CONGRUENCES BETWEEN DERIVATIVES OF ARTIN L-SERIES AT $s = 0$

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Abstract. We conjecture a family of integral congruence relations between the values at zero of different order derivatives of Artin L-series over general number fields. We show this prediction specialises to recover the 'refined class number formula for $\mathbb{G}_m$' independently conjectured for abelian extensions by Mazur and Rubin and by the third author, prove it for Artin L-series of characters that factor through a natural class of Frobenius extensions of $\mathbb{Q}$ and provide concrete supporting evidence in other special cases. We also use techniques of non-commutative Iwasawa theory to prove, modulo only a standard μ-vanishing hypothesis, the analogous family of congruence relations for p-adic Artin L-series over arbitrary totally real number fields.

1. Introduction

1.1. **The main results.** In this article we formulate, and then provide supporting evidence for, a conjectural family of integral congruence relations between the (normalised) values at zero of differing order higher derivatives of the Artin L-series of finite dimensional complex characters over general number fields.

We refer to our conjectural congruences as the non-commutative Class Number Formula Conjecture for $\mathbb{G}_m$ (or 'nCNF($\mathbb{G}_m$)' for brevity in the rest of the Introduction) since, upon specialisation to the L-series of linear characters, they recover the 'refined class number formula for $\mathbb{G}_m$' independently conjectured by Mazur and Rubin [26] and by the third author [32]. In particular, via this connection, our conjecture also simultaneously extends to the L-series of arbitrary finite dimensional complex characters a range of earlier much studied conjectures for Dirichlet L-series that are due to Darmon, to Gross, to Rubin and to Tate among others (see the discussion following Remark 5.4).

Prior to formulating nCNF($\mathbb{G}_m$), there are several preliminary steps that we undertake that are perhaps of some independent interest. Firstly, in §3, we introduce a new, and very natural, notion of 'projective pull-back' for lattices over p-adic group rings. Then, in §4, we define canonical 'Artin-Bockstein maps' in étale cohomology that generalise to non-abelian Galois extensions the classical reciprocity maps of local class field theory. By combining these maps with a canonical arithmetic construction of projective pull-backs, we then obtain a natural generalisation of the 'regulator maps' that are defined in [26] and [32] in terms of local reciprocity maps.

The conjecture nCNF($\mathbb{G}_m$) is then stated precisely as Conjecture 5.1 and uses Artin-Bockstein regulator maps to formulate families of integral congruence relations between the non-commutative Rubin-Stark elements (of differing ranks) defined by the first and third

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author in [7], and hence between the normalised values at zero of higher derivatives (of differing orders) of the corresponding Artin L-series.

The use of non-commutative Rubin-Stark elements is crucial to our approach and means that many of the constructions and results of this article involve the notions of Whitehead order, reduced exterior power and reduced Rubin lattice that were introduced in [6]. In fact, in order to incorporate the notion of projective pull-backs we must strengthen some technical aspects of the theory presented in loc. cit. and, to help the reader, these developments (including a simplified definition of the key notion of Whitehead order) are first presented, together with a brief review of relevant results from [6], in §2.

As mentioned above, for the L-series of linear characters we can show that nCNF($\mathbb{G}_m$) recovers the conjectures of Mazur and Rubin [26] and of the third author [32] concerning Rubin-Stark elements (this will follow from the argument of Theorem 5.5(i)). Further, in other interesting special cases we can show that the congruences in nCNF($\mathbb{G}_m$) have a more explicit interpretation and are thereby able to provide supporting evidence for the conjecture in situations that involve the L-series of non-linear characters. In this way, for example, we can use an approach developed by Johnston and Nickel in [21] to prove the following result (for a precise statement of which see Corollary 5.6).

Theorem A. *The p-component of nCNF($\mathbb{G}_m$) is valid for the Artin L-series of characters that factor through any finite Galois extension L of $\mathbb{Q}$ for which the Galois group is a Frobenius group that has a kernel of order prime to p and a complement that is abelian.*

In other directions, we show that nCNF($\mathbb{G}_m$) incorporates a natural extension to non-abelian Galois extensions of the Rubin-Stark Conjecture formulated in [31] (see Remark 5.2(iii)) and also predicts a family of explicit integrality and congruence restrictions on the Stickelberger elements for non-abelian Galois extensions introduced by Hayes in [18] for which one can provide concrete unconditional supporting evidence (see §5.2.2 and §5.2.3). In the course of proving such results, we are also led to make some new observations regarding the non-commutative Fitting invariants of the Selmer modules of $\mathbb{G}_m$ introduced by Kurihara and the first and third authors (see §5.2.4).

We recall that the definition of non-commutative Rubin-Stark elements in [7] involves both the Dirichlet regulator map and the values at zero of higher derivatives of Artin L-series. In just the same way, for each prime p one can define 'p-adic non-commutative Rubin-Stark elements' by using Gross's p-adic regulator map and the values at zero of higher derivatives of p-adic Artin L-series over totally real number fields. In this way, we are led to formulate (in Conjecture 6.1) a precise analogue of nCNF($\mathbb{G}_m$) for p-adic Artin L-series. By combining deep results of Ritter and Weiss [30] and Kakde [23] in non-commutative Iwasawa theory with Galois-cohomological arguments from [7], we are then able to prove the following result (for a precise version of which see Theorem 6.3).

Theorem B. *The analogue of nCNF($\mathbb{G}_m$) for p-adic Artin L-series is valid for characters that factor through any finite CM Galois extension of a totally real field that validates Iwasawa's μ-invariant conjecture.*

This result can in turn be combined with the known validity of the Gross-Stark Conjecture (due to Dasgupta, Kakde and Ventullo [10]), a technical observation concerning the validity of the Gross-Kuz'min Conjecture and a classical result of Neukirch [27] on the embedding problem to obtain further concrete evidence for nCNF($\mathbb{G}_m$). In particular, this approach will allow us to prove the p-component of nCNF($\mathbb{G}_m$) for the L-series of totally odd characters that factor through a family of finite CM Galois extensions L of $\mathbb{Q}$ for which each of the ramification degree of p in L, the number of p-adic places of L and the order of the commutator subgroups of the Sylow p-subgroups of $\mathrm{Gal}(L/\mathbb{Q})$ are simultaneously unbounded (see Corollary 6.4, Example 6.5 and Remark 6.6), thereby complementing the result of Theorem A.

Finally, to help provide some general context for our approach, we now fix a compact p-adic Lie extension of number fields $\mathcal{K}/K$ of rank one. Then it can be shown that the validity of nCNF($\mathbb{G}_m$) for the L-series of all characters that factor through finite Galois extensions of K in $\mathcal{K}$ implies the validity of the 'Generalized Gross-Stark Conjecture' for $\mathcal{K}/K$ that is formulated in [7, Conj. 9.7] as a derivative formula for the canonical non-commutative Rubin-Stark Euler system. This link will be established elsewhere and in effect shows that nCNF($\mathbb{G}_m$) constitutes a refinement 'at finite level' of the latter conjecture. In addition, it combines with the main result (Theorem 10.15) of loc. cit. to show that nCNF($\mathbb{G}_m$) has an important role to play in attempts to verify the equivariant Tamagawa number conjecture for $\mathbb{G}_m$ (or eTNC($\mathbb{G}_m$) for short in the sequel) for families of non-abelian Galois extensions. Here it is important to note that nCNF($\mathbb{G}_m$) is much more amenable to investigation than is eTNC($\mathbb{G}_m$), as is evidenced in the present article both by the fact that it can often be interpreted comparatively explicitly (as already noted above) and, in addition, can be verified in cases for which eTNC($\mathbb{G}_m$) is not known to be valid (see, for example, Remarks 5.7(ii) and 6.6).

However, whilst this link may perhaps provide some additional motivation to study nCNF($\mathbb{G}_m$), we feel that the aspects discussed in the present article show this conjecture is also itself of some intrinsic interest.

1.2. **General notation.** For each ring R, we write $\zeta(R)$ for its centre and R^{op} for the corresponding opposite ring (so that $\zeta(R) = \zeta(R^{\mathrm{op}})$). By an R-module we shall, unless explicitly stated otherwise, mean a left R-module.

We write $\mathbb{Z}_{(p)}$ for the localization of $\mathbb{Z}$ at a prime number p and for any abelian group, or complex of abelian groups, A we write A_p for the pro-p completion of A and use similar notation for morphisms. We also fix an algebraic closure $\mathbb{Q}_p^c$ of $\mathbb{Q}_p$ and write $\mathbb{C}_p$ for its completion.

For a finite group Γ we write $\mathrm{Ir}(\Gamma)$ and $\mathrm{Ir}_p(\Gamma)$ for the sets of irreducible $\mathbb{C}$-valued and $\mathbb{C}_p$-valued characters of Γ. We write $[\Gamma,\Gamma]$ for the commutator subgroup of Γ and Γ^{ab} for the abelianisation $\Gamma/[\Gamma,\Gamma]$ of Γ. As is usual, we shall refer to a finitely generated $\mathbb{Z}_p[\Gamma]$-module that is free over $\mathbb{Z}_p$ as a '$\mathbb{Z}_p[\Gamma]$-lattice'.

For a natural number t we write $[t]$ for the set of integers i with $1 \le i \le t$. We also write $\{b_{\Gamma,i}\}_{i\in[t]}$ for the standard (ordered) $\mathbb{Z}_p[\Gamma]$-basis of the direct sum $\mathbb{Z}_p[\Gamma]^t$ of t copies of $\mathbb{Z}_p[\Gamma]$.

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2. Whitehead orders and reduced Rubin lattices

We present a simplified definition of the 'Whitehead orders' introduced in [6] and then further develop the theory of reduced Rubin lattices from loc. cit.

2.1. Whitehead orders.

Let R be a Dedekind domain with field of fractions F of characteristic zero, and $\mathcal{A}$ an R-order in a finite dimensional semisimple F-algebra A.

Definition 2.1. *The 'Whitehead order' $\xi(\mathcal{A})$ of $\mathcal{A}$ is the R-submodule of $\zeta(A)$ that is generated by the elements $\mathrm{Nrd}_A(M)$ as M runs over all matrices in $\bigcup_{n\in\mathbb{N}} \mathrm{M}_n(\mathcal{A})$.*

Lemma 2.2. *The R-module $\xi(\mathcal{A})$ has all of the following properties.*

(i) *$\xi(\mathcal{A})$ is an R-order in $\zeta(A)$.*
(ii) *For each $\mathfrak{p} \in \mathrm{Spec}(R)$ one has $\xi(\mathcal{A})_{(\mathfrak{p})} = \xi(\mathcal{A}_{(\mathfrak{p})})$ and $\xi(\mathcal{A})_{\mathfrak{p}} = \xi(\mathcal{A}_{\mathfrak{p}})$.*
(iii) *In $\zeta(A)$ one has $\xi(\mathcal{A}) = \bigcap_{\mathfrak{p}\in\mathrm{Spec}(R)} \xi(\mathcal{A}_{(\mathfrak{p})})$.*

Proof. Since $\xi(\mathcal{A})$ is obviously closed under multiplication and contains 1 (as the reduced norm of the identity matrix), claim (i) is reduced to showing that $\xi(\mathcal{A})$ is finitely generated and such that $F \cdot \xi(\mathcal{A}) = \zeta(A)$. To prove this we note $\zeta(A)$ decomposes as a finite product $\prod_{i\in I} F_i$ of finite degree field extensions F_i of F and that, correspondingly, A decomposes as a product $\prod_{i\in I} A_i$, where each A_i is a central simple F_i-algebra.

For each n the R-module $\mathrm{M}_n(\mathcal{A})$ is finitely generated and so, for every $M \in \mathrm{M}_n(\mathcal{A})$, there exists a monic polynomial $c = c_M$ over R with $c(M) = 0$ in $\mathrm{M}_n(\mathcal{A})$. Hence, writing $M = (M_i)_{i\in I}$ for the decomposition of M in $\mathrm{M}_n(A) = \prod_{i\in I} \mathrm{M}_n(A_i)$, for each i one has $c(M_i) = 0$ in $\mathrm{M}_n(A_i)$. It follows that every eigenvalue λ of the image of M_i in any splitting of $\mathrm{M}_n(A_i)$ satisfies $c(\lambda) = 0$ and so is integral over R. In particular, since $\mathrm{Nrd}_{A_i}(M_i)$ can be computed as the product of such eigenvalues, each term $\mathrm{Nrd}_A(M) = (\mathrm{Nrd}_{A_i}(M_i))_{i\in I}$ is contained in the integral closure of R in the (finite-dimensional) F-algebra $\zeta(A)$, and so $\xi(\mathcal{A})$ is finitely generated, as required.

To prove $F \cdot \xi(\mathcal{A}) = \zeta(A)$, it suffices to show each component F_j of $\prod_{i\in I} F_i$ is contained in $F \cdot \xi(\mathcal{A})$. Now, as $\mathcal{A}$ is an R-order in A, for any $x \in F_j$ one has $r_j \cdot x \in \zeta(\mathcal{A})$ for some $r_j \in R \setminus \{0\}$ and so, for some natural number s (that depends on j but is independent of x), also $x^s = r_j^{-s}\mathrm{Nrd}_{A_j}(r_j x) \in F \cdot \xi(\mathcal{A})$. In particular, for $y \in F_j$ and $a \in [s]$, one has

$$\sum\nolimits_{i=0}^{i=s-1} a^i \binom{s}{i} y^i = y_a := (1 + ay)^s - (ay)^s \in F \cdot \xi(\mathcal{A}).$$

Hence, since $M := (i^{j-1})_{1\le i,j\le s}$ belongs to $\mathrm{GL}_s(F)$ (as $\mathrm{char}(R) = 0$), the element

$$y = \binom{s}{1}^{-1} \binom{s}{1} y = \binom{s}{1}^{-1} \sum\nolimits_{1\le a\le s} (M^{-1})_{1a} y_a$$

belongs to $F \cdot \xi(\mathcal{A})$, as required.

Fix a nonzero $\mathfrak{p} \in \mathrm{Spec}(R)$. Then, since it is clear $\xi(\mathcal{A})_{(\mathfrak{p})} \subseteq \xi(\mathcal{A}_{(\mathfrak{p})})$, the first equality in claim (ii) is reduced to showing that $\xi(\mathcal{A}_{(\mathfrak{p})}) \subseteq \xi(\mathcal{A})_{(\mathfrak{p})}$. To do this we fix $M \in \mathrm{M}_n(\mathcal{A}_{(\mathfrak{p})})$ and $r \in R \setminus \mathfrak{p}$ with $rM \in \mathrm{M}_n(\mathcal{A})$. Then, for each $i \in I$, there exists $s_i \in \mathbb{N}$ such that

$$\mathrm{Nrd}_A(M) = \big(\mathrm{Nrd}_{A_i}(M_i)\big)_{i\in I} = \big(r^{-s_i}\mathrm{Nrd}_{A_i}(rM_i)\big)_{i\in I} = (r^{-s_i})_{i\in I} \cdot \mathrm{Nrd}_A(rM) \tag{1}$$

in $\zeta(A)$. Write $\mathcal{O}_i$ for the integral closure of R in F_i. Then, since claim (i) implies $\xi(\mathcal{A})_{(\mathfrak{p})}$ is an $R_{(\mathfrak{p})}$-order in A, we can fix $n \in \mathbb{N}$ with $\mathfrak{p}^n \cdot \mathcal{O}_{i,(\mathfrak{p})} \subseteq \xi(\mathcal{A})_{(\mathfrak{p})}$ for all $i \in I$. We also fix an element x of R congruent to the image of $r^{-1} \in R_{(\mathfrak{p})}$ in $R_{(\mathfrak{p})}/\mathfrak{p}^n_{(\mathfrak{p})} = R/\mathfrak{p}^n$. Then one has

$$(r^{-s_i})_{i\in I} = (x^{s_i})_{i\in I} + ((r^{-1})^{s_i} - x^{s_i})_{i\in I} = \mathrm{Nrd}_A(x) + ((r^{-1})^{s_i} - x^{s_i})_{i\in I} \in \xi(\mathcal{A})_{(\mathfrak{p})},$$

where the containment is valid as $x \in R \subseteq \mathcal{A}$ and $(r^{-1})^{s_i} - x^{s_i} \in \mathfrak{p}^n \cdot \mathcal{O}_{i,(\mathfrak{p})} \subseteq \xi(\mathcal{A})_{(\mathfrak{p})}$. Given this, the expression (1) implies that $\mathrm{Nrd}_A(M) \in \xi(\mathcal{A})_{(\mathfrak{p})}$, as required.

Finally, we note that the second equality of claim (ii) follows by a similar argument (see [6, Lem. 3.2(ii)]) and that claim (iii) follows directly from the first equality in claim (ii) and a general property of R-lattices. □

Remark 2.3. Lemma 2.2(iii) implies Definition 2.1 coincides with the original definition of $\xi(\mathcal{A})$ via localisations that is given in [6, Def. 3.1]. In particular, this fact implies $\xi(\mathcal{A})$ is, in general, neither contained in nor contains $\zeta(\mathcal{A})$ (cf. [6, Exam. 3.4 and Exam. 3.5]).

2.2. **Reduced Rubin Lattices.** In this section we review, and slightly extend, the theory developed in [6, §4] (where all background details can be found).

2.2.1. Let Γ be a finite group and, for each χ in $\mathrm{Ir}(\Gamma)$, fix a corresponding representation $\rho_\chi : \Gamma \to \mathrm{GL}_{\chi(1)}(\mathbb{C})$. Then, for any subfield F of $\mathbb{C}$, any non-negative integer a and any finitely generated $F[\Gamma]$-module M, the 'a-th reduced exterior power' $\bigwedge_{F[\Gamma]}^a M$ of M is a canonical finitely generated $\zeta(F[\Gamma])$-module. In addition, for each integer s with $0 \leq s \leq a$, there are natural duality pairings

$$\bigwedge\nolimits_{F[\Gamma]}^a M \times \bigwedge\nolimits_{F[\Gamma]^{\mathrm{op}}}^s \mathrm{Hom}_{F[\Gamma]}(M, F[\Gamma]) \to \bigwedge\nolimits_{F[\Gamma]}^{a-s} M, \quad (m, \varphi) \mapsto \varphi(m).$$

We write $\wedge_{j\in[a]} m_j$ for the reduced exterior product in $\bigwedge_{F[\Gamma]}^a M$ of a subset $\{m_j\}_{j\in[a]}$ of M and note that, for any subset $\{\varphi_j\}_{j\in[a]}$ of $\mathrm{Hom}_{F[\Gamma]}(M, F[\Gamma])$, one has

$$(\wedge_{i\in[a]}\varphi_i)(\wedge_{j\in[a]} m_j) = \mathrm{Nrd}_{\mathrm{M}_a(F[\Gamma]^{\mathrm{op}})}((\varphi_i(m_j))_{i,j\in[a]}) \in \zeta(F[\Gamma]). \tag{2}$$

2.2.2. Let now R denote $\mathbb{Z}$ or, for a prime p, either $\mathbb{Z}_{(p)}$ or $\mathbb{Z}_p$, and write F for the field of fractions of R. For a finitely generated $R[\Gamma]$-module M we set

$$M^* := \mathrm{Hom}_{R[\Gamma]}(M, R[\Gamma])$$

and, for a non-negative integer a, define the 'a-th reduced Rubin lattice' of M by setting

$$\bigcap\nolimits_{R[\Gamma]}^a M := \{x \in \bigwedge\nolimits_{F[\Gamma]}^a (F \otimes_R M) : (\wedge_{j\in[a]}\varphi_j)(x) \in \xi(R[\Gamma]),\ \forall\, \{\varphi_j\}_{j\in[a]} \subset M^*\}.$$

If Γ is abelian, then $\bigcap_{\mathbb{Z}[\Gamma]}^a M$ coincides with the module $\wedge_0^a M$ defined in [31]. In general, $\bigcap_{R[\Gamma]}^a M$ is a finitely generated $\xi(R[\Gamma])$-module whose basic properties are described in [6, Th. 4.19]. We recall, in particular, that an injection of $R[\Gamma]$-modules $\iota : M \to M'$ induces an injection $\iota_*^a : \bigcap_{R[\Gamma]}^a M \to \bigcap_{R[\Gamma]}^a M'$ of $\xi(R[\Gamma])$-modules and that for any non-negative integer s with $s \le a$, the reduced exterior product of each subset $\{\varphi_j\}_{j\in[s]}$ of M^* induces a map of $\xi(R[\Gamma])$-modules $\wedge_{j\in[s]}\varphi_j : \bigcap^a_{R[\Gamma]} M \to \bigcap^{a-s}_{R[\Gamma]} M$.

