how the general descent formalism of [12] applies in our setting.

Lemma 4.6. *If the complex $D^\bullet$ is 'semisimple at each ρ in G^*' in the sense of* [11, Def. 3.11]*, then in $\mathcal{P}(\mathbb{Z}_p[G])$ one has*

$$\vartheta_\beta(\mathbf{d}_{\mathbb{Z}_p[G]}(D_0^\bullet)) = (\sum_{\rho\in G^*} \theta^*_{K/k,\Sigma}(\rho)e_\rho)\cdot \mathbf{1}_{\mathbb{Z}_p[G]}$$

where $\beta : H^0(\mathbb{Q}_p[G]\otimes^{\mathbb{L}}_{\mathbb{Z}_p[G]} D_0^\bullet) \to H^1(\mathbb{Q}_p[G]\otimes^{\mathbb{L}}_{\mathbb{Z}_p[G]} D_0^\bullet)$ is the (unique) isomorphism of $\mathbb{Q}_p[G]$-modules with $e_\rho(\mathbb{Q}_p^c[G]\otimes_{\mathbb{Q}_p[G]}\beta) = -1\times\beta^0_\rho$ for each ρ in G^* and $\theta^*_{K/k,\Sigma}(\rho)$ is the 'leading term of $\theta_{K/k,\Sigma}$ at ρ' in the sense of [12, §2].

Proof. We note first that Proposition 4.4 applies to the extension K/k since no place splits completely in k^∞/k. From Proposition 4.1(i) and (iii) we also know that $\mathbb{Z}_p[G]\otimes^{\mathbb{L}}_{\Lambda(\mathcal{G})} D^\bullet$ is canonically isomorphic in $D^{\mathrm{p}}(\mathbb{Z}_p[G])$ to $D_0^\bullet$ and that $D^\bullet$ belongs to the subcategory $D^{\mathrm{p}}_S(\Lambda(K/k))$ of $D^{\mathrm{p}}(\Lambda(K/k))$ that is defined in [12, §1.4]. Hence, if $D^\bullet$ is semisimple at each ρ in G^*, then the equality of Proposition 4.4 implies that the formula of [12, Th. 2.2] (translated from the language of relative algebraic K-theory to that of graded invertible modules in the natural way) can be applied with respect to the data $G=\mathcal{G}, \overline{G}=G, \xi=\theta_{K/k,\Sigma}$ and $C=D^\bullet[1]$. Now for this data the complex $E^\bullet := e_\rho\mathbb{Q}_p^c[\overline{G}]\otimes^{\mathbb{L}}_{\Lambda(G)} C$ is isomorphic to $e_\rho\mathbb{Q}_p^c[\overline{G}]\otimes^{\mathbb{L}}_{\mathbb{Z}_p[\overline{G}]} D_0^\bullet[1]$ and so is acyclic outside degrees -1 and 0 and hence, by the assumption of semisimplicity, the homomorphism $\beta^0_\rho : H^{-1}(E^\bullet)\to H^0(E^\bullet)$ is bijective and the morphism $t(C_\rho) : \mathbf{d}_{\mathbb{Q}_p^c}(E^\bullet)\to \mathbf{1}_{\mathbb{Q}_p^c}$ which occurs in the formula of [12, Th. 2.2] is equal to $\vartheta_{\beta^0_\rho}$. From [12, Lem. 5.5(iv)] one also knows that the integer $r_G(C)(\rho)$ in [12, Th. 2.2] is equal to $-\dim_{\mathbb{Q}_p^c}(H^{-1}(E^\bullet))$ and hence that $(-1)^{r_G(C)(\rho)}t(C_\rho)$ is equal to the morphism $\vartheta_{-1\times\beta^0_\rho} = \vartheta_{e_\rho(\mathbb{Q}_p^c[G]\otimes_{\mathbb{Q}_p[G]}\beta)} = e_\rho(\mathbb{Q}_p^c[G]\otimes_{\mathbb{Q}_p[G]}\vartheta_\beta)$. Given these facts, one finds that $(\sum_{\rho\in G^*}\theta^*_{K/k,\Sigma}(\rho)e_\rho)\cdot\mathbf{1}_{\mathbb{Z}_p[G]}$ and $\vartheta_\beta(\mathbf{d}_{\mathbb{Z}_p[G]}(D_0^\bullet)) = \vartheta_\beta(\mathbf{d}_{\mathbb{Z}_p[G]}(D_0^\bullet[1]))^{-1}$ correspond to the left and right hand sides of the equality of [12, Th. 2.2], and hence that the claimed equality is indeed valid. □

Given Lemma 4.6, it is clear that the validity of (3), and hence of Theorem 3.1, follows directly from the next two results.

Proposition 4.7.

(i) *The complex $D^\bullet$ is semisimple at each ρ in G^*.*

(ii) *The morphism ϑ_β in Lemma 4.6 coincides with the morphism $\vartheta_{F,\Sigma,p}$ in (3).*

Proof. We first introduce some notation. For any complex $A^\bullet$ in $D^-(\Lambda(\mathcal{G}))$ we define an object of $D^-(\mathbb{Q}_p^c\otimes_{\mathbb{Z}_p}\Lambda(\Gamma))$ by setting $A^\bullet_{(\rho)} := \Lambda(\Gamma)\otimes^{\mathbb{L}}_{\Lambda(\mathcal{G})}(e_\rho\mathbb{Q}_p^c[G]\otimes_{\mathbb{Z}_p}A^\bullet)$ where we regard $e_\rho\mathbb{Q}_p^c[G]\otimes_{\mathbb{Z}_p}A^\bullet$ as a complex of $\Lambda(\mathcal{G})$-modules by letting each element σ of $\mathcal{G}$ act on the tensor product as $\sigma_G^{-1}\otimes_{\mathbb{Z}_p}\sigma$ with σ_G the image of σ in G. Then, in terms of the notation used in [11, §3.3], one has $A^\bullet_{(\rho)} = \mathbb{Q}_p^c\otimes_{\mathbb{Z}_p[\rho]}A^\bullet_\rho$. We also note that for each such complex $A^\bullet$ there is a natural isomorphism in $D(\mathbb{Q}_p^c)$ of the form

$$\mathbb{Z}_p\otimes^{\mathbb{L}}_{\Lambda(\Gamma)}A^\bullet_{(\rho)} \cong \mathbb{Z}_p\otimes^{\mathbb{L}}_{\Lambda(\mathcal{G})}(e_\rho\mathbb{Q}_p^c[G]\otimes_{\mathbb{Z}_p}A^\bullet) \cong e_\rho\mathbb{Q}_p^c[G]\otimes^{\mathbb{L}}_{\mathbb{Z}_p[G]}(\mathbb{Z}_p[G]\otimes^{\mathbb{L}}_{\Lambda(\mathcal{G})}A^\bullet). \tag{8}$$

In particular, this isomorphism (with $A^\bullet = D^\bullet$) implies $\mathbb{Z}_p\otimes^{\mathbb{L}}_{\Lambda(\Gamma)}D^\bullet_{(\rho)}$ is isomorphic in $D(\mathbb{Q}_p^c)$ to $e_\rho\mathbb{Q}_p^c[G]\otimes^{\mathbb{L}}_{\mathbb{Z}_p[G]}D_0^\bullet$ and so is acyclic outside degrees 0 and 1. It is thus clear (directly from the definition of semisimplicity) that $D^\bullet$ is semisimple at ρ precisely when the map β^0_ρ is bijective.

In fact we will show that for each ρ in G^* the homomorphism $-1\times\beta^0_\rho$ is induced by the composite

$$\begin{aligned}H^0(\mathbb{Q}_p[G]\otimes^{\mathbb{L}}_{\mathbb{Z}_p[G]} D^\bullet_0) = \mathbb{Q}_p\otimes_{\mathbb{Z}_p} H^0(D^\bullet_0)\cong \mathbb{Q}_p\otimes\mathcal{O}^\times_{F,\Sigma}\\ \xrightarrow{\mathbb{Q}_p\otimes \mathrm{d}_{F,\Sigma}} \mathbb{Q}_p\otimes Y^0_{F,\Sigma}\cong \mathbb{Q}_p\otimes_{\mathbb{Z}_p} H^1(D^\bullet_0) = H^1(\mathbb{Q}_p[G]\otimes^{\mathbb{L}}_{\mathbb{Z}_p[G]} D^\bullet_0).\end{aligned}$$

Note that if this description of $-1\times\beta^0_\rho$ is valid, then claim (ii) follows immediately from a direct comparison of the definitions of β and $\vartheta_{F,\Sigma,p}$ whilst claim (i) is also valid because the required bijectivity of β^0_ρ follows from the bijectivity of $\mathbb{Q}_p\otimes\mathrm{d}_{F,\Sigma}$.

