Annihilating Selmer Modules

James Barrett and David Burns

Abstract. We prove that for a wide class of motives the local and global non-commutative Tamagawa number conjectures of Fukaya and Kato imply explicit restrictions on the Galois structure of Selmer modules (in the sense of Bloch and Kato). We thereby obtain a variety of new, concrete and general conjectures concerning the structures of Selmer modules. In several important cases we prove these conjectures.

Introduction

Let M be a motive defined over a number field k and for each finite abelian extension F of k write M_F for the motive $h^0(\mathrm{Spec}\, F) \otimes_{h^0(\mathrm{Spec}\, k)} M$, regarded as defined over k and with an action of the algebra $\mathbb{Q}[\mathrm{Gal}(F/k)]$ via the first factor in the tensor product. In this article we shall show that for a wide class of M the (very abstract and comparatively inexplicit) local and global non-commutative Tamagawa number conjectures for M_F that are formulated by Fukaya and Kato in [14] predict the existence for each odd prime p of non-trivial elements of $\mathbb{Z}_p[\mathrm{Gal}(F/k)]$ which annihilate the p-adic (Bloch-Kato) Selmer module of M_F and are explicitly constructed by using the values of complex L-functions. In an analogous fashion, our approach also shows that the relevant generalised main conjecture of Iwasawa theory predicts the existence of non-trivial annihilators of the p-adic Selmer module of M_F that are explicitly constructed by using the values of p-adic L-functions.

We further prove that, upon appropriate specialisation, this very general approach gives rise to a wide variety of new and rather concrete results. Such results include an explicit analogue for totally real fields of Brumer's Conjecture, and hence also of Stickelberger's Theorem, and a refinement and generalisation of the main results of Oriat in [24] concerning the structure of certain Galois groups (see Corollary 3.4 and Remarks 3.6), an analogue involving leading terms of Dirichlet L-functions at strictly positive (rather than negative) integers of the refined version of the Coates-Sinnott conjecture studied by Burns and Greither in [8] (see Corollary 3.7) and, in the context of Hasse-Weil L-functions of abelian varieties, a natural 'strong main conjecture' of the kind that Mazur and Tate explicitly ask for in [20, Remark after Conj. 3] (see

Remark 4.4). We also prove, modulo the assumed vanishing of certain μ-invariants, some special cases of the generalised main conjecture of Iwasawa theory and hence deduce that in several significant cases the structural results mentioned above are valid unconditionally (see, for example, Corollaries 3.5 and 3.8). In addition, as a key feature of the proof of these results, which may well itself be of some independent interest, we prove a natural generalisation of the main algebraic result of Snaith in [28] (see §2.1).

The main contents of this article is as follows. In §1 we discuss some preliminaries concerning homological algebra and determinant functors and then in §2 we prove the purely algebraic result (Theorem 2.1) that is central to our approach. In §3 we prove, modulo the assumed vanishing of certain μ-invariants, some important special cases of the generalised main conjecture of Iwasawa theory and also combine this result with an application of Theorem 2.1 to the compactly supported étale cohomology of suitable sheaves to deduce a variety of explicit results on the structures of Galois groups, ideal class groups and wild kernels in higher algebraic K-theory. In §4 we apply Theorem 2.1 to Selmer complexes (in the sense of Nekovář) of certain critical motives. We use this result to show that the conjectures of Fukaya and Kato imply explicit restrictions on the structures of the classical Selmer and Tate-Shafarevic groups of abelian varieties and then, finally, we discuss in greater detail the special case of elliptic curves.

The second named author is very grateful to Kazuya Kato, Masato Kurihara, Jan Nekovář, Otmar Venjakob and Chuck Weibel for illuminating discussions regarding aspects of this project.

1. Preliminaries

All modules are to be regarded, unless explicitly stated otherwise, as left modules. For any noetherian ring Λ we write $D^{\mathrm{p}}(\Lambda)$ for the derived category of perfect complexes of Λ-modules.

1.1. **Admissible complexes.** We first introduce the category of complexes to which our main algebraic result applies. To do this we fix a Dedekind domain R of characteristic 0 with field of fractions F , a finite abelian group G and a direct factor $\mathfrak{A}$ of the group ring $R[G]$ and we set $A := F \otimes_R \mathfrak{A}$. We write $D^{\mathrm{p,ad}}(\mathfrak{A})$ for the full subcategory of $D^{\mathrm{p}}(\mathfrak{A})$ comprising complexes C which satisfy the following four assumptions:

- (ad$_1$) C is an object of $D^{\mathrm{p}}(\mathfrak{A})$;
- (ad$_2$) the Euler characteristic of $A \otimes_{\mathfrak{A}} C$ in the Grothendieck group $K_0(A)$ vanishes;
- (ad$_3$) C is acyclic outside degrees $1, 2$ and 3;
- (ad$_4$) $H^1(C)$ is R-torsion-free.

We shall refer to objects of $D^{\mathrm{p,ad}}(\mathfrak{A})$ as 'admissible complexes of $\mathfrak{A}$-modules'. For important arithmetic examples of such complexes see Lemmas 3.1 and 4.1.

If C is an object of $D^{\mathrm{p,ad}}(\mathfrak{A})$, then we define $e_0 = e_0(C)$ to be the sum over all primitive idempotents of A that annihilate the module $H^2(A \otimes_{\mathfrak{A}} C)$. We write $A_0(C)$ for the F-algebra Ae_0 and $\mathfrak{A}_0(C)$ for the R-order $\mathfrak{A}e_0$ in $A_0(C)$ and let $I_{\mathfrak{A}_0(C)}$ denote the ideal in $\mathfrak{A}$ given by $\mathfrak{A} \cap \mathfrak{A}_0(C) = \{a \in \mathfrak{A} : a = ae_0\}$.

Remark 1.1. The F-algebra A is semisimple and so the assumptions (ad$_2$) and (ad$_3$) combine to imply that the A-module $A \otimes_{\mathfrak{A}} H^2(C) \cong H^2(A \otimes_{\mathfrak{A}} C)$ is isomorphic to

$A \otimes_{\mathfrak{A}} (H^1(C) \oplus H^3(C)) \cong H^1(A \otimes_{\mathfrak{A}} C) \oplus H^3(A \otimes_{\mathfrak{A}} C)$. This isomorphism then combines with the assumption (ad_3) to imply that $e_0(C)$ is also equal to the sum over all primitive idempotents of A that annihilate $H^a(A \otimes_{\mathfrak{A}} C)$ for all degrees a.

1.2. **Determinants.** For any commutative unital noetherian ring Λ we write Det_Λ for the determinant functor of Grothendieck-Knudsen-Mumford introduced in [17]. We recall that Det_Λ is well-defined on $D^{\mathrm{p}}(\Lambda)$ and takes values in the category $\mathcal{P}(\Lambda)$ of graded invertible Λ-modules. We further recall that if X belongs to $D^{\mathrm{p}}(\Lambda)$ and $\Lambda \to \Lambda'$ is a homomorphism of commutative unital noetherian rings, then $\Lambda' \otimes^{\mathbb{L}}_\Lambda X$ belongs to $D^{\mathrm{p}}(\Lambda')$ and there is a natural isomorphism $\Lambda' \otimes_\Lambda \mathrm{Det}_\Lambda(X) \cong \mathrm{Det}_{\Lambda'}(\Lambda' \otimes^{\mathbb{L}}_\Lambda X)$ in $\mathcal{P}(\Lambda')$. In particular, if $\Lambda' \otimes^{\mathbb{L}}_\Lambda X$ is acyclic, then $\mathrm{Det}_{\Lambda'}(\Lambda' \otimes^{\mathbb{L}}_\Lambda X)$ identifies with the unit object $(\Lambda', 0)$ of $\mathcal{P}(\Lambda')$ and so there is a natural composite morphism

$$\iota_{X,\Lambda'} : \mathrm{Det}_\Lambda(X) \to \Lambda' \otimes_\Lambda \mathrm{Det}_\Lambda(X) \cong \mathrm{Det}_{\Lambda'}(\Lambda' \otimes^{\mathbb{L}}_\Lambda X) = (\Lambda', 0).$$

In any such case we will usually identify $\mathrm{Det}_\Lambda(X)$ with the invertible Λ-submodule of Λ' that is equal to the ungraded part of $\iota_{X,\Lambda'}(\mathrm{Det}_\Lambda(X))$. For example, if M is a finitely generated torsion Λ-module of projective dimension at most one and $Q(\Lambda)$ is the total quotient ring of Λ, then for any integer i the complex $M[i]$ is an object of $D^{\mathrm{p}}(\Lambda)$ such that $Q(\Lambda) \otimes_\Lambda M[i]$ is acyclic, the initial Fitting ideal $\mathrm{Fit}_\Lambda(M)$ of M is an invertible Λ-ideal and, regarded as a submodule of $Q(\Lambda)$ as above, the determinant $\mathrm{Det}_\Lambda(M[i])$ is equal to $\mathrm{Fit}_\Lambda(M)^{(-1)^{i+1}}$.

2. Annihilation results

In this section we shall prove a general annihilation result for the cohomology modules of a natural class of complexes over abelian group rings. We fix R, F, G and $\mathfrak{A}$ as in §1.1.

2.1. **Statement of the result.** For any $\mathfrak{A}$-module N we write N_{tor} for its R-torsion submodule and set $N_{\mathrm{tf}} := N/N_{\mathrm{tor}}$ which we regard as embedded in the associated space $F \otimes_R N$. We also set $N^* := \mathrm{Hom}_R(N, R)$ and $N^\vee := \mathrm{Hom}_R(N, F/R)$ and regard each as endowed with the action of $\mathfrak{A}$ given by $(a\theta)(n) = a(\theta(n))$ for all a in $\mathfrak{A}$ and n in N. (This is a well-defined action since $\mathfrak{A}$ is commutative but does not agree with the more usual contragredient action of groups rings on duals). In the sequel we shall often use, without further explicit comment, the fact that for this action one has $\mathrm{Ann}_{\mathfrak{A}}(N^\vee) = \mathrm{Ann}_{\mathfrak{A}}(N)$. We shall say that an R-algebra Λ is 'R-Gorenstein' if (it is R-torsion-free and) for all prime ideals $\wp$ of R the $R_\wp \otimes_R \Lambda$-module $R_\wp \otimes_R \Lambda^*$ is free of rank one. For example, since $\mathfrak{A}$ is a direct factor of $R[G]$, the natural isomorphism $R[G]^* \cong R[G]$ implies that $\mathfrak{A}$ is R-Gorenstein.
In the following result we use the notation $\mathfrak{A}_0(C)$, $A_0(C)$ and $I_{\mathfrak{A}_0(C)}$ defined in §1.1.

Theorem 2.1. *Let C be an admissible complex of $\mathfrak{A}$-modules and set $\mathfrak{A}_0 := \mathfrak{A}_0(C)$ and $A_0 := A_0(C)$. Then $C_0 := \mathfrak{A}_0 \otimes^{\mathbb{L}}_{\mathfrak{A}} C$ belongs to $D^{\mathrm{p}}(\mathfrak{A}_0)$ and $A_0 \otimes_{\mathfrak{A}_0} C_0$ is acyclic and so we may regard $\mathrm{Det}_{\mathfrak{A}_0}(C_0)$ as an invertible $\mathfrak{A}_0$-submodule of A_0 (cf. §1.2). With this identification one has*

$$(I_{\mathfrak{A}_0})^{g(C)+n(C)} \mathrm{Ann}_{\mathfrak{A}}(H^3(C)_{\mathrm{tor}})^{g(C)} \cdot \mathrm{Det}_{\mathfrak{A}_0}(C_0)^{-1} \subseteq \mathrm{Ann}_{\mathfrak{A}}(H^2(C)_{\mathrm{tor}}) \tag{1}$$

and

$$(I_{\mathfrak{A}_0})^{g(C^*)+n(C^*)} \mathrm{Ann}_{\mathfrak{A}}(H^2(C)_{\mathrm{tor}})^{g(C^*)} \cdot \mathrm{Det}_{\mathfrak{A}_0}(C_0) \subseteq \mathrm{Ann}_{\mathfrak{A}}(H^3(C)_{\mathrm{tor}}) \tag{2}$$

where $C^* := R\,\mathrm{Hom}_R(C, R[4])$ *and for any complex of* $\mathfrak{A}$*-modules* D *we write* $g(D)$ *for the minimal number of generators of the* $\mathfrak{A}$*-module* $H^3(D)$ *and define* $n(D) := 1$ *if* $H^1(D)$ *vanishes and* $n(D) := 2$ *otherwise. Further, if* $H^1(C)$ *vanishes and either* $\mathfrak{A}_0$ *is* R*-Gorenstein or* $H^2(C)$ *is* R*-torsion-free, then one has*

$$\mathrm{Fit}_{\mathfrak{A}}((H^2(C)_{\mathrm{tor}})^\vee)\cdot \mathrm{Det}_{\mathfrak{A}_0}(C_0) = \mathrm{Fit}_{\mathfrak{A}}(H^3(C)). \tag{3}$$

Remark 2.2. If $R = \mathbb{Z}_p$, then $\mathfrak{A}_0(C)$ is R-Gorenstein in, for example, both of the following cases:

(i) $e_0(C)$ belongs to $\mathbb{Q}_p[H]$ for any subgroup H of G whose Sylow p-subgroup is cyclic (in particular, this condition is automatically satisfied if $p \nmid |G|$);

(ii) $e_0(C)$ is a primitive idempotent of $\mathbb{Q}_p[G]$.

Remark 2.3. If $H^1(C) = 0$ and $H^a(C)$ is finite for both $a = 2$ and $a = 3$, then Theorem 2.1 can be applied with $e_0(C) = 1$. In this case $\mathfrak{A}_0(C) = \mathfrak{A}$ is R-Gorenstein and the equality (3) refines the main algebraic result (Theorem 2.4) of Snaith in [28].

Remark 2.4. There is a natural generalisation of Theorem 2.1 dealing with the subcategory of $D^{\mathrm{p}}(\mathfrak{A})$ comprising complexes which satisfy the assumptions (ad_1), (ad_2) and (ad_3) but for which (ad_4) is replaced by the weaker assumption that the $\mathfrak{A}$-module $H^1(C)_{\mathrm{tor}}$ has finite projective dimension. For details see [3].

2.2. **The proof.** We assume the notation and hypotheses of Theorem 2.1 and, regarding the complex C as fixed, for each integer a we also set $\tilde{H}^a := H^a(C)$ and $\tilde{H}^a_0 := H^a(C_0)$.

At the outset we note that, since C belongs to $D^{\mathrm{p}}(\mathfrak{A})$ by assumption (ad_1), it is clear that C_0 belongs to $D^{\mathrm{p}}(\mathfrak{A}_0)$. Further in each degree a the A_0-module $H^a(A_0 \otimes_{\mathfrak{A}_0} C_0)$ is isomorphic to $H^a(A_0 \otimes_A (A \otimes_{\mathfrak{A}} C)) \cong e_0(H^a(A \otimes_{\mathfrak{A}} C))$ and so Remark 1.1 implies that $A_0 \otimes_{\mathfrak{A}_0} C_0$ is acyclic.

Since all remaining assertions in Theorem 2.1 can be checked after completion at each prime ideal of R, in the rest of this section we will assume that R is a complete discrete valuation ring of characteristic 0.

The main ingredient in the proof of Theorem 2.1 is provided by the following result.

Proposition 2.5.

(i) *One has* $\mathrm{Ann}_{\mathfrak{A}_0}(\tilde{H}^3_0)^{g(C)} \cdot \mathrm{Det}_{\mathfrak{A}_0}(C_0)^{-1} \subseteq \mathrm{Ann}_{\mathfrak{A}_0}(\tilde{H}^2_0)$.

(ii) *If either* $\mathfrak{A}_0$ *is* R*-Gorenstein or* $\tilde{H}^2_0$ *vanishes, then also*

$$\mathrm{Fit}_{\mathfrak{A}_0}(\tilde{H}^3_0)\mathrm{Det}_{\mathfrak{A}_0}(C_0)^{-1} = \mathrm{Fit}_{\mathfrak{A}_0}((\tilde{H}^2_0)^\vee).$$

Proof. The key to our proof of claim (i) is to choose a representative of C_0 as described in the next result.

Lemma 2.6.