Finally, we note that the argument of [6, Lem. 4.16] implies injectivity of the canonical 'evaluation' map of $\xi(R[\Gamma])$-modules

$$\mathrm{ev}^a_M : \bigcap\nolimits^a_{R[\Gamma]} M \to \prod_{\underline{\varphi}} \xi(R[\Gamma]);\ \ x \mapsto \big((\wedge_{j\in[a]}\varphi_j)(x)\big)_{\underline{\varphi}} \tag{3}$$

where in the direct product $\underline{\varphi} = (\varphi_1, \ldots, \varphi_a)$ runs over all elements of $(M^*)^a$.

2.2.3. We now assume to be given an extension of finite groups of the form

$$1 \to \Delta \to \Gamma \xrightarrow{\pi} \Upsilon \to 1. \tag{4}$$

We write $\pi_* : \mathbb{Z}[\Gamma] \to \mathbb{Z}[\Upsilon]$ for the (surjective) ring homomorphism induced by π and $\xi(\Gamma,\Delta)$ for the ideal of $\xi(\mathbb{Z}[\Gamma])$ defined by the tautological short exact sequence

$$0 \to \xi(\Gamma,\Delta) \xrightarrow{\subseteq} \xi(\mathbb{Z}[\Gamma]) \xrightarrow{\tilde{\pi}_*} \xi(\mathbb{Z}[\Upsilon]) \to 0 \tag{5}$$

in which the surjective map $\tilde{\pi}_*$ is induced by π_* and the result of [6, Lem. 2.7(iv)].

For non-negative integers a and d, the projection maps induced by $\mathbb{Q}\otimes_{\mathbb{Z}} \pi_*$

$$\varpi : \bigwedge\nolimits^a_{\mathbb{Q}[\Gamma]} \mathbb{Q}[\Gamma]^d \to \bigwedge\nolimits^a_{\mathbb{Q}[\Upsilon]} \mathbb{Q}[\Upsilon]^d \quad \text{and} \quad \varpi' : \mathrm{Hom}_{\mathbb{Q}[\Gamma]}(\mathbb{Q}[\Gamma]^d, \mathbb{Q}[\Gamma]) \to \mathrm{Hom}_{\mathbb{Q}[\Upsilon]}(\mathbb{Q}[\Upsilon]^d, \mathbb{Q}[\Upsilon])$$

are such that, for all $\{\varphi_i\}_{i\in[a]} \subset \mathrm{Hom}_{\mathbb{Q}[\Gamma]}(\mathbb{Q}[\Gamma]^d, \mathbb{Q}[\Gamma])$ and $x \in \bigwedge^a_{\mathbb{Q}[\Gamma]} \mathbb{Q}[\Gamma]^d$, one has

$$(\wedge_{i\in[a]}\varpi'(\varphi_i))(\varpi(x)) = (\mathbb{Q}\otimes_{\mathbb{Z}}\tilde{\pi}_*)((\wedge_{i\in[a]}\varphi_i)(x)).$$

Hence, as $\varpi'\big(\mathrm{Hom}_{\mathbb{Z}[\Gamma]}(\mathbb{Z}[\Gamma]^d, \mathbb{Z}[\Gamma])\big) = \mathrm{Hom}_{\mathbb{Z}[\Upsilon]}(\mathbb{Z}[\Upsilon]^d, \mathbb{Z}[\Upsilon])$, the map ϖ restricts to a map

$$\varrho^{a,d}_{\Gamma,\Delta} : \bigcap\nolimits^a_{\mathbb{Z}[\Gamma]} \mathbb{Z}[\Gamma]^d \to \bigcap\nolimits^a_{\mathbb{Z}[\Upsilon]} \mathbb{Z}[\Upsilon]^d \quad \text{with} \quad \varrho^{a,d}_{\Gamma,\Delta}\big(\xi(\Gamma,\Delta)\cdot \bigcap\nolimits^a_{\mathbb{Z}[\Gamma]} \mathbb{Z}[\Gamma]^d\big) = 0.$$

One has $\varrho^{0,d}_{\Gamma,\Delta} = \tilde{\pi}_*$. If $0 \le a < a'$, then for $\{\theta_j\}_{j\in[a]} \subset \mathrm{Hom}_{\mathbb{Z}[\Gamma]}(\mathbb{Z}[\Gamma]^d, \mathbb{Z}[\Gamma])$, there exists a commutative diagram of $\xi(\mathbb{Z}[\Gamma])$-modules

$$\begin{array}{ccc} \bigcap^{a'}_{\mathbb{Z}[\Gamma]} \mathbb{Z}[\Gamma]^d & \xrightarrow{\wedge_{j\in[a]}\theta_j} & \bigcap^{a'-a}_{\mathbb{Z}[\Gamma]} \mathbb{Z}[\Gamma]^d \\ {\scriptstyle \varrho^{a',d}_{\Gamma,\Delta}}\downarrow & & \downarrow{\scriptstyle \varrho^{a'-a,d}_{\Gamma,\Delta}} \\ \bigcap^{a'}_{\mathbb{Z}[\Upsilon]} \mathbb{Z}[\Upsilon]^d & \xrightarrow{\wedge_{j\in[a]}\varpi'(\theta_j)} & \bigcap^{a'-a}_{\mathbb{Z}[\Upsilon]} \mathbb{Z}[\Upsilon]^d. \end{array} \tag{6}$$

The following result analyses the cokernel of the maps $\varrho^{a,d}_{\Gamma,\Delta}$.

Lemma 2.4. *Write $\bigwedge^a_{\mathbb{Z}[\Gamma]^{\mathrm{op}}}(\mathbb{Z}[\Gamma]^{d,*})$ for the $\xi(\mathbb{Z}[\Gamma])$-submodule of $\bigcap^a_{\mathbb{Z}[\Gamma]^{\mathrm{op}}}(\mathbb{Z}[\Gamma]^{d,*})$ generated by $\{\wedge_{i\in[a]}x_i : x_i \in \mathbb{Z}[\Gamma]^{d,*}\}$. Then there exists a natural exact sequence of $\xi(\mathbb{Z}[\Gamma])$-modules*

$$\mathrm{cok}(\varrho^{a,d}_{\Gamma,\Delta}) \hookrightarrow \mathrm{Ext}^1_{\xi(\mathbb{Z}[\Gamma])}\big(\bigwedge\nolimits^a_{\mathbb{Z}[\Gamma]^{\mathrm{op}}}(\mathbb{Z}[\Gamma]^{d,*}), \xi(\Gamma,\Delta)\big) \xrightarrow{\kappa^{a,d}_{\Gamma,\Delta}} \mathrm{Ext}^1_{\xi(\mathbb{Z}[\Gamma])}\big(\bigwedge\nolimits^a_{\mathbb{Z}[\Gamma]^{\mathrm{op}}}(\mathbb{Z}[\Gamma]^{d,*}), \xi(\mathbb{Z}[\Gamma])\big).$$

Proof. Set $M = \bigwedge^a_{\mathbb{Z}[\Gamma]^{\rm op}}(\mathbb{Z}[\Gamma]^{d,*}), \overline{M} = \bigwedge^a_{\mathbb{Z}[\Upsilon]^{\rm op}}(\mathbb{Z}[\Upsilon]^{d,*})$, $\Lambda = \xi(\mathbb{Z}[\Gamma]), \overline{\Lambda} = \xi(\mathbb{Z}[\Upsilon]), \Lambda^0 = \xi(\Gamma,\Delta)$ and $\varrho = \varrho^{a,d}_{\Gamma,\Delta}$. Write e for the idempotent $|\Delta|^{-1}\sum_{\delta\in\Delta}\delta$ of $\zeta(\mathbb{Q}[\Gamma])$ and, for any torsion-free Λ-module X, set $X[e] := \{x \in 1\otimes X \subset \mathbb{Q}\otimes_{\mathbb{Z}} X : e(x) = 0\}$. Then, after identifying $\mathbb{Z}[\Gamma]e$ with $\mathbb{Z}[\Upsilon]$, and hence Λe with $\overline{\Lambda}$, in the natural way, there exists a short exact sequence of Λ-modules $0 \to M[e] \to M \xrightarrow{m\mapsto e(m)} \overline{M} \to 0$. Upon applying the functor $\mathrm{Hom}_\Lambda(-,\overline{\Lambda})$ to this sequence, we therefore obtain an exact sequence

$$0 \to \mathrm{Hom}_\Lambda(\overline{M},\overline{\Lambda}) \to \mathrm{Hom}_\Lambda(M,\overline{\Lambda}) \to \mathrm{Hom}_\Lambda(M[e],\overline{\Lambda}). \tag{7}$$

The last term here vanishes since it is both $\mathbb{Z}$-free and spans the $\mathbb{Q}$-space

$$\mathbb{Q}\otimes_{\mathbb{Z}}\mathrm{Hom}_\Lambda(M[e],\overline{\Lambda}) = \mathrm{Hom}_{\zeta(\mathbb{Q}[\Gamma])}((1-e)(\mathbb{Q}\otimes_{\mathbb{Z}} M),\zeta(\mathbb{Q}[\Upsilon])) = (0)$$

(where the last equality is valid since $(1-e)\zeta(\mathbb{Q}[\Upsilon]) = (0)$). In addition, the second term in (7) identifies with $\mathrm{Hom}_{\overline{\Lambda}}(\overline{M},\overline{\Lambda})$, and hence with $\bigcap^a_{\overline{\Lambda}}\mathbb{Z}[\Upsilon]^d$ (by [6, Rem. 4.18]) and so the sequence identifies $\bigcap^{a'}_{\mathbb{Z}[\Upsilon]}\mathbb{Z}[\Upsilon]^d$ with $\mathrm{Hom}_\Lambda(M,\overline{\Lambda})$. Since $\bigcap^a_{\mathbb{Z}[\Gamma]}\mathbb{Z}[\Gamma]^d$ similarly identifies with $\mathrm{Hom}_\Lambda(M,\Lambda)$, the claimed exact sequence is thus obtained by applying the functor $\mathrm{Hom}_\Lambda(M,-)$ to (5) (so that the map $\kappa^{a,d}_{\Gamma,\Delta}$ is induced by the inclusion $\Lambda^0\subseteq\Lambda$). □

Remark 2.5. Lemma 2.4 has the following explicit consequences (where we continue to use the notation of its proof). If Λ is Gorenstein, then $\mathrm{Ext}^1_\Lambda(M,\Lambda) = (0)$ and so ϱ is surjective if and only if $\mathrm{Ext}^1_\Lambda(M,\Lambda^0) = (0)$. If M is Λ-projective (as is the case if Λ is hereditary, but is not always true - see Example 2.7 below), then $\mathrm{Ext}^1_\Lambda(M,\Lambda^0) = (0)$ and so ϱ is surjective. If p does not divide $|\Delta|$, then Λ^0_p is a direct summand of Λ_p (as a Λ_p-module) so that $\kappa^{a,d}_{\Gamma,\Delta,p}$ is injective and hence ϱ_p is surjective.

Remark 2.6. One can also directly show $\varrho^{a,d}_{\Gamma,\Delta}$ is surjective if Υ (but not necessarily Γ) is abelian. This is clear if $a > d$, since then $\bigcap^a_{\mathbb{Z}[\Upsilon]}\mathbb{Z}[\Upsilon]^d = (0)$. If $a \le d$, it follows from the existence of a canonical surjective map $\bigcap^a_{\mathbb{Z}[\Gamma]}\mathbb{Z}[\Gamma]^d \to \Lambda^t$, where t is the binomial coefficient $\binom{d}{a}$ (see [6, Th. 4.19(vi)]), the natural isomorphism $\bigcap^a_{\mathbb{Z}[\Upsilon]}\mathbb{Z}[\Upsilon]^d = \bigwedge^a_{\mathbb{Z}[\Upsilon]}\mathbb{Z}[\Upsilon]^d \cong \mathbb{Z}[\Upsilon]^t$ and the fact that the map $\Lambda \to \xi(\mathbb{Z}[\Upsilon]) = \mathbb{Z}[\Upsilon]$ induced by $\tilde{\pi}_*$ is surjective (by [6, Lem. 3.2(v)]).

Example 2.7. The $\xi(\mathbb{Z}[\Gamma])$-module $\bigwedge^a_{\mathbb{Z}[\Gamma]}\mathbb{Z}[\Gamma]^d$ need not be projective. To describe an example, we fix an odd prime p and consider the Heisenberg group

$$\Gamma := \langle \gamma_1,\gamma_2,\gamma_3 \mid \gamma_1^p = \gamma_2^p = \gamma_3^p = 1, \gamma_3\gamma_2 = \gamma_2\gamma_3\gamma_1, \gamma_1\gamma_2 = \gamma_2\gamma_1, \gamma_1\gamma_3 = \gamma_3\gamma_1\rangle.$$

This is the unique non-abelian group of order p^3 and exponent p (cf. [20, §4.4]), its centre is $\Xi := \langle\gamma_1\rangle = [\Gamma,\Gamma]$ and the subgroup $\Xi' := \langle\gamma_1,\gamma_2\rangle = \langle\gamma_1\rangle\times\langle\gamma_2\rangle$ is normal; it has p^2 linear characters inflated from Γ/Ξ, and $p-1$ irreducible characters of degree p, each of the form $\mathrm{Ind}^G_{\Xi'}(\mathrm{Inf}^{\Xi'}_{\Xi}(\phi))$ with ϕ a non-trivial linear character of Ξ. Using this, an explicit computation of reduced norms shows that, for each $M = (m_{ij}) \in \mathrm{M}_d(\mathbb{Q}_p[\Gamma])$, one has

$$\mathrm{Nrd}_{\mathbb{Q}_p[\Gamma]}(M) = e\cdot\det(M_1) + (1-e)\cdot\det(M_2) = \det(M_2) + e(\det(M_1)-\det(M_2)), \tag{8}$$

with $e:=(1/p)\sum_{\gamma\in\Xi}\gamma\in\zeta(\mathbb{Q}[\Gamma])$, M_1 the image of M in $\mathrm{M}_d(\mathbb{Q}_p[\Gamma/\Xi])$ and, for $\omega\in\mathbb{Q}_p^c\setminus\{1\}$ with $\omega^p=1$ and $\pi:\mathbb{Q}[\Xi']\to\mathbb{Q}[\Xi]$ the natural projection, one has

$$M_2:=\left(\prod_{k=0}^{k=p-1}\left(\sum_{l=0}^{l=p-1}\omega^{kl}\pi(m_{ijl})\right)\right)_{ij}\in\mathrm{M}_d(\mathbb{Q}_p[\Xi]),$$

where $m_{ijl}\in\mathbb{Q}_p[\Xi']$ is such that $m_{ij}=\sum_{l=0}^{l=p-1}m_{ijl}\,\gamma_3^l$ in $\mathbb{Q}_p[\Gamma]$. For each M it can be checked that $e(\det(M_1)-\det(M_2))$ belongs to the maximal ideal of the local ring $\mathbb{Z}_p[\Gamma]e$ and so (8) implies that $e\notin\xi(\mathbb{Z}_p[\Gamma])$. It follows that the (semi-perfect) ring $\xi(\mathbb{Z}[\Gamma])_p=\xi(\mathbb{Z}_p[\Gamma])$ has a local component Λ for which $e\Lambda\neq(0)\neq(1-e)\Lambda$. Then, for $a\in[d-1]$, the $(\mathbb{Q}_p\cdot\Lambda)$-component of $\mathbb{Q}_p\cdot\bigwedge^a_{\mathbb{Z}[\Gamma]}\mathbb{Z}[\Gamma]^d$ is not free (by [6, Lem. 4.16] and the argument of [6, Th. 4.19(vi)]) and so the $\xi(\mathbb{Z}[\Gamma])$-module $\bigwedge^a_{\mathbb{Z}[\Gamma]}\mathbb{Z}[\Gamma]^d$ cannot be projective.

3. Projective pullbacks

As preparation for our main constructions, we fix an extension of finite groups of the form (4) and introduce a notion of 'projective pullbacks relative to π' for the category of $\mathbb{Z}_p[\Upsilon]$-lattices.

For convenience, in the sequel we shall refer to a map of $\mathbb{Z}_p[\Upsilon]$-modules as 'admissible' if it is injective and its cokernel is a lattice. We also fix a $\mathbb{Z}_p[\Upsilon]$-lattice M.

Definition 3.1. A 'projective hull' of M is an admissible map of $\mathbb{Z}_p[\Upsilon]$-lattices $\iota:M\to P$ in which P is projective and the following condition is satisfied: for any admissible map $\iota':M\to P'$ in which P' is projective, there exists an admissible map $\kappa:P\to P'$ for which $\iota'=\kappa\circ\iota$. Two projective hulls $\iota:M\to P$ and $\iota':M\to P'$ are said to be 'equivalent' if there exists an isomorphism of $\mathbb{Z}_p[\Upsilon]$-modules $\kappa:P\to P'$ such that $\iota'=\kappa\circ\iota$.

We set $M^*:=\mathrm{Hom}_{\mathbb{Z}_p}(M,\mathbb{Z}_p)$, regarded as a $\mathbb{Z}_p[\Upsilon]$-lattice via the contragredient action.