It therefore suffices to prove the above description of $-1\times\beta^0_\rho$. To do this we fix a place v in Σ and write $c_v: Y_{F,\Sigma}\to\bigoplus_{w|v}\mathbb{Z}$ for the homomorphism sending each element of $Y_{F,\Sigma}$ to its respective coefficient at each place w of F above v. Then it is enough to prove that for each ρ in G^* the composite homomorphism

$$(\mathcal{O}^\times_{F,\Sigma})_\rho\cong H^0(D^\bullet_0)_\rho\xrightarrow{(-1)\times\beta^0_\rho} H^1(D^\bullet_0)_\rho\cong (Y^0_{F,\Sigma})_\rho\xrightarrow{\subseteq}(Y_{F,\Sigma})_\rho\xrightarrow{(c_v)_\rho}(\bigoplus_{w|v}\mathbb{Z})_\rho \tag{9}$$

is induced by $\mathbb{Q}_p\otimes\mathrm{d}_{F,\Sigma}$. Here, and in the sequel, for any finitely generated $\mathbb{Z}[G]$-module, resp. finitely generated $\mathbb{Z}_p[G]$-module, N we set $N_\rho := e_\rho\mathbb{Q}^c_p[G]\otimes_{\mathbb{Z}[G]}N$, resp. $N_\rho := e_\rho\mathbb{Q}^c_p[G]\otimes_{\mathbb{Z}_p[G]}N$.

To prove that (9) is induced by $\mathbb{Q}_p\otimes\mathrm{d}_{F,\Sigma}$ we write j for the open immersion $U_{k,\Sigma}\to C_k$, let Z_Σ denote the complement of $U_{k,\Sigma}$ in C_k and write $i: Z_\Sigma\to C_k$ for the natural closed immersion. If y is a place of any subfield of K, then we write $\kappa(y)$ for the residue field of y and set $Z_y := \mathrm{Spec}(\kappa(y))$. If y belongs to Σ we also write $i_y: Z_y\to C_k$ for the natural closed immersion. Then for any quotient $\mathcal{Q}$ of $\mathcal{G}$ there is a natural exact sequence of $\Lambda(\mathcal{Q})$-sheaves $0\to j_!(\Lambda(\mathcal{Q})^\#)\to\Lambda(\mathcal{Q})^\#\to i_*i^*(\Lambda(\mathcal{Q})^\#)\to 0$ on C_k and hence a composite morphism in $D^{\mathrm{p}}(\Lambda(\mathcal{Q}))$ of the form

$$\begin{aligned}\lambda_{\mathcal{Q}}: R\Gamma_{\text{ét}}(C_k, j_!(\Lambda(\mathcal{Q})^\#))^\vee[-2]\to R\Gamma_{\text{ét}}(Z_\Sigma, i^*(\Lambda(\mathcal{Q})^\#))^\vee[-1]\\ \cong\bigoplus_{y\in\Sigma}R\Gamma_{\text{ét}}(Z_y, i_y^*(\Lambda(\mathcal{Q})^\#))^\vee[-1]\to R\Gamma_{\text{ét}}(Z_v, i_v^*(\Lambda(\mathcal{Q})^\#))^\vee[-1],\end{aligned}$$

where for each complex C in $D^-(\Lambda(\mathcal{Q}))$ we set $C^\vee := R\,\mathrm{Hom}_{\Lambda(\mathcal{Q})}(C,\Lambda(\mathcal{Q}))$ (endowed with the natural contragredient $\mathcal{Q}$-action) and the last arrow is the 'projection to the summand at v' morphism. Now, if E is the subfield of K with $\mathcal{Q}=G_{E/k}$, then the complexes $R\Gamma_{\text{ét}}(Z_v, i_v^*(\Lambda(\mathcal{Q})^\#))^\vee[-1]$ and $\mathcal{E}(\mathcal{Q})^\bullet_v := R\Gamma_{\text{ét}}(Z_v, i_v^*(\Lambda(\mathcal{Q})^\#))$ are both canonically isomorphic to the direct sum over the set $S_v(E)$ of places w of E above v of the complexes

$$\Lambda(\mathcal{Q}_w/\mathcal{Q}^0_w)^\#\xrightarrow{1-\sigma_w}\Lambda(\mathcal{Q}_w/\mathcal{Q}^0_w)^\#$$

where the first term occurs in degree 0, $\mathcal{Q}_w$ and $\mathcal{Q}^0_w$ are the decomposition and inertia groups of w in $\mathcal{Q}$ and σ_w is the frobenius automorphism in $\mathcal{Q}_w/\mathcal{Q}^0_w\cong G_{\kappa(w)/\kappa(v)}$. Since $R\Gamma_{\text{ét}}(C_k, j_!(\Lambda(\mathcal{Q})^\#))^\vee[-2]$ is equal to $D^\bullet$, resp. is naturally isomorphic to $D^\bullet_0$, if $\mathcal{Q}=\mathcal{G}$, resp. $\mathcal{Q}=G$, the morphisms $\lambda_{\mathcal{G}}$ and λ_G therefore combine to induce a morphism of exact triangles in $D(\mathbb{Q}_p^c)$ of the form

$$\begin{array}{ccccccc} D^\bullet_{(\rho)} & \xrightarrow{\gamma-1} & D^\bullet_{(\rho)} & \longrightarrow & e_\rho\mathbb{Q}_p^c[G]\otimes^{\mathbb{L}}_{\mathbb{Z}_p[G]} D^\bullet_0 & \longrightarrow & D^\bullet_{(\rho)}[1] \\ \lambda_{\mathcal{G}}\downarrow & & \lambda_{\mathcal{G}}\downarrow & & \lambda_G\downarrow & & \lambda_{\mathcal{G}}[1]\downarrow \\ \mathcal{E}(\mathcal{G})^\bullet_{v,(\rho)} & \xrightarrow{\gamma-1} & \mathcal{E}(\mathcal{G})^\bullet_{v,(\rho)} & \longrightarrow & e_\rho\mathbb{Q}_p^c[G]\otimes^{\mathbb{L}}_{\mathbb{Z}_p[G]} \mathcal{E}(G)^\bullet_v & \longrightarrow & \mathcal{E}(\mathcal{G})^\bullet_{v,(\rho)}[1]. \end{array} \tag{10}$$

The upper triangle in this diagram is induced by the isomorphism $\mathbb{Z}_p\otimes^{\mathbb{L}}_{\Lambda(\Gamma)} D^\bullet_{(\rho)}\cong e_\rho\mathbb{Q}_p^c[G]\otimes^{\mathbb{L}}_{\mathbb{Z}_p[G]} D^\bullet_0$ already used above and lower triangle by (8) with $A^\bullet=\mathcal{E}(\mathcal{G})^\bullet_v$ and the natural descent isomorphism $\mathbb{Z}_p[G]\otimes^{\mathbb{L}}_{\Lambda(\mathcal{G})}\mathcal{E}(\mathcal{G})^\bullet_v\cong\mathcal{E}(G)^\bullet_v$.
To compute the cohomology of $\mathcal{E}(G)^\bullet_v$ we use the quasi-isomorphism

$$a_v:\mathcal{E}(G)^\bullet_v\to\bigoplus_{w\in S_v(F)} R\Gamma_{\text{ét}}(Z_v,\Lambda(G_w/G_w^0))$$

that is induced by the morphisms (of complexes of $\Lambda(G_w/G_w^0)$-modules)

$$\begin{array}{ccc} \Lambda(G_w/G_w^0)^{\#} & \xrightarrow{1-\sigma_w} & \Lambda(G_w/G_w^0)^{\#} \\ \| & & \downarrow{\scriptstyle -\sigma_w} \\ \Lambda(G_w/G_w^0) & \xrightarrow{1-\sigma_w} & \Lambda(G_w/G_w^0). \end{array} \tag{11}$$