(i) C_0 *is isomorphic in* $D^{\mathrm{p}}(\mathfrak{A}_0)$ *to a complex of finitely generated free* $\mathfrak{A}_0$*-modules* $F_0^\bullet$ *of the form* $F_0^1 \to F_0^2 \to F_0^3$ *in which* F_0^1 *occurs in degree* 1 *and the rank of* F_0^3 *is equal to* $g(C)$.

(ii) C_0 *is acyclic outside degrees* 2 *and* 3 *and the modules* $\tilde{H}^a_0$ *are finite for* $a = 2, 3$.

Proof. Since C is both perfect (by assumption (ad_1)) and acyclic outside degrees $1, 2$ and 3 (by assumption (ad_3)) a standard argument shows that it is isomorphic in

$D^{\rm p}(\mathfrak{A})$ to a complex of the form $F^\bullet : F^1 \xrightarrow{d^1} F^2 \xrightarrow{d^2} F^3$ where F^1 occurs in degree 1, F^2 and F^3 are finitely generated free $\mathfrak{A}$-modules with $\mathrm{rk}_{\mathfrak{A}}(F^3)$ equal to the minimal number $g(C)$ of generators of $H^3(C)$ as an $\mathfrak{A}$-module, and F^1 is a finitely generated $\mathfrak{A}$-module that has finite projective dimension (cf. [SGA4$\frac{1}{2}$, Rapport, Lem. 4.7]). Now $\ker(d^1) \cong H^1(C)$ is R-free (by assumption (ad$_4$)) and $\mathrm{im}(d^1)$ is a submodule of the free R-module F^2 and so the tautological exact sequence $0 \to \ker(d^1) \to F^1 \to \mathrm{im}(d^1) \to 0$ implies that F^1 is R-free. It follows that F^1 is a projective $\mathfrak{A}$-module since any finitely generated $\mathfrak{A}$-module that is both R-free and of finite projective dimension is projective (cf. [1, Th. 8]). In addition, since $A \otimes_{\mathfrak{A}} F^2$ and $A \otimes_{\mathfrak{A}} F^3$ are both free A-modules and the Euler characteristic of $A \otimes_{\mathfrak{A}} F^\bullet$ in $K_0(A)$ vanishes (by assumption (ad$_2$)) the A-module $A \otimes_{\mathfrak{A}} F^1$ is also free. Hence, since $\mathfrak{A}$ is a product of local rings, this implies that F^1 is a (finitely generated) free $\mathfrak{A}$-module. Thus, since $C_0 = \mathfrak{A}_0 \otimes^{\mathbb{L}}_{\mathfrak{A}} C$ is isomorphic in $D^{\rm p}(\mathfrak{A}_0)$ to the complex $F_0^\bullet := \mathfrak{A}_0 \otimes_{\mathfrak{A}} F^\bullet$, we have now proved claim (i). Regarding claim (ii), the description of C_0 given in claim (i) makes it clear that C_0 is acyclic outside degrees $1, 2$ and 3 and that $\tilde{H}^3_0$ is canonically isomorphic to $\mathrm{cok}(\mathfrak{A}_0 \otimes_{\mathfrak{A}} d^2) \cong \mathfrak{A}_0 \otimes_{\mathfrak{A}} \mathrm{cok}(d^2) \cong \mathfrak{A}_0 \otimes_{\mathfrak{A}} \tilde{H}^3$. To compute explicitly the groups $\tilde{H}^1_0$ and $\tilde{H}^2_0$ we proceed in the following way. For any $\mathfrak{A}_0$-submodule I of A and $R[G]$-module M we endow the tensor product $I \otimes_R M$ with the action of $\mathfrak{A}_0 \times R[G]$ under which each a_0 in $\mathfrak{A}_0$ acts as $i \otimes_R m \mapsto a_0 i \otimes_R m$ and each g in G as $i \otimes_R m \mapsto ig^{-1} \otimes_R g(m)$. In particular, since $\mathfrak{A}_0$ is R-free, for any projective $\mathfrak{A}$-module Q the tensor product $\mathfrak{A}_0 \otimes_R Q$ is a cohomologically-trivial G-module and so the map $a \otimes_R q \mapsto \sum_{g \in G} g(a \otimes_R q)$ induces an isomorphism of $\mathfrak{A}_0$-modules $\mathfrak{A}_0 \otimes_{\mathfrak{A}} Q = H_0(G, \mathfrak{A}_0 \otimes_R Q) \cong H^0(G, \mathfrak{A}_0 \otimes_R Q)$. Such isomorphisms give rise to a convergent cohomological spectral sequence of the form $H^b(G, \mathfrak{A}_0 \otimes_R \tilde{H}^a) \Rightarrow \tilde{H}^{b+a}_0$. Since $\tilde{H}^a$ vanishes for $a < 1$ (by assumption (ad$_3$)) this spectral sequence induces a canonical isomorphism of $\mathfrak{A}_0$-modules $\tilde{H}^1_0 \cong H^0(G, \mathfrak{A}_0 \otimes_R \tilde{H}^1)$ and also a canonical exact sequence (of low degree terms)

$$H^1(G, \mathfrak{A}_0 \otimes_R \tilde{H}^1) \to \tilde{H}^2_0 \to H^0(G, \mathfrak{A}_0 \otimes_R \tilde{H}^2) \to H^2(G, \mathfrak{A}_0 \otimes_R \tilde{H}^1). \tag{4}$$

Now in each degree a the space $F \otimes_R H^0(G, \mathfrak{A}_0 \otimes_R \tilde{H}^a) \cong F \otimes_R H_0(G, \mathfrak{A}_0 \otimes_R \tilde{H}^a) = F \otimes_R (\mathfrak{A}_0 \otimes_{\mathfrak{A}} \tilde{H}^a)$ is isomorphic to $A_0 \otimes_A (H^a(A \otimes_{\mathfrak{A}} C)) \cong e_0(H^a(A \otimes_{\mathfrak{A}} C))$ and so vanishes by Remark 1.1. The modules $H^0(G, \mathfrak{A}_0 \otimes_R \tilde{H}^a)$ and $H_0(G, \mathfrak{A}_0 \otimes_R \tilde{H}^a) = \mathfrak{A}_0 \otimes_{\mathfrak{A}} \tilde{H}^a$ are therefore finite. This implies in particular that $\tilde{H}^1_0 \cong H^0(G, \mathfrak{A}_0 \otimes_R \tilde{H}^1)$ and $\tilde{H}^3_0 \cong \mathfrak{A}_0 \otimes_{\mathfrak{A}} \tilde{H}^a$ are finite and, since the first term in the exact sequence (4) is obviously finite, also that $\tilde{H}^2_0$ is finite. Finally we note that $\tilde{H}^1_0 \cong H^0(G, \mathfrak{A}_0 \otimes_R \tilde{H}^1)$ is a submodule of $\mathfrak{A}_0 \otimes_R \tilde{H}^1$ which is R-free (by assumption (ad$_4$) and the fact that $\mathfrak{A}_0$ is R-free), and hence that in fact the finite module $\tilde{H}^1_0$ vanishes. □

Returning to the proof of Proposition 2.5(i), Lemma 2.6(i) allows us to replace C_0 by the complex $F_0^\bullet$. Having made this change, the proof of Proposition 2.5(i) then proceeds exactly as with the proof of the main algebraic result (Theorem 2.4) of Snaith in [28]. To explain this we write d^1_0 for the differential of $F_0^\bullet$ in degree 1 and B^2_0 for the group $\mathrm{im}(d^1_0)$ of coboundaries of $F^\bullet_0$ in degree 2. Then, since the group $\ker(d^1_0) = H^1(F^\bullet_0) \cong H^1(C_0)$ vanishes (by Lemma 2.6(ii)), there exists a homomorphism of $\mathfrak{A}_0$-modules $\eta_2 : B^2_0 \cong F^1_0$ such that $d^1_0 \circ \eta_2 = \mathrm{id}_{B^2_0}$. Given the existence of η_2, and modulo the obvious translation between cohomology and homology complexes, the inclusion of Proposition 2.5(i) is proved by exactly mimicking the arguments of [28, §2.5, Lem. 2.6, Prop. 2.7] with $F_0^\bullet$ now playing the role of the complex which occurs in the second line of [28, Th. 2.4].

Turning to claim (ii) of Proposition 2.5 we observe that, since C_0 belongs to $D^{\rm p}(\mathfrak{A}_0)$, is acyclic outside degrees 2 and 3 and such that $H^a(C_0)$ is finite for both $a = 2, 3$ (by Lemma 2.6), one can choose an exact sequence of $\mathfrak{A}_0$-modules

$$0 \to \tilde{H}_0^2 \to Q \xrightarrow{d} Q' \to \tilde{H}_0^3 \to 0 \tag{5}$$

which is such that both Q and Q' are finite and of projective dimension at most one and there exists an isomorphism ι in $D^{\rm p}(\mathfrak{A}_0)$ between C_0 and the complex $Q \xrightarrow{d} Q'$ (where the modules are placed in degrees 2 and 3, and the cohomology is identified with $\tilde{H}_0^2$ and $\tilde{H}_0^3$ by using the maps in (5)) for which $H^i(\iota)$ is the identity map in each degree i. This implies that $\mathrm{Fit}_{\mathfrak{A}_0}(Q)$ and $\mathrm{Fit}_{\mathfrak{A}_0}(Q')$ are invertible ideals of $\mathfrak{A}_0$ and that

$$\mathrm{Det}_{\mathfrak{A}_0}(C_0)^{-1} = \mathrm{Fit}_{\mathfrak{A}_0}(Q)\,\mathrm{Fit}_{\mathfrak{A}_0}(Q')^{-1}. \tag{6}$$

Thus, if $\mathfrak{A}_0$ is R-Gorenstein, then Proposition 2.5(ii) follows from the equality $\mathrm{Fit}_{\mathfrak{A}_0}(Q)\,\mathrm{Fit}_{\mathfrak{A}_0}(Q')^{-1} = \mathrm{Fit}_{\mathfrak{A}_0}((\tilde{H}_0^2)^\vee)\mathrm{Fit}_{\mathfrak{A}_0}(\tilde{H}_0^3)^{-1}$ which results from applying [8, Lem. 5] to the sequence (5). On the other hand, if $\tilde{H}_0^2 = 0$, then (5) implies that the projective dimension of the $\mathfrak{A}_0$-module $\tilde{H}_0^3$ is at most one and hence that one can take $Q = 0$ and $Q' = \tilde{H}_0^3$ in which case the equality of Proposition 2.5(ii) follows directly from (6). This completes the proof of Proposition 2.5. □

We now return to the proof of Theorem 2.1. We first observe that the inclusion (1) is obtained by simply substituting the inclusions of the next result into that of Proposition 2.5(i).

Lemma 2.7.

(i) $I_{\mathfrak{A}_0} \cdot \mathrm{Ann}_{\mathfrak{A}}(\tilde{H}^3_{\rm tor}) \subseteq \mathrm{Ann}_{\mathfrak{A}}(\tilde{H}^3_0)$.

(ii) $(I_{\mathfrak{A}_0})^{n(C)} \cdot \mathrm{Ann}_{\mathfrak{A}}(\tilde{H}^2_0) \subseteq \mathrm{Ann}_{\mathfrak{A}}(\tilde{H}^2_{\rm tor})$ *where $n(C)$ is the integer defined in Theorem 2.1.*

Proof. From Remark 1.1 we know that the module $\tilde{H}^3_{\rm tf} \subset H^3(A \otimes_{\mathfrak{A}} C)$ is annihilated by e_0 and hence that the module $\mathfrak{A}_0 \otimes_{\mathfrak{A}} \tilde{H}^3_{\rm tf}$ is annihilated by left multiplication by $\mathfrak{A} \cap \mathfrak{A}_0 = I_{\mathfrak{A}_0}$. Claim (i) is therefore an easy consequence of the isomorphism $\tilde{H}^0_3 \cong \mathfrak{A}_0 \otimes_{\mathfrak{A}} \tilde{H}^3$ and exact sequence $\mathfrak{A}_0 \otimes_{\mathfrak{A}} \tilde{H}^3_{\rm tor} \to \mathfrak{A}_0 \otimes_{\mathfrak{A}} \tilde{H}^3 \to \mathfrak{A}_0 \otimes_{\mathfrak{A}} \tilde{H}^3_{\rm tf} \to 0$.

For any $\mathfrak{A}$-submodule I of A and $\mathfrak{A}$-module M we regard the tensor product $I \otimes_R M$ as a module over $\mathfrak{A} \times R[G]$ in the same way as in the proof of Lemma 2.6(ii). Then we claim that in each degree a the Tate cohomology group $\hat{H}^a(G, \mathfrak{A}_0 \otimes_R M)$ is an $\mathfrak{A}$-module that is annihilated by $I_{\mathfrak{A}_0}$. To show this we write α for the projection $\mathfrak{A} \to \mathfrak{A}_0$ and note that, since $\mathfrak{A}_0$ is R-free, there is a natural exact sequence of $\mathfrak{A} \times R[G]$-modules $0 \to \mathrm{Ker}(\alpha) \otimes_R M \to \mathfrak{A} \otimes_R M \to \mathfrak{A}_0 \otimes_R M \to 0$. Now $\mathfrak{A} \otimes_R M$ is a cohomologically-trivial G-module since $\mathfrak{A}$ is a direct factor of $R[G]$ and so this sequence induces an isomorphism of $\mathfrak{A}$-modules $\hat{H}^a(G, \mathfrak{A}_0 \otimes_R M) \cong \hat{H}^{a+1}(G, \mathrm{Ker}(\alpha) \otimes_R M)$ in each degree a. Finally we note that the latter module is annihilated by $I_{\mathfrak{A}_0}$ as $I_{\mathfrak{A}_0} \cdot \mathrm{Ker}(\alpha) = 0$.

The fact that $H^2(G, \mathfrak{A}_0 \otimes_R \tilde{H}^1)$ is annihilated by $I_{\mathfrak{A}_0}$ combines with the exact sequence (4) to imply that there is an inclusion $I_{\mathfrak{A}_0} \cdot \mathrm{Ann}_{\mathfrak{A}}(\tilde{H}^2_0) \subseteq \mathrm{Ann}_{\mathfrak{A}}(H^0(G, \mathfrak{A}_0 \otimes_R \tilde{H}^2))$, resp. an equality $\mathrm{Ann}_{\mathfrak{A}}(\tilde{H}^2_0) = \mathrm{Ann}_{\mathfrak{A}}(H^0(G, \mathfrak{A}_0 \otimes_R \tilde{H}^2))$ if $\tilde{H}^1 = 0$. Now, as observed in the proof of Lemma 2.6(ii), the module $H^0(G, \mathfrak{A}_0 \otimes_R \tilde{H}^2)$ is finite and hence equal to $H^0(G, (\mathfrak{A}_0 \otimes_R \tilde{H}^2)_{\rm tor}) = H^0(G, \mathfrak{A}_0 \otimes_R \tilde{H}^2_{\rm tor})$. Further, since $\mathfrak{A}$ is a direct factor of $R[G]$, there is a natural isomorphism of $\mathfrak{A}$-modules $\tilde{H}^2_{\rm tor} \cong H^0(G, \mathfrak{A} \otimes_R \tilde{H}^2_{\rm tor})$. The natural exact sequence

$$0 \to H^0(G, \mathrm{Ker}(\alpha) \otimes_R \tilde{H}^2_{\rm tor}) \to H^0(G, \mathfrak{A} \otimes_R \tilde{H}^2_{\rm tor}) \to H^0(G, \mathfrak{A}_0 \otimes_R \tilde{H}^2_{\rm tor})$$

therefore combines with the fact that $I_{\mathfrak{A}_0}$ annihilates $H^0(G, \mathrm{Ker}(\alpha) \otimes_R \tilde{H}^2_{\rm tor})$ to imply that $I_{\mathfrak{A}_0} \cdot \mathrm{Ann}_{\mathfrak{A}}(H^0(G, \mathfrak{A}_0 \otimes_R \tilde{H}^2)) \subseteq \mathrm{Ann}_{\mathfrak{A}}(H^0(G, \mathfrak{A} \otimes_R \tilde{H}^2_{\rm tor})) = \mathrm{Ann}_{\mathfrak{A}}(\tilde{H}^2_{\rm tor})$. After putting everything together, and recalling the definition of the integer $n(C)$, we obtain claim (ii). □