Lemma 3.2. *Let $\iota:M\to P$ an admissible map of $\mathbb{Z}_p[\Upsilon]$-lattices in which P is projective. Then ι is a projective hull if and only if $\mathrm{Hom}_{\mathbb{Z}_p}(\iota,\mathbb{Z}_p)$ is a projective cover of M^*. In particular, every $\mathbb{Z}_p[\Upsilon]$-lattice has a unique equivalence class of projective hulls.*

Proof. Set $\Lambda:=\mathbb{Z}_p[\Upsilon]$. Then Λ is Gorenstein and has the following properties: every Λ-lattice M is reflexive (that is, the natural map $M\to(M^*)^*$ is bijective); a map θ of Λ-lattices is admissible if and only if its $\mathbb{Z}_p$-dual θ^* is surjective; the $\mathbb{Z}_p$-dual of a finitely generated projective Λ-module is finitely generated projective. These properties combine to imply θ is an admissible map from M to a finitely generated projective module if and only if θ^* is a surjective map from a finitely generated projective module to M^*.

Using this equivalence, one checks easily that $\iota:M\to P$ is a projective hull of M if and only if ι^* is a projective cover of M^*. Given this correspondence, the final assertion follows directly from the fact $\mathbb{Z}_p[\Upsilon]$ is semi-perfect and so every finitely generated module has a projective cover that is unique up to isomorphism (cf. [8, Prop. (6.20), Th. (6.23)]). □

We can now give the main definition of this section.

Definition 3.3. A 'projective pullback relative to π' of M is a pair $(\iota_{M,Q},\kappa_{P,Q})$ comprising a projective hull $\iota_{M,Q}:M\to Q$ of M and a projective cover $\kappa_{P,Q}:P\to Q$ of Q considered as a $\mathbb{Z}_p[\Gamma]$-module via π (so that P is a projective $\mathbb{Z}_p[\Gamma]$-module). Two such pairs $(\iota_{M,Q},\kappa_{P,Q})$

and $(\iota'_{M,Q'},\kappa'_{P',Q'})$ are 'equivalent' if there exist bijective maps μ and $\widetilde{\mu}$ within a commutative diagram of $\mathbb{Z}_p[\Gamma]$-modules of the form

$$\begin{array}{ccccc} & \xrightarrow{\iota_{M,Q}} & Q & \xleftarrow{\kappa_{P,Q}} & P \\ M & & \mu\downarrow & & \downarrow\widetilde{\mu} \\ & \xrightarrow[\iota'_{M,Q'}]{} & Q' & \xleftarrow[\kappa'_{P',Q'}]{} & P'. \end{array} \tag{9}$$

Lemma 3.4. *There exists a unique equivalence class $\pi^*(M)$ of projective pullbacks of M relative to π. For $(\iota_{M,Q},\kappa_{P,Q})$ in $\pi^*(M)$, the map $(\kappa_{P,Q})_\Delta : P_\Delta \to Q$ is bijective.*

Proof. For $(\iota_{M,Q},\kappa_{P,Q})$ and $(\iota'_{M,Q'},\kappa'_{P',Q'})$ in $\pi^*(M)$, Lemma 3.2 implies the existence of an isomorphism μ that makes the triangle in (9) commute. Since $\mu\circ\kappa_{P,Q}$ is a projective cover of Q', there is an isomorphism $\widetilde{\mu}$ making the square in (9) commute (cf. [8, Prop. (6.20)]).

Set $\kappa = \kappa_{P,Q}$ and write $\eta : P \twoheadrightarrow P_\Delta$ for the canonical map, so that $\kappa = \kappa_\Delta\circ\eta$. To prove the bijectivity of κ_Δ, it is enough to prove its injectivity, and to do this we use the projectivity of Q (as a $\mathbb{Z}_p[\Upsilon]$-module) to fix a section σ of κ_Δ. Then, if κ_Δ is not injective, one has $\sigma(Q) \neq P_\Delta$ and so the full pre-image $\eta^{-1}(\sigma(Q))$ of $\sigma(Q)$ under η is a proper submodule of P. However, since $\kappa(\eta^{-1}(\sigma(Q))) = \kappa_\Delta(\sigma(Q)) = Q = \kappa(P)$, this last assertion contradicts the fact κ is essential. □

The above argument can be extended to show, more conceptually, that $\pi^*(M)$ is the initial object of a category with objects (ι,κ), where ι is an admissible map from M to a projective $\mathbb{Z}_p[\Upsilon]$-lattice Q, and $\kappa : P \to Q$ a surjective map of $\mathbb{Z}_p[\Gamma]$-lattices, with P projective. (This aspect will be considered more fully in the upcoming thesis of the second author). For our immediate purposes, however, the significance of Lemma 3.4 is that the diagrams (9) imply all constructions that we make in the sequel are independent, in a natural sense, of the choice of representative of $\pi^*(M)$. In particular, by fixing a representative of $\pi^*(M)$ of the form $(\iota_{M,P_\Delta},\kappa_P)$, with κ_P the canonical map $P \to P_\Delta$, we can, and will, identify $\pi^*(M)$ with the projective $\mathbb{Z}_p[\Gamma]$-module P and regard M as a submodule of $\pi^*(M)_\Delta = P_\Delta$ (via ι_{M,P_Δ}). We shall freely use this approach in the rest of this section.

In the next definition, we fix a $\mathbb{Z}_p[\Gamma]$-module N and a map of $\mathbb{Z}_p[\Upsilon]$-modules $\theta : M \to N_\Delta$.

Definition 3.5. A 'pullback of θ relative to π' is a map of $\mathbb{Z}_p[\Gamma]$-modules $\tilde{\theta} : \pi^*(M) \to N$ for which θ is the restriction to M of the induced map $\tilde{\theta}_\Delta : \pi^*(M)_\Delta \to N_\Delta$ (such a map $\tilde{\theta}$ exists if and only if θ factors through ι for any given (ι,κ) in $\pi^*(M)$). Two pullbacks $\tilde{\theta}$ and $\tilde{\theta}'$ of θ will be identified if they are respectively defined via elements (ι,κ) and (ι',κ') of $\pi^*(M)$ and one has $\tilde{\theta}_\Delta = \tilde{\theta}'_\Delta \circ \mu$ for an isomorphism μ as in (9).

Remark 3.6. Fix $(\iota_{M,Q},\kappa_{P,Q})$ in $\pi^*(M)$ and an isomorphism $\mu : P \oplus R \cong \mathbb{Z}_p[\Gamma]^d$ of $\mathbb{Z}_p[\Gamma]$-modules. Then $M \xrightarrow{(\iota_{M,Q},0)} Q \oplus R_\Delta \xrightarrow{\mu_\Delta} \mathbb{Z}_p[\Upsilon]^d$ is admissible and $\kappa_{P,Q}$ is the restriction through μ of the ring homomorphism $\pi^d_* : \mathbb{Z}_p[\Gamma]^d \to \mathbb{Z}_p[\Upsilon]^d$ induced by π. Conversely, given any admissible $\iota : M \to \mathbb{Z}_p[\Upsilon]^d$, there exist direct summands P and Q of $\mathbb{Z}_p[\Gamma]^d$ and $\mathbb{Z}_p[\Upsilon]^d$ such that $\iota(M) \subseteq Q$, the map $M \to Q$ induced by ι is a projective hull and the restriction of π^d_* to P is a projective cover of Q (as a $\mathbb{Z}_p[\Gamma]$-module). Given such a map ι, constructing a pullback of $\theta : M \to N_\Delta$ relative to π is therefore equivalent to constructing a map of $\mathbb{Z}_p[\Gamma]$-modules $\overline{\theta} : \mathbb{Z}_p[\Gamma]^d \to N$ such that θ is the restriction of $\overline{\theta}_\Delta$ through ι.

The next definition concerns the reduced Rubin lattice $\bigcap^a_{\mathbb{Z}_p[\Gamma]} N$ associated to a finitely generated $\mathbb{Z}_p[\Gamma]$-module N and a non-negative integer a (cf. §2.2). In particular, we recall that $\bigcap^a_{\mathbb{Z}_p[\Gamma]} N$ is a lattice over the Whitehead order $\xi(\mathbb{Z}_p[\Gamma])$ of $\mathbb{Z}_p[\Gamma]$, a simplified definition of which is provided in §2.1.

Definition 3.7. For any non-negative integer a, the projection $\pi^*(M) \twoheadrightarrow \pi^*(M)_\Delta$ induces a map $\varrho^a_{M,\pi} : \bigcap^a_{\mathbb{Z}_p[\Gamma]} \pi^*(M) \to \bigcap^a_{\mathbb{Z}_p[\Upsilon]} \pi^*(M)_\Delta$ of $\xi(\mathbb{Z}_p[\Gamma])$-modules. We define a $\xi(\mathbb{Z}_p[\Upsilon])$-submodule of $\bigcap^a_{\mathbb{Z}_p[\Upsilon]} M$ by setting $\left(\bigcap^a_{\mathbb{Z}_p[\Upsilon]} M\right)^\pi := \left(\bigcap^a_{\mathbb{Z}_p[\Upsilon]} M\right) \cap \mathrm{im}(\varrho^a_{M,\pi})$.

Remarks 3.8. (i) Given $(\iota_{M,Q}, \kappa_{P,Q}) \in \pi^*(M)$, one can compute $\left(\bigcap^a_{\mathbb{Z}_p[\Upsilon]} M\right)^\pi$ as the set of elements of $\bigcap^a_{\mathbb{Z}_p[\Upsilon]} M$ whose image under the (injective) map $\bigcap^a_{\mathbb{Z}_p[\Upsilon]} M \to \bigcap^a_{\mathbb{Z}_p[\Upsilon]} Q$ induced by $\iota_{M,Q}$ is contained in the image of the map $\bigcap^a_{\mathbb{Z}_p[\Gamma]} P \to \bigcap^a_{\mathbb{Z}_p[\Upsilon]} Q$ induced by $\kappa_{P,Q}$. (This description is easily seen to be independent of the choice of $(\iota_{M,Q}, \kappa_{P,Q})$.)
(ii) If Υ (but not necessarily Γ) is abelian, then $\left(\bigcap^a_{\mathbb{Z}_p[\Upsilon]} M\right)^\pi = \bigcap^a_{\mathbb{Z}_p[\Upsilon]} M$ for all a and M. This follows easily from the first observation in Remark 3.6 and the fact that, for each $d \geq a$, the map $\bigcap^a_{\mathbb{Z}_p[\Gamma]} \mathbb{Z}_p[\Gamma]^d \to \bigcap^a_{\mathbb{Z}_p[\Upsilon]} \mathbb{Z}_p[\Upsilon]^d$ induced by π^d_* is surjective (cf. Remark 2.6).

4. Artin-Bockstein maps

We now fix a finite Galois extension of number fields L/K of group G and a normal subgroup H of G. We set $E := L^H$ and $\overline{G} := G/H \cong \mathrm{Gal}(E/K)$ and (as a particular case of (4)) consider the group extension

$$1 \to H \to G \xrightarrow{\pi} \overline{G} \to 1. \tag{10}$$

For a finite set S of places of K we write S_E for the set of places of E lying above S, $Y_{E,S}$ for the free abelian group on S_E and $X_{E,S}$ for the submodule of $Y_{E,S}$ comprising elements whose coefficients sum to zero. If S contains the set S_K^∞ of archimedean places, we write $\mathcal{O}_{E,S}$ for the subring of E comprising elements integral at all places outside S_E. For a finite set T of places of K with $T \cap S = \emptyset$, we write $U_S^T(E)$ for the (finite index) subgroup of $\mathcal{O}_{E,S}^\times$ comprising all elements congruent to 1 modulo all places in T_E and $\mathrm{Cl}_S^T(E)$ for the ray class group of $\mathcal{O}_{E,S}$ modulo $\prod_{w\in T_E} w$. We write $\mathcal{S}_S^T(E)$ for the '(S-relative T-trivialized) transpose Selmer group' for $\mathbb{G}_m$ over E defined in [4] and recall that it lies in a canonical short exact sequence of the form

$$0 \to \mathrm{Cl}_S^T(E) \to \mathcal{S}_S^T(E) \xrightarrow{\varrho_{E,S}} X_{E,S} \to 0.$$

We further recall from loc. cit. the existence of a complex of G-modules

$$C_{L,S}^T := \mathsf{R}\,\mathrm{Hom}_{\mathbb{Z}}(\mathsf{R}\Gamma_{c,T}((\mathcal{O}_{L,S})_{\mathcal{W}}, \mathbb{Z}), \mathbb{Z})[-2].$$

This complex is defined up to canonical isomorphism in the derived category $D(\mathbb{Z}[G])$ of G-modules, acyclic outside degrees zero and one and such that $H^0(C_{L,S}^T) = U_S^T(L)$ and $H^1(C_{L,S}^T) = \mathcal{S}_S^T(L)$ and there exists a canonical 'projection formula' isomorphism in $D(\mathbb{Z}[\overline{G}])$

$$\mathbb{Z}[\overline{G}] \otimes^{\mathsf{L}}_{\mathbb{Z}[G]} C_{L,S}^T \cong C_{E,S}^T. \tag{11}$$

The construction of $C^T_{L,S}$ in [4, §2] is motivated by the theory of Weil-étale cohomology for varieties over finite fields developed by Lichtenbaum in [25] and, if S contains all p-adic places of K, then $\mathbb{Z}_p \otimes_{\mathbb{Z}} C^T_{L,S}$ can be described in terms of the compactly-supported p-adic cohomology of $\mathbb{Z}_p$ on $\mathrm{Spec}(\mathcal{O}_{L,S})$.

4.1. **Artin-Bockstein maps and pullbacks.** Setting $I(G,H) := \ker(\pi_*)$, the tautological exact sequence $0 \to I(G,H) \to \mathbb{Z}[G] \xrightarrow{\pi_*} \mathbb{Z}[\overline{G}] \to 0$ combines with the isomorphism (11) to give a canonical exact triangle

$$I(G,H) \otimes^{\mathsf{L}}_{\mathbb{Z}[G]} C^T_{L,S} \to C^T_{L,S} \to C^T_{E,S} \to I(G,H) \otimes^{\mathsf{L}}_{\mathbb{Z}[G]} C^T_{L,S}[1] \tag{12}$$

in $D(\mathbb{Z}[G])$ and hence a canonical map of G-modules

$$\begin{aligned} U^T_S(E) = H^0(C^T_{E,S}) &\to H^1(I(G,H) \otimes^{\mathsf{L}}_{\mathbb{Z}[G]} C^T_{L,S}) \\ &= I(G,H) \otimes_{\mathbb{Z}[G]} H^1(C^T_{L,S}) = I(G,H) \otimes_{\mathbb{Z}[G]} \mathcal{S}^T_S(L). \end{aligned} \tag{13}$$

We assume $|S| > 1$. For $v \in S$ we fix a place $w_v \in \{v\}_L$, write $w_{v,E}$ for the restriction of w_v to E and G_v for the decomposition subgroup of w_v in G, and consider the map

$$\mathcal{S}^T_S(L) \xrightarrow{\varrho_{L,S}} X_{L,S} \to \mathbb{Z}[G] \cdot w_v \cong \mathbb{Z}[G] \otimes_{\mathbb{Z}[G_v]} \mathbb{Z},$$

in which the second arrow is the natural projection and the isomorphism sends w_v to $1 \otimes 1$. In particular, if $N(G_v)$ is the normal closure of G_v in G, and we set $I_v(G) := I(G, N(G_v))$, then this map combines with (13) for $H = N(G_v)$ to give a canonical map

$$U^T_S(L^{N(G_v)}) \to I_v(G) \otimes_{\mathbb{Z}[G_v]} \mathbb{Z} \twoheadrightarrow I_v(G) \otimes_{\mathbb{Z}[N(G_v)]} \mathbb{Z} = I_v(G)/I_v(G)^2. \tag{14}$$

If G is abelian, then $G_v = N(G_v)$, $I_v(G)/I_v(G)^2$ identifies with $\mathbb{Z}[G/G_v] \otimes_{\mathbb{Z}} G_v$ and the argument of [4, Lem. 5.20] shows that, if we set $F = L^{N(G_v)}$, then the map (14) is induced by the composite $U^T_S(F) \subset F^\times_{w_v,F} \to G_v$, where the arrow denotes the local reciprocity map.

We now fix a place v in S that splits completely in E (so that $N(G_v) \subseteq H$ and hence $I_v(G) \subseteq I(G,H)$) and consider the composite 'Artin-Bockstein' map

$$\varphi_v : U^T_S(E) \subseteq U^T_S(L^{N(G_v)}) \to I_v(G)/I_v(G)^2 \to I_v(G)_H,$$

where the first arrow is the map (14) and the second the canonical projection. In the next result we fix a prime p and show that if S contains the set $S^{\mathrm{ram}}_{L/K}$ of places (of K) that ramify in L, then a description of the composite map (13) on the level of complexes gives a canonical pullback of $\mathbb{Z}_p \otimes_{\mathbb{Z}} \varphi_v$ relative to $\mathbb{Z}_p \otimes_{\mathbb{Z}} \pi$.

Lemma 4.1. *Assume that $|S| > 1$ and $S^{\mathrm{ram}}_{L/K} \subseteq S$, and that p is a prime for which $U^T_S(L)_p$ is torsion-free. Then each choice of place v_0 in $S \setminus \{v\}$ specifies a pullback $\pi^*(\varphi_v)_p$ of $\mathbb{Z}_p \otimes_{\mathbb{Z}} \varphi_v$ relative to $\mathbb{Z}_p \otimes_{\mathbb{Z}} \pi$.*

Proof. Set $U_F := U^T_S(F)_p$ for $F \in \{E, L\}$, $\mathcal{S} := \mathcal{S}^T_S(L)_p$, $S_0 := S \setminus \{v_0\}$ and write ϱ for the composite $\mathcal{S} \to X_{L,S,p} \to Y_{L,S_0,p}$, where the first map is $\varrho_{L,S,p}$ and the second the natural projection. Set $n := |S| - 1$, label (and thereby order) the places of S_0 as $\{v_i\}_{i \in [n]}$ and for each i set $w_i := w_{v_i}$.