We also identify the groups $H^0_{\text{ét}}(Z_v,\Lambda(G_w/G_w^0))$ and $H^1_{\text{ét}}(Z_v,\Lambda(G_w/G_w^0))$ with $H^0_{\text{ét}}(Z_w,\mathbb{Z}_p)\cong\mathbb{Z}_p$ and $H^1_{\text{ét}}(Z_w,\mathbb{Z}_p)\cong\mathrm{Hom}_{\mathrm{cont}}(G_{\kappa(w)^c/\kappa(w)},\mathbb{Z}_p)\cong\mathbb{Z}_p$ in the natural way, where the last map evaluates each homomorphism at the topological generator $\gamma^{\deg_k(w)}$ of $G_{\kappa(w)^c/\kappa(w)}$. We write $\tilde\beta^0_\rho$ for the image under $\mathbb{Q}_p^c\otimes_{\mathbb{Z}_p[\rho]}-$ of the Bockstein homomorphism in degree 0 of the triple $(\mathcal{E}(G)^\bullet_{v,\rho},\mathbb{Z}_p,\gamma)$. Then the homomorphisms β^0_ρ and $\tilde\beta^0_\rho$ are defined by using the maps that are induced on cohomology by the upper and lower exact triangle in diagram (10) and so, by passing to cohomology in this diagram, we obtain a commutative diagram

$$\begin{array}{ccccccc} (\mathcal{O}^\times_{F,\Sigma})_\rho & \xrightarrow{\cong} & H^0(D^\bullet_0)_\rho & \xrightarrow{\beta^0_\rho} & H^1(D^\bullet_0)_\rho & \xrightarrow{\cong} & (Y^0_{F,\Sigma})_\rho \\ \downarrow{\scriptstyle H^0(\lambda_G)'} & & \downarrow{\scriptstyle H^0(\lambda_G)} & & \downarrow{\scriptstyle H^1(\lambda_G)} & & \downarrow{\scriptstyle (c_v)_\rho} \\ (\bigoplus_{w|v}\mathbb{Z})_\rho & \xrightarrow{H^0(a_v)^{-1}} & H^0(\mathcal{E}(G)^\bullet_v)_\rho & \xrightarrow{\tilde\beta^0_\rho} & H^1(\mathcal{E}(G)^\bullet_v)_\rho & \xrightarrow{H^1(a_v)} & (\bigoplus_{w|v}\mathbb{Z})_\rho \end{array} \tag{12}$$

in which $H^0(\lambda_G)'$ is induced by the map $\mathcal{O}^\times_{F,\Sigma}\to\bigoplus_{w|v}\mathbb{Z}$ given by $(\mathrm{val}_w)_{w|v}$.
We now write κ for the constant field of k. Then the argument of Rapaport and Zink in [33, 1.2] (see also [5, Rem. 7]) implies that the map $\tilde\beta^0_\rho$ in (12) is induced by taking cup product with the pull back ϕ to $H^1_{\text{ét}}(Z_w,\mathbb{Z}_p)\cong\mathbb{Z}_p$ of the element of $H^1_{\text{ét}}(\mathrm{Spec}(\kappa),\mathbb{Z}_p)\cong\mathrm{Hom}_{\mathrm{cont}}(\Gamma,\mathbb{Z}_p)$ which sends γ to 1. Since $\phi(\gamma^{\deg_k(w)})=\deg_k(w)$ the composite homomorphism in the lower row of (12) is thus equal to multiplication by $-\deg_k(w)$ (the negative sign occurs here because (11) induces the identity map, resp. multiplication by -1, on cohomology in degree 0, resp. 1). From the commutativity of (12) we therefore obtain the required description of (9) and hence complete the proof of Proposition 4.7. $\square$

In the next result we fix an embedding $\mathbb{Q}^c \to \mathbb{Q}_p^c$ and use this to identify the groups $G^\wedge$ and G^*.

Lemma 4.8. *The element $\sum_{\rho\in G^*}\theta^*_{K/k,\Sigma}(\rho)e_\rho$ in Lemma 4.6 is equal to the element $Z^*_{F/k,\Sigma}(1)$ in (3).*

Proof. Recalling the definition (2) of $Z^*_{F/k,\Sigma}(1)$ it is enough to prove that for each ρ in G^* one has $\theta^*_{K/k,\Sigma}(\rho) = \lim_{t\to 1}(1-t)^{-r_{\check\rho}}Z_\Sigma(\check\rho,t)$, where $r_{\check\rho}$ is the algebraic order of $Z_\Sigma(\check\rho,t)$ at $t=1$. To do this we identify $\Lambda(k^\infty/k)$ with the power series ring $\mathbb{Z}_p[[T]]$ via the correspondence $\gamma = 1+T$. Then for each ρ in G^* the element $\theta^*_{K/k,\Sigma}(\rho)$ is defined to be the leading term $\Phi_\rho(\theta_{K/k,\Sigma})^*(0)$ at $T=0$ of the image of $\theta_{K/k,\Sigma}$ under the homomorphism $\Phi_\rho : Q(\Lambda(K/k))^\times \to Q(\mathcal{O}[[T]])^\times$ that is defined in [12, (5)].
We write k_ρ for the subfield of F fixed by $\ker(\rho)$, set $K_\rho := k_\rho k^\infty$ and regard ρ as an element of $\mathrm{Hom}(G_{K_\rho/k},\mathbb{Q}_p^{c\times})$ in the obvious way. We write Σ_ρ for the subset of Σ comprising those places which ramify in k_ρ/k (or equivalently, in K_ρ/k). To deal with the possibility that Σ_ρ is empty we also fix a place v_0 of k that does not belong to Σ and is inert in the (cyclic) extension k_ρ/k and set $\Sigma' := \Sigma\cup\{v_0\}$ and $\Sigma'_\rho := \Sigma_\rho\cup\{v_0\}$. Then from (6) one has

$$\Phi_\rho(\theta_{K/k,\Sigma}) = \Phi_\rho((1-\mathrm{Fr}_{K_\rho}(v_0)^{-1})^{-1}\prod_{v\in\Sigma\setminus\Sigma_\rho}(1-\mathrm{Fr}_{K_\rho}(v)^{-1}))\Phi_\rho(\theta_{K_\rho/k,\Sigma'_\rho}). \tag{13}$$

But, recalling the explicit definition of Φ_ρ, one computes for each $v\in\Sigma\setminus\Sigma_\rho\cup\{v_0\}$ that $\Phi_\rho(1-\mathrm{Fr}_{K_\rho}(v)^{-1}) = 1-\check\rho(\mathrm{Fr}_{k_\rho}(v))(1+T)^{-\deg_k(v)}$ and so $\Phi_\rho(1-\mathrm{Fr}_{K_\rho}(v)^{-1})^*(0) = c_{v,\rho}$ with

$$c_{v,\rho} := \begin{cases} 1-\check\rho(\mathrm{Fr}_{k_\rho}(v)), & \text{if } \check\rho(\mathrm{Fr}_{k_\rho}(v))\neq 1,\\ \deg_k(v), & \text{otherwise.}\end{cases}$$

We next recall that the algebraic order of $Z_{\Sigma'_\rho}(\check\rho,t)$ at $t=1$ is $\dim_{\mathbb{Q}_p^c}(e_\rho(\mathbb{Q}_p^c\otimes Y^0_{F,\Sigma'_\rho}))$ (cf. [39, p. 111]) and claim that $e_\rho(\mathbb{Q}_p^c\otimes Y^0_{F,\Sigma'_\rho})$ vanishes. Indeed $e_\rho(\mathbb{Q}_p^c\otimes Y^0_{F,\Sigma'_\rho})$ is a subspace of $e_\rho(\mathbb{Q}_p^c\otimes Y_{F,\Sigma'_\rho}) \cong \bigoplus_{x\in\Sigma'_\rho} e_\rho\mathbb{Q}_p^c[G/G_x]$ and for each x in Σ'_ρ the space $e_\rho\mathbb{Q}_p^c[G/G_x]$ vanishes since x has non-trivial decomposition group in k_ρ/k and so ρ is not trivial on G_x. The value of the series $\Phi_\rho(\theta_{K_\rho/k,\Sigma'_\rho})$ at $T=0$ is thus equal to $\rho(\theta_{k_\rho/k,\Sigma'_\rho}) = \rho(Z_{k_\rho/k,\Sigma'_\rho}(1)) = Z_{\Sigma'_\rho}(\check\rho,1)\notin\{0,\infty\}$ (where we have extended ρ in the obvious way to a homomorphism $\mathbb{Q}_p[G_{k_\rho/k}]\to\mathbb{Q}_p^c$) and so $\Phi_\rho(\theta_{K_\rho/k,\Sigma'_\rho})^*(0) = Z_{\Sigma'_\rho}(\check\rho,1)$. From (13) one therefore has

$$\theta^*_{K/k,\Sigma}(\rho) = (1-\check\rho(\mathrm{Fr}_{k_\rho}(v_0)))^{-1}(\prod_{v\in\Sigma\setminus\Sigma_\rho}c_{v,\rho})Z_{\Sigma'_\rho}(\check\rho,1) = \lim_{t\to 1}(1-t)^{-r_{\check\rho}}Z_\Sigma(\check\rho,t),$$

where the last equality follows from an explicit analysis of Euler factors. □

This completes our proof of Theorem 3.1.

5. Explicit consequences

In this section we derive several explicit consequences of Theorem 3.1 including Corollary 1.2. As in §3.1 we fix a finite abelian Galois extension F of k. We set $G := G_{F/k}$.