To prove the inclusion (2) of Theorem 2.1 we note first that $C^* = R\,\mathrm{Hom}_R(C, R[-4])$ belongs to $D^{\rm p,ad}(\mathfrak{A})$. Indeed, since $\mathfrak{A}$ is R-Gorenstein, if P is a finitely generated projective, resp. free, $\mathfrak{A}$-module, then P^* is also projective, resp. free, and this implies that C^* satisfies the assumptions (ad$_1$) and (ad$_2$). Further, by computing the cohomology of C^* via the natural spectral sequence $\mathrm{Ext}^a_R(H^b(C), R) \Rightarrow H^{a-b+4}(C^*)$ one finds that C^* also satisfies the assumptions (ad$_3$) and (ad$_4$) (so C^* is an admissible complex of $\mathfrak{A}$-modules) and indeed that there are natural isomorphisms of $\mathfrak{A}$-modules $H^1(C^*) \cong (\tilde{H}^3)^*, H^2(C^*)_{\rm tor} \cong (\tilde{H}^3_{\rm tor})^\vee, H^2(C^*)_{\rm tf} \cong (\tilde{H}^2)^*$, $H^3(C^*)_{\rm tor} \cong (\tilde{H}^2_{\rm tor})^\vee$ and $H^3(C^*)_{\rm tf} \cong (\tilde{H}^1)^*$. In particular, since $\mathrm{Ann}_A(A \otimes_{\mathfrak{A}} M) = \mathrm{Ann}_A(A \otimes_{\mathfrak{A}} M^*)$ for any $\mathfrak{A}$-module M, the sum $e_0(C^*)$ of all primitive idempotents of A that annihilate $H^2(A \otimes_{\mathfrak{A}} C^*) \cong A \otimes_{\mathfrak{A}} H^2(C^*)_{\rm tf} \cong A \otimes_{\mathfrak{A}} (\tilde{H}^2)^*$ is equal to e_0. Further, under the identifications described in §1.2, the sublattice $\mathrm{Det}_{\mathfrak{A}_0}(\mathfrak{A}_0 \otimes^{\mathbb{L}}_{\mathfrak{A}} C^*)$ of A_0 is equal to $\mathrm{Det}_{\mathfrak{A}_0}(C_0)^{-1}$ and so the inclusion (2) follows directly from (1) upon replacing C by C^* and using the equalities $\mathrm{Ann}_{\mathfrak{A}}(H^3(C^*)_{\rm tor}) = \mathrm{Ann}_{\mathfrak{A}}((\tilde{H}^2_{\rm tor})^\vee) = \mathrm{Ann}_{\mathfrak{A}}(\tilde{H}^2_{\rm tor})$ and $\mathrm{Ann}_{\mathfrak{A}}(H^2(C^*)_{\rm tor}) = \mathrm{Ann}_{\mathfrak{A}}((\tilde{H}^3_{\rm tor})^\vee) = \mathrm{Ann}_{\mathfrak{A}}(\tilde{H}^3_{\rm tor})$.

To complete the proof of Theorem 2.1 we now need only note that the equality (3) is obtained by substituting the next result into the equality of Proposition 2.5(ii).

We fix an algebraic closure F^c of F and for each homomorphism $\rho : G \to F^{c\times}$ we write e_ρ for the associated idempotent $\frac{1}{|G|}\sum_{g\in G} \rho(g^{-1})g$ of $F^c[G]$.

Lemma 2.8. *Assume $\tilde{H}^1$ vanishes.*

(i) *Then* $\mathrm{Fit}_{\mathfrak{A}_0}(\tilde{H}^3_0) = \mathrm{Fit}_{\mathfrak{A}}(\tilde{H}^3)$.

(ii) *Further, if either $\mathfrak{A}_0$ is R-Gorenstein or $\tilde{H}^2$ is R-free, then* $\mathrm{Fit}_{\mathfrak{A}_0}((\tilde{H}^2_0)^\vee) = \mathrm{Fit}_{\mathfrak{A}}((\tilde{H}^2_{\rm tor})^\vee)e_0$.

Proof. The isomorphism $\tilde{H}^3_0 \cong \mathfrak{A}_0 \otimes_{\mathfrak{A}} \tilde{H}^3$ discussed in the proof of Lemma 2.6(ii) implies $\mathrm{Fit}_{\mathfrak{A}_0}(\tilde{H}^3_0) = \mathrm{Fit}_{\mathfrak{A}}(\tilde{H}^3)e_0$. Thus, as $1 = e_0 + (1 - e_0)$, to prove claim (i) it suffices to prove that $\mathrm{Fit}_{\mathfrak{A}}(\tilde{H}^3)(1 - e_0)$ vanishes, or equivalently that in $F^c \otimes_F A$ one has $\mathrm{Fit}_{\mathfrak{A}}(\tilde{H}^3)e_\rho = 0$ for each ρ in $\mathrm{Hom}(G, F^{c\times})$ with $e_\rho e_0 = 0$. To do this we follow an argument used in the proof of [6, Th. 8.2(ii)]. We thus fix such a ρ, set $\mathfrak{A}_\rho := (R_\rho \otimes_R \mathfrak{A})e_\rho$ with R_ρ the ring generated over R by the values of ρ, and choose a resolution of the $\mathfrak{A}$-module $\tilde{H}^3$ of the form $\mathfrak{A}^m \xrightarrow{\theta} \mathfrak{A}^n \to \tilde{H}^3 \to 0$. This sequence induces an exact sequence

$$\mathfrak{A}^m_\rho \xrightarrow{\theta_\rho} \mathfrak{A}^n_\rho \to \tilde{H}^3_\rho \to 0,$$

where $\theta_\rho = \mathfrak{A}_\rho \otimes_{\mathfrak{A}} \theta$ and $\tilde{H}^3_\rho := \mathfrak{A}_\rho \otimes_{\mathfrak{A}} \tilde{H}^3$, and this sequence in turn implies that the ideal $\mathrm{Fit}_{\mathfrak{A}}(\tilde{H}^3)\cdot \mathfrak{A}_\rho = \mathrm{Fit}_{\mathfrak{A}_\rho}(\tilde{H}^3_\rho)$ is equal to the image I_{θ_ρ} of $\wedge^n_{\mathfrak{A}_\rho}\theta_\rho$ in $\wedge^n_{\mathfrak{A}_\rho}(\mathfrak{A}^n_\rho) \cong \mathfrak{A}_\rho$. Now, since $\tilde{H}^1$ vanishes, Remark 1.1 implies that the A-modules $A\otimes_{\mathfrak{A}}\tilde{H}^2$ and $A\otimes_{\mathfrak{A}}\tilde{H}^3$ are isomorphic and hence that, since $e_\rho e_0 = 0$, the $\mathfrak{A}_\rho$-rank of $\tilde{H}^3_\rho$ is at least 1. The last displayed sequence therefore implies that the $\mathfrak{A}_\rho$-rank of $\mathrm{im}(\theta_\rho)$ is at most $n-1$. Since $\wedge^n_{\mathfrak{A}_\rho}\mathfrak{A}^n_\rho \cong \mathfrak{A}_\rho$ is R_ρ-free, this in turn implies that $I_{\theta_\rho} = 0$, as required.

To prove claim (ii) we first note that, as $\tilde{H}^1$ vanishes, the exact sequence (4) gives an isomorphism of finite $\mathfrak{A}_0$-modules $\tilde{H}^2_0 \cong H^0(G, \mathfrak{A}_0 \otimes_R \tilde{H}^2) = H^0(G, \mathfrak{A}_0 \otimes_R \tilde{H}^2_{\mathrm{tor}})$ and hence also $(\tilde{H}^2_0)^\vee \cong H_0(G, (\mathfrak{A}_0 \otimes_R \tilde{H}^2_{\mathrm{tor}})^\vee)$. Thus, if $\tilde{H}^2$ is R-free, then both $\tilde{H}^2_{\mathrm{tor}}$ and $\tilde{H}^2_0$ vanish so the equality of claim (ii) is clear. To deal with the case that $\mathfrak{A}_0$ is R-Gorenstein we use an observation of Parker [25, Lem. 4.3.6]. Indeed, since $\mathfrak{A}_0$ is also R-free, in this case there are isomorphisms of $\mathfrak{A}_0 \times R[G]$-modules of the form $(\mathfrak{A}_0 \otimes_R \tilde{H}^2_{\mathrm{tor}})^\vee \cong \mathfrak{A}_0^* \otimes_R (\tilde{H}^2_{\mathrm{tor}})^\vee \cong \mathfrak{A}_0 \otimes_R (\tilde{H}^2_{\mathrm{tor}})^\vee$ so that the $\mathfrak{A}_0$-module $(\tilde{H}^2_0)^\vee$ is isomorphic to $H_0(G, \mathfrak{A}_0 \otimes_R (\tilde{H}^2_{\mathrm{tor}})^\vee) \cong \mathfrak{A}_0 \otimes_{\mathfrak{A}} (\tilde{H}^2_{\mathrm{tor}})^\vee$. Thus one has $\mathrm{Fit}_{\mathfrak{A}_0}((\tilde{H}^2_0)^\vee) = \mathrm{Fit}_{\mathfrak{A}_0}(\mathfrak{A}_0 \otimes_{\mathfrak{A}} (\tilde{H}^2_{\mathrm{tor}})^\vee) = \mathrm{Fit}_{\mathfrak{A}}((\tilde{H}^2_{\mathrm{tor}})^\vee)e_0$, as required. □

3. Compact support cohomology

In this section we apply Theorem 2.1 in the context of the compactly supported étale cohomology of certain sheaves. To do this we let k be a number field, p an *odd* prime and Σ a finite set of places of k containing all archimedean places and all places above p. We also fix a $\mathbb{Z}_p$-order $\mathfrak{A}$ and an étale (pro-)sheaf of projective $\mathfrak{A}$-modules T on $\mathrm{Spec}(\mathcal{O}_{k,\Sigma})$. Fixing an algebraic closure k^c of k and writing $G_{k,\Sigma}$ for the Galois group over k of the maximal extension of k inside k^c that is unramified outside Σ we may regard T as a projective $\mathfrak{A}$-module that is endowed with a continuous action of $G_{k,\Sigma}$ that commutes with the given action of $\mathfrak{A}$. We set $T^*(1) := \mathrm{Hom}_{\mathbb{Z}_p}(T, \mathbb{Z}_p(1))$, regarded as endowed with the following commuting actions of $\mathfrak{A}$ and $G_{k,\Sigma}$: for each $a \in \mathfrak{A}$, $\sigma \in G_{k,\Sigma}$, $f \in T^*(1)$ and $t \in T$ one has $a(f)(t) = f(a(t))$ and $\sigma(f)(t) = \sigma(f(\sigma^{-1}(t)))$. We also set $V := \mathbb{Q}_p \otimes_{\mathbb{Z}_p} T$, $V^*(1) = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} T^*(1)$, $W := V/T$ and $W^*(1) := V^*(1)/T^*(1) \cong \mathrm{Hom}_{\mathbb{Z}_p}(T, (\mathbb{Q}_p/\mathbb{Z}_p)(1))$, each endowed with the actions of $\mathfrak{A}$ and $G_{k,\Sigma}$ that are induced from the respective actions on T and $T^*(1)$.

If R denotes either $\mathcal{O}_{k,\Sigma}, k$ or k_v for some place v and $\mathcal{F}$ is an étale (pro-)sheaf on $\mathrm{Spec}(R)$, then we abbreviate the complex $R\Gamma_{\text{ét}}(\mathrm{Spec}(R), \mathcal{F})$ and in each degree a the group $H^a_{\text{ét}}(\mathrm{Spec}(R), \mathcal{F})$ to $R\Gamma(R, \mathcal{F})$ and $H^a(R, \mathcal{F})$ respectively. For any étale (pro-)sheaf $\mathcal{F}$ on $\mathrm{Spec}(\mathcal{O}_{k,\Sigma})$ we also define the compact support cohomology complex $R\Gamma_c(\mathcal{O}_{k,\Sigma}, \mathcal{F})$ by means of the exact triangle

$$R\Gamma_c(\mathcal{O}_{k,\Sigma}, \mathcal{F}) \to R\Gamma(\mathcal{O}_{k,\Sigma}, \mathcal{F}) \to \bigoplus_{v\in\Sigma} R\Gamma(k_v, \mathcal{F}) \to R\Gamma_c(\mathcal{O}_{k,\Sigma}, \mathcal{F})[1] \tag{7}$$

where the second arrow is the direct sum of the natural localisation morphisms. For each integer a we then set $H^a_c(\mathcal{O}_{k,\Sigma}, \mathcal{F}) := H^a(R\Gamma_c(\mathcal{O}_{k,\Sigma}, \mathcal{F}))$. Note that this definition of cohomology with compact support differs from that given in [21, Chap. II] since Milne uses Tate cohomology at each archimedean place. Nevertheless in each degree $a \geq 2$ the group $H^a_c(\mathcal{O}_{k,\Sigma}, \mathcal{F})$ defined here agrees with that defined by Milne. In particular, if T is as above, then by taking the inverse limit over n of the global duality isomorphism of [21, Chap. II, Cor. 3.3] for the sheaf $F = T/p^n$, one obtains a canonical isomorphism of $\mathfrak{A}$-modules

$$H^a_c(\mathcal{O}_{k,\Sigma},T) \cong H^{3-a}(\mathcal{O}_{k,\Sigma},W^*(1))^\vee \tag{8}$$

for $a \in \{2,3\}$.
For any abelian group A we write $A_{\rm cotor}$ for the quotient of A by its maximal divisible subgroup. We recall that if A is a torsion group, then $(A_{\rm cotor})^\vee$ is naturally isomorphic to $(A^\vee)_{\rm tor}$.

3.1. **The complexes.** With $\mathcal{F}$ denoting either T, V or W as above, for each non-archimedean place v of k we write $H^1_f(k_v,\mathcal{F})$ for the finite support cohomology group that is defined by Bloch and Kato in [5]. We recall in particular that $H^1_f(k_v,V)$ is a subspace of $H^1(k_v,V)$ and that $H^1_f(k_v,T)$, resp. $H^1_f(k_v,W)$, is defined to be the pre-image, resp. image, of $H^1_f(k_v,V)$, under the natural map $H^1(k_v,T)\to H^1(k_v,V)$, resp. $H^1(k_v,V)\to H^1(k_v,W)$. If v is archimedean then, since p is odd, one has $H^1(k_v,\mathcal{F})=0$ and so we set $H^1_f(k_v,\mathcal{F})=0$. Then the global finite support cohomology group $H^1_f(k,\mathcal{F})$ of Bloch and Kato is defined by the natural exact sequence

$$0\to H^1_f(k,\mathcal{F}) \xrightarrow{\subseteq} H^1(\mathcal{O}_{k,\Sigma},\mathcal{F}) \to \bigoplus_{v\in\Sigma}\frac{H^1(k_v,\mathcal{F})}{H^1_f(k_v,\mathcal{F})}, \tag{9}$$

and hence there is an induced localisation map

$$\lambda_\mathcal{F}: H^1_f(k,\mathcal{F}) \to \bigoplus_{v\in\Sigma} H^1_f(k_v,\mathcal{F}).$$

We also recall that Bloch and Kato [5] define the Selmer group $\mathrm{Sel}(T)$ and Tate-Shafarevic group $\text{Ш}(T)$ of T to be equal to $H^1_f(k,W)$ and the cokernel of the natural homomorphism $H^1_f(k,V)\to H^1_f(k,W)=\mathrm{Sel}(T)$ respectively.