Fix a projective cover of $\mathbb{Z}_p[G]$-modules $\varpi_1 : P \to \ker(\varrho)$ and a $\mathbb{Z}_p[G]$-module P' of minimal rank so $P \oplus P'$ is free, of rank d_0 say. Fix an identification $P \oplus P' = \mathbb{Z}_p[G]^{d_0}$ and write $\varpi_2 = (\varpi_1, 0_{P'})$ for the induced surjection $\mathbb{Z}_p[G]^{d_0} \to \ker(\varrho)$. Set $d := n + d_0$, $b_i := b_{G,i}$ for $i \in [d]$, b_i^* for the element of $\mathbb{Z}_p[G]^{d,*}$ that is dual to b_i and $\varpi : \mathbb{Z}_p[G]^d \to \mathcal{S}$ for the map of $\mathbb{Z}_p[G]$-modules that sends b_i to a choice of pre-image of w_i under ϱ if $i \in [n]$ and to $\varpi_2(b_{i-n})$ if $i \in [d] \setminus [n]$. Then ϖ is surjective and one has

$$\varrho(\varpi(b_i)) = w_i \text{ if } i \in [n] \quad \text{and} \quad \varrho(\varpi(b_i)) = 0 \text{ if } i \in [d] \setminus [n]. \tag{15}$$

In addition, as U_L is $\mathbb{Z}_p$-free and $S^{\mathrm{ram}}_{L/K} \subseteq S$, the argument of [5, Prop. 3.2] shows $C = C^T_{L,S,p}$ is represented by a complex $\mathbb{Z}_p[G]^d \xrightarrow{\phi} \mathbb{Z}_p[G]^d$ in such a way that ϖ induces an isomorphism $\hat{\varpi} : \mathrm{cok}(\phi) \cong H^1(C) = \mathcal{S}$ and there is an induced identification $\hat{\iota} : U_L \cong \ker(\phi)$. The exact triangle (12) is then induced by the exact sequence of complexes (with vertical differentials)

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I(G,H)_p^d & \xrightarrow{\subset} & \mathbb{Z}_p[G]^d & \xrightarrow{\pi_{*,p}} & \mathbb{Z}_p[\overline{G}]^d & \longrightarrow & 0 \\ & & {\scriptstyle\phi}\big\downarrow & & {\scriptstyle\phi}\big\downarrow & & {\scriptstyle\phi_H}\big\downarrow & & \\ 0 & \longrightarrow & I(G,H)_p^d & \xrightarrow{\subset} & \mathbb{Z}_p[G]^d & \xrightarrow{\pi_{*,p}} & \mathbb{Z}_p[\overline{G}]^d & \longrightarrow & 0 \end{array}$$

and so the Snake Lemma implies that the p-completion of (13) sends $u \in U_E$ to $\pi(\phi(\iota(u)'))$. Here ι and π are the injection $U_E \to \mathbb{Z}_p[\overline{G}]^d$ and surjection $I(G,H)_p^d \to I(G,H)_p \otimes_{\mathbb{Z}_p[G]} \mathcal{S}$ induced by $\hat{\iota}$ and $\hat{\varpi}$ and $\iota(u)'$ is any element of $\mathbb{Z}_p[G]^d$ with $\pi_{*,p}(\iota(u)') = \iota(u)$ in $\ker(\phi_H) \subseteq \mathbb{Z}_p[\overline{G}]^d$. In particular, if $v = v_i$ splits completely in E, then (15) implies $\mathrm{im}(b_i^* \circ \phi) \subseteq I_v(G)_p \subseteq I(G,H)_p$ and so (the final assertion of Remark 3.6 and) this computation implies $b_i^* \circ \phi$ defines a pullback $\pi^*(\varphi_v)_p$ of $\mathbb{Z}_p \otimes_{\mathbb{Z}} \varphi_v$ relative to $\mathbb{Z}_p \otimes_{\mathbb{Z}} \pi$ (and the embedding ι). □

Remark 4.2. The pullback $\pi^*(\varphi_v)_p$ constructed above is independent of the representative of C since if another complex $\mathbb{Z}_p[G]^d \xrightarrow{\phi'} \mathbb{Z}_p[G]^d$ is used, then $b_i^* \circ \phi'$ defines the same pullback (in the sense of Definition 3.5) as does $b_i^* \circ \phi$. To see this, note that if $\hat{\iota}' : U_L \cong \ker(\phi')$ and $\varpi' : \mathbb{Z}_p[G]^d \to \mathcal{S}$ are the maps associated to ϕ', then [5, Prop. 3.2(iv)] implies the existence of an exact commutative diagram of $\mathbb{Z}_p[G]$-modules

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & U_L & \xrightarrow{\hat{\iota}} & \mathbb{Z}_p[G]^d & \xrightarrow{\phi} & \mathbb{Z}_p[G]^d & \xrightarrow{\varpi} & \mathcal{S} & \longrightarrow & 0 \\ & & \| & & {\scriptstyle\kappa'}\big\downarrow & & \big\downarrow{\scriptstyle\kappa} & & \| & & \\ 0 & \longrightarrow & U_L & \xrightarrow{\hat{\iota}'} & \mathbb{Z}_p[G]^d & \xrightarrow{\phi'} & \mathbb{Z}_p[G]^d & \xrightarrow{\varpi'} & \mathcal{S} & \longrightarrow & 0 \end{array} \tag{16}$$

in which κ' and κ are bijective and the matrix of κ with respect to $\{b_i\}_{i \in [d]}$ has the form

$$\left(\begin{array}{c|c} I_n & * \\ \hline 0 & M_\kappa \end{array} \right), \tag{17}$$

where I_n is the $n \times n$ identity matrix and $M_\kappa \in \mathrm{GL}_{d-n}(\mathbb{Z}_p[G])$. The commutativity of (16) then combines with the shape of this block matrix to imply that the automorphism κ'_Δ of $\mathbb{Z}_p[\overline{G}]^d$ is such that $(b_i^* \circ \phi')_\Delta \circ \kappa'_\Delta = (b_i^* \circ \phi)_\Delta$, as required.

4.2. **Reduced Artin-Bockstein maps.** In this section we assume $|S|>1$ and $S^{\rm ram}_{L/K}\subseteq S$. We fix a place v_0 in S and then label the places in $S_0:=S\setminus\{v_0\}$ as in the proof of Lemma 4.1 (so that $n=|S_0|$). For $F\in\{E,L\}$ we define

$$\Sigma(F)=\Sigma_{S_0}(F):=\{v\in S_0: v \text{ splits completely in } F\}\quad\text{and}\quad r(F):=|\Sigma(F)|.$$

We also assume, as we may, the labelling of S_0 is such that $\Sigma(F)=\{v_i\}_{i\in[r(F)]}$ for both $F=E$ and $F=L$. For each subset Σ of $S\setminus S_K^\infty$, we then define ideals of $\xi(\mathbb{Z}[G])$ by setting

$$\begin{aligned}\iota_\Sigma(G)&:=\xi(\mathbb{Z}[G])\cdot\{(\wedge_{v\in\Sigma}\theta_v)(x): d\geq|\Sigma|,\, x\in{\textstyle\bigcap}^{|\Sigma|}_{\mathbb{Z}[G]}\mathbb{Z}[G]^d,\, \theta_v\in\operatorname{Hom}_G(\mathbb{Z}[G]^d,I_v(G))\}\\ \iota^H_\Sigma(G)&:=\xi(\mathbb{Z}[G])\cdot\{(\wedge_{v\in\Sigma}\theta_v)(x): d\geq|\Sigma|,\, x\in\ker(\varrho^{|\Sigma|,d}_{G,H}),\, \theta_v\in\operatorname{Hom}_G(\mathbb{Z}[G]^d,I_v(G))\},\end{aligned}$$

where $\varrho^{|\Sigma|,d}_{G,H}$ is the natural map ${\bigcap}^{|\Sigma|}_{\mathbb{Z}[G]}\mathbb{Z}[G]^d\to{\bigcap}^{|\Sigma|}_{\mathbb{Z}[\overline{G}]}\mathbb{Z}[\overline{G}]^d$ (cf. §2.2.3). The first inclusion in the next result implies that the action of $\xi(\mathbb{Z}[G])$ on $\iota_\Sigma(G)/\iota^H_\Sigma(G)$ factors through $\xi(\mathbb{Z}[\overline{G}])$.

Lemma 4.3. *One has* $\xi(G,H)\iota_\Sigma(G)\subseteq\iota^H_\Sigma(G)\subseteq\xi(G,H)$.

Proof. If one sets $(\Gamma,\Delta)=(G,H)$ and $a=a'=|\Sigma|$ in the commutative diagram (6) (in §2.2.3), then $\varrho^{a'-a,d}_{\Gamma,\Delta}$ is the map $\tilde{\pi}_*:\xi(\mathbb{Z}[G])\to\xi(\mathbb{Z}[\overline{G}])$, and so one deduces that $\iota^H_\Sigma(G)\subseteq\xi(G,H)$. Since $\xi(G,H)\cdot{\bigcap}^{|\Sigma|}_{\mathbb{Z}[G]}\mathbb{Z}[G]^d\subseteq\ker(\varrho^{|\Sigma|,d}_{G,H})$ one also has $\xi(G,H)\iota_\Sigma(G)\subseteq\iota^H_\Sigma(G)$. □

Finally, for any abelian group A we consider the direct product

$$A^{[G,S,T]}:=\prod_{\underline{\varphi}}A,$$

where $\underline{\varphi}$ runs over all elements of $\operatorname{Hom}_G(U^T_S(L),\mathbb{Z}[G])^{r(L)}_p$.

Lemma 4.4. *Assume $|S|>1$, $S^{\rm ram}_{L/K}\subseteq S$ and $U^T_S(L)_p$ is torsion-free and write Σ for $\Sigma_{S_0}(E)\setminus\Sigma_{S_0}(L)$. Then the reduced exterior product of the maps $\{\pi^*(\varphi_v)_p\}_{v\in\Sigma}$ induces a well-defined homomorphism of $\xi(\mathbb{Z}_p[\overline{G}])$-modules*

$$\operatorname{Rec}_\pi:\big({\textstyle\bigcap}^{r(E)}_{\mathbb{Z}_p[\overline{G}]}U^T_S(E)_p\big)^\pi\to\big(\iota_\Sigma(G)/\iota^H_\Sigma(G)\big)^{[G,S,T]}_p.$$

Proof. We set $U:=U^T_S(L)_p$, $r:=r(L)$, $r':=r(E)$ and $r^*:=r'-r(=|\Sigma|)$. Using the construction in Lemma 4.1 we fix an embedding $\hat{\imath}:U\to\mathbb{Z}_p[G]^d$, write ι for the induced map $U^H\to(\mathbb{Z}_p[G]^d)^H\cong\mathbb{Z}_p[\overline{G}]^d$ and, for $j\in[r']\setminus[r]$ fix a pullback $\phi_j:=b^*_j\circ\phi$ of $\mathbb{Z}_p\otimes_\mathbb{Z}\varphi_{v_j}$ relative to $\mathbb{Z}_p\otimes_\mathbb{Z}\pi$.

Then, as $\operatorname{cok}(\hat{\imath})$ is torsion-free, the map $\operatorname{Hom}_{\mathbb{Z}_p[G]}(\mathbb{Z}_p[G]^d,\mathbb{Z}_p[G])\to\operatorname{Hom}_{\mathbb{Z}_p[G]}(U,\mathbb{Z}_p[G])$ induced by restriction through $\hat{\imath}$ is surjective and so we can fix a preimage $\theta^{\hat{\imath}}$ under this map of any given θ in $\operatorname{Hom}_{\mathbb{Z}_p[G]}(U,\mathbb{Z}_p[G])$.

For x in $\big({\bigcap}^{r'}_{\mathbb{Z}_p[\overline{G}]}U^T_S(E)_p\big)^\pi$ we choose $\hat{x}$ in ${\bigcap}^{r'}_{\mathbb{Z}_p[G]}\mathbb{Z}_p[G]^d$ with $\iota_*(x)=\varrho^{r',d}_{G,H}(\hat{x})$. For each $\underline{\varphi}=(\varphi_1,\ldots,\varphi_r)$ in $\operatorname{Hom}_{\mathbb{Z}_p[G]}(U,\mathbb{Z}_p[G])^r$ we then define an element of $\xi(\mathbb{Z}_p[G])$ by setting

$$x_{\underline{\varphi}}:=(\wedge^{j=r}_{j=1}\varphi^{\hat{\imath}}_j)((\wedge^{j=r'}_{j=r+1}\phi_j)(\hat{x}))=\operatorname{Nrd}_{\mathbb{Q}_p[G]}(-1)^{rr^*}\cdot(\wedge^{j=r'}_{j=r+1}\phi_j)((\wedge^{j=r}_{j=1}\varphi^{\hat{\imath}}_j)(\hat{x})).$$

Since $(\wedge_{j=1}^{j=r}\varphi_j^{\hat\iota})(\hat x) \in \bigcap_{\mathbb{Z}_p[G]}^{r^*}\mathbb{Z}_p[G]^d$, one has $x_{\underline{\varphi}} \in \iota_\Sigma(G)_p$. To obtain a map of the required sort it is thus enough to show that the projection $\mathrm{Rec}_\pi(x)$ of the element $(x_{\underline{\varphi}})_{\underline{\varphi}}$ of $\iota_\Sigma(G)^{[G,S,T]}$ to $\big(\iota_\Sigma(G)/\iota_\Sigma^H(G)\big)_p^{[G,S,T]}$ is independent of the choices of $\hat\iota$, $\{\phi_j\}_{j\in[r']\setminus[r]}$, $\hat x$ and $\{\varphi_j^{\hat\iota}\}_{j\in[r]}$.

Firstly, we let $\hat\iota' : U \to \mathbb{Z}_p[G]^d$ be an alternative choice of embedding as in (16) with ϕ_j' the corresponding pullback of φ_{v_j} for $j \in [r'] \setminus [r]$. Then the commutativity of the second square in (16) combines with the shape of the matrix (17) to imply that

$$\phi_j' \circ \kappa' = b_j^* \circ \phi' \circ \kappa' = (b_j^* \circ \kappa) \circ \phi = b_j^* \circ \phi = \phi_j$$

for $j \in [r'] \setminus [r] \subseteq [n]$ and so

$$\begin{aligned}(\wedge_{j=1}^{j=r}\varphi_j^{\hat\iota'})((\wedge_{j=r+1}^{j=r'}\phi_j')((\wedge_{\mathbb{Q}_p[G]}^{r'}\kappa'_{\mathbb{Q}_p})(\hat x))) &= (\wedge_{j=1}^{j=r}(\varphi_j^{\hat\iota'}\circ\kappa'))((\wedge_{j=r+1}^{j=r'}(\phi_j'\circ\kappa'))(\hat x))\\ &= (\wedge_{j=1}^{j=r}(\varphi_j^{\hat\iota'}\circ\kappa'))((\wedge_{j=r+1}^{j=r'}\phi_j)(\hat x)),\end{aligned}$$

with $\kappa'_{\mathbb{Q}_p} := \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \kappa'$. Since the first square in (16) implies $\varrho_{G,H}^{r',d}((\wedge_{\mathbb{Q}_p[G]}^{r'}\kappa'_{\mathbb{Q}_p})(\hat x)) = \iota'_*(x)$ and that $\varphi_j^{\hat\iota'}\circ\kappa'$ restricts through $\hat\iota$ to give φ_j for $j\in[r]$, it is thus enough to show that, if $\hat\iota$ and ϕ are fixed, then $\mathrm{Rec}_\pi(x)$ is unchanged if one makes alternative choices $\hat x'$ and $\{\tilde\varphi_j^{\hat\iota}\}_{j\in[r]}$ of $\hat x$ and $\{\varphi_j^{\hat\iota}\}_{j\in[r]}$. To see this we note that (6) implies

$$\varrho_{G,H}^{r^*,d}\big((\wedge_{j=1}^{j=r}\varphi_j^{\hat\iota})(\hat x) - (\wedge_{j=1}^{j=r}\tilde\varphi_j^{\hat\iota})(\hat x')\big) = (\wedge_{j=1}^{j=r}(\varphi_j^{\hat\iota})^H)(\iota_*(x)) - (\wedge_{j=1}^{j=r}(\tilde\varphi_j^{\hat\iota})^H)(\iota_*(x))$$

and that the latter difference vanishes since both individual terms depend only on x and the maps φ_j. It therefore follows that $(\wedge_{j=1}^{j=r}\varphi_j^{\hat\iota})(\hat x) - (\wedge_{j=1}^{j=r}\tilde\varphi_j^{\hat\iota})(\hat x')$ belongs to $\ker(\varrho_{G,H}^{r^*,d})$ and so is sent by $\wedge_{j=r+1}^{j=r'}\phi_j$ to an element of $\iota_\Sigma^H(G)$, as required. □

Remark 4.5. For a finite group Γ, the projection map $I(\Gamma)/I(\Gamma)^2 \to I(\Gamma^{\mathrm{ab}})/I(\Gamma^{\mathrm{ab}})^2$ is bijective as both quotients identify with Γ^{ab}. In the setting of Lemma 4.4, however, the corresponding map $\iota_\Sigma(G)/\iota_\Sigma^H(G) \to \iota_\Sigma(G^{\mathrm{ab}})/\iota_\Sigma^{H'}(G^{\mathrm{ab}})$, with H' the image of H in G^{ab}, need not be injective (see Remark 5.4).

Remark 4.6. If G is abelian, then $\iota_\Sigma(G) = \prod_{v\in\Sigma} I_v(G)$ and $\iota_\Sigma^H(G) = \iota_\Sigma(G)\cdot\{h-1 : h \in H\}$ (see Lemma 5.8) and so the observation made just after (14) implies Rec_π recovers the reciprocity maps that occur in the conjectures formulated in [26] and [32].

5. The non-commutative class number conjecture

5.1. **Statement of the conjecture.** In this section we assume $|S| > 1$ and $S_{L/K}^{\mathrm{ram}} \subseteq S$ and order the places of $S = \{v_i\}_{0\le i\le n}$ as in §4.2.

Let F denote either E or L and set $\Gamma = \mathrm{Gal}(F/K)$. Then, for each non-negative integer a, the 'a-th derived Stickelberger function' for $F/K, S$ and T is the $\zeta(\mathbb{C}[\Gamma])$-valued meromorphic function

$$\theta_{F/K}^a(z) = \theta_{F/K,S,T}^a(z) := \sum\nolimits_{\chi\in\mathrm{Ir}(\Gamma)}(z^{-a\chi(1)}L_{S,T}(\check\chi, z))\cdot e_\chi,$$

where $L_{S,T}(\check\chi,z)$ is the S-truncated T-modified Artin L-function for the contragredient $\check\chi$ of χ and e_χ the primitive central idempotent $\chi(1)|\Gamma|^{-1}\cdot\sum_{\gamma\in\Gamma}\chi(\gamma)\gamma^{-1}$ of $\mathbb{C}[\Gamma]$. (In the case

$a = 0$ we usually abbreviate $\theta^a_{F/K}(z)$ to $\theta_{F/K}(z)$ and simply refer to this function as the 'Stickelberger function for $F/K, S$ and T'.)

Then, since all places in $\Sigma(F)$ split completely in F, an explicit analysis of the functional equation of Artin L-functions (as in [34, Chap. I, Prop. 3.4]) shows that $\theta^{r(F)}_{F/K}(z)$ is holomorphic at $z = 0$ and its value $\theta^{r(F)}_{F/K}(0)$ is easily seen to belong to $\zeta(\mathbb{R}[\Gamma])$.