5.1. **Chinburg's conjectures.** In [5, Th. 4.1] it is shown that [loc. cit., Conj. C(F/k)] implies the validity of Chinburg's $\Omega(3)$-Conjecture for F/k. Corollary 1.2(i) is therefore a direct consequence of Theorem 3.1. To prove Corollary 1.2(ii) we now assume that F/k is tamely ramified. In this case Chinburg has proved the validity of his '$\Omega(2)$-Conjecture' (this follows upon combining [13, §4.2, Th. 4] with [16, Cor. 4.10]). In view of Corollary 1.2(i) we therefore need only recall that Chinburg has also proved that if the $\Omega(2)$-Conjecture and $\Omega(3)$-Conjecture are both valid for an extension F/k, then the $\Omega(1)$-Conjecture is automatically valid for F/k (this is a consequence of [13, §4.1, Th. 2 and the remarks which follow it]).

5.2. **Refined Stark conjectures.** The fact that Theorem 3.1 is equivalent to the function field case of [6, Conj. C(F/k)] follows directly from [5, Lem. 1(i), Lem. 2]. Corollary 1.2(iii) therefore follows immediately upon combining Theorem 3.1 with the main result (Th. 3.1) of [6]. We assume henceforth that F/k is unramified outside a finite non-empty set of places Σ of k. Then all of the assertions listed in Corollary 1.2(iv) relate to explicit congruences for the 'Stickelberger element' $\theta^{\rm stick}_{F/k,\Sigma} := \sum_{\chi,g} L_\Sigma(\chi,0)\check{\chi}(g)g$ where χ runs over $G^\wedge$ and g over G, and all of these congruences are easily verified whenever $L_\Sigma(\chi,0)=0$ for all χ. However, the results of [6, Cor. 4.1 and Cor. 4.3] show that [6, Conj. C(F/k)] implies the explicit congruences of [6, (14) and (18)] which specialise to recover all of the above congruences for $\theta^{\rm stick}_{F/k,\Sigma}$ but in general amount to a family of non-trivial congruence relations between the values at $s=0$ of higher order derivatives of the functions $L_\Sigma(\chi,s)$. Theorem 3.1 therefore implies that all of these congruences are valid unconditionally. This completes the proof of Corollary 1.2.

5.3. **Module structures and Fitting ideals.** The main result (Theorem 3.1) of Hayward and the present author in [10] is that, under suitable conditions, the validity of [6, Conj. C(F/k)] implies a family of explicit restrictions on the structures of $\mathcal{O}^\times_{F,\Sigma}$ and $\mathrm{Cl}(\mathcal{O}_{F,\Sigma})$ as G-modules. Theorem 3.1 therefore implies that all of these results are valid unconditionally.
In another direction, Macias Castillo [28] has shown that for many cases in which the element $\theta^{\rm stick}_{F/k,\Sigma}$ discussed in §5.2 vanishes the validity of [6, Conj. C(F/k)] implies a natural generalisation of Stickelberger's theorem in which the values at $s=0$ of higher order derivatives of the functions $L_\Sigma(\chi,s)$ are used to construct elements of $\mathbb{Z}[G]$ which annihilate the divisor class group Cl^0_F. Theorem 3.1 therefore now implies that all of these annihilation results are valid unconditionally.
The next result shows that Theorem 3.1 also gives explicit information about the Fitting ideals of certain natural Weil-étale cohomology groups. In this result we write κ for the constant field of k and recall that $q_k := |\kappa|$.

Proposition 5.1. *Let $\mathbb{Z}'$ be any finitely generated subring of $\mathbb{Q}$ for which the G-module $\mathbb{Z}'\otimes\kappa^\times$ is cohomologically-trivial (this is automatically the case if, for example, the highest common factor of q_k-1 and $|G|$ is invertible in $\mathbb{Z}'$). Then one has*

$$Z_{F/k,\Sigma}(1)\cdot \mathrm{Ann}_{\mathbb{Z}'[G]}(\mathbb{Z}'\otimes\kappa^\times) = \mathrm{Fit}_{\mathbb{Z}'[G]}(\mathbb{Z}'\otimes H^1_{\mathrm{W\acute{e}t}}(\mathcal{O}_{F,\Sigma},\mathbb{G}_m)).$$

Proof. For each character χ in $G^\wedge$ the algebraic order at $t = 1$ of $Z_\Sigma(\chi, t)$ is equal to $\dim_{\mathbb{Q}^c}(e_{\check\chi}(\mathbb{Q}^c \otimes \mathcal{O}^\times_{F,\Sigma}))$ (cf. [39, p. 111]). Since $\mathbb{Q}^c \otimes \mathcal{O}^\times_{F,\Sigma}$ is isomorphic as a $\mathbb{Q}^c[G]$-module to $\mathbb{Q}^c \otimes H^0(D^\bullet_{F,\Sigma})$ this formula implies in particular that $Z_{F/k,\Sigma}(1) = Z^*_{F/k,\Sigma}(1)e_0$ with $e_0 := \sum_\chi e_\chi$ where χ runs over all elements of $G^\wedge$ for which the module $e_\chi(\mathbb{Q}^c \otimes H^0(D^\bullet_{F,\Sigma}))$ vanishes. Thus, if in terms of the notation of [7, Th. 8.2.1], one takes $\Lambda = \mathbb{Z}', \mathcal{L}^*_\tau = Z^*_{F/k,\Sigma}(1), \Psi^\bullet = \mathbb{Z}'[G] \otimes^{\mathbb{L}}_{\mathbb{Z}[G]} D^\bullet_{F,\Sigma}$ and $\lambda_\tau = \vartheta_{F,\Sigma}$, then the validity of the (in general conjectural) equality of [7, Th. 8.2.1(i)] follows from the image under $\mathbb{Z}'[G] \otimes_{\mathbb{Z}[G]} -$ of the equality of Theorem 3.1, and the element $\mathcal{L}_\tau$ which occurs in [7, Th. 8.2.1(ii)] is equal to $Z_{F/k,\Sigma}(1)$. Since $\kappa^\times$ identifies with $H^0(D^\bullet_{F,\Sigma})_{\rm tor}$ our assumption that $\mathbb{Z}' \otimes \kappa^\times$ is cohomologically-trivial therefore implies that [7, Th. 8.2.1(iii)] applies in this setting to give an equality $Z_{F/k,\Sigma}(1) \cdot \mathrm{Fit}_{\mathbb{Z}'[G]}(\mathbb{Z}' \otimes \kappa^\times) = \mathrm{Fit}_{\mathbb{Z}'[G]}(\mathbb{Z}' \otimes H^1(D^\bullet_{F,\Sigma}))$. The claimed equality now follows because the module $\kappa^\times$ is cyclic, and hence $\mathrm{Fit}_{\mathbb{Z}'[G]}(\mathbb{Z}' \otimes \kappa^\times) = \mathrm{Ann}_{\mathbb{Z}'[G]}(\mathbb{Z}' \otimes \kappa^\times)$, and because the duality theorem in Weil-étale cohomology gives a natural isomorphism $H^1(D^\bullet_{F,\Sigma}) \cong H^1_{\rm W\acute{e}t}(\mathcal{O}_{F,\Sigma}, \mathbb{G}_m)$ (cf. [27, proof of Th. 6.5] and [5, Lem. 1(i)]). ☐

Appendix A. On Geometric Main Conjectures

David Burns, King Fai Lai and Ki-Seng Tan[1]

Throughout this appendix we refer to the main body of this article (the notation of which we often assume) as the reference [B].

The main results of Crew in [17] are proved using certain deep results of Katz and Messing [22] and Bloch and Illusie [21] concerning crystalline cohomology. In this appendix we combine [B, Prop. 4.1] with a strengthening of the main result of Kueh, Lai and Tan in [25] to show that all of the 'main conjectures of geometric Iwasawa theory' discussed in [17], and hence all of the results in [B], can also be deduced from Weil's description of the Zeta function of a finite abelian extension of global function fields of characteristic p in terms of ℓ-adic homology for any prime $\ell \neq p$, thereby avoiding any use of crystalline cohomology (or Drinfeld modules).

We thus fix an abelian extension K/k of global function fields of characteristic p that is unramified outside a finite non-empty set of places Σ of k and write k^∞ for the constant $\mathbb{Z}_p$-extension of k. Just as in the proof of [B, Prop 4.4] we shall assume that $G_{K/k} = H \times J$ for a finite (abelian) group H of order $\hbar$ prime to p and J a non-zero free $\mathbb{Z}_p$-module of finite rank. We therefore obtain a finite abelian extension of k by setting $k' := K^J$.