Lemma 3.1. *Set $C_T := R\Gamma_c(\mathcal{O}_{k,\Sigma},T)$, $\mathfrak{A}_0 := \mathfrak{A}_0(C_T)$ and $I_T := I_{\mathfrak{A}_0}$.*

(i) *Then C_T is an object of $D^{\rm p,ad}(\mathfrak{A})$.*
(ii) *One has $\mathrm{Ann}_\mathfrak{A}(H^3(C_T)_{\rm tor}) = \mathrm{Ann}_\mathfrak{A}(H^0(k,W^*(1))_{\rm cotor})$.*
(iii) *If $\mathrm{cok}(\lambda_T)$, resp. $\mathrm{cok}(\lambda_{T^*(1)})$, is finite, then $\text{Ш}(T)$ is annihilated by the ideal $\mathrm{Ann}_\mathfrak{A}(H^2(C_T)_{\rm tor})$, resp. $\mathrm{Ann}_\mathfrak{A}(H^2(C_{T^*(1)})_{\rm tor})$.*
(iv) *In all cases the module $\mathrm{Sel}(T^*(1))$, and hence also $\text{Ш}(T)$, is annihilated by the ideal $I_T\cdot\mathrm{Ann}_\mathfrak{A}(H^2(C_T)_{\rm tor})$.*

Proof. It is well known that C_T satisfies the assumptions (ad$_1$), (ad$_2$) and (ad$_3$) of §1.1 (see, for example, [14, 1.6.5 and 2.1.3]). Further, the long exact cohomology sequence of the natural exact triangle

$$C_T \to R\Gamma_c(\mathcal{O}_{k,\Sigma},V)\to R\Gamma_c(\mathcal{O}_{k,\Sigma},W)\to C_T[1]$$

identifies $H^1(C_T)_{\rm tor}$ with $H^0_c(\mathcal{O}_{k,\Sigma},W)_{\rm cotor}$ and the latter group vanishes because (the long exact cohomology sequence of (7) implies that) $H^0_c(\mathcal{O}_{k,\Sigma},\mathcal{F})$ vanishes for all sheaves $\mathcal{F}$. It follows that C_T also satisfies the assumption (ad$_4$) and hence belongs to $D^{\rm p,ad}(\mathfrak{A})$. This proves claim (i).
The duality isomorphism (8) with $a=3$ restricts to give an isomorphism $H^3(C_T)_{\rm tor}\cong (H^0(\mathcal{O}_{k,\Sigma},W^*(1))^\vee)_{\rm tor} \cong (H^0(\mathcal{O}_{k,\Sigma},W^*(1))_{\rm cotor})^\vee$. Claim (ii) follows immediately from the latter isomorphism and the equality $H^0(\mathcal{O}_{k,\Sigma},W^*(1)) = H^0(k,W^*(1))$.

By comparing (9) to the long exact cohomology sequence of (7) with $\mathcal{F} = T$ one finds that the localisation homomorphism λ_T fits into a natural exact sequence

$$H^1_f(k, T) \xrightarrow{\lambda_T} \bigoplus_{v\in\Sigma} H^1_f(k_v, T) \xrightarrow{\tilde\lambda_T} H^2(C_T) \to \mathrm{cok}(\tilde\lambda_T) \to 0. \tag{10}$$

Now for each place v in Σ, the Pontryagin dual of the tautological exact sequence

$$0 \to H^1_f(k_v, T) \xrightarrow{\subseteq} H^1(k_v, T) \to \frac{H^1(k_v, V)}{H^1_f(k_v, V)}$$

combines with the local duality isomorphism $H^1(k_v, T)^\vee \cong H^1(k_v, W^*(1))$ and the definition of $H^1_f(k_v, W^*(1))$ to imply that $H^1_f(k_v, T)^\vee$ is naturally isomorphic to the quotient $H^1(k_v, W^*(1))/H^1_f(k_v, W^*(1))$. Hence, upon taking the Pontryagin dual of (10), and using the global duality isomorphism (8) with $a = 2$, one obtains an exact sequence

$$0 \to \mathrm{cok}(\tilde\lambda_T)^\vee \to H^1(\mathcal{O}_{k,\Sigma}, W^*(1)) \xrightarrow{\tilde\lambda_T^\vee} \bigoplus_{v\in\Sigma} \frac{H^1(k_v, W^*(1))}{H^1_f(k_v, W^*(1))}$$

in which $\tilde\lambda_T^\vee$ identifies with the sum of the natural localisation maps. Since the Selmer group $\mathrm{Sel}(T^*(1))$ is defined to be $\ker(\tilde\lambda_T^\vee)$ we thus obtain an isomorphism $\mathrm{cok}(\tilde\lambda_T) \cong \mathrm{Sel}(T^*(1))^\vee$ and so (10) induces an exact sequence

$$0 \to \mathrm{cok}(\lambda_T) \to H^2(C_T) \to \mathrm{Sel}(T^*(1))^\vee \to 0. \tag{11}$$

From the definition of $\text{Ш}(T^*(1))$ as the cokernel of the natural map $H^1_f(k, V^*(1)) \to H^1_f(k, W^*(1)) = \mathrm{Sel}(T^*(1))$ one obtains a natural short exact sequence

$$0 \to \mathbb{Q}_p/\mathbb{Z}_p \otimes_{\mathbb{Z}_p} H^1_f(k, T^*(1)) \to \mathrm{Sel}(T^*(1)) \to \text{Ш}(T^*(1)) \to 0. \tag{12}$$

Since $\text{Ш}(T^*(1))$ is finite (cf. [13, Chap. II, 5.3.5]) this sequence induces a natural isomorphism $\text{Ш}(T^*(1)) \cong \mathrm{Sel}(T^*(1))_{\mathrm{cotor}}$. This in turn implies that $\text{Ш}(T^*(1))^\vee$ is isomorphic to $(\mathrm{Sel}(T^*(1))_{\mathrm{cotor}})^\vee \cong (\mathrm{Sel}(T^*(1))^\vee)_{\mathrm{tor}}$. One also knows, by the main result of Flach in [12], that $\text{Ш}(T^*(1))^\vee$ is isomorphic to $\text{Ш}(T)$. Hence, if $\mathrm{cok}(\lambda_T)$ is finite, then (11) induces a surjection $H^2(C_T)_{\mathrm{tor}} \twoheadrightarrow (\mathrm{Sel}(T^*(1))^\vee)_{\mathrm{tor}} \cong \text{Ш}(T^*(1))^\vee \cong \text{Ш}(T)$. Claim (iii) therefore follows from this surjection and the analogous surjection that is obtained by repeating this argument with T replaced by $T^*(1)$.

Given the surjective map $H^2(C_T) \twoheadrightarrow \mathrm{Sel}(T^*(1))^\vee$ in (11), to prove claim (iv) it suffices to show that $I_T \cdot \mathrm{Ann}_{\mathfrak{A}}(H^2(C_T)_{\mathrm{tor}})$ annihilates $H^2(C_T)$. But the definition of $e_0 = e_0(C_T)$ ensures that the ideal $I_T = \mathfrak{A} \cap \mathfrak{A}e_0$ annihilates $H^2(C_T)_{\mathrm{tf}} \subset H^2(A \otimes_{\mathfrak{A}} C_T)$ and so the tautological exact sequence $0 \to H^2(C_T)_{\mathrm{tor}} \to H^2(C_T) \to H^2(C_T)_{\mathrm{tf}} \to 0$ makes it clear that $I_T \cdot \mathrm{Ann}_{\mathfrak{A}}(H^2(C_T)_{\mathrm{tor}})$ annihilates $H^2(C_T)$. $\square$

Remark 3.2. We consider the hypotheses of Lemma 3.1(iii) under the assumption that $H^0(k_v, V)$ vanishes for all places v. Then $H^1(k_v, V)$ also vanishes for all places $v \nmid p$, whilst the exponential map of Bloch and Kato induces an isomorphism between $\bigoplus_{v|p} H^1_f(k_v, V)$ and the tangent space $t_p(V)$ of V. Thus, if $t_p(V)$ vanishes, then the module $\bigoplus_{v\in\Sigma} H^1_f(k_v, T)$, and hence also its quotient $\mathrm{cok}(\lambda_T)$, is finite. Finally we note that for any given representation V the spaces $H^0(k_v, V(-r))$ and $t_p(V(-r))$ vanish for all sufficiently large integers r.

3.2. **Global Zeta isomorphisms.** For any Galois extension of fields F/E we set $G_{F/E} := \mathrm{Gal}(F/E)$. We set $G_k := G_{k^c/k}$ and recall that $G_{k,\Sigma}$ denotes the Galois group over k of the maximal extension of k inside k^c that is unramified outside Σ.
Let M be a motive over k and F a finite abelian extension of k inside k^c. We fix an odd prime p and a full G_k-stable sublattice T in the p-adic realisation V of M and set $\mathfrak{A} := \mathbb{Z}_p[G]$ with $G := G_{F/k}$. We also fix a finite set of places Σ of k containing all archimedean places, all which ramify in F/k, all at which M has bad reduction and all above p. Then $T_F := \mathfrak{A} \otimes_{\mathbb{Z}_p} T$ is an étale sheaf of free $\mathfrak{A}$-modules on $\mathrm{Spec}(\mathcal{O}_{k,\Sigma})$ and we set $C(T_F) := R\Gamma_c(\mathcal{O}_{k,\Sigma}, T_F)$ and $\mathfrak{A}_0 := \mathfrak{A}_0(C(T_F))$. We regard $M_F := h^0(\mathrm{Spec}(F)) \otimes_{h^0(\mathrm{Spec}(k))} M$ as a motive defined over k and with an action of $\mathbb{Q}[G]$ via the first factor in the tensor product. Then, in terms of the notation of Theorem 2.1, the 'non-commutative Tamagawa number conjecture' of [14, Conj. 2.3.2] for the motive M_F conjectures an explicit generator of the fractional $\mathfrak{A}_0$-ideal $\mathrm{Det}_{\mathfrak{A}_0}(C(T_F)_0)$ in terms of an element of $\mathbb{Q}_p[G]$ obtained by multiplying the leading term at $s = 0$ of the $\mathbb{C}[G]$-valued complex L-function of M_F by suitable regulators and periods. This means that when combined with Theorem 2.1 and the descriptions of Lemma 3.1 the relevant case of [14, Conj. 2.3.2] predicts explicit annihilators of the $\mathbb{Z}_p[G]$-modules $Ш(T_F)$ and $\mathrm{Sel}(T^*(1)_F)$ in terms of the values of complex L-functions. In a very similar fashion, one can use Theorem 2.1 and Lemma 3.1 to show that the relevant (generalised) main conjecture of Iwasawa theory predicts the existence of explicit annihilators for the $\mathbb{Z}_p[G]$-modules $Ш(T_F)$ and $\mathrm{Sel}(T^*(1)_F)$ that are constructed from the values of p-adic L-functions. By means of an explicit example, in the next subsection we shall consider in detail the Iwasawa-theoretic approach in the context of Tate motives.

3.3. **Tate motives.** For any CM-field E we write E^+ for its maximal (totally) real subfield. In this subsection we fix a totally real field k and a finite abelian CM-extension F of k inside k^c, set $G := G_{F/k}$ and write τ for the unique non-trivial element of G_{F/F^+}. We write $\mu(F,p)$ for the Iwasawa-theoretic (p-adic) μ-invariant of F and recall that Iwasawa has conjectured in [15] that $\mu(F,p) = 0$. We fix a finite set of places Σ of k containing all archimedean places, all which ramify in F/k and all above p and for each integer r set $\mathbb{Z}_p(r)_F := \mathbb{Z}_p[G] \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(r)$. We regard $\mathbb{Z}_p(r)_F$ as an étale sheaf of free $\mathbb{Z}_p[G]$-modules on $\mathrm{Spec}(\mathcal{O}_{k,\Sigma})$ in the natural way and then, following Lemma 3.1, we obtain an object of $D^{\mathrm{p,ad}}(\mathbb{Z}_p[G])$ by setting

$$C_r(F/k) := R\Gamma_c(\mathcal{O}_{k,\Sigma}, \mathbb{Z}_p(r)_F). \tag{13}$$

We write $e_{F/k,r}$ for the idempotent $e_0(C_r(F/k))$ that is defined in §1.1. For each homomorphism $\rho : G \to \mathbb{Q}_p^{c\times}$ we write $\tilde{\rho}$ for the induced homomorphism of rings $\mathbb{Z}_p[G] \to \mathbb{Q}_p^c$.
The following result will be proved in §3.4.

Theorem 3.3. *Assume that F contains a primitive p-th root of unity and that $\mu(F,p)$ vanishes. Let r be any integer. If a homomorphism $\rho : G \to \mathbb{Q}_p^{c\times}$ satisfies $\tilde{\rho}(e_{F/k,r}) = 1$, then $\rho(\tau) = (-1)^{r-1}$. Further, in $\mathbb{Q}_p[G]$ one has*

$$\mathrm{Det}_{\mathbb{Z}_p[G]e_{F/k,r}}(C_r(F/k)_0)^{-1} = \mathbb{Z}_p[G] \cdot \mathcal{L}_{F/k,\Sigma,r} \tag{14}$$

where $\mathcal{L}_{F/k,\Sigma,r}$ *is the unique element of* $\mathbb{Q}_p[G]e_{F/k,r}$ *with the following property: for each homomorphism* $\rho : G \to \mathbb{Q}_p^{c\times}$ *with* $\tilde{\rho}(e_{F/k,r}) = 1$ *one has* $\tilde{\rho}(\mathcal{L}_{F/k,\Sigma,r}) = L_{p,\Sigma}(r, \omega^{1-r}\cdot\rho)$*, where* ω *is the Teichmüller character* $G \to \mathbb{Q}_p^\times$ *and* $L_{p,\Sigma}(r, \omega^{1-r}\cdot\rho)$ *is the value at* $s = r$ *of the* Σ*-truncated* p*-adic* L*-function of the character* $\omega^{1-r}\cdot\rho$*.*

In the next two subsections we combine this result with Theorem 2.1 and Lemma 3.1 to derive some interesting consequences regarding the explicit structure of Galois groups, ideal class groups and wild kernels in higher algebraic K-theory.
For any torsion abelian group A we write A_p for its p-primary part.

3.3.1. We write $H(F)$ for the Hilbert class field of F and F_∞ for the cyclotomic $\mathbb{Z}_p$-extension of F. We also write $\mathrm{Pic}(\mathcal{O}_F)'$ for the subgroup of $\mathrm{Pic}(\mathcal{O}_F)$ that corresponds under the Artin isomorphism $\mathrm{Pic}(\mathcal{O}_F) \cong G_{H(F)/F}$ to the group $G_{H(F)/H(F)\cap F_\infty}$. For any extension E of F with $E \subset F^c$ we let $M_\Sigma(E)$ denote the maximal abelian pro-p extension of E inside F^c for which $M_\Sigma(E)/E$ is unramified outside the set of places lying above Σ.

Corollary 3.4. *Assume that* F *contains a primitive* p*-th root of unity, that* $\mu(F,p)$ *vanishes and that* F *validates Leopoldt's conjecture at* p*. Then the Tate-Shafarevic group* $Ш(\mathbb{Z}_p(1)_F)$ *is canonically isomorphic to* $\mathrm{Pic}(\mathcal{O}_F)_p$*. Further, if we set* $e_+ := (1+\tau)/2$ *and* $e_G := (\sum_{g\in G} g)/|G|$ *and let* I_G *denote the augmentation ideal of* $\mathbb{Z}_p[G]$*, then* $e_{F/k,0} = e_+ - e_G$*,*

$$I_G^2 \cdot \mathcal{L}_{F/k,\Sigma,1} \subseteq \mathrm{Ann}_{\mathbb{Z}_p[G]}(G_{M_\Sigma(F)/F_\infty})e_+ \subseteq \mathbb{Z}_p \otimes \mathrm{Ann}_{\mathbb{Z}[G]}(\mathrm{Pic}(\mathcal{O}_F)') \tag{15}$$

and

$$I_G^3 \cdot \mathcal{L}_{F/k,\Sigma,1} \subseteq \mathbb{Z}_p \otimes \mathrm{Ann}_{\mathbb{Z}[G]}(\mathrm{Pic}(\mathcal{O}_F)). \tag{16}$$

If the Sylow p*-subgroup of* G *is cyclic, then also*

$$I_G \cdot \mathcal{L}_{F/k,\Sigma,1} = \mathrm{Fit}_{\mathbb{Z}_p[G]}(G_{M_\Sigma(F)/F_\infty})e_+ \subseteq \mathbb{Z}_p \otimes \mathrm{Fit}_{\mathbb{Z}[G]}(\mathrm{Pic}(\mathcal{O}_F)') \tag{17}$$

and

$$I_G^2 \cdot \mathcal{L}_{F/k,\Sigma,1} \subseteq \mathbb{Z}_p \otimes \mathrm{Fit}_{\mathbb{Z}[G]}(\mathrm{Pic}(\mathcal{O}_F)). \tag{18}$$

Proof. The canonical isomorphism $Ш(\mathbb{Z}_p(1)_F) \cong \mathrm{Pic}(\mathcal{O}_F)_p$ is proved (unconditionally) by Flach in [12]. For the rest of the proof we set $C := C_1(F/k)e_+$ and we use the integers $n(C)$ and $g(C)$ defined in Theorem 2.1.
Then the first assertion of Theorem 3.3 (with $r = 1$) implies that $e_{F/k,1} = e_+e_{F/k,1}$ and hence also $C_1(F/k)_0 = C_0$. Further, under the assumption that F validates Leopoldt's conjecture at p, a standard computation (as, for example, in the proof of [8, Lem. 3]) shows that C is acyclic outside degrees 2 and 3 (so $n(C) = 1$) and that there are natural isomorphisms $H^2(C) \cong e_+G_{M_\Sigma(F)/F}$, $H^2(C)_{\mathrm{tor}} \cong e_+G_{M_\Sigma(F)/F_\infty}$ and also $H^3(C) \cong \mathbb{Z}_p$ (so $H^3(C)_{\mathrm{tor}} = 0$, $g(C) = 1$ and $e_{F/k,0} = e_+ - e_G$). Upon substituting these facts and the equality (14) with $r = 1$ into (1), resp. (3), and noting that $\mathrm{Fit}_{\mathbb{Z}_p[G]}(\mathbb{Z}_p) = \mathrm{Ann}_{\mathbb{Z}_p[G]}(\mathbb{Z}_p) = I_G$, one obtains the first inclusion of (15) and the equality in (17), but in the latter case with $G_{M_\Sigma(F)/F_\infty}$ replaced by $(G_{M_\Sigma(F)/F_\infty})^\vee$. However, if the Sylow p-subgroup of G is cyclic, then for any finite $\mathbb{Z}_p[G]$-module M one has $\mathrm{Fit}_{\mathbb{Z}_p[G]}(M^\vee) = \mathrm{Fit}_{\mathbb{Z}_p[G]}(M)$ and so this does indeed prove the equality in (17).