The 'non-commutative Rubin-Stark element' associated to $F/K, S, T$ is then defined (in [7, Def. 6.5]) to be the unique element $\varepsilon_{F/K} = \varepsilon^{\Sigma(F)}_{F/K,S,T}$ of $\bigwedge^{r(F)}_{\mathbb{R}[\Gamma]}(\mathbb{R}\cdot U^T_S(F))$ that satisfies

$$\big(\textstyle\bigwedge^{r(F)}_{\mathbb{R}[\Gamma]} R_{F,S}\big)(\varepsilon_{F/K}) = \theta^{r(F)}_{F/K}(0)\cdot \wedge_{i\in[r(F)]}(w_{i,F} - w_{0,F}), \tag{18}$$

where $R_{F,S}$ denotes the Dirichlet regulator isomorphism $\mathbb{R}\cdot U^T_S(F) \xrightarrow{\sim} \mathbb{R}\cdot X_{F,S}$.

In the sequel we fix a prime p and an isomorphism of fields $j : \mathbb{C} \cong \mathbb{C}_p$ and identify $\varepsilon_{F/K}$ with its image under the induced embedding $\bigwedge^{r(F)}_{\mathbb{R}[\Gamma]}(\mathbb{R}\cdot U^T_S(F)) \to \bigwedge^{r(F)}_{\mathbb{C}_p[\Gamma]}(\mathbb{C}_p\cdot U^T_S(F)_p)$. We also set $\Sigma_{L,E} := \Sigma(E)\setminus\Sigma(L)$ and use the composite homomorphism

$$\mathrm{ev}_\pi : \textstyle\bigcap^{r(L)}_{\mathbb{Z}_p[G]} U^T_S(L)_p \to \xi(\mathbb{Z}_p[G])^{[G,S,T]} \to \big(\xi(\mathbb{Z}_p[G])/\iota^H_{\Sigma_{L,E}}(G)_p\big)^{[G,S,T]}$$

in which the first map is $\mathrm{ev}^{r(L)}_M$ with $M = U^T_S(L)_p$ and the second is the natural projection.

We can now state the central conjecture of this article.

Conjecture 5.1. *Assume $|S| > 1$, $S^{\mathrm{ram}}_{L/K} \subseteq S$ and $U^T_S(L)_p$ is torsion-free. Then one has*

$$\varepsilon_{E/K} \in \big(\textstyle\bigcap^{r(E)}_{\mathbb{Z}_p[\overline{G}]} U^T_S(E)_p\big)^\pi \tag{19}$$

and

$$\mathrm{ev}_\pi(\varepsilon_{L/K}) = \mathrm{Nrd}_{\mathbb{Q}[G]}(-1)^{r(L)(r(E)-r(L))}\cdot \mathrm{Rec}_\pi(\varepsilon_{E/K}). \tag{20}$$

Remarks 5.2. (i) The containment (19) implies $\varepsilon_{E/K}$ belongs to the domain of Rec_π and hence that the right hand side of the equality (20) is well-defined.
(ii) If G is abelian, then Remark 3.8 implies that (19) recovers the p-primary part of the 'Rubin-Stark Conjecture' [31, Conj. B′], whilst (20) coincides with the 'refined class number formula for $\mathbb{G}_m$' that is conjectured (for abelian extensions) by Mazur and Rubin and by the third author (for details see Theorem 5.5 below). For this reason we refer to Conjecture 5.1 as the 'non-commutative class number formula conjecture for $\mathbb{G}_m$'.
(iii) Assume $U^T_S(L)$ is torsion-free. Then (19) (with $E = L$ and for all p) combines with the general result of [6, Th. 4.19(iii)] to predict $\varepsilon_{L/K}$ belongs to $\bigcap^{r(L)}_{\mathbb{Z}[G]} U^T_S(L)$. This prediction extends the Rubin-Stark Conjecture to general Galois extensions and will be referred to as the 'non-commutative Rubin-Stark Conjecture'.
(iv) If Tate's formulation [34, Chap. I, Conj. 5.1] of Stark's Conjecture is valid for L/K, then the validity of Conjecture 5.1 is independent of the choice of isomorphism $j : \mathbb{C} \cong \mathbb{C}_p$. We therefore do not explicitly indicate the choice of j either in the statement of Conjecture 5.1 or in the arguments that follow.
(v) The elements $\varepsilon_{F/K}$, and thus also $\mathrm{ev}_\pi(\varepsilon_{L/K})$, are independent of the choice of place $v_0 \in S\setminus\Sigma(E)$ and this is also true for the map Rec_π (and hence for the validity of (20)).

Indeed, whilst the computation of Rec_π given in the proof of Lemma 4.4 chooses a map ϖ as in (16) and hence *a priori* relies on the choice of v_0, if ϖ' is constructed just as ϖ but with respect to a different choice of v_0 in $S \setminus \Sigma(E)$, then one has $\varpi = \kappa \circ \varpi'$ for an automorphism κ of $\mathbb{Z}_p[G]^d$ that is represented with respect to the standard basis by a block matrix of the form $\left(\begin{array}{c|c} I_n & * \\ \hline 0 & I_{d-n} \end{array}\right)$. This fact in turn allows one to construct an analogue of diagram (16) that combines with the argument of Lemma 4.4 to show Rec_π is unchanged if one replaces ϖ by ϖ', as required.
(vi) If $r(L) = 0$, then the equality (18) with $F = L$ implies that $\varepsilon_{L/K}$ is equal to $\theta_{L/K}(0)$ and so coincides with the Stickelberger element for L/K that is introduced by Hayes in [18].

5.2. **Special cases.** We now provide evidence in support of special cases of Conjecture 5.1.

5.2.1. *Abelian and Frobenius extensions.*

If G has an abelian Sylow p-subgroup and a normal p-complement, then the algebra $\mathbb{Z}_p[G]$ is a direct product of matrix rings over commutative local $\mathbb{Z}_p$-algebras (this is proved by Demeyer and Janusz in [12, p. 390, Cor.]). In this section we fix such a group G and a corresponding direct product decomposition

$$\mathbb{Z}_p[G] = \prod_{i\in I} \mathrm{M}_{n_i}(R_i), \tag{21}$$

where the index set I is finite and each R_i is a commutative local $\mathbb{Z}_p$-algebra.

Lemma 5.3. *For each exact sequence of groups of the form (10) the following claims are valid for every non-negative integer a and natural number d.*

(i) *The map $\varrho^{a,d}_{G,H}$ is surjective.*
(ii) *For each $i \in I$, there exists an ideal $J_i = J_i(H)$ of R_i that is independent of a and d and such that*
$$\ker(\varrho^{a,d}_{G,H}) = J \cdot {\bigcap}^a_{\mathbb{Z}_p[G]} \mathbb{Z}_p[G]^d,$$
where $J = J(H)$ denotes the ideal $\bigoplus_{i\in I} J_i$ of $\xi(\mathbb{Z}_p[G])$.
(iii) *If the order of H is a power of p, then J is contained in* $\mathrm{Jac}(\xi(\mathbb{Z}_p[G]))$.
(iv) *If, for some $i \in I$, the ring R_i is a Dedekind domain, then J_i is* 0 *or* R_i.

Proof. The decomposition (21) implies, via a standard Morita equivalence argument, that $\xi(\mathbb{Z}_p[G]) = \prod_{i\in I} R_i$ and also

$${\bigcap}^a_{\mathbb{Z}_p[G]} \mathbb{Z}_p[G]^d = \bigoplus\nolimits_{i\in I} {\bigwedge}^{an_i}_{R_i} R_i^{dn_i} \tag{22}$$

(cf. [6, Th. 4.19(vii)]). In particular, there exists an ideal $J_i = J_i(H)$ of R_i for each $i \in I$ such that $\ker(\pi_*) = \prod_{i\in I} \mathrm{M}_{n_i}(J_i)$ and hence

$${\bigcap}^a_{\mathbb{Z}_p[G]} \mathbb{Z}_p[\overline{G}]^d = \bigoplus\nolimits_{i\in I} {\bigwedge}^{an_i}_{R_i} (R_i/J_i)^{dn_i} \quad \text{and} \quad \ker(\varrho^{a,d}_{G,H}) = J \cdot {\bigcap}^a_{\mathbb{Z}_p[G]} \mathbb{Z}_p[G]^d, \tag{23}$$

with $J := \bigoplus_{i\in I} J_i$. Given this description, claims (i) and (ii) are both clear.

Claim (iii) is true since if the order of H is a power of p, then $\ker(\pi_*) \subseteq \mathrm{Jac}(\mathbb{Z}_p[G])$ and so $J_i \subseteq \mathrm{Jac}(R_i)$ for all $i \in I$.

Claim (iv) is true since the first equality in (23) implies that each ring R_i/J_i is $\mathbb{Z}_p$-free. □

Remark 5.4. In the setting of Lemma 4.4, Lemma 5.3(ii) implies that

$$\iota_\Sigma^H(G)_p = J \cdot \iota_\Sigma(G)_p. \tag{24}$$

Hence, if the order of H is a power of p, then for each $i \in I$ Lemma 5.3(iii) combines with Nakayama's Lemma to imply that the i-component of $\big(\iota_\Sigma(G)/\iota_\Sigma^H(G)\big)_p$ vanishes if and only if the i-component of $\iota_\Sigma(G)_p$ vanishes (in particular, if $n_i > 1$ in the decomposition (21), then this observation justifies Remark 4.5). For a concrete example of this, one need only fix a prime $\ell \equiv 1 \,(\mathrm{mod}\, p^2)$ and take G to be a semidirect product $\mathbb{Z}/\ell \rtimes \mathbb{Z}/p^3$, where the image of $\mathbb{Z}/p^3$ in $\mathrm{Aut}(\mathbb{Z}/\ell)$ has order p^2, with H the central subgroup of G of order p.

Before stating the main result of this section, we recall that the conjecture formulated (for abelian extensions) by Kurihara and the first and third of us in [4, Conj. 5.4] is a strengthening of the conjecture formulated, independently, by Mazur and Rubin in [26, Conj. 5.2] and by the third of us in [32, Conj. 3], and hence refines earlier conjectures that are due to Darmon, to Gross, to Rubin and to Tate among others (for more details see [4, Rem. 5.6, Rem. 5.7 and Th. 5.10]).

We further recall that G is said to be a 'Frobenius group' if it has a proper non-trivial 'Frobenius complement' subgroup A such that $A \cap gAg^{-1} = \{1\}$ for all $g \in G \setminus A$, in which case G contains a unique normal subgroup N, known as the 'Frobenius kernel', such that G is a semidirect product $N \rtimes A$.

Theorem 5.5. *Fix data p, L/K, G and H as above and assume either*

(i) *G is abelian, or*
(ii) *G is a Frobenius group with a kernel N and complement A, such that N is a subgroup of H of order prime to p and A is abelian.*

Then the validity of Conjecture 5.1 is implied by the p-component of [4, Conj. 5.4] *for the data L/K and H in case (i), and for the data L'/K and H', where L' is the maximal abelian extension of K in L and H' the image of H in $\mathrm{Gal}(L'/K)$, in case (ii).*

From [4, Cor. 1.2] we therefore directly obtain the following evidence for Conjecture 5.1.

Corollary 5.6. *Conjecture 5.1 is valid if $K = \mathbb{Q}$ and either G is abelian, or G and H satisfy the hypotheses of Theorem 5.5(ii).*

Remark 5.7. Assume G satisfies the hypotheses of Theorem 5.5(ii). Then it can be shown that A is cyclic and $|N| \equiv 1 \,(\mathrm{mod}\, |A|)$. In addition, if N is solvable, then G is solvable and so Šafarevič's Theorem implies the existence of infinitely many Galois extensions $L/\mathbb{Q}$ with $\mathrm{Gal}(L/\mathbb{Q})$ isomorphic to G (cf. [28, Chap. IX, §5]). For illustrative purposes, we record two families of such groups G that have been studied by Johnston and Nickel (and for details of other interesting families, see [21, §2.3] and [22, §2.2]).
(i) ([21, Exam. 2.16]) Let q be a prime power and $\mathbb{F}_q$ the finite field with q elements. The group $G := \mathrm{Aff}(q)$ of affine transformations on $\mathbb{F}_q$ is the group of transformations of the form $x \mapsto ax + b$ with a and b in $\mathbb{F}_q$. Then G is a Frobenius group with kernel the subgroup of transformations $x \mapsto x + b$ for $b \in \mathbb{F}_q$. In addition, in this case, every character in $\mathrm{Ir}(G)$ is either linear or rational-valued and, if $L/\mathbb{Q}$ is Galois of group G, then the p-component of $\mathrm{eTNC}(\mathbb{G}_m)$ for $L/\mathbb{Q}$ is verified in [21, Th. 4.6].

(ii) ([22, Exam. 2.11]) Let p and q be primes, f and n natural numbers such that $q > n > 1$ and p divides $q^f - 1$. Then there exists a Frobenius group G that satisfies the hypotheses of Theorem 5.5(ii) and is such that $|G| = pq^{fn(n-1)/2}$, N is nilpotent of class $n-1$ and there are characters in $\mathrm{Ir}(G)$ that are neither linear nor rational-valued. In particular, Stark's Conjecture (and hence also $\mathrm{eTNC}(\mathbb{G}_m)$) is not known to be valid for any Galois extension $L/\mathbb{Q}$ of group G.

The proof of Theorem 5.5 will occupy the rest of this section. We will assume the hypotheses of case (ii) since the argument in this case also incorporates a proof of the result under the conditions of case (i). At the outset we also note that, in case (ii), G has an abelian Sylow p-subgroup and a normal p-complement N and, as proved by Johnston and Nickel in [21, Prop. 2.13], for every index $i \in I$ for which one has $n_i > 1$ in the decomposition (21) the ring R_i is a Dedekind domain. The subgroup H is normal in G (since $N \subset H$ and G/N is abelian) and we write $\overline{G}$ for the (abelian) quotient group G/H.

In this case, Lemma 5.3(i) (or, as $\overline{G}$ is abelian, Remark 3.8) combines with the observation in Remark 5.2(ii) to imply that (19) is equivalent to the validity of the p-component of the Rubin-Stark Conjecture for the abelian extension E/K.

Further, the fact $\overline{G}$ is abelian also combines with Lemma 5.3(iv) to imply that, if $n_i > 1$, then $J_i = R_i$ and so the equality (24) implies that the i-component of the quotient module $(\iota_\Sigma(G)/\iota_\Sigma^H(G))_p$ vanishes. To verify the equality (20) it is therefore enough to restrict to components of $(\iota_\Sigma(G)/\iota_\Sigma^H(G))_p$ that correspond to indices i for which $n_i = 1$.

We note next that N is equal to the commutator subgroup $[G,G]$ and has order prime to p. It follows that the idempotent $e_N := |N|^{-1}\sum_{g\in N} g$ belongs to $\zeta(\mathbb{Z}_p[G])$ and that the product of the algebras R_i over all indices i (in (21)) for which $n_i = 1$ identifies with $\mathbb{Z}_p[G]e_N$, and hence with $\mathbb{Z}_p[G/N] = \mathbb{Z}_p[G^{\mathrm{ab}}]$. With respect to these identifications the direct sum $e_N(\iota_\Sigma(G)/\iota_\Sigma^H(G))_p$ of the corresponding components of $(\iota_\Sigma(G)/\iota_\Sigma^H(G))_p$ identifies with $(\iota_\Sigma(G^{\mathrm{ab}})/\iota_\Sigma^{H'}(G^{\mathrm{ab}}))_p$, with H' the image of H in G^{ab}.

We set $L' := L^N$. Then $\Sigma(L) \subseteq \Sigma(L')$ and, if there exists a place v' in $\Sigma(L') \setminus \Sigma(L)$ the element $e_N(\varepsilon_{L/K})$ vanishes and, in addition, the image of $I_v(G)$ in $\mathbb{Z}_p[G^{\mathrm{ab}}]$ is zero and so the module $e_N(\iota_\Sigma(G)/\iota_\Sigma^H(G))_p$ vanishes. In this case, therefore, the validity of the e_N-component of (20) is clear.

In the sequel we can thus assume that $\Sigma(L) = \Sigma(L')$, and hence that $\Sigma := \Sigma(E)\setminus\Sigma(L)$ is equal to $\Sigma' := \Sigma(E) \setminus \Sigma(L')$. In this case one has $e_N(\varepsilon_{L/K}) = \varepsilon_{L'/K}$ and there is a natural identification $e_N(\iota_\Sigma(G)/\iota_\Sigma^H(G))_p = (\iota_{\Sigma'}(G^{\mathrm{ab}})/\iota_{\Sigma'}^{H'}(G^{\mathrm{ab}}))_p$. In addition, by using the descent isomorphism (11) (for $G \to G^{\mathrm{ab}}$), it can be checked that the composite homomorphism

$$\Big({\bigcap}_{\mathbb{Z}_p[\overline{G}]}^{r(E)} U_S^T(E)_p\Big) \xrightarrow{\mathrm{Rec}_\pi} \big(\iota_\Sigma(G)/\iota_\Sigma^H(G)\big)_p^{[G,S,T]} \xrightarrow{\times e_N} \iota_{\Sigma'}(G^{\mathrm{ab}})/\iota_{\Sigma'}^{H'}(G^{\mathrm{ab}}))_p^{[G,S,T]}$$

coincides with $\mathrm{Rec}_{\pi'}$, with π' the natural map $G^{\mathrm{ab}} \to G^{\mathrm{ab}}/H' = \overline{G}$. In this case, therefore, the e_N-component of the equality (20) is equivalent to the corresponding equality with L/K and H replaced by L'/K and H'.

Via the above observations, the proof of Theorem 5.5 is reduced to verifying (19) and (20) in the case that G is abelian. In the remainder of the argument, we will therefore assume that G is abelian and also set $W := \Sigma(E) \setminus \Sigma(L)$. For each v in W we write $w_{v,E}$

for the restriction of w_v to E and set $\mathcal{J}_W := \prod_{v\in W} I_v(G)$ and $J_W := \prod_{v\in W} I_{w_{v,E}}(H)$. We also write $I(H)$ for the augmentation ideal of $\mathbb{Z}[H]$.

Lemma 5.8. *If G is abelian, then the following claims are valid.*

(i) $\xi(\mathbb{Z}[G]) = \mathbb{Z}[G]$, $\xi(\mathbb{Z}[\overline{G}]) = \mathbb{Z}[\overline{G}]$ *and* $\xi(G,H) = I(G,H)$.

(ii) *For each natural number d and non-negative integer a with $a \le d$ the homomorphism $\varrho^{a,d}_{G,H}$ is surjective and has kernel $I(H)\cdot\bigwedge^a_{\mathbb{Z}_p[G]}\mathbb{Z}_p[G]^d$.*

(iii) $\iota_W(G) = \mathcal{J}_W$ *and* $\iota^H_W(G) = I(H)\cdot\mathcal{J}_W$.

(iv) *There is a natural identification $\iota_W(G)/\iota^H_W(G) = \mathbb{Z}[\overline{G}]\otimes_{\mathbb{Z}} J_W/(I(H)\cdot J_W)$.*

Proof. Claim (i) follows directly from [6, Lem. 3.2(iii)].