A.1. **Class groups.** We first present a proof of [B, Prop 4.4] using ℓ-adic cohomology for primes $\ell \neq p$. To do this we note that if G is any group of the form $H \times \Gamma$, then there is a natural ring isomorphism $\Lambda(G) \cong \mathbb{Z}_p[H] \otimes_{\mathbb{Z}_p} \Lambda(\Gamma)$. We may therefore write each element of $\Lambda(G)$ uniquely in the form $x = \sum_{h \in H} hx_h$ with x_h in $\Lambda(\Gamma)$ for each h in H. For each finite extension $\mathcal{O}$ of $\mathbb{Z}_p$ we set $\Lambda_{\mathcal{O}}(\Gamma) := \mathcal{O} \otimes_{\mathbb{Z}_p} \Lambda(\Gamma)$ and for each homomorphism $\omega : H \to \mathcal{O}^\times$ we then define the 'ω-projection of x' by setting

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$x_\omega := \sum_{h\in H}\omega(h)x_h \in \Lambda_{\mathcal{O}}(\Gamma)$. The association $x \mapsto x_\omega$ induces a homomorphism $\tilde{\omega} : \Lambda(G) \to \Lambda_{\mathcal{O}}(\Gamma)$ and hence a functor $\mathcal{P}(\omega) := \tilde{\omega}_*$ from $\mathcal{P}(\Lambda(G))$ to $\mathcal{P}(\Lambda_{\mathcal{O}}(\Gamma))$ and for each G-module N we write N_ω for the corresponding 'ω-eigenspace' $\Lambda_{\mathcal{O}}(\Gamma)\otimes_{\Lambda(G),\tilde{\omega}} N$. We recall that if Γ is topologically isomorphic to $\mathbb{Z}_p^e$ for some integer $e > 0$, then $\Lambda_{\mathcal{O}}(\Gamma)$ is isomorphic to a power series ring in e-variables over $\mathcal{O}$ and so is a unique factorization domain (cf. [4, Chap. 7, §3.9, Prop. 8]).

We note at the outset that, since $p \nmid \hbar$, it is enough to prove the validity for each homomorphism $\omega : H \to \mathcal{O}^\times$ of the image under $\mathcal{P}(\omega)$ of the equality of [B, Prop. 4.4] for the pair $(K/k, \Sigma)$. In this regard the following result is a key step (it is a natural generalisation of the main result of [25] and will be proved in §A.3).

Theorem A.1. *Let n be a natural number. Then there exists a finite set of places Σ' and an abelian extension L/k with all of the following properties: $\Sigma \subseteq \Sigma'$ and $|\Sigma'| > n$; L contains Kk^∞; the set of places which ramify in L/k is equal to Σ'; $G_{L/k}$ is isomorphic to $H \times \Gamma$ where H is as above and Γ is topologically isomorphic to $\mathbb{Z}_p^e$ for some $e > 1$; for each homomorphism $\omega : H \to \mathcal{O}^\times$ the element $(\theta_{L/k,\Sigma'})_\omega$ is an irreducible element of the unique factorization domain $\Lambda_{\mathcal{O}}(\Gamma)$.*

We fix n, Σ' and L as in Theorem A.1 and set $\theta_\omega := (\theta_{L/k,\Sigma'})_\omega$ and $R := \Lambda_{\mathcal{O}}(\Gamma)$. We recall that the Brumer-Stark conjecture, as proved by Deligne in [39, Chap. V, Th. 1.2], implies that θ_ω annihilates $X_{L/k,\omega}$. (In this regard we recall the remarks made at the beginning of §4.2 regarding differences in notation. We also remark in passing that all of the results in [39, Chap. V] rely solely on the interpretation of Zeta functions in terms of ℓ-adic homology for any prime $\ell \neq p$.) Thus, since θ_ω is irreducible, one has $\mathrm{ch}_R(X_{L/k,\omega}) = R\cdot\theta_\omega^{n_{\Sigma',\omega}}$ for some non-negative integer $n_{\Sigma',\omega}$. Now each place v in Σ' ramifies in L/k and no place splits completely in k^∞/k and so the modules $\Lambda(G/G_v)$ which occur in the formula of [B, Rem. 4.3] (with K and Σ replaced by L and Σ') are pseudo-null. Since this is also true of the module $\mathbb{Z}_p$ (as $e > 1$) this formula implies that

$$\vartheta_{\mathrm{can}}(\mathbf{d}_R(D^\bullet_{L/k,\Sigma',\omega}))^{-1} = (\mathrm{ch}_R(X_{L/k,\omega}),0) = \theta_\omega^{n_{\Sigma',\omega}}\cdot\mathbf{1}_R. \tag{A.1}$$

With $K' := Kk^\infty$ and $G' := G_{K'/k}$ we write Γ' for the quotient of Γ with $G' = H\times\Gamma'$ and set $R' := \Lambda_{\mathcal{O}}(\Gamma')$ and $S := \Lambda_{\mathcal{O}}(J)$. Then Γ' is a free $\mathbb{Z}_p$-module of finite rank and, as J is a quotient of Γ', there are projection homomorphisms $\pi : R \to S$, $\pi_1 : R \to R'$ and $\pi_2 : R' \to S$ with $\pi = \pi_2\circ\pi_1$. Now each place v in $\Sigma'\setminus\Sigma$ is unramified in K'/k so $\mathrm{ch}_{\Lambda(G')}(\Lambda(G'/G'_v)) = (1-\mathrm{Fr}_{K'}(v)^{-1})$. Hence, since all places have infinite residue degree in K'/k, [B, Rem. 4.3] implies that

$$\begin{aligned}
\pi_*(\vartheta_{\mathrm{can}}(\mathbf{d}_R(D^\bullet_{L/k,\Sigma',\omega}))^{-1}) &= \pi_{2,*}(\pi_{1,*}(\vartheta_{\mathrm{can}}(\mathbf{d}_R(D^\bullet_{L/k,\Sigma',\omega}))^{-1}))\\
&= \pi_{2,*}(\vartheta_{\mathrm{can}}(\mathbf{d}_{R'}(D^\bullet_{K'/k,\Sigma',\omega}))^{-1})\\
&= \pi_{2,*}(\vartheta_{\mathrm{can}}(\mathbf{d}_{R'}(D^\bullet_{K'/k,\Sigma,\omega}))^{-1}\prod_{v\in\Sigma'\setminus\Sigma}(1-\mathrm{Fr}_{K'}(v)^{-1})_\omega)\\
&= \vartheta_{\mathrm{can}}(\mathbf{d}_S(D^\bullet_{K/k,\Sigma,\omega}))^{-1}\prod_{v\in\Sigma'\setminus\Sigma}(1-\mathrm{Fr}_K(v)^{-1})_\omega.
\end{aligned}$$

In addition, [B,(6)] implies $\pi_*(\theta_\omega\cdot\mathbf{1}_R)=(\theta_{K/k,\Sigma})_\omega\prod_{v\in\Sigma'\setminus\Sigma}(1-\mathrm{Fr}_K(v)^{-1})_\omega\cdot\mathbf{1}_S$. Projecting (A.1) under π_* thus gives an equality

$$\vartheta_{\mathrm{can}}(\mathbf{d}_S(D^\bullet_{K/k,\Sigma,\omega}))^{-1}\prod_{v\in\Sigma'\setminus\Sigma}(1-\mathrm{Fr}_K(v)^{-1})_\omega = (\theta_{K/k,\Sigma})_\omega^{n_{\Sigma',\omega}}\prod_{v\in\Sigma'\setminus\Sigma}(1-\mathrm{Fr}_K(v)^{-1})_\omega^{n_{\Sigma',\omega}}\cdot\mathbf{1}_S.$$

If now $n_{\Sigma',\omega}=1$, then the image under $\mathcal{P}(\omega)$ of the equality of [B, Prop. 4.4] for the pair $(K/k,\Sigma)$ is simply obtained by cancelling each term $(1-\mathrm{Fr}_K(v)^{-1})_\omega$ from the last displayed equality (which is permissible since each such element is a non zero-divisor in S).