The remaining inclusions of (15) and (17) follow from the existence of a surjection $e_+G_{M_\Sigma(F)/F_\infty} \twoheadrightarrow e_+\mathrm{Pic}(\mathcal{O}_F)'_p$ (obtained by restricting the natural surjection $G_{M_\Sigma(F)/F} \twoheadrightarrow \mathrm{Pic}(\mathcal{O}_F)_p$). Indeed, this surjection implies that

$$\begin{aligned}
\mathrm{Ann}_{\mathbb{Z}_p[G]}(G_{M_\Sigma(F)/F_\infty})e_+ &= \mathrm{Ann}_{\mathbb{Z}_p[G]e_+}(e_+G_{M_\Sigma(F)/F_\infty}) \\
&\subseteq \mathrm{Ann}_{\mathbb{Z}_p[G]e_+}(e_+\mathrm{Pic}(\mathcal{O}_F)'_p) \\
&= \mathrm{Ann}_{\mathbb{Z}_p[G]}(\mathrm{Pic}(\mathcal{O}_F)'_p)e_+ \\
&\subset \mathrm{Ann}_{\mathbb{Z}_p[G]}(\mathrm{Pic}(\mathcal{O}_F)'_p) \\
&= \mathbb{Z}_p \otimes \mathrm{Ann}_{\mathbb{Z}[G]}(\mathrm{Pic}(\mathcal{O}_F)')
\end{aligned}$$

and so proves the second inclusion of (15). The inclusion in (17) follows in just the same way since, if the Sylow p-subgroup of G is cyclic, then any surjection of finite $\mathbb{Z}_p[G]$-modules $M \twoheadrightarrow N$ implies an inclusion $\mathrm{Fit}_{\mathbb{Z}_p[G]}(M) \subseteq \mathrm{Fit}_{\mathbb{Z}_p[G]}(N)$.
We next note that, since the group $\Gamma := G_{H(F)\cap F_\infty/F}$ is isomorphic (as a $\mathbb{Z}_p[G]$-module) to a quotient of $\mathbb{Z}_p \cong G_{F_\infty/F}$, one has $I_G \subseteq \mathrm{Ann}_{\mathbb{Z}_p[G]}(\Gamma) = \mathrm{Fit}_{\mathbb{Z}_p[G]}(\Gamma)$. The tautological short exact sequence $0 \to \mathrm{Pic}(\mathcal{O}_F)'_p \to \mathrm{Pic}(\mathcal{O}_F)_p \to \Gamma \to 0$ therefore combines with standard properties of Fitting ideals to imply that both $I_G \cdot \mathrm{Ann}_{\mathbb{Z}[G]}(\mathrm{Pic}(\mathcal{O}_F)') \subseteq \mathrm{Ann}_{\mathbb{Z}_p[G]}(\mathrm{Pic}(\mathcal{O}_F)_p) = \mathbb{Z}_p \otimes \mathrm{Ann}_{\mathbb{Z}[G]}(\mathrm{Pic}(\mathcal{O}_F))$ and also $I_G \cdot \mathrm{Fit}_{\mathbb{Z}[G]}(\mathrm{Pic}(\mathcal{O}_F)') \subseteq \mathrm{Fit}_{\mathbb{Z}_p[G]}(\mathrm{Pic}(\mathcal{O}_F)_p) = \mathbb{Z}_p \otimes \mathrm{Fit}_{\mathbb{Z}[G]}(\mathrm{Pic}(\mathcal{O}_F))$. The inclusions of (16) and (18) now follow by combining these inclusions with (15) and (17) respectively. □

Corollary 3.5. *If F is abelian over $\mathbb{Q}$ and contains a primitive p-th root of unity, then all of the inclusions (15), (16), (17) and (18) are valid unconditionally.*

Proof. This follows immediately from Corollary 3.4 and the fact that if F is abelian over $\mathbb{Q}$, then Ferrero and Washington [11] have proved that $\mu(F,p)$ vanishes and Brumer has proved that F validates Leopoldt's conjecture at p (cf. [31, Th. 5.25]). □

Remarks 3.6.
(i) If F_∞/F is totally ramified (as is the case, for example, if either $\mathrm{disc}(F)$ is prime to p or F is a subfield of $\mathbb{Q}_\infty$), then $H(F) \cap F_\infty = F$ so $\mathrm{Pic}(\mathcal{O}_F)' = \mathrm{Pic}(\mathcal{O}_F)$ and the inclusions (15) and (17) are strictly stronger than (16) and (18) respectively.
(ii) The inclusions of Corollary 3.4 are a natural counterpart to Brumer's Conjecture and hence, if F is abelian over $\mathbb{Q}$ (in which case Corollary 3.5 applies), to Stickelberger's Theorem. Indeed, whilst the latter uses values of Dirichlet L-functions at $s = 0$ to produce annihilators of $\mathrm{Pic}(\mathcal{O}_F)$ inside $\mathbb{Z}_p[G](1-e_+)$, Corollary 3.4 uses values of p-adic L-functions at $s = 1$ to produce annihilators of $\mathrm{Pic}(\mathcal{O}_F)$ inside $\mathbb{Z}_p[G]e_+$. Further, if the 'p-adic Stark conjecture at $s = 1$' of Serre (for a precise statement of which see [9, Conj. 5.2, Rem. 5.3]) is valid for all characters of G, then the element $\mathcal{L}_{F/k,\Sigma,1}$ can be re-expressed explicitly in terms of the values of Dirichlet L-functions at $s = 1$.
(iii) The equality in (17) specialises to give a strengthening of the main results of Oriat in [24]. To explain this we assume $k = \mathbb{Q}$, $G = G_{F/\mathbb{Q}}$ is cyclic (so Corollary 3.5 applies) and Σ is the set comprising ∞, p and the primes which ramify in $F/\mathbb{Q}$ and we fix an *injective* homomorphism $\rho : G \to \mathbb{Q}_p^{c\times}$. Now if ℓ belongs to $\Sigma \setminus \{p,\infty\}$, then ℓ ramifies in $F/\mathbb{Q}$ and so (since ρ is injective) the Euler factor of ρ at ℓ is trivial, and hence $\tilde{\rho}(\mathcal{L}_{F/k,\Sigma,1}) = L_{p,\Sigma}(1,\rho) = L_p(1,\rho)$. We next fix a generator g of G, set

$\zeta := \rho(g)$ and $\mathcal{O} := \mathbb{Z}_p[\zeta]$ and regard $\tilde\rho$ as a homomorphism $\mathbb{Z}_p[G] \to \mathcal{O}$. Then $1-g$ is a generator over $\mathbb{Z}_p[G]$ of I_G (since G is cyclic) and so the image under $\tilde\rho$ of the equality in (17) is $L_p(1,\rho)(1-\zeta)\mathcal{O} = \mathrm{Fitt}_{\mathcal{O}}(\mathcal{O} \otimes_{\mathbb{Z}_p[G],\tilde\rho} e_+ G_{M_\Sigma(F)/F_\infty})$. This equality implies the containments of [24, Th. A and Th. B] since the module $G_{M_{\{p\}}(F^+)/(F^+)_\infty}$ is (easily seen to be) a quotient of $e_+ G_{M_\Sigma(F)/F_\infty}$ and, if $|G|$ is not a power of p, then $1-\zeta$ is a unit of $\mathcal{O}$.

3.3.2. For each integer $r > 1$ we write $K^w_{2r-2}(\mathcal{O}_F)$ for the 'wild kernel' of higher algebraic K-theory that is defined by Banaszak in [2]. For each integer a we also write $W(a)$ for the G_k-module $\mathbb{Q}_p/\mathbb{Z}_p(a)$. Finally we recall that for any torsion abelian group A we write A_p for its p-primary part.

Corollary 3.7. *Assume that F contains a primitive p-th root of unity and that $\mu(F,p)$ vanishes. Fix an integer $r>1$ and for each integer a set $I_{G,a} := \mathbb{Z}_p[G] \cap \mathbb{Z}_p[G]e_{F/k,a}$.*

(i) *Then the (Bloch-Kato) Tate-Shafarevic group $Ш(\mathbb{Z}_p(r)_F)$ is canonically isomorphic to $K^w_{2r-2}(\mathcal{O}_F)_p$ and the ideal*

$$I^4_{G,r}\mathrm{Ann}_{\mathbb{Z}_p[G]}(H^0(F,W(1-r)))\cdot\mathcal{L}_{F/k,\Sigma,r} + \mathrm{Ann}_{\mathbb{Z}_p[G]}(H^0(F,W(r)))\cdot\mathcal{L}_{F/k,\Sigma,1-r}$$

is contained in $\mathbb{Z}_p \otimes \mathrm{Ann}_{\mathbb{Z}[G]}(K^w_{2r-2}(\mathcal{O}_F))$.

(ii) *If the group $H^2(\mathcal{O}_{F,\Sigma}, W(1-r))$ is finite, as is conjectured by Schneider (cf.* [27, p. 192]*), then the term $I^4_{G,r}$ can be omitted from the displayed expression in claim (i) and the resulting ideal has finite index in $\mathbb{Z}_p \otimes \mathrm{Ann}_{\mathbb{Z}[G]}(K^w_{2r-2}(\mathcal{O}_F))$.*

Proof. In this proof we shall use the following notation: for each non-negative integer a and each integer b we write $Ш^a(G_{F,\Sigma},\mathbb{Z}_p(b))$ for the kernel of the natural localisation homomorphism

$$H^a(\mathcal{O}_{F,\Sigma},\mathbb{Z}_p(b)) \to \bigoplus_{w\in\Sigma(F)} H^a(F_w,\mathbb{Z}_p(b)) \tag{19}$$

- this is the Tate-Shafarevic group in degree a of the $G_{F,\Sigma}$-module $\mathbb{Z}_p(b)$, as defined by Neukirch, Schimdt and Wingberg in [23, (8.6.2)].

For each integer a we also set $e_{(a)} := (1-(-1)^a\tau)/2 \in \mathbb{Z}_p[G]$, $\mathfrak{A}_{(a)} := e_{(a)}\mathbb{Z}_p[G]$, $T_{(a)} := e_{(a)}\mathbb{Z}_p(a)_F$ and so (following Lemma 3.1(i)) we obtain an object of $D^{\mathrm{p,ad}}(\mathfrak{A}_{(a)})$ by setting

$$C_{(a)} := R\Gamma_c(\mathcal{O}_{k,\Sigma},T_{(a)}) \cong e_{(a)}C_a(F/k).$$

Then the first assertion of Theorem 3.3 implies $e_{F/k,a} = e_{(a)}e_{F/k,a}$ and hence also that $C_a(F/k)_0 = (C_{(a)})_0$.

Next we note that if $a \neq 0$, then $H^0(k_v,T_{(a)})$ vanishes for each non-archimedean place v in Σ and thus, since Tate cohomology groups at archimedean places vanish in all degrees (as p is odd), the global duality theorem [21, Chap. II, Cor. 3.3] induces a canonical isomorphism $Ш^1(G_{F,\Sigma},\mathbb{Z}_p(a)) \cong H^2(\mathcal{O}_{F,\Sigma},W(1-a))^\vee$. Using this isomorphism an explicit computation of the groups $H^i(C_a(F/k))$ via the long exact cohomology sequence of (7) with $\mathcal{F} = \mathbb{Z}_p(a)_F$ shows that the complex $C_a(F/k)$ is acyclic outside of degrees 1, 2 and 3 and that there is a canonical (duality) isomorphism $H^3(C_a(F/k)) \cong H^0(F,W(1-a))^\vee$ and a canonical exact sequence

$$0 \to H^0(G_{\mathbb{C}/\mathbb{R}}, \prod_{F\to\mathbb{C}}(2\pi i)^a\mathbb{Z}_p) \to H^1(C_a(F/k)) \to H^2(\mathcal{O}_{F,\Sigma},W(1-a))^\vee \to 0$$

where on the module $\prod_{F\to\mathbb{C}}(2\pi i)^a\mathbb{Z}_p$ the group $G_{\mathbb{C}/\mathbb{R}}$ acts diagonally and G acts via F. Thus, since $e_{(a)}$ annihilates $H^0(G_{\mathbb{C}/\mathbb{R}},\prod_{F\to\mathbb{C}}(2\pi i)^a\mathbb{Z}_p)$ and acts as the identity on $H^0(F,W(1-a))^\vee$ there are canonical isomorphisms

$$H^i(C_{(a)})\cong\begin{cases} e_{(a)}H^2(\mathcal{O}_{F,\Sigma},W(1-a))^\vee, & \text{if } i=1,\\ H^0(F,W(1-a))^\vee, & \text{if } i=3.\end{cases} \tag{20}$$

By substituting the equality (14) with $r=a$ into the inclusion (1) with $\mathfrak{A}=\mathfrak{A}_{(a)}$ and $C=C_{(a)}$ (and noting that in this case (20) implies $H^3(C)=H^3(C)_{\rm tor}, g(C)=1$ and $n(C)=2$), multiplying the resulting inclusion by $I_{G,a}=I_{\mathfrak{A}_0(C_{(a)})}$ and applying Lemma 3.1(iv) with $T=T_{(a)}$ and $\mathfrak{A}=\mathfrak{A}_{(a)}$ we therefore deduce that

$$\begin{aligned} I^4_{G,a}\mathrm{Ann}_{\mathbb{Z}_p[G]}(H^0(F,W(1-a)))\cdot\mathcal{L}_{F/k,\Sigma,a} &\subseteq \mathrm{Ann}_{\mathfrak{A}_{(a)}}(\text{Ш}(T_{(a)}))\\ &= e_{(a)}\mathrm{Ann}_{\mathbb{Z}_p[G]}(\text{Ш}(\mathbb{Z}_p(a)_F))\\ &= e_{(a)}\mathrm{Ann}_{\mathbb{Z}_p[G]}(\text{Ш}(\mathbb{Z}_p(1-a)_F)).\end{aligned} \tag{21}$$

Here the last equality follows from the isomorphism $\text{Ш}(\mathbb{Z}_p(1-a)_F)\cong\text{Ш}(\mathbb{Z}_p(a)_F)^\vee$. Now if the module $H^2(\mathcal{O}_{F,\Sigma},W(1-a))$ is finite, as is conjectured by Schneider in [27, p. 192], and proved by Banaszak [2, Lem. 1] for $a<0$, then (20) implies that $H^i(C_{(a)})$ is finite for both $i=1$ and $i=3$. But $C_{(a)}$ belongs to $D^{\rm p,ad}(\mathfrak{A}_{(a)})$ and so is acyclic outside degrees $1,2$ and 3 and such that the Euler characteristic of $\mathbb{Q}_p\otimes_{\mathbb{Z}_p}C_{(a)}$ in $K_0(e_{(a)}\mathbb{Q}_p[G])$ vanishes. Thus, if Schneider's conjecture is valid, then the groups $H^i(C_{(a)})$ are finite in all degrees i and hence $e_{F/k,a}=e_{(a)}$ so $I_{G,a}=\mathfrak{A}_{(a)}$ and the term $I^4_{G,a}=\mathfrak{A}_{(a)}$ can be omitted from the left hand side of (21). We may therefore apply (21) with both $a=r$ and $a=1-r<0$ to deduce that the displayed expression in claim (i) is contained in $e_{(r)}\mathrm{Ann}_{\mathbb{Z}_p[G]}(\text{Ш}(\mathbb{Z}_p(r)_F))\oplus e_{(1-r)}\mathrm{Ann}_{\mathbb{Z}_p[G]}(\text{Ш}(\mathbb{Z}_p(r)_F))=\mathrm{Ann}_{\mathbb{Z}_p[G]}(\text{Ш}(\mathbb{Z}_p(r)_F))$ and so to complete the proof of claim (i) it suffices to show that $\text{Ш}(\mathbb{Z}_p(r)_F)$ is canonically isomorphic to $K^w_{2r-2}(\mathcal{O}_F)_p$. To show this we note first that, since $r>1$, one has $H^1_f(k_v,\mathbb{Z}_p(r)_F)=H^1(k_v,\mathbb{Z}_p(r)_F)$ for each place v in Σ (by [5, Exam. 3.9]) and hence (9) implies $H^1_f(k,\mathbb{Z}_p(r)_F)=H^1(\mathcal{O}_{k,\Sigma},\mathbb{Z}_p(r)_F)$. Taking this into account, the exact sequence (11) combines with the long exact cohomology sequence of (7) with $\mathcal{F}=\mathbb{Z}_p(r)_F$ to imply that $\mathrm{Sel}(\mathbb{Z}_p(1-r)_F)^\vee$ is isomorphic to $\text{Ш}^2(G_{F,\Sigma},\mathbb{Z}_p(r))$. Further, since $H^2(\mathcal{O}_{F,\Sigma},\mathbb{Z}_p(r))$ is finite it follows that $\mathrm{Sel}(\mathbb{Z}_p(1-r)_F)^\vee$ is finite and hence equal to $\text{Ш}(\mathbb{Z}_p(1-r)_F)^\vee\cong\text{Ш}(\mathbb{Z}_p(r)_F)$. It is therefore enough to show that $\text{Ш}^2(G_{F,\Sigma},\mathbb{Z}_p(r))$ is canonically isomorphic to $K^w_{2r-2}(\mathcal{O}_F)_p$.