To prove claim (ii) we write $\left[{d \atop a}\right]$ for the set of permutations τ of $[d]$ with $\tau(1) < \ldots < \tau(a)$ and $\tau(a+1) < \ldots < \tau(d)$. Then the $\mathbb{Z}_p[\Upsilon]$-module $\bigcap^a_{\mathbb{Z}_p[\Upsilon]}\mathbb{Z}_p[\Upsilon]^d = \bigwedge^a_{\mathbb{Z}_p[\Upsilon]}\mathbb{Z}_p[\Upsilon]^d$ is free with basis $\{\wedge_{j\in[a]} b_{\Upsilon,\tau(j)} : \tau \in \left[{d \atop a}\right]\}$ (cf. the argument of [6, Th. 4.19(vi)]). Claim (ii) now follows easily upon comparing this explicit description with $\Upsilon = G$ and $\Upsilon = \overline{G}$ and noting $I(G,H) = \mathbb{Z}[G]\cdot I(H)$.

Turning to claim (iii), we set $a := |W|$ and fix an integer d with $d \ge a$. Then the above explicit descriptions imply $\iota_W(G)$ is generated as a G-module by the elements

$$(\wedge_{v\in W}\theta_v)\big(\wedge_{j\in[a]} b_{G,\tau(j)}\big) = \det\big(\big(\theta_v(b_{G,\tau(j)})\big)_{v\in W, j\in[a]}\big)$$

as θ_v ranges over $\mathrm{Hom}_G(\mathbb{Z}[G]^d, I_v(G))$ and τ over $\left[{d \atop a}\right]$. Now, since each term $\theta_v(b_{G,\tau(j)})$ belongs to $I_v(G)$, it is clear that each such determinant belongs to $\mathcal{J}_W$. Conversely, if we use the given ordering of S_0 to relabel the places in W as $\{v_i\}_{i\in[a]}$, then for any product $x := \prod_{i\in[a]} x_i$ of elements x_i of $I_{v_i}(G)$ one has $x = (\wedge_{i\in[a]}\theta_{x,i})\big(\wedge_{j\in[a]} b_{G,\tau_0(j)}\big)$, where we set $\theta_{x,i} := x_i\cdot b^*_{G,i} \in \mathrm{Hom}_G(\mathbb{Z}[G]^d, I_{v_i}(G))$ and write τ_0 for the identity permutation in $\left[{d \atop a}\right]$. These observations combine to imply the first equality in claim (iii) and, given this, the second equality in claim (iii) then follows directly from the equality $\ker(\varrho^{a,d}_{G,H}) = I(H)\cdot\bigwedge^a_{\mathbb{Z}_p[G]}\mathbb{Z}_p[G]^d$ proved in claim (ii).

Before proving claim (iv) we note that, for each $v \in W$, there is a natural isomorphism $\mathbb{Z}[\overline{G}]\otimes_{\mathbb{Z}} I_{w_{v,E}}(H) \simeq \mathbb{Z}[G]I_{w_{v,E}}(H) = I_v(G)$. These isomorphisms combine to give a canonical isomorphism $\mathbb{Z}[\overline{G}]\otimes_{\mathbb{Z}} J_W \simeq \mathbb{Z}[G]J_W = \mathcal{J}_W$ and hence also an identification

$$\mathcal{J}_W/(I(H)\cdot\mathcal{J}_W) = \mathbb{Z}[\overline{G}]\otimes_{\mathbb{Z}} J_W/(I(H)\cdot J_W)$$

(cf. [4, Prop. 4.9]). Given this identification, claim (iv) follows directly from claim (iii). □

We now set $r := r(L)$, $r' := r(E)$, $U_L := U^T_S(L)_p$ and $U_E := U^T_S(E)_p$. We also fix an embedding $\hat{\iota} : U_L \to \mathbb{Z}_p[G]^d$ of $\mathbb{Z}_p[G]$-modules (which, for simplicity, we henceforth suppress from notation) and an endomorphism ϕ of the $\mathbb{Z}_p[G]$-module $\mathbb{Z}_p[G]^d$, as are used in the explicit description of the map Rec_π given in the proof of Lemma 4.4.

As we observed earlier, in this case the predicted containment (19) coincides with the p-component of the Rubin-Stark Conjecture. In addition, if for each element $\underline{\varphi} = (\varphi_1, \ldots, \varphi_r)$ of $\mathrm{Hom}_{\mathbb{Z}_p[G]}(U_L, \mathbb{Z}_p[G])^r$ we set $\Phi := \wedge_{i\in[r]}\varphi_i$, then claims (iii) and (iv) of Lemma 5.8 show that the equality (20) implies $\Phi(\varepsilon_{L/K})$ belongs to $\mathcal{J}_{W,p}$ and further that (20) itself is valid if and only if for every such element Φ one has

$$\Phi(\varepsilon_{L/K}) \equiv (-1)^{r(r'-r)}\cdot\Phi\big((\wedge_{i=r+1}^{i=r'}\phi_i)(\hat\varepsilon_E)\big) \ \text{ modulo } \ I(H)\cdot\mathcal{J}_{W,p},$$

where $\hat\varepsilon_E$ is any choice of element of $\bigcap^{r'}_{\mathbb{Z}_p[G]}\mathbb{Z}_p[G]^d = \bigwedge^{r'}_{\mathbb{Z}_p[G]}\mathbb{Z}_p[G]^d$ with the property that $\varrho^{r',d}_{G,H}(\hat\varepsilon_E) = \varepsilon_{E/K}$. After taking account of [4, Th. 5.10], these observations imply that this special case of (20) is equivalent to the p-component of [4, Conj. 5.4] provided that for all elements Φ as above one has

$$\begin{aligned}(25)\quad \Phi\big((\wedge_{i=r+1}^{i=r'}\phi_j)(\hat\varepsilon_E)\big) &= \Phi^H\big(\mathrm{Rec}_W(\varepsilon_{E/K})\big)\\ &\in \mathbb{Z}_p[\overline{G}]\otimes_{\mathbb{Z}_p}\big(J_W/(I(H)\cdot J_W)\big)_p = \mathcal{J}_{W,p}/(I(H)\cdot\mathcal{J}_W)_p.\end{aligned}$$

Here Φ^H is the element of $\bigwedge^r_{\mathbb{Z}_p[\overline{G}]}\mathrm{Hom}_{\mathbb{Z}_p[\overline{G}]}(U_E,\mathbb{Z}_p[\overline{G}])$ that is obtained from Φ via the recipe of [4, Def. 4.10] and we also use the canonical reciprocity homomorphism

$$\mathrm{Rec}_W : \bigcap\nolimits^{r'}_{\mathbb{Z}_p[\overline{G}]}U_E \to \big(\bigcap\nolimits^{r}_{\mathbb{Z}_p[\overline{G}]}U_E\big)\otimes_{\mathbb{Z}_p}\big(J_W/(I(H)\cdot J_W)\big)_p$$

defined in [4, (23)]. It is therefore enough to note that the required equality (25) follows directly from the argument of [4, Lem. 5.20] (in which the endomorphism ψ corresponds to our fixed map ϕ) and the general result of [4, Prop. 4.11] (which describes the explicit relation between the maps Φ and Φ^H). This completes the proof of Theorem 5.5.

5.2.2. *The case $E = K$.*

If $E = K$, then Conjecture 5.1 has an explicit interpretation (even if G is non-abelian). To show this we recall that n denotes $|S|-1$ and we assume $U_S^T(L)_p$ is torsion-free. Then, in this case, the analytic class number formula implies that, for each $\mathbb{Z}_p$-basis $\{u_a\}_{a\in[n]}$ of $U := U_S^T(K)_p$, there exists an element μ of $\mathbb{Z}_p^\times$ such that

$$\mu\cdot|\mathrm{Cl}_S^T(K)|\cdot(\mathbb{C}_p\otimes_{\mathbb{R}}\bigwedge\nolimits^n_{\mathbb{R}}R_{K,S})(\wedge_{a\in[n]}u_a) = \theta^n_{K/K,S,T}(0)\cdot\wedge_{a\in[n]}(v_a - v_0).$$

This implies $\varepsilon_{K/K}$ is equal to $\mu\cdot|\mathrm{Cl}_S^T(K)|\cdot\wedge_{a=1}^{a=n}u_a$ and so belongs to $\bigwedge^n_{\mathbb{Z}_p}U = \bigcap^n_{\mathbb{Z}_p}U$. Since the general result of [6, Th. 4.19(vi)] implies that $\varrho^{n,d}_{G,G}$ is surjective for every d, one also has $\big(\bigcap^n_{\mathbb{Z}_p}U\big)^\pi = \bigcap^n_{\mathbb{Z}_p}U$ and so the containment (19) is valid if $E = K$.

To make (20) more explicit in this case we use the approach of Lemma 4.1 to fix an embedding of $\mathbb{Z}_p$-modules $\iota : U \to \mathbb{Z}_p^d$ and, for each $j\in[n]$, an associated pullback $\hat\varphi_j = \pi^*(\varphi_{v_j})$ of φ_{v_j} with respect to the homomorphism π from G to the trivial group. Writing $\{b_i\}_{i\in[d]}$ for the canonical basis of $\mathbb{Z}_p^d$, one thus has $\iota_*(\wedge_{a\in[n]}u_a) = \sum_{\sigma\in\left[{d\atop n}\right]}c_\sigma\cdot\wedge_{i\in[n]}b_{\sigma(i)}$ for unique elements c_σ of $\mathbb{Z}_p$. The argument in Lemma 4.4 then shows that each individual summand in the expression

$$\mathrm{Reg}_{L/K} = \mathrm{Reg}_{L/K,S,T} := \sum\nolimits_{\sigma\in\left[{d\atop n}\right]}\mu\cdot c_\sigma\cdot\mathrm{Nrd}_{\mathbb{Q}_p[G]}\big(\big(\hat\varphi_j(b_{G,\sigma(i)})\big)_{i,j\in[n]}\big)$$

belongs to $\iota_{S_0}(G)_p$ and is independent of the choice of basis $\{u_a\}_{a\in[n]}$ and that the image of $\mathrm{Reg}_{L/K}$ in $\big(\iota_{S_0}(G)/\iota^G_{S_0}(G)\big)_p$ is independent of the choices of embedding ι and pullbacks $\{\hat\varphi_j\}_{j\in[n]}$.

If $r(L) > 0$, then $\mathrm{Reg}_{L/K} = 0$ (as each matrix $\big(\hat\varphi_j(b_{G,\sigma(i)})\big)_{i,j\in[n]}$ has a column of zeroes). However, if $r(L) = 0$, so $\varepsilon_{L/K} = \theta_{L/K}(0)$, then (20) with $E = K$ predicts

$$\theta_{L/K}(0) \equiv |\mathrm{Cl}_S^T(K)| \cdot \mathrm{Reg}_{L/K} \ \ \text{modulo} \ \iota_{S_0}^G(G)_p,$$

and hence also that $\theta_{L/K}(0) \in \iota_{S_0}(G)_p$.

If G is cyclic, then the argument of Lemma 5.8 shows that these predictions recover the conjecture formulated (for such extensions) by Tate in [35].

5.2.3. *The cases* $\Sigma(L) = \Sigma(E)$ *and* $\Sigma(L) = \emptyset$.

If $\Sigma(L) = \Sigma(E)$, or equivalently $r(E) = r(L)$, then $\Sigma_{L,E} = \emptyset$ so $\iota_{\Sigma_{L,E}}(G) = \xi(\mathbb{Z}_p[G])$ and $\iota^H_{\Sigma_{L,E}}(G) = \xi(G,H)_p$. Thus, in this case, the validity of (20) follows directly from (6) with $a' = a = r(E)$ and the fact that the projection map $\mathbb{R}\otimes_{\mathbb{Z}}\pi_* : \mathbb{R}[G] \to \mathbb{R}[\overline{G}]$ sends $\theta^{r(L)}_{L/K}(0)$ to $\theta^{r(L)}_{E/K}(0) = \theta^{r(E)}_{E/K}(0)$.

If $\Sigma(L) = \emptyset$, then $\varepsilon_{L/K} = \theta_{L/K}(0)$ and $r(L) = 0$ so $\bigcap^{r(L)}_{\mathbb{Z}_p[G]} U_S^T(L)_p = \xi(\mathbb{Z}_p[G])$ (by [6, Th. 4.19(i)]). In this case, therefore, the containment (19) with $E = L$ predicts that $\theta_{L/K}(0)$ belongs to $\xi(\mathbb{Z}_p[G])$. If true, this containment combines with the results of [6, Lem. 3.5(i), (iv)] to imply that, for any element x of $\zeta(\mathbb{Q}_p[G])$ one has

$$x \in \delta(\mathbb{Z}_p[G]) \Longrightarrow \theta_{L/K}(0)\cdot x \in \mathbb{Z}_p[G],$$

where $\delta(\mathbb{Z}_p[G])$ is the 'ideal of denominators' of $\xi(\mathbb{Z}_p[G])$ defined in loc. cit. Evidence in support of the latter implication is provided by recent results of Ellerbrock and Nickel [13, Th. 1, Th. 2] which combine to imply that for any integer t one has

$$p^t \in \delta(\mathbb{Z}_p[G]) \Longrightarrow \theta_{L/K}(0)\cdot p^t \in \mathbb{Z}_p[G].$$

5.2.4. *The link to refined Rubin-Stark Conjectures.* In this section we assume that Stark's Conjecture (cf. Remark 5.2(iv)) is valid for L/K. We also assume $S^{\mathrm{ram}}_{E/K} \subseteq S$ and $U_S^T(E)_p$ is torsion-free and fix an exact sequence $h = h_{E,S,T}$ of $\mathbb{Z}_p[\overline{G}]$-modules of the form

$$\mathbb{Z}_p[\overline{G}]^d \xrightarrow{\phi} \mathbb{Z}_p[\overline{G}]^d \to \mathcal{S}_S^T(E)_p \to 0$$

specified in (16) (with L replaced by E). We further set $r := r(E)$ and recall that the r-th Fitting invariant $\mathrm{Fit}^r_{\mathbb{Z}_p[\overline{G}]}(h)$ of h is an ideal of $\xi(\mathbb{Z}_p[\overline{G}])$ defined in [6, Def. 3.14]. As a first step, we formulate a conjectural description of this ideal.

Conjecture 5.9. *If* $S^{\mathrm{ram}}_{E/K} \subseteq S$ *and* $U_S^T(E)_p$ *is torsion-free, then one has*

$$\mathrm{Fit}^r_{\mathbb{Z}_p[\overline{G}]}(h) = \{(\textstyle\bigwedge_{i\in[r]}\varphi_i)(\varepsilon_{E/K}) : (\varphi_1,\cdots,\varphi_r) \in \mathrm{Hom}_{\overline{G}}(U_S^T(E),\mathbb{Z}[\overline{G}])^r_p\}.$$

If $\overline{G}$ is abelian, then this conjecture was first formulated in [4, Conj. 7.3]. In the general case it is motivated by the observations made in [6, §8.3], and refines the non-commutative Rubin-Stark Conjecture from Remark 5.2(iii). Its connection to (the containment (19) in) Conjecture 5.1 is explained by the next result. In this result we use the idempotent of $\zeta(\mathbb{Q}[\overline{G}])$ obtained by setting

$$e = e_{E,S,T} := \textstyle\sum_\chi e_\chi,$$

where in the sum χ runs over all characters in $\mathrm{Ir}(\overline{G})$ for which $e_\chi(\varepsilon_{E/K})$ does not vanish.

Lemma 5.10. *If $\xi(\mathbb{Z}_p[\overline{G}])e$ is a principal ideal ring, then Conjecture 5.9 implies (19).*

Proof. We first recall from [7, Prop. 6.16(ii) and Rem. 6.17] that there exists a unit c of $\zeta(\mathbb{Q}_p[\overline{G}])e$ such that

$$\varepsilon_{E/K} = c \cdot (\wedge_{i\in[d]\setminus[r]}(b^*_{\overline{G},i} \circ \phi))(\wedge_{j\in[d]} b_{\overline{G},j}). \tag{26}$$

For all elements $\underline{\varphi} := (\varphi_1, \cdots, \varphi_r)$ of $\mathrm{Hom}_{\overline{G}}(U_S^T(E), \mathbb{Z}[\overline{G}])_p^r$, one therefore has

$$\begin{aligned}(\wedge_{i\in[r]}\varphi_i)(\varepsilon_{E/K}) &= c \cdot (\wedge_{i\in[r]}\varphi_i)\big((\wedge_{i\in[d]\setminus[r]}(b^*_{\overline{G},i} \circ \phi))(\wedge_{j\in[d]} b_{\overline{G},j})\big)\\ &= c \cdot \mathrm{Nrd}_{\mathbb{Q}_p[\overline{G}]}((-1)^{r(d-r)}) \cdot \big((\wedge_{i\in[r]}\varphi_i) \wedge (\wedge_{i\in[d]\setminus[r]}(b^*_{\overline{G},i} \circ \phi))\big)(\wedge_{j\in[d]} b_{\overline{G},j})\\ &= c \cdot \mathrm{Nrd}_{\mathbb{Q}_p[\overline{G}]}((-1)^{r(d-r)}) \cdot \mathrm{Nrd}_{\mathbb{Q}_p[\overline{G}]}(M(\underline{\varphi})),\end{aligned}$$

with $M(\underline{\varphi})$ the matrix in $\mathrm{M}_{d,d}(\mathbb{Z}_p[\overline{G}])$ specified by

$$M(\underline{\varphi})_{ij} = \begin{cases} \varphi_i(b_{\overline{G},j}), & \text{if } i \in [r]\\ (b^*_{\overline{G},i} \circ \phi)(b_{\overline{G},j}), & \text{if } i \in [d] \setminus [r].\end{cases}$$

These matrices $M(\underline{\varphi})$ are the only matrices that can have non-zero reduced norm amongst all that can be obtained from the matrix of ϕ with respect to the basis $\{b_{\overline{G},i}\}_{i=1}^d$ by replacing all entries in any set of r columns by arbitrary elements of $\mathbb{Z}_p[\overline{G}]$. Hence, by the general result of [6, Prop. 4.21(ii)], one has

$$\mathrm{Fit}^r_{\mathbb{Z}_p[\overline{G}]}(h) = \xi(\mathbb{Z}_p[\overline{G}])e \cdot \{\mathrm{Nrd}_{\mathbb{Q}_p[\overline{G}]}(M(\underline{\varphi})) : \underline{\varphi} \in \mathrm{Hom}_{\overline{G}}(U_S^T(E), \mathbb{Z}[\overline{G}])_p^r\} \subseteq \xi(\mathbb{Z}_p[\overline{G}]).$$

Thus, since $\mathrm{Nrd}_{\mathbb{Q}_p[\overline{G}]}((-1)^{r(d-r)}) \in \xi(\mathbb{Z}_p[\overline{G}])^\times$, the above expression for $(\wedge_{i\in[r]}\varphi_i)(\varepsilon_{E/K})$ combines with Conjecture 5.9 to imply that $c \cdot \mathrm{Fit}^r_{\mathbb{Z}_p[\overline{G}]}(h) = \mathrm{Fit}^r_{\mathbb{Z}_p[\overline{G}]}(h)$. In addition, since $\xi(\mathbb{Z}_p[\overline{G}])e$ is assumed to be a principal ideal ring, the $\xi(\mathbb{Z}_p[\overline{G}])e$-module $\mathrm{Fit}^r_{\mathbb{Z}_p[\overline{G}]}(h)$ is free of rank one and so this equality implies that c belongs to $\xi(\mathbb{Z}_p[\overline{G}])e$.