It is thus enough to prove that $n_{\Sigma',\omega}=1$ for some Σ' as in Theorem A.1. To do this we argue by contradiction and so assume that $n_{\Sigma',\omega}\neq 1$ for *every* such Σ'. We also choose a further set Σ'' as in Theorem A.1 with $\Sigma'\subsetneq\Sigma''$ (this is always possible by varying the parameters n and Σ) and set $\Gamma_k:=G_{k^\infty/k}$. Then upon projecting (A.1), and the analogous equality with Σ'' replacing Σ', under $\mathcal{P}(R)\to\mathcal{P}(\Lambda_{\mathcal{O}}(\Gamma_k))$ we obtain equalities

$$\vartheta_{\mathrm{can}}(\mathbf{d}_{\Lambda_{\mathcal{O}}(\Gamma_k)}(D^\bullet_{k'^\infty/k,\Sigma^*,\omega}))^{-1}=(\theta'_{\Sigma^*})_\omega^{n_{\Sigma^*,\omega}}\cdot\mathbf{1}_{\Lambda_{\mathcal{O}}(\Gamma_k)} \tag{A.2}$$

with $\theta'_{\Sigma^*}:=\theta_{k'^\infty/k,\Sigma^*}$ for both $\Sigma^*=\Sigma'$ and $\Sigma^*=\Sigma''$. We now identify $\Lambda_{\mathcal{O}}(\Gamma_k)$ with the power series ring $\mathcal{O}[[T]]$ as in the proof of [B, Lem. 4.8]. Then one has $(\theta'_{\Sigma''})_\omega=(\theta'_{\Sigma'})_\omega\prod_{v\in\Sigma''\setminus\Sigma'}(1-\zeta_v^{-1}(1+T)^{-\deg_k(v)})$ with $\zeta_v:=\omega(\mathrm{Fr}_{k'}(v))\in\mathcal{O}^\times$ whilst, since each place in $\Sigma''\setminus\Sigma'$ is unramified and of infinite residue degree in k'^∞/k, the formula of [B, Rem. 4.3] implies

$$\vartheta_{\mathrm{can}}(\mathbf{d}_{\Lambda_{\mathcal{O}}(\Gamma_k)}(D^\bullet_{k'^\infty/k,\Sigma'',\omega}))^{-1} = \vartheta_{\mathrm{can}}(\mathbf{d}_{\Lambda_{\mathcal{O}}(\Gamma_k)}(D^\bullet_{k'^\infty/k,\Sigma',\omega}))^{-1}\prod_{v\in\Sigma''\setminus\Sigma'}(1-\zeta_v^{-1}(1+T)^{-\deg_k(v)}).$$

Thus (A.2) implies

$$(\theta'_{\Sigma'})_\omega^{n_{\Sigma',\omega}-n_{\Sigma'',\omega}}=u\prod_{v\in\Sigma''\setminus\Sigma'}(1-\zeta_v^{-1}(1+T)^{-\deg_k(v)})^{n_{\Sigma'',\omega}-1}$$

for some unit u of $\mathcal{O}[[T]]$. Since $n_{\Sigma'',\omega}\neq 1$ and $(\theta'_{\Sigma'})_\omega\in\mathcal{O}[[T]]$ this equality implies ($n_{\Sigma'',\omega}>1$ and $n_{\Sigma',\omega}\neq n_{\Sigma'',\omega}$ and hence) that the set of zeroes of $\theta'_{\Sigma'}$ in $\mathbb{Q}_p^c$ is equal to $\bigcup_{v\in\Sigma''\setminus\Sigma'}\{\zeta-1:\zeta^{\deg_k(v)}=\zeta_v\}$. By fixing Σ' and varying Σ'' this gives an obvious contradiction.

A.2. **The general case.** We now assume to be given a continuous homomorphism $\rho:\pi_1(U_{k,\Sigma})\to\mathrm{GL}_e(\mathbb{Z}_p)$ with abelian image. We fix an abelian extension K/k with $K\subseteq M_\Sigma(k)$ and $G:=G_{K/k}$ of the form $H\times\mathbb{Z}_p^d$ with H finite of order prime to p and such that ρ factors through the surjection $\pi_1(U_{k,\Sigma})\cong G_{M_\Sigma(K)/k}\twoheadrightarrow G$. We write $\hat\rho$ both for the ring homomorphism $\Lambda(G)\to\Lambda(G)\otimes_{\mathbb{Z}_p}M_e(\mathbb{Z}_p)=M_e(\Lambda(G))$ which sends each g in G to $g\otimes\rho(g)$ and also for the induced homomorphism $\Lambda(G)[[T]]\to M_e(\Lambda(G))[[T]]=M_e(\Lambda(G)[[T]])$ which sends $\sum_{m\geq 0}c_mT^m$ to $\sum_{m\geq 0}\hat\rho(c_m)T^m$.

We set $T_\rho := \mathbb{Z}_p^e$, regarded as a module over G, and hence over $G_{M_\Sigma(K)/k}$, via ρ. We endow $\Lambda(G)^\#(\rho) := \Lambda(G)^\# \otimes_{\mathbb{Z}_p} T_\rho$ with the (left) action of $G \times G_{M_\Sigma(K)/k}$ obtained by letting G act via multiplication on the left hand factor and $G_{M_\Sigma(K)/k}$ act diagonally. We regard $\Lambda(G)^\#(\rho)$ as an étale sheaf of free $\Lambda(G)$-modules on $U_{k,\Sigma}$ in the natural way and set $C^\bullet_{K/k,\rho} := R\Gamma_{\text{ét}}(C_k, j_{k,\Sigma!}(\Lambda(G)^\#(\rho)))$. We recall that $x \mapsto x^\sim$ denotes the $\mathbb{Z}_p$-linear involution of $\Lambda(G)$ which inverts elements of G.

Proposition A.2. *The $\Lambda(G)$-module $H^2(C^\bullet_{K/k,\rho})$ is both finitely generated and torsion and one has* $(\mathrm{ch}_{\Lambda(G)}(H^2(C^\bullet_{K/k,\rho})), 0) = \det(\hat\rho(\theta^\sim_{K/k,\Sigma})) \cdot \mathbf{1}_{\Lambda(G)}$.

Proof. For any $\Lambda(G)$-module M we regard $M\langle\rho\rangle := M \otimes_{\mathbb{Z}_p} T_\rho$ as a (left) $\Lambda(G)$-module by means of the diagonal G-action. Then the exact functor $M \mapsto M\langle\rho\rangle$ induces a functor $D^{\mathrm{p}}(\Lambda(G)) \to D^{\mathrm{p}}(\Lambda(G))$ which we write as $C^\bullet \mapsto C^\bullet\langle\rho\rangle$.
The association $g \otimes t \mapsto g \otimes g(t)$ induces an isomorphism of $\Lambda(G)$-modules $\Lambda(G)^\#(\rho) \cong \Lambda(G)^\#\langle\rho\rangle$ under which the diagonal action of $G_{M_\Sigma(K)/k}$ on $\Lambda(G)^\#(\rho)$ corresponds to an action solely on the first factor of $\Lambda(G)^\#\langle\rho\rangle$. This in turn induces an isomorphism in $D^{\mathrm{p}}(\Lambda(G))$ of the form $C^\bullet_{K/k,\rho} \cong C^\bullet_{K/k,\Sigma}\langle\rho\rangle$, where $C^\bullet_{K/k,\Sigma}$ is the complex defined in [B,§4.1], and hence an isomorphism of $\Lambda(G)$-modules $H^2(C^\bullet_{K/k,\rho}) \cong H^2(C^\bullet_{K/k,\Sigma})\langle\rho\rangle$. This shows that $H^2(C^\bullet_{K/k,\rho})$ is a finitely generated torsion $\Lambda(G)$-module and also makes clear that the required equality follows from [B, Prop. 4.4] (as proved in §A.1) and the following general fact: if f is a characteristic element of a finitely generated torsion $\Lambda(G)$-module M (such as, by [B, Rem. 4.5], $f = \theta^\sim_{K/k,\Sigma}$ and $M = H^2(C^\bullet_{K/k,\Sigma})$), then $\det(\hat\rho(f))$ is a characteristic element of $M\langle\rho\rangle$.
To prove the latter assertion the structure theory of finitely generated torsion $\Lambda(G)$-modules allows us to assume that $M = \Lambda(G)/(f)$ for a non zero-divisor f of $\Lambda(G)$ and in this case there is an exact sequence $\Lambda(G)\langle\rho\rangle \xrightarrow{(\times f)\langle\rho\rangle} \Lambda(G)\langle\rho\rangle \to M\langle\rho\rangle \to 0$. Now the standard $\mathbb{Z}_p$-basis $\{t_i : 1 \le i \le e\}$ of $T_\rho = \mathbb{Z}_p^e$ gives a $\Lambda(G)$-basis $\{1 \otimes_{\mathbb{Z}_p} t_i : 1 \le i \le e\}$ of $\Lambda(G)\langle\rho\rangle$ and, with respect to this basis, the matrix of $(\times f)\langle\rho\rangle$ is $\hat\rho(f)$. The above exact sequence therefore implies that $\det(\hat\rho(f)) \cdot \Lambda(G) = \mathrm{Fit}_{\Lambda(G)}(M\langle\rho\rangle)$ and hence, by the observation made in §2.2(iv), that $\det(\hat\rho(f))$ is a characteristic element of $M\langle\rho\rangle$. □

We write $|\cdot|_p$ for the absolute value on $\mathbb{Q}_p^c$. Then, since ρ is continuous, if a series f in $\Lambda(G)[[T]]$ is continuous on the closed unit disc $|T|_p \le 1$ so is the series $\hat\rho(f)$ and one has $\hat\rho(f)(1) = \hat\rho(f(1))$. In particular, if we set

$$L_\Sigma(T) := \prod_{v \notin \Sigma} (1 - \mathrm{Fr}_K(v) \cdot T^{\deg_k(v)})^{-1} \in \Lambda(G)[[T]],$$

then Lemma A.3 below implies that the series

$$L_\Sigma(\rho, T) := \det(\hat\rho(L_\Sigma(T))) = \prod_{v \notin \Sigma} \det(1 - \rho(\mathrm{Fr}_K(v)) \cdot T^{\deg_k(v)})^{-1} \in \Lambda(G)[[T]]$$

is continuous on $|T|_p \le 1$ and satisfies $L_\Sigma(\rho, 1) = \det(\hat\rho(\theta^\sim_{K/k,\Sigma}))$. This last equality implies that Proposition A.2 recovers the main conjectures considered by Crew in [17].