To prove this we follow [2] in setting $W^a(F_w):=H^0_{\text{ét}}(F_w,W(a))$ for each place w in $\Sigma(F)$ and integer a. Then for each such w there is a local duality isomorphism $H^2_{\text{ét}}(F_w,\mathbb{Z}_p(r))\cong W^{1-r}(F_w)^\vee$ and an obvious isomorphism $W^{1-r}(F_w)^\vee\cong W^{r-1}(F_w)$. By combining these isomorphisms with the fact that the canonical chern class map $K_{2r-2}(\mathcal{O}_{F,\Sigma})_p\to H^2(\mathcal{O}_{F,\Sigma},\mathbb{Z}_p(r))$ is bijective (see Remark 3.9(i) below) one finds that the homomorphism (19) with $a=2$ and $b=r$ identifies with the composite

$$\kappa_p: K_{2r-2}(\mathcal{O}_{F,\Sigma})_p\xrightarrow{\lambda'_p}K_{2r-2}(F)_p\xrightarrow{\lambda_p}\bigoplus_{w\in S(F)}W^{r-1}(F_w)\xrightarrow{\pi_{\Sigma(F)}}\bigoplus_{w\in\Sigma(F)}W^{r-1}(F_w)$$

where λ'_p is the localisation map in K-theory, λ_p the map in [2, Th. 2], $S(F)$ the set of all places of F and for any subset S' of $S(F)$ we write $\pi_{S'}$ for the projection

$\bigoplus_{w\in S(F)} W^{r-1}(F_w) \to \bigoplus_{w\in S'} W^{r-1}(F_w)$. But from the exact localisation sequence of K-theory one knows that λ'_p is injective and has image equal to $\ker(\pi_{S(F)\setminus\Sigma(F)}\circ\lambda_p)$ (cf. [2, §III.1]). This implies that $\mathrm{im}(\lambda'_p)\cap\ker(\pi_{\Sigma(F)}\circ\lambda_p) = \ker(\lambda_p)$ and hence that λ'_p induces an isomorphism $Ш^2(G_{F,\Sigma},\mathbb{Z}_p(r)) = \ker(\kappa_p) \cong \ker(\lambda_p) =: K^w_{2r-2}(\mathcal{O}_F)_p$, as required to complete the proof of claim (i).

The first assertion of claim (ii) follows immediately from what we observed above regarding Schneider's conjecture. The second assertion of claim (ii) is then valid because for each $a\in\{r,1-r\}$ the ideal $\mathrm{Ann}_{\mathbb{Z}_p[G]}(H^0(F,W(1-a)))\cdot\mathcal{L}_{F/k,\Sigma,a}$ has finite index in $e_{(a)}\mathbb{Z}_p[G]$ (since if $\tilde\rho(e_{(a)})\neq 0$, then $\tilde\rho(\mathcal{L}_{F/k,\Sigma,a}) = L_{p,\Sigma}(a,\omega^{1-a}\cdot\rho)\neq 0$). □

Corollary 3.8. *If F is abelian over $\mathbb{Q}$ and contains a primitive p-th root of unity, then the inclusion of Corollary 3.7(i) is valid unconditionally.*

Proof. This is clear since if F is abelian over $\mathbb{Q}$, then $\mu(F,p)$ vanishes (by [11]). □

Remarks 3.9.

(i) For all odd primes p and all integers $r>1$ the Quillen-Lichtenbaum Conjecture from [19] predicts that the (G-equivariant) p-adic étale Chern class homomorphism $\mathrm{ch}_{2r-2,p} : K_{2r-2}(\mathcal{O}_{F,\Sigma})_p \to H^2(\mathcal{O}_{F,\Sigma},\mathbb{Z}_p(r))$ is bijective. The bijectivity of the maps $\mathrm{ch}_{2,p}$ was proved by Tate in [29]. It is also known, by work of Suslin, that the Quillen-Lichtenbaum Conjecture is in general a consequence of the conjecture of Bloch and Kato relating Milnor K-theory to étale cohomology. Following fundamental work of Voevodsky and Rost, Weibel has recently completed the proof of the Bloch-Kato Conjecture and hence also shown that $\mathrm{ch}_{2r-2,p}$ is bijective for all odd primes p and all integers $r>1$ (cf. [32]).

(ii) Fix an integer $r>1$. Then, since $\mathfrak{A}_{(1-r)}$ is R-Gorenstein, we may also apply (3) to the complex $C = C_{(1-r)}$ that occurs in the proof of Corollary 3.7. This shows that $\mathrm{Ann}_{\mathbb{Z}_p[G]}(H^0(F,W(r)))\cdot\mathcal{L}_{F/k,\Sigma,1-r}$ is equal to the Fitting ideal of an explicit étale cohomology module and hence is contained in $\mathbb{Z}_p\otimes\mathrm{Fit}_{\mathbb{Z}[G]}(K_{2r-2}(\mathcal{O}_{F,\Sigma}))$. In particular, in this way one can recover the result of [8, Cor. 5.2].

3.4. **The proof of Theorem 3.3.** To prove the first assertion of Theorem 3.3 it is clearly enough to show that any idempotent of $\mathbb{Q}_p[G]$ which annihilates $H^2_c(\mathcal{O}_{k,\Sigma},\mathbb{Q}_p(r)_F)$ must also annihilate the space $\bigoplus_{v|\infty} H^0(k_v,\mathbb{Q}_p(r)_F) \cong H^0(G_{\mathbb{C}/\mathbb{R}},\prod_{F\to\mathbb{C}}\mathbb{Q}_p(r))$, where on the product term $G_{\mathbb{C}/\mathbb{R}}$ acts diagonally and G acts via F.

If $r=0$, then the required claim follows from the isomorphism of $\mathbb{Q}_p[G]$-modules $\mathrm{Hom}_{\mathbb{Q}_p}(H^2_c(\mathcal{O}_{k,\Sigma},\mathbb{Q}_p(0)_F),\mathbb{Q}_p) \cong H^1(\mathcal{O}_{k,\Sigma},\mathbb{Q}_p(1)_F) \cong \mathbb{Q}_p\otimes_{\mathbb{Z}_p}\mathcal{O}^\times_{F,\Sigma}$ coming from global duality and Kummer theory and the fact that $\mathbb{Q}_p\otimes_{\mathbb{Z}_p}\mathcal{O}^\times_{F,\Sigma}$ contains a $\mathbb{Q}_p[G]$-submodule isomorphic to $H^0(G_{\mathbb{C}/\mathbb{R}},\prod_{F\to\mathbb{C}}\mathbb{Q}_p) \cong \prod_w\mathbb{Q}_p$ where w runs over all archimedean places of F. If $r\neq 0$, then the cohomology sequence of the exact triangle (7) with $\mathcal{F} = \mathbb{Z}_p(r)_F$ combines with the fact that $H^0(\mathcal{O}_{k,\Sigma},\mathbb{Q}_p(r)_F)$ vanishes to induce an injective homomorphism from $\bigoplus_{v|\infty} H^0(k_v,\mathbb{Q}_p(r)_F)$ to $H^1_c(\mathcal{O}_{k,\Sigma},\mathbb{Q}_p(r)_F)$ and this implies the required claim since, by Remark 1.1 and Lemma 3.1(i), any idempotent which annihilates $H^2_c(\mathcal{O}_{k,\Sigma},\mathbb{Q}_p(r)_F)$ must also annihilate $H^1_c(\mathcal{O}_{k,\Sigma},\mathbb{Q}_p(r)_F)$. This completes the proof of the first assertion of Theorem 3.3.

To continue we must introduce some additional notation. For any subfield F' of F we write F'_∞ for the cyclotomic $\mathbb{Z}_p$-extension of F'. For any quotient $\mathcal{G}$ of the (abelian) group $G_{F_\infty/k}$ we write $\Lambda(\mathcal{G})$ for the completed group ring $\varprojlim_U \mathbb{Z}_p[\mathcal{G}/U]$ where U runs over all open subgroups of $\mathcal{G}$ and we write $Q(\mathcal{G})$ for the total quotient ring of $\Lambda(\mathcal{G})$. For each integer r we write $\Lambda(\mathcal{G})^{\#}(r)$ for the set $\Lambda(\mathcal{G})$ upon which $\Lambda(\mathcal{G})$ acts via multiplication and $G_{k,\Sigma}$ acts in the following way: each σ in $G_{k,\Sigma}$ acts as multiplication by the element $\chi_{\rm cyc}(\sigma)^r\bar{\sigma}^{-1}$ where $\chi_{\rm cyc}$ is the cyclotomic character $G_{k,\Sigma}\to\mathbb{Z}_p^\times$ and $\bar{\sigma}$ denotes the image of σ in $\mathcal{G}$. We write τ for the unique non-trivial element of G_{F_∞/F_∞^+} and for each integer r set $e_{(r)} := (1-(-1)^r\tau)/2 \in \Lambda(G_{F_\infty/k})$. If E is the subfield of F_∞ with $G_{E/k}=\mathcal{G}$ then, following Nekovář [22] and Fukaya and Kato [14, §2.1.1], we obtain an object of $D^{\rm p}(\Lambda(\mathcal{G}))$ by setting

$$C_{E/k,r} := R\Gamma_c(\mathcal{O}_{k,\Sigma}, e_{(r)}\Lambda(\mathcal{G})^{\#}(r)). \tag{22}$$

We next fix a topological generator γ of $\Gamma_k := G_{k_\infty/k}$ and a pre-image $\hat{\gamma}$ of γ under the surjection $G_{F_\infty/k}\to\Gamma_k$. We write F^s for the subfield of F_∞ that corresponds by Galois theory to the subgroup of $G_{F_\infty/k}$ that is generated (topologically) by $\hat{\gamma}$. By choosing $\hat{\gamma}$ appropriately we may (and will henceforth) assume that F^s contains a primitive p-th root of unity. Then one has $F^s\cap k_\infty = k$ and $F^sk_\infty = F_\infty$ and so we may identify $G_{F_\infty/k}$ with the direct product $G_{F^s/k}\times\Gamma_k$. This decomposition allows us to identify $\Lambda(G_{F_\infty/k})$ with the power series ring $\mathbb{Z}_p[G_{F^s/k}][[T]]$ by means of the correspondence $\gamma\leftrightarrow T+1$ and also implies that any continuous homomorphism $\psi: G_{F_\infty/k}\to\mathbb{Q}_p^{c\times}$ can be written uniquely as a product $\psi=\psi_s\times\psi_w$ with ψ_s a homomorphism $G_{F^s/k}\to\mathbb{Q}_p^{c\times}$ and ψ_w a continuous homomorphism $\Gamma_k\to\mathbb{Q}_p^{c\times}$. Each such ψ extends by linearity to give a ring homomorphism $\tilde{\psi}:\Lambda(G_{F_\infty/k})\to\mathbb{Q}_p^c$ and if we write $\mathfrak{p}(\psi)$ for the associated prime ideal $\ker(\tilde{\psi})$ of $\Lambda(G_{F_\infty/k})$, then $\tilde{\psi}$ extends to a well-defined homomorphism (also denoted by $\tilde{\psi}$) from the localisation $\Lambda(G_{F_\infty/k})_{\mathfrak{p}(\psi)}$ to $\mathbb{Q}_p^c$. In particular, the constructions of Deligne and Ribet that are used by Wiles in [33, p. 501f.] imply that there are elements G_Σ and H of $\Lambda(G_{F_\infty/k})$ such that for all homomorphisms ψ as above the quotient $f_\Sigma := G_\Sigma/H$ belongs to $\Lambda(G_{F_\infty/k})_{\mathfrak{p}(\psi)}$ and is such that $\tilde{\psi}(f_\Sigma)=f_{\Sigma,\psi}(0)$ where $f_{\Sigma,\psi} := G_{\Sigma,\psi}/H_\psi$ with $G_{\Sigma,\psi}$ the Deligne-Ribet power series of ψ in $\mathcal{O}_\psi[[T]]$ and $H_\psi(T)$ equal to 1, resp. to $\psi_w(\gamma)(1+T)-1$, if ψ_s is non-trivial, resp. trivial. We recall also that for each continuous homomorphism $\kappa: G_{F_\infty/k}\to\mathbb{Q}_p^{c\times}$ for which κ_s is trivial one has

$$f_{\Sigma,\kappa\psi}(0) = f_{\Sigma,\psi}(\kappa(\gamma)-1). \tag{23}$$

(For more details about the construction of f_Σ, G_Σ and H see for example [8, §3].)
We regard each homomorphism $G\to\mathbb{Q}_p^{c\times}$ as a (continuous) homomorphism $G_{F_\infty/k}\to\mathbb{Q}_p^{c\times}$ in the obvious way. For each integer a we also write ${\rm tw}_a$ for the endomorphism of $Q(G_{F_\infty/k})$ that is induced by the map which sends each element g of $G_{F_\infty/k}$ to $\chi_{\rm cyc}(g)^a g\in\Lambda(G_{F_\infty/k})$.
The equality proved in the next result is the relevant (generalised) main conjecture of Iwasawa theory.

Proposition 3.10. *We set $C_{(r)} := C_{F_\infty/k,r}$ and $R_r := e_{(r)}\Lambda(G_{F_\infty/k})$ and write $Q(R_r)$ for the total quotient ring of R_r. Then the complex $Q(R_r)\otimes_{R_r}C_{(r)}$ is acyclic and so* *we may regard* $\mathrm{Det}_{R_r}(C_{(r)})$ *as an invertible* R_r*-submodule of* $Q(R_r)$ *(cf. §1.2). With respect to this identification one has* $\mathrm{Det}_{R_r}(C_{(r)})^{-1} = R_r \cdot \mathrm{tw}_{1-r}(f_\Sigma)$.

Proof. It is well known (and straightforward to prove) that $H^a(C_{(r)}) = 0$ if $a \notin \{2, 3\}$ and that $H^2(C_{(r)})$ and $H^3(C_{(r)})$ are canonically isomorphic to $Y_\Sigma(F_\infty^+) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(r-1)$ and $\mathbb{Z}_p(r-1)$ respectively, where $Y_\Sigma(F_\infty^+) := G_{M_\Sigma(F_\infty^+)/F_\infty^+}$ (for more details of this calculation see, for example, the proof of [8, Lem. 3]). Since both $Y_\Sigma(F_\infty^+)$ and $\mathbb{Z}_p$ are finitely generated torsion $\Lambda(\Gamma_k)$-modules it is therefore clear that $Q(R) \otimes_R C_{(r)}$ is acyclic.