We now fix an element $\tilde{c}$ of $\xi(\mathbb{Z}_p[G])$ with $\tilde{c} \cdot e = c$ and, for each index $i \in [r]$, a pre-image $\tilde{\phi}_i$ of $b^*_{\overline{G},i} \circ \phi$ under the natural surjective map $\mathrm{End}_{\mathbb{Z}_p[G]}(\mathbb{Z}_p[G]^d) \to \mathrm{End}_{\mathbb{Z}_p[\overline{G}]}(\mathbb{Z}_p[\overline{G}]^d)$. Then the equality (26) implies that the element

$$\varepsilon_{E/K} = c \cdot (\wedge_{i\in[d]\setminus[r]}(b^*_{\overline{G},i} \circ \phi))(\wedge_{j\in[d]} b_{\overline{G},j}) = \varrho^{r,d}_{G,H}\big(\tilde{c} \cdot (\wedge_{i\in[d]\setminus[r]}\tilde{\phi}_i)(\wedge_{j\in[d]} b_{G,j})\big)$$

belongs to $\varrho^{r,d}_{G,H}\big(\bigcap^r_{\mathbb{Z}_p[G]}\mathbb{Z}_p[G]^d\big)$, as required to verify (19). □

6. Non-commutative p-adic class number formulas

In this section we fix an odd prime p and formulate a precise analogue of Conjecture 5.1 that concerns the values at zero of derivatives of p-adic Artin L-series. We then prove this conjecture modulo Iwasawa's μ-invariant conjecture and use this result to obtain further evidence for Conjecture 5.1 in the setting of Galois extensions L/K in which L is a CM field and K is totally real.

6.1. **Statement of the conjecture and main result.** We assume K is totally real and L is CM and set $G := \mathrm{Gal}(L/K)$. We write L^+ for the maximal totally real subfield of L, τ for the unique non-trivial element of $\mathrm{Gal}(L/L^+)$ and $\mathrm{Ir}_p^{\pm}(G)$ for the subsets of $\mathrm{Ir}_p(G)$ comprising characters χ for which one has $\chi(\tau) = \pm\chi(1)$. For a $\mathbb{Z}_p[G]$-module M and an element m of M, we set $M^{\pm} := (1 \pm \tau)(M)$ and $m^{\pm} := ((1 \pm \tau)/2)(m) \in M^{\pm}$.

We also fix a finite set S of places of K that contains both $S_{L/K}^{\mathrm{ram}}$ (and hence S_K^{∞}) and the set S_K^p of p-adic places, and an auxiliary finite set T of places of K that is disjoint from S.

For ψ in $\mathrm{Ir}_p^+(G)$ we write $L_{p,S,T}(\psi, s)$ for the S-truncated T-modified p-adic Artin L-series of ψ, as constructed by Greenberg in [16]. For a non-negative integer a we then define the 'a-th derived non-abelian p-adic Stickelberger series' for $L/K, S$ and T by setting

$$\theta_{L/K}^{p,a}(z) = \theta_{L/K,S,T}^{p,a}(z) := \sum\nolimits_{\rho \in \mathrm{Ir}_p^-(G)} e_\rho \cdot z^{-\rho(1)a} L_{p,S,T}(\check{\rho} \cdot \omega_K, z),$$

where ω_K is the p-adic Teichmüller character of K.

We recall (from [3, Th. 3.1]) that for every character ρ in $\mathrm{Ir}_p^-(G)$, one has

$$\mathrm{ord}_{z=0} L_{p,S,T}(\check{\rho} \cdot \omega_K, z) \geq \dim_{\mathbb{C}_p}(\mathrm{Hom}_{\mathbb{C}_p[G]}(V_{\check{\rho}}, \mathbb{C}_p \cdot Y_{L,S,p}^-)),$$

where $V_{\check{\rho}}$ is a $\mathbb{C}_p[G]$-module of character $\check{\rho}$. In particular, if we write $\Sigma(L)$ for the subset of S comprising places that split completely in L/K, and set $r(L) := |\Sigma(L)|$, then this inequality implies that $\theta_{L/K}^{p,r(L)}(z)$ is p-adic holomorphic at $z = 0$ and it is easily checked that its value $\theta_{L/K}^{p,r(L)}(0)$ at $z = 0$ belongs to $\zeta(\mathbb{Q}_p[G])$.

We next recall that in [17, §1] Gross defines for each place w of L a local p-adic absolute value by means of the composite

$$|| \cdot ||_{w,p} : L_w^\times \xrightarrow{r_w} G_{L_w^{\mathrm{ab}}/L_w} \xrightarrow{\chi_{L_w}} \mathbb{Z}_p^\times \xrightarrow{x \mapsto x^{-1}} \mathbb{Z}_p^\times,$$

where L_w^{ab} is the maximal abelian extension of L_w in L_w^c, r_w the local reciprocity map and χ_{L_w} the cyclotomic character. We write

$$R_{L,S,p}^{\mathrm{Gross}} : \mathbb{Q}_p \otimes_{\mathbb{Z}_p} U_S(L)_p^- \to \mathbb{Q}_p \otimes_{\mathbb{Z}_p} Y_{L,S,p}^- \tag{27}$$

for the map of $\mathbb{Q}_p[G]$-modules that sends each u in $U_S(L)^-$ to $\sum_{w \in S_L} \log_p ||u||_{w,p} \cdot w$.

We can now state an analogue of Conjecture 5.1 for p-adic Artin L-series.

Conjecture 6.1. *Let p be an odd prime for which $U_S^T(L)_p$ is torsion-free. Fix a CM Galois extension E of K in L and set $H := \mathrm{Gal}(L/E)$ and $\overline{G} := \mathrm{Gal}(E/K)$. Then there exist elements*

$$\varepsilon_{L/K}^p \in \bigcap\nolimits_{\mathbb{Z}_p[G]}^{r(L)} U_S^T(L)_p^- \quad \text{and} \quad \varepsilon_{E/K}^p \in \big(\bigcap\nolimits_{\mathbb{Z}_p[\overline{G}]}^{r(E)} U_S^T(E)_p^-\big)^\pi$$

that satisfy

$$\big(\bigwedge\nolimits_{\mathbb{Q}_p[\mathrm{Gal}(F/K)]}^{r(F)} R_{F,S,p}^{\mathrm{Gross}}\big)(\varepsilon_{F/K}^p) = \theta_{F/K}^{p,r(F)}(0) \cdot \wedge_{v \in \Sigma(F)} w_{v,F}^-, \tag{28}$$

for both $F = L$ and $F = E$ and also

$$\mathrm{ev}_\pi(\varepsilon_{L/K}^p) = \mathrm{Nrd}_{\mathbb{Q}[G]}(-1)^{r(L)(r(E)-r(L))} \cdot \mathrm{Rec}_\pi(\varepsilon_{E/K}^p).$$

Remark 6.2. The equality (28) is the precise analogue of the (minus part of the) equality (18) that defines the non-commutative Rubin-Stark element $\varepsilon_{F/K}$, and so Conjecture 6.1 can be interpreted as a 'non-commutative p-adic Class Number Conjecture for $\mathbb{G}_m$' (for L/K). In general, however, if the map $R^{\rm Gross}_{F,S,p}$ fails to be bijective, then (28) does not uniquely determine $\varepsilon^p_{F/K}$. Nevertheless, if the Gross-Kuz'min Conjecture is valid for every character in $\mathrm{Ir}_p^-(\mathrm{Gal}(F/K))$, then $R^{\rm Gross}_{F,S,p}$ is bijective and so the known validity of the Gross-Stark Conjecture (as established by Dasgupta, Kakde and Ventullo in [10]) combines with (28) and the argument of [3, Cor. 3.8] to imply that $\varepsilon^p_{F/K} = \varepsilon^-_{F/K}$.

To state our main result concerning Conjecture 6.1, we write $\mu_p(L)$ for the Iwasawa μ-invariant of the cyclotomic $\mathbb{Z}_p$-extension of L. We further recall that Iwasawa [19] has conjectured $\mu_p(L) = 0$.

Theorem 6.3. *If $\mu_p(L) = 0$, then Conjecture 6.1 is valid.*

This result will be proved in §6.2 and combines with Remark 6.2 to give unconditional evidence in support of Conjecture 5.1 that goes beyond the result of Theorem 5.5. Such results include the following (which will be proved in §6.3) in which, in contrast to Theorem 5.5, the order of the commutator subgroup $[G,G]$ can be divisible by an arbitrarily large power of p.

Corollary 6.4. *Conjecture 5.1 is valid if $K = \mathbb{Q}$, the fields E and L are CM and $[G,G]$ is a p-subgroup of the decomposition group in G of any, and hence every, p-adic place of L.*

Example 6.5. Corollary 6.4 applies in cases for which the order of $[G,G]$, the index $[G : G_p]$ and the ramification degree of p in L can each be arbitrarily large. To give an example, fix powers m and n of distinct odd primes p and q for which the subgroup of $(\mathbb{Z}/n)^\times$ generated by p does not contain $-1 \pmod{n}$ and set $E := \mathbb{Q}(e^{2\pi i/mn})$. Then, for each p-adic place v of E, one has $E_v = (E^+)_v$ and the field $F_v := E_v(\sqrt[m]{p})$ is a Galois extension of $\mathbb{Q}_p$ for which the commutator subgroup of $\mathrm{Gal}(F_v/\mathbb{Q}_p)$ is equal to $\mathrm{Gal}(F_v/E_v)$ (since $p \notin (E_v^\times)^p$) and so has order m. Then, as m is odd, a result of Neukirch [27, §3, Cor. 3] implies the existence of a cyclic (totally real) extension L^+ of E^+ of degree m that is Galois over $\mathbb{Q}$ and such that F_v is equal to the completion of L^+ at a p-adic place of L^+. Further, setting $\Gamma := \mathrm{Gal}(L^+/\mathbb{Q})$, the group $[\Gamma,\Gamma]$ is equal to $\mathrm{Gal}(L^+/E^+)$ and is contained in the decomposition subgroup in Γ of every p-adic place of L^+. Now set $L := EL^+$ and $G := \mathrm{Gal}(L/\mathbb{Q})$. Then L is a CM field, the order of $[G,G]$ is equal to m and, for any p-adic place w of L, the absolute ramification degree of w is divisible by $[\mathbb{Q}(e^{2\pi i/m}) : \mathbb{Q}]$ and the index of G_w in G is equal to the number of p-adic places of $\mathbb{Q}(e^{2\pi i/n})$.

Remark 6.6. Fix an odd prime p and a finite CM Galois extension L of a totally real field K of group G. Then, if G is abelian, the 'minus part' $\mathrm{eTNC}(L/K)^-_p$ of the p-component of $\mathrm{eTNC}(\mathbb{G}_m)$ for L/K has been shown by Bullach, Daoud and the first and fourth authors [2, Th. B(a)] to follow from the seminal work of Dasgupta and Kakde [9] on the Brumer-Stark Conjecture (and see also the related work of Atsuta and Kataoka [1] and Dasgupta, Kakde and Silliman [11]). However, if G is not abelian, then the strongest result concerning $\mathrm{eTNC}(L/K)^-_p$ is due to Nickel [29, Th. 2] and assumes, amongst other things, that the Sylow p-subgroups of $\mathrm{Gal}(L/K)$ are abelian. Now, if one takes L as constructed in Example 6.5

and $K = \mathbb{Q}$, then G has a unique Sylow p-subgroup P and the fixed field of P in L contains $e^{2\pi i/p}$ but not $e^{2\pi i/p^2}$. By using this fact, one can check that the commutator subgroup $[P, P]$ has order m/p and hence that P is not abelian for $m > p$.

6.2. **The proof of Theorem 6.3.** We fix notation E, H and $\overline{G}$ as in Theorem 6.3 and set

$$r := r(L) \quad \text{and} \quad r' := r(E).$$

We also assume, as we may, that S is labelled so v_0 is archimedean, $\Sigma(L) = \{v_i\}_{i\in[r]}$ and $\Sigma(E) = \{v_i\}_{i\in[r']}$. In particular, we note that, since v_0 does not split completely in E, the sets $\Sigma(E)$ and $\Sigma(L)$ coincide with the sets that are denoted by the same notation in §4.2.

We next write $\mathcal{K}_\infty$ for the cyclotomic $\mathbb{Z}_p$-extension of L, set $\mathcal{G} := \mathrm{Gal}(\mathcal{K}_\infty/K)$ and write $\Omega(\mathcal{K}_\infty)$ for the set of finite Galois extensions of K in $\mathcal{K}_\infty$. For each F in $\Omega(\mathcal{K}_\infty)$ we set

$$U_F := U_S^T(F)_p^-, \quad \mathcal{S}_F := \mathcal{S}_S^T(F)_p^- \quad \text{and} \quad \mathcal{G}_F := \mathrm{Gal}(F/K).$$

We also write $\Lambda(\mathcal{G}_\infty)$ for the Iwasawa algebra $\varprojlim_F \mathbb{Z}_p[\mathcal{G}_F]$, where F runs over $\Omega(\mathcal{K}_\infty)$ and the transition morphisms are the natural projections $\mathbb{Z}_p[\mathcal{G}_{F'}] \to \mathbb{Z}_p[\mathcal{G}_F]$ for $F \subset F'$. We recall that the total quotient ring $Q(\mathcal{G}_\infty)$ of $\Lambda(\mathcal{G}_\infty)$ is semisimple and we write $\mathrm{Nrd}_{Q(\mathcal{G}_\infty)}$ for its reduced norm.

We finally set

$$\theta_{\mathcal{K}_\infty} := (\theta^T_{F/K,S}(0))_F \in \varprojlim_{F\in\Omega(\mathcal{K}_\infty)} \zeta(\mathbb{Q}_p[\mathcal{G}_F])^-,$$

where the transition morphisms are the natural projections $\zeta(\mathbb{Q}_p[\mathcal{G}_{F'}]) \to \zeta(\mathbb{Q}_p[\mathcal{G}_F])$ for $F \subset F'$, and define an object of $D(\Lambda(\mathcal{G}_\infty))$ by setting

$$C_{\mathcal{K}_\infty} := \varprojlim_{F\in\Omega(\mathcal{K}_\infty)} C^T_{F,S,p},$$

where the transition morphisms for $F \subset F'$ are induced by the corresponding cases of (11).

6.2.1. The following argument reinterprets in the special case of CM extensions of totally real fields various general constructions and results of [7]. In particular, it relies crucially on an interpretation of the main conjecture of non-commutative Iwasawa theory for totally real fields that is established in loc. cit. It then also involves a close analysis (in the setting of CM fields) of various technical results from loc. cit. in order to provide a link to the Artin-Bockstein maps that underlie the definition of the map Rec_π in Theorem 6.3.

At the outset we use [7, Prop. 8.2] (and, in particular, claim (iii) of the latter result) to fix a representative

$$\Lambda(\mathcal{G}_\infty)^{d,-} \xrightarrow{\phi} \Lambda(\mathcal{G}_\infty)^{d,-}$$

of $C^-_{\mathcal{K}_\infty}$ with the following property: for each CM field F in $\Omega(\mathcal{K}_\infty)$ the exact sequence of $\mathbb{Z}_p[\mathcal{G}_F]$-modules

$$0 \to U_F \xrightarrow{\hat{\iota}_F} \mathbb{Z}_p[\mathcal{G}_F]^{d,-} \xrightarrow{\phi_F} \mathbb{Z}_p[\mathcal{G}_F]^{d,-} \to \mathcal{S}_F \to 0 \tag{29}$$

that is induced by natural isomorphism $\mathbb{Z}_p[\mathcal{G}_F] \otimes^{\mathsf{L}}_{\Lambda(\mathcal{G}_\infty)} C^-_{\mathcal{K}_\infty} \cong C^{T,-}_{F,S,p}$ in $D(\mathbb{Z}_p[\mathcal{G}_F])$ is of the form specified in the proof of Lemma 4.1.

We write $\{b_i\}_{i\in[d]}$ for the standard basis of $\Lambda(\mathcal{G}_\infty)^d$, $\{b_{F,i}\}_{i\in[d]}$ for its image in $\mathbb{Z}_p[\mathcal{G}_F]^d$ and x_F for the element $\wedge_{j\in[d]} b^-_{F,j}$ of $\bigcap^d_{\mathbb{Z}_p[\mathcal{G}_F]} \mathbb{Z}_p[\mathcal{G}_F]^d$. Then, via [7, Cor. 7.9] and the argument of

[7, Prop. 7.8], the known validity, modulo the vanishing of $\mu_p(L)$, of the main conjecture of non-commutative Iwasawa theory (due, independently, to Ritter and Weiss [30] and Kakde [23]), implies the existence of an element u of $K_1(\Lambda(\mathcal{G}_\infty))$ such that

$$\theta_{\mathcal{K}_\infty} = \mathrm{Nrd}_{Q(\mathcal{G}_\infty)}(u) \cdot \big((\wedge_{i=1}^{i=d}\phi_{F,i})(x_F)\big)_{F\in\Omega(\mathcal{K}_\infty)}. \tag{30}$$

With F denoting either L or E we use $\hat{\iota}_{F,*}$ to identify U_F with a submodule of $\mathbb{Z}_p[\mathcal{G}_F]^{d,-}$ and define an element of $\bigcap_{\mathbb{Z}_p[\mathcal{G}_F]}^{r(F)}\mathbb{Z}_p[\mathcal{G}_F]^{d,-}$ by setting

$$\varepsilon_F^p = \varepsilon_{F/K}^p := \mathrm{Nrd}_{\mathbb{Q}_p[\mathcal{G}_F]}(-1)^{r(F)(d-r(F))}\mathrm{Nrd}_{\mathbb{Q}_p[\mathcal{G}_F]}(u_F)\cdot(\wedge_{a=r(F)+1}^{a=d}\phi_{F,a})(x_F), \tag{31}$$

where u_F denotes the image of u in $K_1(\mathbb{Z}_p[\mathcal{G}_F])$.