Lemma A.3. *The series $L_\Sigma(T)$ is continuous on the closed unit disc $|T|_p \le 1$ and satisfies $L_\Sigma(1) = \theta^\sim_{K/k,\Sigma}$.*

Proof. Fix a place v of k with $v \notin \Sigma$ and set $\delta_v(T) := (1 - \mathrm{Fr}_K(v) \cdot (q_k T)^{\deg_k(v)})$. Then $\delta_v(T)$ is continuous and non-vanishing on $|T|_p \le 1$ and $\delta_v(1) \in \Lambda(G)^\times$ and so it is enough to prove $L^v_\Sigma(T) := \delta_v(T) L_\Sigma(T)$ is continuous on $|T|_p \le 1$ and satisfies $L^v_\Sigma(1) = \delta_v(1)\theta^\sim_{K/k,\Sigma}$. The topology of $\Lambda(G)$ implies $L^v_\Sigma(T)$ is continuous on $|T|_p \le 1$ if for each open subgroup U of G all but finitely many of the coefficients of $L^v_\Sigma(T)$ are in the kernel of the projection $\pi_U : \Lambda(G) \to \mathbb{Z}_p[G/U]$. This property is satisfied because [39, Chap. V, Prop. 2.15] implies that $\pi_U(L^v_\Sigma(T))$ belongs to the polynomial ring $\mathbb{Z}_p[G/U][T]$ (indeed, in the notation of loc. cit., where G plays the role of our G/U, one has $\pi_U(L^v_\Sigma(T)) = \Theta^{\{v\}}_\Sigma(T)^\sim$ where $x \mapsto x^\sim$ acts coefficient-wise on $\mathbb{Z}_p[G/U][T]$). One also has $\pi_U(L^v_\Sigma(1)) = \pi_U(L^v_\Sigma(T))(1) = (1 - \mathrm{Fr}_{K^U}(v) \cdot q_k^{\deg_k(v)})\theta^\sim_{K^U/k,\Sigma} = \pi_U(\delta_v(1)\theta^\sim_{K/k,\Sigma})$ for each such U and hence that $L^v_\Sigma(1) = \delta_v(1)\theta^\sim_{K/k,\Sigma}$, as required. □

A.3. **The proof of Theorem A.1.** Recalling that $G_{K/k} = H \times J$ and $k' := K^J$ we identify $G_{k'/k}$ with H in the natural way. We write $\mathcal{O}$ for the extension of $\mathbb{Z}_p$ generated by the set of $\hbar$-th roots of unity in $\mathbb{Q}_p^c$ and define $H^* := \mathrm{Hom}(H, \mathbb{Q}_p^{c\times}) = \mathrm{Hom}(H, \mathcal{O}^\times)$. For a $\mathbb{Z}_p$-module M we set $\mathcal{O}M := \mathcal{O} \otimes_{\mathbb{Z}_p} M$.

A.3.1. *Norm residue pairings.* For the moment we assume given a finite non-empty set Σ' of places of k and a profinite Galois extension E of k with $k' \subset E$, $G_{E/k'}$ abelian and E/k unramified outside Σ'. We write $I_{E/k'}$ for the augmentation ideal of $\Lambda(E/k')$, set $Y' = \mathbb{Z}_p \otimes \mathrm{Hom}_{\mathbb{Z}}(Y^0_{k',\Sigma'}, \mathbb{Z})$ and $U' := \mathbb{Z}_p \otimes \mathcal{O}^\times_{k',\Sigma'}$ and note that both Y' and U' are torsion-free (the latter since $p = \mathrm{char}(k)$). We note also that the local norm residue maps induce a pairing $\langle\ ,\ \rangle_{E/k} : U' \times Y' \to G_{E/k'} \cong I_{E/k'}/I^2_{E/k'}$ and hence an $\mathcal{O}$-linear homomorphism

$$\Phi_{E/k} : \mathcal{O}U' \otimes_{\mathcal{O}} \mathcal{O}Y' \to \mathcal{O}G_{E/k'} \cong \mathcal{O}(I_{E/k'}/I^2_{E/k'}). \tag{A.3}$$

It is easily seen that if $\mathcal{O}U' \otimes_{\mathcal{O}} \mathcal{O}Y'$ is endowed with the diagonal action of H and $I_{E/k'}/I^2_{E/k'}$ with the natural conjugation action of H, then $\Phi_{E/k}$ is a homomorphism of $\mathcal{O}[H]$-modules. Thus, if E/k is abelian, then H acts trivially on $I_{E/k'}/I^2_{E/k'}$ and so the eigenspaces $U'_{\omega'}$ and $Y'_{\omega^{-1}}$ are orthogonal with respect to $\langle\ ,\ \rangle_{E/k}$ if $\omega' \neq \omega$.
Since $p \nmid \hbar$ there are decompositions $\mathcal{O}U' = \bigoplus_{\omega\in H^*} U'_\omega$ and $\mathcal{O}Y' = \bigoplus_{\omega\in H^*} Y'_\omega$. For each ω we write r_ω for the (common) $\mathcal{O}$-rank of U'_ω and $Y'_{\omega^{-1}}$. We choose $\mathcal{O}$-bases $\{c_i^\omega\}_i$ and $\{d_j^{\omega^{-1}}\}_j$ of U'_ω and $Y'_{\omega^{-1}}$ and define the discriminant $\det_{E/k,\omega}$ of the restriction of $\langle\ ,\ \rangle_{E/k}$ to $U'_\omega \times Y'_{\omega^{-1}}$ by setting $\mathrm{disc}_{E/k,\omega} := \det((\Phi_{E/k}(c_i^\omega \otimes d_j^{\omega^{-1}}))_{1\le i,j\le r_\omega})$. We regard $\mathrm{disc}_{E/k,\omega}$ as an element of $\mathcal{O}(I^{r_\omega}_{E/k'}/I^{r_\omega+1}_{E/k'})$ in the natural way and note that it is unique to within multiplication by elements of $\mathcal{O}^\times$.
If now $G_{E/k'}$ is a non-zero free $\mathbb{Z}_p$-module of rank d, we fix a $\mathbb{Z}_p$-basis $\{\sigma_i : 1 \le i \le d\}$ of $G_{E/k'}$, set $t_i = \sigma_i - 1$ and identify $\Lambda_{\mathcal{O}}(G_{E/k'})$ with the power series ring $R_d := \mathcal{O}[[t_1, ..., t_d]]$. Then for each non-negative integer m the augmentation quotient $\mathcal{O}(I^m_{E/k'}/I^{m+1}_{E/k'})$ identifies with the space of all degree m homogeneous polynomials in R_d and, with respect to this identification, we let $f_{E/k,\omega}$ be the degree r_ω homogenous polynomial that represents $\mathrm{disc}_{E/k,\omega}$.
We can now state the main result of this subsection.

Proposition A.4. *There exists a finite (non-empty) set of places Σ' of k and an abelian extension L of k with all of the following properties.*

(i) *Σ' contains Σ and at least two places which split completely over k', and the order of $\mathrm{Cl}(\mathcal{O}_{k',\Sigma'})$ is prime to p.*

(ii) *L contains Kk^∞, L/k is ramified precisely at the places in Σ', $G_{L/k'}$ is isomorphic to $\mathbb{Z}_p^d$ for some natural number d and $G_{L/k}$ is naturally isomorphic to the product $H \times G_{L/k'}$.*

(iii) *The discriminant polynomials $\{f_{L/k,\omega} : \omega \in H^*\}$ defined above are algebraically independent and irreducible over $\mathbb{Q}_p^c$.*

Proof. We choose two k-places v_1, v_2 which split completely over k'. We replace Σ by $\Sigma \cup \{v_1, v_2\}$ and then simply use [25, Cor. 2.1] to find a set Σ' that satisfies the conditions in claim (i).