Next we note that, by Lemma 3.11 below, it is enough to prove the claimed equality $\mathrm{Det}_{R_r}(C_{(r)})^{-1} = R_r \cdot \mathrm{tw}_{1-r}(f_\Sigma)$ in the case $r = 1$. To do this we set $R := R_1$ and $C := C_{(1)}$ and $\Xi := \mathrm{Det}_R(C)^{-1} \subset Q(R)$. Recalling the reduction techniques of [7, Lem. 6.1], it is thus enough to prove that after localising at each height one prime ideal $\mathfrak{p}$ of R one has $\Xi_\mathfrak{p} = R_\mathfrak{p} \cdot f_\Sigma$. We note that the residue characteristic of any such prime ideal $\mathfrak{p}$ is either 0 or p.

We assume first that the residue characteristic of $\mathfrak{p}$ is p. Then the explicit descriptions of $H^a(C)$ given above combine with [7, Lem. 6.3] and the (assumed) vanishing of $\mu(F, p)$ to imply that $R_\mathfrak{p} \otimes_R C$ is acyclic and hence that $\Xi_\mathfrak{p} = R_\mathfrak{p}$. The required equality is therefore valid in this case because the vanishing of $\mu(F, p)$ also implies that both G_Σ and H, and hence also $f_\Sigma = G_\Sigma/H$, are units of $R_\mathfrak{p}$ (indeed, in the terminology of [8], this is equivalent to asserting that both G_Σ and H are 'associated to distinguished polynomials' in $\Lambda(G_{F_\infty/k})$ and hence is proved in [8, Th. 3.1iii)]).

If now the residue characteristic of $\mathfrak{p}$ is 0, then $R_\mathfrak{p}$ is both a discrete valuation ring and a $\mathbb{Q}_p$-algebra and decomposes as a product $\prod_\kappa R_{\mathfrak{p},\kappa}$ where κ runs over all homomorphisms $G_{F_\infty/k_\infty} \to \mathbb{Q}_p^{c\times}$ which satisfy $\kappa(\tau) = 1$, and each algebra $R_{\mathfrak{p},\kappa} := e_\kappa R_\mathfrak{p}$ is a principal ideal domain. This implies that in each degree i the $R_\mathfrak{p}$-module $H^i(C_\mathfrak{p})$ is both torsion and of projective dimension at most one and so one has

$$\begin{aligned}\Xi_\mathfrak{p} = \mathrm{Det}_{R_\mathfrak{p}}(C_\mathfrak{p})^{-1} &= \prod_{i\in\mathbb{Z}} \mathrm{Det}_{R_\mathfrak{p}}(H^i(C_\mathfrak{p})[-i])^{-1} \\ &= \prod_{i\in\mathbb{Z}} \mathrm{Fit}_{R_\mathfrak{p}}(H^i(C_\mathfrak{p}))^{(-1)^i} = \mathrm{Fit}_{R_\mathfrak{p}}(Y_\Sigma(F_\infty^+)_\mathfrak{p})\mathrm{Fit}_{R_\mathfrak{p}}((\mathbb{Z}_p)_\mathfrak{p})^{-1},\end{aligned}$$

where the second equality follows from [17, Rem. b) after Th. 2], the third from the observation made at the end of §1.2 and the last from the explicit description of the modules $H^i(C)$ given above. The required equality $\Xi_\mathfrak{p} = R_\mathfrak{p} \cdot f_\Sigma$ is thus reduced to proving that for each κ as above the $R_{\mathfrak{p},\kappa}$-ideal $\mathrm{Fit}_{R_{\mathfrak{p},\kappa}}(e_\kappa(Y_\Sigma(F_\infty^+)_\mathfrak{p}))\mathrm{Fit}_{R_{\mathfrak{p},\kappa}}(e_\kappa(\mathbb{Z}_p)_\mathfrak{p})^{-1}$ is generated by the element $e_\kappa(G_\Sigma)/e_\kappa(H) = G_{\Sigma,\kappa}/H_\kappa$. But the explicit description of H_κ makes it clear that $\mathrm{Fit}_{R_{\mathfrak{p},\kappa}}(e_\kappa(\mathbb{Z}_p)_\mathfrak{p}) = R_{\mathfrak{p},\kappa} \cdot H_\kappa$ whilst the equality $\mathrm{Fit}_{R_{\mathfrak{p},\kappa}}(e_\kappa(Y_\Sigma(F_\infty^+)_\mathfrak{p})) = R_{\mathfrak{p},\kappa} \cdot G_{\Sigma,\kappa}$ coincides precisely with the main conjecture of Iwasawa theory that is proved by Wiles in [33] (see, for example, [8, (3)]). □

Lemma 3.11. *Assume the notation and hypotheses of Proposition 3.10. If* $\mathrm{Det}_{R_1}(C_{(1)})^{-1} = R_1 \cdot f_\Sigma$, *then* $\mathrm{Det}_{R_r}(C_{(r)})^{-1} = R_r \cdot \mathrm{tw}_{1-r}(f_\Sigma)$ *for all integers* r.

Proof. We set $\mathcal{G} := G_{F_\infty/k}, \Lambda := \Lambda(\mathcal{G}), Q := Q(\Lambda)$, $R := R_1 = e_{(1)}\Lambda$ and for each integer a also $\Xi_a := \mathrm{Det}_{R_a}(C_{(a)})^{-1} \subset Q(\Lambda)$. For any Λ-module M and integer a we write $\Lambda \otimes_a M$ for the tensor product $\Lambda \otimes M$ which has $\lambda'\lambda \otimes_a m = \lambda' \otimes_a \mathrm{tw}_a(\lambda)m$

for all λ, λ' in Λ and m in M. We regard $\Lambda \otimes_a M$ as an Λ-module via multiplication on the left. Then for any invertible R-ideal J in $Q(R)$ and any integer a the map $\lambda \otimes_a j \mapsto \lambda \mathrm{tw}_{-a}(j)$ induces an isomorphism of Λ-modules $\Lambda \otimes_a J \cong \mathrm{tw}_{-a}(J)$. Hence there are natural isomorphisms

$$\begin{aligned}\mathrm{tw}_{-a}(\Xi_1) \cong \Lambda \otimes_a \mathrm{Det}_{R_{(1)}}(C_{(1)})^{-1} \cong \mathrm{Det}_{R_{a+1}}(\Lambda \otimes_a C_{(1)})^{-1}\\ \cong \mathrm{Det}_{R_{a+1}}(\mathbb{Z}_p(a) \otimes_{\mathbb{Z}_p} C_{(1)})^{-1}.\end{aligned}$$

Here we regard $\mathbb{Z}_p(a) \otimes_{\mathbb{Z}_p} C_{(1)}$ as a complex of Λ-modules via the diagonal action of $\mathcal{G}$ (which acts on $\mathbb{Z}_p(a)$ since F contains a primitive p-th root of unity) and the last displayed isomorphism is induced in the following way: if we choose a topological generator ξ of $\mathbb{Z}_p(a)$, then the association $\lambda \otimes_a c \mapsto \xi \otimes_{\mathbb{Z}_p} \lambda c$ induces an isomorphism $\Lambda \otimes_a C_{(1)} \cong \mathbb{Z}_p(a) \otimes_{\mathbb{Z}_p} C_{(1)}$ in $D^{\mathrm{p}}(\Lambda)$. Setting $D_{(m)} := R\Gamma_c(\mathcal{O}_{k,\Sigma}, \Lambda^{\#}(m))$ for each integer m, we next recall that there is a standard isomorphism $\mathbb{Z}_p(a) \otimes_{\mathbb{Z}_p} D_{(1)} \cong D_{(a+1)}$ in $D^{\mathrm{p}}(\Lambda)$, and note that this isomorphism restricts to give an isomorphism in $D^{\mathrm{p}}(R_{a+1})$ of the form

$$\mathbb{Z}_p(a) \otimes_{\mathbb{Z}_p} C_{(1)} = \mathbb{Z}_p(a) \otimes_{\mathbb{Z}_p} e_{(1)} D_{(1)} = e_{(a+1)}(\mathbb{Z}_p(a) \otimes_{\mathbb{Z}_p} D_{(1)}) \cong e_{(a+1)} D_{(a+1)} = C_{(a+1)}.$$

Upon combining the last two displayed isomorphisms we obtain an isomorphism $\mathrm{tw}_{-a}(\Xi_1) \cong \mathrm{Det}_{R_{a+1}}(C_{(a+1)})^{-1}$ which in turn implies an equality $\mathrm{tw}_{-a}(\Xi_1) = \Xi_{a+1}$. In particular, if $\Xi_1 = R_1 \cdot f_\Sigma$, then for any integer r one has $\Xi_r = \mathrm{tw}_{1-r}(\Xi_1) = \mathrm{tw}_{1-r}(R_1 \cdot f_\Sigma) = \mathrm{tw}_{1-r}(R_1) \cdot \mathrm{tw}_{1-r}(f_\Sigma) = R_{-r} \cdot \mathrm{tw}_{1-r}(f_\Sigma)$. This implies the claimed result since clearly $e_{(-r)} = e_{(r)}$ and so $R_{-r} = R_r$. □

To conclude the proof of Theorem 3.3 we now set $e_r := e_{F/k,r}, \mathfrak{A}_r := \mathbb{Z}_p[G]e_r$ and $C_{r,0} := C_r(F/k)_0$ where $C_r(F/k)$ is the complex defined in (13), and let R_r be as in Proposition 3.10. Then the first assertion of Theorem 3.3 (already proved at the beginning of this subsection) implies $e_r e_{(r)} = e_r$ and hence that there is a natural isomorphism in $D^{\mathrm{p}}(\mathfrak{A}_r)$ of the form

$$\mathfrak{A}_r \otimes^{\mathbb{L}}_{e_{(r)}\mathbb{Z}_p[G]} C_{F/k,r} \cong \mathfrak{A}_r \otimes^{\mathbb{L}}_{\mathbb{Z}_p[G]} C_r(F/k) =: C_{r,0},$$

where $C_{F/k,r}$ is the complex defined in (22). The canonical descent isomorphism $e_{(r)}\mathbb{Z}_p[G] \otimes^{\mathbb{L}}_{R_r} C_{F_\infty/k,r} \cong C_{F/k,r}$ therefore induces a natural isomorphism in $\mathcal{P}(\mathfrak{A}_{(r)})$ of the form

$$\begin{aligned}\mathfrak{A}_r \otimes_{R_r} \mathrm{Det}_{R_r}(C_{F_\infty/k,r})^{-1} &\cong \mathfrak{A}_r \otimes_{e_{(r)}\mathbb{Z}_p[G]} \mathrm{Det}_{e_{(r)}\mathbb{Z}_p[G]}(e_{(r)}\mathbb{Z}_p[G] \otimes^{\mathbb{L}}_{R_r} C_{F_\infty/k,r})^{-1}\\ &\cong \mathfrak{A}_r \otimes_{e_{(r)}\mathbb{Z}_p[G]} \mathrm{Det}_{e_{(r)}\mathbb{Z}_p[G]}(C_{F/k,r})^{-1}\\ &\cong \mathrm{Det}_{\mathfrak{A}_r}(\mathfrak{A}_r \otimes^{\mathbb{L}}_{e_{(r)}\mathbb{Z}_p[G]} C_{F/k,r})^{-1}\\ &\cong \mathrm{Det}_{\mathfrak{A}_r}(C_{r,0})^{-1}.\end{aligned}$$

This isomorphism combines with the equality of Proposition 3.10, the acyclicity of $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} C_{r,0}$ and the descent formula of [7, Lem. 8.1] to imply that in $\mathbb{Q}_p[G]$ one has

$$\mathrm{Det}_{\mathfrak{A}_r}(C_{r,0})^{-1} = \mathbb{Z}_p[G] \cdot \mathcal{L}_r$$

where $\mathcal{L}_r$ is the (unique) element of $\mathbb{Q}_p[G]e_r$ with $\tilde{\rho}(\mathcal{L}_r) = \tilde{\rho}(\mathrm{tw}_{1-r}(f_\Sigma))$ for each homomorphism $\rho : G \to \mathbb{Q}_p^{c\times}$ with $\tilde{\rho}(e_r) = 1$. To deduce the required equality (14) from here it is thus enough to show that $\mathcal{L}_r = \mathcal{L}_{F/k,\Sigma,r}$ and this is true because for each homomorphism $\rho : G \to \mathbb{Q}_p^{c\times}$ with $\tilde\rho(e_r) = 1$ (and hence also $(\chi_{\rm cyc}^{1-r}\rho)(\tau) = 1$) one has

$$\begin{aligned}\tilde\rho(\mathcal{L}_r) = \tilde\rho(\mathrm{tw}_{1-r}(f_\Sigma)) = (\widetilde{\chi_{\rm cyc}^{1-r}\rho})(f_\Sigma) = f_{\Sigma,\chi_{\rm cyc}^{1-r}\rho}(0)\\ = f_{\Sigma,\omega^{1-r}\rho}(\chi_{\rm cyc}(\gamma)^{1-r} - 1) = L_{p,\Sigma}(r,\omega^{1-r}\rho) = \tilde\rho(\mathcal{L}_{F/k,\Sigma,r}).\end{aligned}$$

Here the second equality follows from the definition of tw_{1-r}, the third from the definition of f_Σ, the fourth from the general property (23) and the fact that $\omega = \omega_s = (\chi_{\rm cyc})_s$ and the fifth from the fundamental interpolation property of the function $f_{\Sigma,\omega^{1-r}\rho}(T)$. This completes the proof of Theorem 3.3.

4. Selmer complexes for critical motives

In this section we use the notation F, T, Σ and T_F introduced in §3.2. We recall (from Bloch and Kato [5, Prop. 5.4]) that if T is the p-adic Tate module of an elliptic curve E that is defined over k and has finite (classical) Tate-Shafarevic group $Ш(E_{/F})$ over F, then the (Bloch-Kato) Tate-Shafarevic group $Ш(T_F)$ that is defined in §3.1 is canonically isomorphic to $Ш(E_{/F})_p$. However, in this case the idempotent $e_0(R\Gamma_c(\mathcal{O}_{k,\Sigma}, T_F))$ that occurs in Theorem 2.1 can be shown to be equal to 0 and so the approach of §3.2 doesn't give anything new. With this in mind, we now explain how the Selmer complexes of Nekovář provide a more useful way to apply Theorem 2.1 in the setting of such critical motives.

For simplicity we will assume henceforth that $k = \mathbb{Q}$ (but we reassure the reader that there are analogous results for any number field k). We fix an algebraic closure $\mathbb{Q}_p^c$ of $\mathbb{Q}_p$ and set $G_{\mathbb{Q}_p} := G_{\mathbb{Q}_p^c/\mathbb{Q}_p}$.

4.1. The complexes.

We fix a critical motive M over $\mathbb{Q}$ and an odd prime p, write V for the p-adic realisation of M and assume that V has good ordinary reduction at p. We recall that under these hypotheses there exists a unique $G_{\mathbb{Q}_p}$-stable $\mathbb{Q}_p$-subspace V^0 of V with the property that the natural map $D_{\rm dR}(\mathbb{Q}_p, V^0) \to D_{\rm dR}(\mathbb{Q}_p, V) \twoheadrightarrow t_p(V)$ induces an identification $D_{\rm dR}(\mathbb{Q}_p, V^0) \cong t_p(V)$ (cf. [26]). We fix a full $G_{\mathbb{Q}}$-stable $\mathbb{Z}_p$-sublattice T of V and thereby obtain a full $G_{\mathbb{Q}_p}$-stable $\mathbb{Z}_p$-sublattice of V^0 by setting $T^0 := T \cap V^0$.