Then, with this definition of ε_F^p, the result of [7, Prop. 6.16(i)] combines with the exactness of (29) to imply a containment

$$\varepsilon_F^p \in \bigcap\nolimits_{\mathbb{Z}_p[\mathcal{G}_F]}^{r(F)} U_F. \tag{32}$$

In particular, if $F = E$, then the equality

$$\varepsilon_E^p = \varrho_{G,H,p}^{r',d}\big(\mathrm{Nrd}_{\mathbb{Q}_p[G]}(-1)^{r'(d-r')}\mathrm{Nrd}_{\mathbb{Q}_p[G]}(u_L)\cdot(\wedge_{a=r'+1}^{a=d}\phi_{L,a})(x_L)\big) \tag{33}$$

implies that

$$\varepsilon_E^p \in \mathrm{im}(\varrho_{G,H,p}^{r',d}) \cap \bigcap\nolimits_{\mathbb{Z}_p[\overline{G}]}^{r(E)} U_E = \big(\bigcap\nolimits_{\mathbb{Z}_p[\overline{G}]}^{r(E)} U_E\big)^{\pi}. \tag{34}$$

It follows that ε_E^p belongs to the domain of Rec_π and we next claim that

$$\mathrm{ev}_\pi(\varepsilon_L^p) = \mathrm{Nrd}_{\mathbb{Q}_p[G]}(-1)^{r(r'-r)}\cdot \mathrm{Rec}_\pi(\varepsilon_E^p). \tag{35}$$

To verify this it is enough to prove that for every $\underline{\varphi} = (\varphi_1,\dots,\varphi_r)$ in $\mathrm{Hom}_{\mathbb{Z}_p[G]}(U_L,\mathbb{Z}_p[G])$, the projection of $\mathrm{Nrd}_{\mathbb{Q}_p[G]}(-1)^{r(r'-r)}\cdot(\wedge_{j\in[r]}\varphi_j)(\varepsilon_L^p)$ to $\big(\iota_{\Sigma_{L,E}}(G)/\iota_{\Sigma_{L,E}}^H(G)\big)_p$ is equal to the $\underline{\varphi}$-component of $\mathrm{Rec}_\pi(\varepsilon_E^p)$. This is true since the equality (33) combines with the explicit description of Rec_π that is given in Lemma 4.4 to imply that the $\underline{\varphi}$-component of $\mathrm{Rec}_\pi(\varepsilon_E^p)$ is equal to the projection to $\big(\iota_{\Sigma_{L,E}}(G)/\iota_{\Sigma_{L,E}}^H(G)\big)_p$ of the element

$$\begin{aligned}
&(\wedge_{j=1}^{j=r}\varphi_j)\big((\wedge_{j=r+1}^{j=r'}\phi_{L,j})(\mathrm{Nrd}_{\mathbb{Q}_p[G]}(-1)^{r'(d-r')}\mathrm{Nrd}_{\mathbb{Q}_p[G]}(u_L)\cdot(\wedge_{a=r'+1}^{a=d}\phi_{L,a})(x_L))\big)\\
&= \mathrm{Nrd}_{\mathbb{Q}_p[G]}(-1)^{r'(d-r')+(r'-r)(d-r')}\mathrm{Nrd}_{\mathbb{Q}_p[G]}(u_L)\cdot(\wedge_{j=1}^{j=r}\varphi_j)\big((\wedge_{a=r+1}^{a=d}\phi_{L,a})\big)(x_L)\big)\\
&= \mathrm{Nrd}_{\mathbb{Q}_p[G]}(-1)^{r(r'-r)}\cdot(\wedge_{j=1}^{j=r}\varphi_j)\big(\mathrm{Nrd}_{\mathbb{Q}_p[G]}(-1)^{r(d-r)}\mathrm{Nrd}_{\mathbb{Q}_p[G]}(u_L)\cdot(\wedge_{a=r+1}^{a=d}\phi_{L,a})(x)\big)\\
&= \mathrm{Nrd}_{\mathbb{Q}_p[G]}(-1)^{r(r'-r)}\cdot(\wedge_{j=1}^{j=r}\varphi_j)(\varepsilon_L^p),
\end{aligned}$$

where the second equality follows from the fact that

$$r'(d-r') + (r'-r)(d-r') \equiv r(r'-r) + r(d-r) \quad \text{modulo } 2.$$

6.2.2. Given the containments (32), (34) and equality (35), the proof of Theorem 6.3 is reduced to showing that the element $\varepsilon_F^p = \varepsilon_{F/K}^p$ defined in (31) validates the equality (28).

To do this we set $\mathcal{G} := \mathcal{G}_F$ and $r^* := r(F)$. We also set $\lambda_{F,S}^p := \bigwedge_{\mathbb{Q}_p[\mathcal{G}]}^{r^*}(\mathbb{Q}_p \otimes_{\mathbb{Z}_p} R_{F,S,p}^{\mathrm{Gross}})$ and use the natural projection map $\varrho : Y_{F,S,p}^- \to Y_{F,\Sigma(F),p}^-$.

We note that the exactness of (29) combines with [7, Prop. 6.16(ii)] to imply that if χ is any character in $\mathrm{Ir}_p^-(\mathcal{G})$ for which $e_\chi(\varepsilon_F^p)$ is non-zero, then the map $e_\chi(\mathbb{Q}_p^c \cdot \varrho)$ is bijective. This implies, in particular, that the element $\lambda_{F,S}^p(\varepsilon_F^p)$ belongs to the image of the inclusion $\mathbb{Q}_p \cdot \bigcap_{\mathbb{Z}_p[\mathcal{G}]}^{r^*} Y_{F,\Sigma(F),p}^- \to \mathbb{Q}_p \cdot \bigcap_{\mathbb{Z}_p[\mathcal{G}]}^{r^*} Y_{F,S,p}^-$.

We write $\{(w_i^-)^*\}_{i\in[r^*]}$ for the $\mathbb{Z}_p[\mathcal{G}]^-$-basis of $\mathrm{Hom}_{\mathbb{Z}_p[\mathcal{G}]}(Y_{F,\Sigma(F),p}^-, \mathbb{Z}_p[\mathcal{G}]^-)$ that is dual to the basis $\{w_i^-\}_{i\in[r^*]}$ of $Y_{F,\Sigma(F),p}^-$ and note that this basis gives rise to an isomorphism of $\xi(\mathbb{Z}_p[\mathcal{G}])$-modules

$$\wedge_{i\in[r^*]}(w_i^-)^* : \bigcap\nolimits_{\mathbb{Z}_p[\mathcal{G}]}^{r^*} Y_{F,\Sigma(F),p}^- \to \bigcap\nolimits_{\mathbb{Z}_p[\mathcal{G}]}^{0} Y_{F,\Sigma(F),p}^- = \xi(\mathbb{Z}_p[\mathcal{G}])^-$$

that sends $\wedge_{i\in[r^*]} w_i^-$ to the identity element $(1-\tau)/2$ of the ring $\xi(\mathbb{Z}_p[\mathcal{G}])^-$.

Given these observations, the required equality (28) will follow if the composite map

$$(\mathbb{Q}_p \cdot \wedge_{i\in[r^*]}(w_i^-)^*) \circ \big(\bigwedge\nolimits_{\mathbb{Q}_p[\mathcal{G}]}^{r^*}(\mathbb{Q}_p \cdot \varrho)\big) \circ \bigwedge\nolimits_{\mathbb{Q}_p[\mathcal{G}]}^{r^*}(\mathbb{Q}_p \cdot \lambda_{F,S}^p) \tag{36}$$

sends ε_F^p to $\theta_{F/K}^{p,r^*}(0)$. To verify this we use the following technical result.

Lemma 6.7. *Write $\mathcal{K}'_\infty$ for the cyclotomic $\mathbb{Z}_p$-extension of F, set $\mathcal{G}'_\infty := \mathrm{Gal}(\mathcal{K}'_\infty/K)$ and fix a topological generator γ of $\mathrm{Gal}(\mathcal{K}'_\infty/F)$.*

(i) *For each $j \in [r^*]$, there exists a (unique) homomorphism*

$$\widehat{\phi}_j = (\widehat{\phi}_{j,F'})_{F'} \in \varprojlim\nolimits_{F'\in\Omega(\mathcal{K}'_\infty)} \mathrm{Hom}_{\mathbb{Z}_p[\mathcal{G}_{F'}]}\big(\mathbb{Z}_p[\mathcal{G}_{F'}]^d, \mathbb{Z}_p[\mathcal{G}_{F'}]\big) = \mathrm{Hom}_{\Lambda(\mathcal{G}'_\infty)}(\Lambda(\mathcal{G}'_\infty)^d, \Lambda(\mathcal{G}'_\infty))$$

with $b_j^* \circ \phi = (\gamma - 1)(\widehat{\phi}_j)$.

(ii) *The composite map (36) is equal to $\mathrm{Nrd}_{\mathbb{Q}_p[\mathcal{G}]}(\log_p(\kappa_K(\gamma))^{r^*}) \cdot \wedge_{j\in[r^*]}\widehat{\phi}_{j,F}$.*

Proof. If a map $\widehat{\phi}_j$ with the property stated in claim (i) exists, then it is clearly unique. On the other hand, for each j in $[r^*]$, our ordering of S guarantees that the map $b_{F,j}^* \circ \phi_F$ vanishes and this directly implies the existence of $\widehat{\phi}_j$.

Turning to claim (ii) we note that [7, (9.3.4)] implies the composite map (36) can be computed as the normalised reduced exterior product

$$\mathrm{Nrd}_{\mathbb{Q}_p[\mathcal{G}]}(\log_p(\kappa_K(\gamma))^{r^*}) \cdot \wedge_{j\in[r^*]}(\varpi_j \circ \beta).$$

Here β is the connecting homomorphism $H^0(C_{F,S,p}^{T,-}) = \ker(\phi_F) \to \mathrm{cok}(\phi')$ that is associated to the short exact sequence of complexes (with vertical differentials)

$$\begin{array}{ccccc}
\Lambda(\mathcal{G}'_\infty)^{d,-} & \xhookrightarrow{x\mapsto(\gamma-1)x} & \Lambda(\mathcal{G}'_\infty)^{d,-} & \twoheadrightarrow & \mathbb{Z}_p[\mathcal{G}]^{d,-} \\
\big\downarrow \phi' & & \big\downarrow \phi' & & \big\downarrow \phi_E \\
\Lambda(\mathcal{G}'_\infty)^{d,-} & \xhookrightarrow{x\mapsto(\gamma-1)x} & \Lambda(\mathcal{G}'_\infty)^{d,-} & \twoheadrightarrow & \mathbb{Z}_p[\mathcal{G}]^{d,-}
\end{array}$$

in which the endomorphism ϕ' is induced by ϕ. In addition, ϖ_j is the composite homomorphism $\mathrm{cok}(\phi') \to \mathrm{cok}(\phi_F) = \mathcal{S}_F \xrightarrow{\varrho_{F,S,p}} Y^-_{F,S,p} \xrightarrow{\varrho} Y^-_{F,\Sigma(F),p} \to \mathbb{Z}_p[\mathcal{G}]^-$ in which the first arrow is the natural projection map and the last sends each element to its coefficient at w_j^-.

To deduce the result of claim (ii) from this description of (36) it is thus enough to note that an explicit computation of the connecting homomorphism in the above diagram combines with the defining equality $\phi_j = (\gamma-1)(\widehat{\phi}_j)$ of $\widehat{\phi}_j$ to show that $\varpi_j \circ \beta = \widehat{\phi}_{j,F}$. □

Via the above result, the verification that ε^p_F validates (28) is reduced to showing that

$$(\wedge_{j\in[r^*]}\widehat{\phi}_{j,F})(\varepsilon^p_F) = \mathrm{Nrd}_{\mathbb{Q}_p[\mathcal{G}]}(\log_p(\kappa_K(\gamma))^{-r^*})\cdot \theta^{p,r^*}_{F/K}(0).$$

To do this, we use the fact that

$$(37)\quad \big(\wedge_{j\in[r^*]}(b^*_{F',j}\circ\phi_{F'})\big)_{F'} = \big(\wedge_{j\in[r^*]}(\gamma-1)(\widehat{\phi}_{j,F'})\big)_{F'} = \mathrm{Nrd}_{Q(\mathcal{G}'_\infty)}(\gamma-1)^{r^*}\cdot\big(\wedge_{j\in[r^*]}\widehat{\phi}_{j,F'}\big)_{F'},$$

where in each case F' runs over $\Omega(\mathcal{K}'_\infty)$. Here the first equality follows directly from the definition of $\widehat{\phi}_j$ and the second is justified by the argument of [7, (9.1.2)].

Writing π_F for the natural projection map $\varprojlim_{F'\in\Omega(\mathcal{K}'_\infty)}\zeta(\mathbb{Q}_p[\mathcal{G}_{F'}]) \to \zeta(\mathbb{Q}_p[\mathcal{G}])$, we now obtain the required equality via the computation

$$\begin{aligned}(\wedge_{j\in[r^*]}\widehat{\phi}_{j,F})(\varepsilon^p_F) &= (\wedge_{j\in[r^*]}\widehat{\phi}_{j,F})\big(\mathrm{Nrd}_{\mathbb{Q}_p[\mathcal{G}]}(-1)^{r^*(d-r^*)}\mathrm{Nrd}_{\mathbb{Q}_p[\mathcal{G}]}(u_F)\cdot(\wedge_{i=r^*+1}^{i=d}\phi_{F,i})(x_F)\big)\\ &= \mathrm{Nrd}_{\mathbb{Q}_p[\mathcal{G}]}(u_F)\big((\wedge_{j\in[r^*]}\widehat{\phi}_{j,F})\wedge(\wedge_{i=r^*+1}^{i=d}\phi_{F,i})\big)(x_F)\\ &= \pi_F(\mathrm{Nrd}_{Q(\mathcal{G}'_\infty)}(\gamma-1)^{-r^*}\cdot\theta_{\mathcal{K}'_\infty})\\ &= \mathrm{Nrd}_{\mathbb{Q}_p[\mathcal{G}]}(\log_p(\kappa_K(\gamma))^{-r^*})\cdot\theta^{p,r^*}_{F/K}(0).\end{aligned}$$

Here the first equality follows from the explicit definition (31) of ε^p_F, the third from (30) and the relations (37), and the fourth directly from the equality of [7, (9.3.3)].

This completes the proof of Theorem 6.3.

6.3. **The proof of Corollary 6.4.** Let F denote either L or E. Then, since the archimedean place of $\mathbb{Q}$ splits in F^+, the element $\varepsilon^{1+\tau}_{F/K}$ is trivial and so one has

$$\varepsilon_{F/K} = (1/2)((1-\tau)+(1-\tau))(\varepsilon_{F/K}) = (1/2)(1-\tau)(\varepsilon_{F/K}) = \varepsilon^-_{F/K}$$

in $\bigwedge^{r(F)}_{\mathbb{R}[\mathrm{Gal}(F/K)]}(\mathbb{R}\cdot U^T_S(F))$. It is therefore enough to prove Conjecture 5.1 after replacing $\varepsilon_{L/K}$ and $\varepsilon_{E/K}$ by $\varepsilon^-_{L/K}$ and $\varepsilon^-_{E/K}$ respectively.

In addition, by the discussion just before the statement of Corollary 6.4, if the Gross-Kuz'min Conjecture is valid for the pair (L,p), and hence also for (E,p), then $\varepsilon^-_{F/K} = \varepsilon^p_{F/K}$ and so, assuming $\mu_p(L)=0$, the validity of Conjecture 5.1 in this case follows directly from the claims made in Theorem 6.3. To complete the proof it is therefore enough to show that the stated hypotheses on L imply both that $\mu_p(L)=0$ (so that the result of Theorem 6.3 is unconditional) and that the Gross-Kuz'min Conjecture is valid for (L,p).

Now, since the fixed field L' of $[G,G]$ in L is an abelian extension of $\mathbb{Q}$ one has $\mu_p(L')=0$ (by [14]) and the Gross-Kuz'min Conjecture for (L',p) is valid (by [15]). Then, as the degree $|[G,G]|$ of L/L' is (by assumption) a power of p, the vanishing of $\mu_p(L')$ combines with Nakayama's Lemma to imply $\mu_p(L)=0$. In addition, since (by assumption) each

p-adic place of L' has full decomposition subgroup in $\mathrm{Gal}(L/L')$, the following observation (which originates in the unpublished preprint [33] of one of us) implies the validity of the Gross-Kuz'min Conjecture for (L,p). This completes the proof of Corollary 6.4.

Lemma 6.8. *Let L/F be a finite extension of number fields such that $|S_L^p| = |S_F^p|$. Then the validity of the Gross-Kuz'min Conjecture for (F,p) implies its validity for (L,p).*

Proof. We write k_∞ for the cyclotomic $\mathbb{Z}_p$-extension of a number field k. Then the Gross-Kuz'min Conjecture for (F,p) asserts that the maximal pro-p abelian extension $F\langle p\rangle$ of F that is unramified outside S_F^p and such all p-adic places of F_∞ are totally split in $F\langle p\rangle$ is a finite extension of F_∞. To study these extensions we use the exact commutative diagram

$$\begin{array}{ccccccc}(\mathcal{O}^\times_{L,\Sigma_L})_p & \xrightarrow{\theta_L} & \widetilde{\bigoplus}_{v\in S_L^p} D_v(L) & \longrightarrow & \mathrm{Gal}(L\langle p\rangle/L_\infty) & \longrightarrow & \mathrm{Cl}(\mathcal{O}_{L,\Sigma_L})_p \\ \downarrow{\scriptstyle \theta_1} & & \downarrow{\scriptstyle \theta_2} & & \downarrow{\scriptstyle \theta_3} & & \\ (\mathcal{O}^\times_{F,\Sigma_F})_p & \xrightarrow{\theta_F} & \widetilde{\bigoplus}_{v\in S_F^p} D_v(F) & \longrightarrow & \mathrm{Gal}(F\langle p\rangle/F_\infty). & & \end{array}$$

Here, for $E\in\{L,F\}$, we write Σ_E for $S_E^\infty\cup S_E^p$, $D_v(E)$ for the decomposition subgroup of each v in S_E^p in $\mathrm{Gal}(E\langle p\rangle/E)$ (so that $D_v(E)\cong\mathbb{Z}_p$), θ_E for the map induced by the local reciprocity maps $E_v^\times\to\mathrm{Gal}(E_{\infty,v}/E_v)\cong D_v(E)$ for all $v\in S_E^p$, $\widetilde{\bigoplus}_{v\in S_E^p}D_v(E)$ for the set of $(g_v)_v$ in $\bigoplus_{v\in S_E^p}D_v(E)$ with $\prod_v g_v\in\mathrm{Gal}(E\langle p\rangle/E_\infty)$ and all unlabelled arrows are the obvious maps. In particular, the exactness of each row follows from class field theory (see, for example, the proof of [24, Prop. 7.5]). In addition, θ_1 is the natural norm map and θ_2 and θ_3 the natural restrictions maps.

Now $\mathrm{cok}(\theta_1)$ and $\mathrm{Cl}(\mathcal{O}_{L,\Sigma_L})_p$ are finite and, since $|S_L^p|=|S_F^p|$, θ_2 is injective and $\mathrm{cok}(\theta_2)$ is finite. Hence, if $\mathrm{Gal}(F\langle p\rangle/F_\infty)$ is finite, then an application of the Five Lemma implies that $\mathrm{Gal}(L\langle p\rangle/L_\infty)$ is also finite, as required. □

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