We next choose an abelian extension F of k which is ramified precisely at the places in Σ' and is a finite degree extension of Kk^∞. Then, following [36, Lem. 3.3], there exists a Galois extension M/k with the following properties: $F \subseteq M$, M/k is ramified precisely at the places in Σ', $G_{M/k'}$ is a free $\mathbb{Z}_p$-module of finite rank and the homomorphism $\Phi_{M/k}$ defined as in (A.3) is both injective and has image a direct summand of $\mathcal{O}G_{M/k'}$. We write L for the maximal abelian extension of k inside M. Then L contains F (and hence Kk^∞), is ramified precisely at the places in Σ' and Lemma A.5 below implies that $G_{L/k'}$ is a (non-zero) free $\mathbb{Z}_p$-module of finite rank. Since $G_{L/k}$ is abelian and the order of $H \cong G_{k'/k}$ is prime to p the isomorphism $G_{L/k} \cong H \times G_{L/k'}$ is also clear. This proves claim (ii).

Lemma A.5 also induces an identification $\Phi_{L/k} = \Phi^H_{M/k}$ and implies that $\Phi_{L/k}$ is injective and its image is a direct summand of $\mathcal{O}G_{L/k'}$. Hence if for each ω we fix $\mathcal{O}$-bases $\{c_i^\omega\}_i$ and $\{d_j^{\omega^{-1}}\}_j$ as above, then $\bigcup_{\omega \in H^*}\{\Phi_{L/k}(c_i^\omega \otimes d_j^{\omega^{-1}}) \mid 1 \le i, j \le r_\omega\}$ can be extended to an $\mathcal{O}$-basis $\tau_1, \tau_2, ..., \tau_d$ of $\mathcal{O}G_{L/k'}$. Set $s_i = \tau_i - 1$ for $1 \le i \le d$. Then $R_d = \mathcal{O}[[s_1, ..., s_d]]$ and for each ω there is a subset $\{s^\omega_{i,j} \mid 1 \le i, j \le r_\omega\}$ of $\{s_1, ..., s_d\}$ with $f_{L/k,\omega} = \det(s^\omega_{i,j})_{1 \le i,j \le r_\omega}$ and $s^{\omega'}_{i',j'} \neq s^\omega_{i,j}$ unless $\omega' = \omega$, $i' = i$ and $j' = j$. The polynomials $f_{L/k,\omega}$ for $\omega \in H^*$ are thus irreducible over $\mathbb{Q}_p^c$ (by [42, Vol. 1, p. 94]) and, since they are in different variables for different ω, they are also algebraically independent over $\mathbb{Q}_p^c$. $\square$

Lemma A.5. *The restriction map $G_{M/k'} \to G_{L/k'}$ induces an isomorphism $G^H_{M/k'} \cong G_{L/k'}$, where we regard $G_{M/k'}$ as a $\mathbb{Z}_p[H]$-module in the natural way.*

Proof. We set $\Pi := G_{M/k'}$ and write I_H for the augmentation ideal of $\mathbb{Z}_p[H]$. Since $p \nmid \hbar$ there is a direct sum decomposition $\Pi = \Pi^H \oplus I_H(\Pi)$. We set $\Pi' := G_{M/k}$. Then $I_H(\Pi) \subseteq [\Pi', \Pi']$ and so, without loss of generality, we may assume that $I_H(\Pi)$ vanishes. Then Π is central in Π' and, as $\Pi'/\Pi \cong H$ is abelian, also $[\Pi', \Pi'] \subseteq \Pi$. Now if $g_1, g_2 \in \Pi'$, then $g_1 g_2 g_1^{-1} = x g_2$ with $x = g_1 g_2 g_1^{-1} g_2^{-1} \in [\Pi', \Pi'] \subseteq \Pi$. Since $g_2^\hbar \in \Pi$ we have $x^\hbar g_2^\hbar = (x g_2)^\hbar = g_1 g_2^\hbar g_1^{-1} = g_2^\hbar$ and so $x^\hbar = \mathrm{id}$. But every element of the abelian group $[\Pi', \Pi']$ is a product of elements of the form x so $[\Pi', \Pi']$ is a torsion subgroup of Π and hence trivial. $\square$

A.3.2. *Stickelberger elements.* We fix notation as in Proposition A.4, set $\Gamma := G_{L/k'}$ and identify $G_{L/k}$ with $H \times \Gamma$. For each $\omega \in H^*$ we set $f_\omega := f_{L/k,\omega}$ and also use the notation introduced just prior to Theorem A.1 to set $\theta_\omega := (\theta_{L/k,\Sigma'})_\omega \in \Lambda_{\mathcal{O}}(\Gamma)$. Then the image of the equality of [38, Prop. 7.1] under the natural projection $\Lambda_{\mathcal{O}}(G_{L/k}) \to \Lambda_{\mathcal{O}}(\Gamma)$ gives an equality $\theta_{L/k',\Sigma'} = \prod_{\omega\in H^*}\theta_\omega$. Hence, if for each ω we write ξ_ω for the homogeneous polynomial in R_d with $\theta_\omega = \xi_\omega$ + higher degree terms in R_d, then

$$\theta_{L/k',\Sigma'} = \prod_{\omega\in H^*}\xi_\omega + \text{higher degree terms in } R_d. \tag{A.4}$$

Further, the first step of the argument in [38, §7.2] shows that for each ω in H^* there exists an ω' in H^* and a non-zero element c_ω of the field of fractions of $\mathcal{O}$ with

$$\xi_\omega = c_\omega f_{\omega'} \in R_d. \tag{A.5}$$

We write r for the $\mathbb{Z}$-rank of $\mathcal{O}^\times_{k',\Sigma'}$. Then the discriminant of the pairing $\langle\ ,\ \rangle_{L/k'}$ can be regarded as an element of $I^r_{L/k'}/I^{r+1}_{L/k'}$ and as such is equal to $v\prod_{\omega\in H^*}f_\omega$ for some v in $\mathcal{O}^\times$ (since the eigenspaces $U'_{\omega'}$ and $Y'_{\omega^{-1}}$ are mutually orthogonal whenever $\omega' \neq \omega$). Thus, after taking account of [38, Lem. 6.2] and the fact that $p \nmid |\operatorname{Cl}(\mathcal{O}_{k',\Sigma'})|$, the refined class number formula of Gross (as proved in this case, using elementary methods, by Tan in [36]) implies $\theta_{L/k',\Sigma'} = u\prod_{\omega\in H^*}f_\omega$ + higher degree terms in R_d for some u in $\mathcal{O}^\times$. Comparing this equation to (A.4) and (A.5) one finds that $\prod_{\omega\in H^*}c_\omega f_{\omega'} = u\prod_{\omega\in H^*}f_\omega$. Since the polynomials $\{f_\omega : \omega\in H^*\}$ are algebraically independent, this implies $\{f_{\omega'} : \omega\in H^*\} = \{f_\omega : \omega\in H^*\}$ and hence that $\prod_{\omega\in H^*}c_\omega = u \in \mathcal{O}^\times$. But, since each polynomial $f_{\omega'}$ is irreducible, the equality (A.5) also implies that each c_ω belongs to $\mathcal{O}$ and hence to $\mathcal{O}^\times$ (because $\prod_{\omega\in H^*}c_\omega \in \mathcal{O}^\times$). From (A.5) it thus follows that the polynomial ξ_ω is irreducible in $\mathcal{O}[s_1,...,s_d]$.

We now suppose that $\theta_\omega = \theta_1\theta_2$ in R_d where θ_i begins with the homogeneous polynomial ξ_i for $i = 1,2$. Then $\xi_\omega = \xi_1\xi_2$ and so, since ξ_ω is irreducible in $\mathcal{O}[s_1,...,s_d]$, we must have, say, $\xi_1 \in \mathcal{O}^\times$. But then the series θ_1 begins with an element of $\mathcal{O}^\times$ and so is a unit of R_d. This proves θ_ω is irreducible. (In fact, by a similar argument, one can also show that the elements $\{\theta_\omega : \omega\in H^*\}$ are pairwise coprime in $\Lambda_{\mathcal{O}}(\Gamma)$).

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King's College London
Dept. of Mathematics
London WC2R 2LS
United Kingdom
david.burns@kcl.ac.uk

School of Mathematics and Statistics
University of Sydney
NSW 2006
Australia
kflai@math.usyd.edu.au

Department of Mathematics
National Taiwan University
Taipei 10764
Taiwan
tan@math.ntu.edu.tw