In the next result we assume to be given a finite abelian Galois extension $F/\mathbb{Q}$ and set $T_F := \mathbb{Z}_p[G_{F/\mathbb{Q}}] \otimes_{\mathbb{Z}_p} T$, resp. $T_F^0 := \mathbb{Z}_p[G_{F/\mathbb{Q}}] \otimes_{\mathbb{Z}_p} T^0$, which we regard as a module over $\mathbb{Z}_p[G_{F/\mathbb{Q}}] \times \mathbb{Z}_p[G_{\mathbb{Q}}]$, resp. $\mathbb{Z}_p[G_{F/\mathbb{Q}}] \times \mathbb{Z}_p[G_{\mathbb{Q}_p}]$, in the following way: $G_{F/\mathbb{Q}}$ acts via multiplication on the left and each $\sigma \in G_{\mathbb{Q}}$, resp. $\sigma \in G_{\mathbb{Q}_p}$, acts as $x \otimes_{\mathbb{Z}_p} t \mapsto x\bar\sigma^{-1} \otimes_{\mathbb{Z}_p} \sigma(t)$ for each $x \in \mathbb{Z}_p[G_{F/\mathbb{Q}}]$ and $t \in T$, resp. $t \in T^0$, where $\bar\sigma$ denotes the image of σ in $G_{F/\mathbb{Q}}$. We fix a finite set of places Σ of $\mathbb{Q}$ that contains both p and the archimedean place as well as all places which ramify in $F/\mathbb{Q}$ and all at which M has bad reduction and set $U_\Sigma := \mathrm{Spec}(\mathbb{Z}_\Sigma)$. Then T_F, resp. T_F^0, can be regarded as an étale (pro-)sheaf of free $\mathbb{Z}_p[G_{F/\mathbb{Q}}]$-modules on U_Σ, resp. on $\mathrm{Spec}(\mathbb{Q}_p)$, and so we may use the Selmer complex $\mathrm{SC}(U_\Sigma, T_F, T_F^0)$ of Nekovář that is considered by Fukaya and Kato in [14, 4.1.2.]. We also set $V_F := \mathbb{Q}_p \otimes_{\mathbb{Z}_p} T_F$, $V_F^*(1) := \mathbb{Q}_p \otimes_{\mathbb{Z}_p} T_F^*(1)$, $W_F := V_F/T_F$, $W_F^*(1) := V_F^*(1)/T_F^*(1)$, $V_F^0 := \mathbb{Q}_p \otimes_{\mathbb{Z}_p} T_F^0$ and $(V_F^0)^*(1) := \mathbb{Q}_p \otimes_{\mathbb{Z}_p} (T_F^0)^*(1)$, each linear dual being endowed with the actions of $\mathbb{Z}_p[G_{F/\mathbb{Q}}] \times \mathbb{Z}_p[G_{\mathbb{Q},\Sigma}]$, resp. $\mathbb{Z}_p[G_{F/\mathbb{Q}}] \times \mathbb{Z}_p[G_{\mathbb{Q}_p}]$, that are induced from the given actions on T_F and T_F^0 in the manner described at the beginning of §3.

Proposition 4.1. *We set* $C := \mathrm{SC}(U_\Sigma, T_F, T_F^0)[-1]$ *and assume that* V_F *satisfies the following condition:*

($*$) *the spaces* $H^0(\mathbb{Q}_p, V_F/V_F^0)$, $H^0(\mathbb{Q}_p, (V_F^0)^*(1))$ *and* $H^0(\mathbb{Q}_\ell, V_F)$ *for each prime* $\ell \notin \Sigma$ *all vanish.*

Then each of the following assertions are valid.

(i) *If* $H^0(\mathbb{Z}_\Sigma, W_F^*(1))$ *vanishes, then* $H^1(C)$ *vanishes and* C *is an object of* $D^{\mathrm{p,ad}}(\mathbb{Z}_p[G_{F/\mathbb{Q}}])$.

(ii) *There exists a (finitely generated)* $\mathbb{Z}_p[G_{F/\mathbb{Q}}]$*-submodule* B *of* $H^3(C)$ *and a surjective homomorphism of* $\mathbb{Z}_p[G_{F/\mathbb{Q}}]$*-modules of the form* $\beta : B \twoheadrightarrow \mathrm{Sel}(T_F^*(1))^\vee$ *where* $\mathrm{Sel}(T_F^*(1))$ *is the (Bloch-Kato) Selmer group that is defined in §3.1 and* $\ker(\beta)$ *is finite. In particular, there are inclusions* $\mathrm{Ann}_{\mathfrak{A}}(H^3(C)_{\mathrm{tor}}) \subseteq \mathrm{Ann}_{\mathfrak{A}}(\text{Ш}(T_F^*(1))) = \mathrm{Ann}_{\mathfrak{A}}(\text{Ш}(T_F))$ *and* $\mathrm{Ann}_{\mathfrak{A}}(H^3(C)) \subseteq \mathrm{Ann}_{\mathfrak{A}}(\mathrm{Sel}(T_F^*(1)))$.

(iii) $\mathrm{Ann}_{\mathfrak{A}}(H^0(\mathbb{Z}_\Sigma, W_F)) \subseteq \mathrm{Ann}_{\mathfrak{A}}(H^2(C)_{\mathrm{tor}})$.

(iv) *The idempotent* $e_0(C)$ *in Theorem 2.1 (with* C *as above and* $\mathfrak{A} = \mathbb{Z}_p[G_{F/\mathbb{Q}}]$*) is equal to the sum of all primitive idempotents of* $\mathbb{Q}_p[G_{F/\mathbb{Q}}]$ *which annihilate the space* $H^1_f(\mathbb{Q}, V_F)$.

Proof. We set $\mathfrak{A} = \mathbb{Z}_p[G_{F/\mathbb{Q}}]$ and $A = \mathbb{Q}_p[G_{F/\mathbb{Q}}]$.

Now, since p is odd, the $\mathfrak{A}$-module $H^0(G_{\mathbb{C}/\mathbb{R}}, T_F)$ is projective and the complex $R\Gamma(\mathbb{Q}_p, T_F^0)$ is an object of $D^{\mathrm{p}}(\mathfrak{A})$. It is also well known that $R\Gamma_c(U_{k,\Sigma}, T_F)$ is an object of $D^{\mathrm{p}}(\mathfrak{A})$ and so, as C is defined by means of an exact triangle

$$R\Gamma_c(U_{k,\Sigma}, T_F)[-1] \to C \to H^0(G_{\mathbb{C}/\mathbb{R}}, T_F)[-1] \oplus R\Gamma(\mathbb{Q}_p, T_F^0)[-1] \to R\Gamma_c(U_{k,\Sigma}, T_F)$$

(cf. [14, (4.1)]), it is clear that C belongs to $D^{\mathrm{p}}(\mathfrak{A})$. In addition, an easy analysis of the long exact cohomology sequence of this exact triangle shows that C is acyclic outside degrees $2, 3$ and 4 and that there is a surjective homomorphism

$$H^0(\mathbb{Z}_\Sigma, W_F^*(1))^\vee \cong H^3_c(\mathbb{Z}_\Sigma, T_F) \twoheadrightarrow H^4(C), \tag{24}$$

where the isomorphism is by (8). Hence if $H^0(\mathbb{Z}_\Sigma, W_F^*(1))$ vanishes, then $H^4(C)$ vanishes and so C is acyclic outside degrees 2 and 3. It is also known that the Euler characteristic of $A \otimes_{\mathfrak{A}} C$ in $K_0(A)$ vanishes (see, for example, [14, (4.2)]). In particular, we have by now shown that C satisfies all of the assumptions (ad_1), (ad_2), (ad_3) and (ad_4) and proved claim (i).

To prove claim (ii) we write $\tilde{C}$ for the complex $\mathrm{SC}(T_F, T_F^0)$ of $\mathfrak{A}$-modules that is defined in [14, (4.3)]. Then the long exact cohomology sequence of the exact triangle [14, (4.5)] gives a surjective homomorphism of $\mathfrak{A}$-modules

$$H^3(C) \twoheadrightarrow H^2(\tilde{C}) \tag{25}$$

which has finite kernel. We also recall that in [14, 4.2.28] Fukaya and Kato define Selmer modules $\mathrm{Sel}_{(i)}(W_F^*(1))$ for $i = 1, 2$ and that $\mathrm{Sel}_{(2)}(W_F^*(1))$ coincides with the group $\mathrm{Sel}(T_F^*(1))$ that occurs in §3.1. In addition, since (by our assumption ($*$)) both of the spaces $H^0(\mathbb{Q}_p, V_F/V_F^0)$ and $H^0(\mathbb{Q}_p, (V_F^0)^*(1))$ vanish, the result of [14, Lem. 4.2.32] applies in our case. When combined with the definitions of the groups $\mathrm{Sel}_{(i)}(W_F^*(1))$ the latter result implies that there exists an injection $\mathrm{Sel}(T_F^*(1)) = \mathrm{Sel}_{(2)}(W_F^*(1)) \hookrightarrow$

$\mathrm{Sel}_{(1)}(W_F^*(1))$ with finite cokernel and hence also a surjective homomorphism with finite kernel of the form

$$\mathcal{X} \twoheadrightarrow \mathrm{Sel}(T_F^*(1))^\vee$$

where we set $\mathcal{X} := \mathrm{Sel}_{(1)}(W_F^*(1))^\vee$. Now, from the proof of [14, Prop. 4.2.35(2)], one also knows that local and global duality combine to induce an isomorphism between $\mathcal{X}$ and the kernel of a natural homomorphism $H^2(\tilde{C}) \to H^2(\mathbb{Q}_p, T_F^0)$ and in particular therefore imply the existence of an injective homomorphism of $\mathfrak{A}$-modules $\iota : \mathcal{X} \hookrightarrow H^2(\tilde{C})$. We therefore obtain the first assertion of claim (ii) by simply taking B to be the pre-image under (25) of $\mathrm{im}(\iota)$. The second assertion of claim (ii) then follows from the obvious inclusions $\mathrm{Ann}_{\mathfrak{A}}(H^3(C)) \subseteq \mathrm{Ann}_{\mathfrak{A}}(B) \subseteq \mathrm{Ann}_{\mathfrak{A}}(\mathrm{Sel}(T_F^*(1)))$ and $\mathrm{Ann}_{\mathfrak{A}}(H^3(C)_{\mathrm{tor}}) \subseteq \mathrm{Ann}_{\mathfrak{A}}(B_{\mathrm{tor}}) \subseteq \mathrm{Ann}_{\mathfrak{A}}(Ш(T_F^*(1))) = \mathrm{Ann}_{\mathfrak{A}}(Ш(T_F))$ where the last inclusion follows from the fact that, since β has finite kernel, it restricts to give a surjection $B_{\mathrm{tor}} \twoheadrightarrow (\mathrm{Sel}(T_F^*(1))^\vee)_{\mathrm{tor}} \cong (\mathrm{Sel}(T_F^*(1))_{\mathrm{cotor}})^\vee \cong Ш(T_F^*(1))^\vee \cong Ш(T_F)$.
We next set $C' := \mathrm{SC}(U_\Sigma, T_F^*(1), (T/T^0)_F^*(1))[-1]$. Then the long exact cohomology sequence of the exact triangle of [14, Prop. 4.4.1(1)] gives an exact sequence

$$\bigoplus_{\ell \notin \Sigma} H^0(\mathbb{Q}_\ell, T_F) \to H^2(C) \to H^2(R\,\mathrm{Hom}_{\mathbb{Z}_p}(C', \mathbb{Z}_p[-5])).$$

The assumption that $H^0(\mathbb{Q}_\ell, V_F)$ vanishes for each prime $\ell \notin \Sigma$ implies both that the space $H^0(\mathbb{Z}_\Sigma, V_F)$ vanishes, and hence that the module $H^0(\mathbb{Z}_\Sigma, W_F)$ is $\mathbb{Z}_p$-torsion, and also that the module $\bigoplus_{\ell \notin \Sigma} H^0(\mathbb{Q}_\ell, T_F)$ vanishes. From the spectral sequence $\mathrm{Ext}^a_{\mathbb{Z}_p}(H^b(C'), \mathbb{Z}_p) \Rightarrow H^{a-b+5}(R\,\mathrm{Hom}_{\mathbb{Z}_p}(C', \mathbb{Z}_p[-5]))$ we also know that $H^2(R\,\mathrm{Hom}_{\mathbb{Z}_p}(C', \mathbb{Z}_p[-5]))_{\mathrm{tor}}$ is naturally isomorphic to $(H^4(C')_{\mathrm{tor}})^\vee$. The above exact sequence thus induces an injective homomorphism of $\mathfrak{A}$-modules

$$H^2(C)_{\mathrm{tor}} \hookrightarrow (H^4(C')_{\mathrm{tor}})^\vee = H^4(C')^\vee \hookrightarrow H^0(\mathbb{Z}_\Sigma, W_F) \tag{26}$$

where the equality and second injection both follow from the surjection (24) with T_F replaced by $T_F^*(1)$ and T_F^0 by $(T/T^0)_F^*(1)$. It is therefore clear that $\mathrm{Ann}_{\mathfrak{A}}(H^0(\mathbb{Z}_\Sigma, W_F)) \subseteq \mathrm{Ann}_{\mathfrak{A}}(H^2(C)_{\mathrm{tor}})$, as required to prove claim (iii).
Finally we note that our assumption $(*)$ combines with the argument used to prove [14, Lem. 4.1.8] (which is valid in this case since V is de Rham as a representation of $G_{\mathbb{Q}_p}$), to imply that the space $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} H^2(C) = H^1(\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathrm{SC}(U_\Sigma, T_F, T_F^0))$ is isomorphic to $H^1(C_f(\mathbb{Q}, V_F))$ where $C_f(\mathbb{Q}, V_F)$ is the complex defined in [14, §2.4.2]. Claim (iv) therefore follows from the isomorphism $H^1(C_f(\mathbb{Q}, V_F)) \cong H^1_f(\mathbb{Q}, V_F)$ of [14, (2.6)]. □

Remark 4.2. It can be shown that the result of Proposition 4.1 extends in a straightforward manner to the setting of those p-adic representations (over arbitrary number fields) that are both de Rham and also satisfy the 'condition of Dąbrowski-Panchishkin' discussed in [9, §6.2].

4.2. **p-adic Zeta functions.** We fix M, F, T_F and C as in §4.1 and also set $\mathfrak{A} = \mathbb{Z}_p[G_{F/\mathbb{Q}}]$, $A = \mathbb{Q}_p[G_{F/\mathbb{Q}}]$ and $\mathfrak{A}_0 = \mathfrak{A}_0(C)(= \mathfrak{A}e_0(C))$. We recall that in [14, Th. 4.1.12] Fukaya and Kato have proved that the (local and global) non-commutative Tamagawa number conjectures for the pair $(M_F, \mathfrak{A})$ combine to predict that the fractional $\mathfrak{A}_0$-ideal $\mathrm{Det}_{\mathfrak{A}_0}(C_0) \subset A$ is generated by an element $\zeta_\beta(U_\Sigma, T_F, T_F^0)$ of A with the following property: at each homomorphism $\rho : G_{F/k} \to \mathbb{Q}_p^{c\times}$ with $\tilde{\rho}(e_0(C)) = 1$ the image of $\zeta_\beta(U_\Sigma, T_F, T_F^0)$ under $\tilde{\rho}$ is equal to the value at $s=0$ of the $\mathbb{C}[G_{F/\mathbb{Q}}]$-valued complex L-function of M_F multiplied by a suitable combination of natural regulators, periods and Euler factors. When combined with Proposition 4.1 and the inclusion (2) this observation predicts the existence of explicit conjectural annihilators for the $\mathfrak{A}$-modules $Ш(T_F)$ and $\mathrm{Sel}(T_F^*(1))$. By means of an explicit example, in the next subsection we shall consider in greater detail the case of abelian varieties.

4.3. **Abelian varieties.** In this subsection we fix an abelian variety A that is defined over $\mathbb{Q}$ and has good ordinary reduction at p and write A^t for the dual abelian variety. We write T for the p-adic Tate module of A^t and set $V := \mathbb{Q}_p \otimes_{\mathbb{Z}_p} T$ and $W := V/T$. We also fix a finite abelian extension F of $\mathbb{Q}$ and use the same notation Σ, $T_F, T_F^0, V_F, V^0, V_F^0$ and W_F as in §4.1 (in this case V^0 can in fact be identified with the $\mathbb{Q}_p$-space spanned by the p-adic Tate module of the formal group of A^t [9, Exam. 6.3]). We regard the motive $M_F := h^1(A_{/F})(1)$ as defined over $\mathbb{Q}$ and with coefficients $\mathbb{Q}[G_{F/\mathbb{Q}}]$.

We fix an isomorphism of fields $\mathbb{C} \cong \mathbb{C}_p$ and henceforth use this to identify the groups $\mathrm{Hom}(G_{F/\mathbb{Q}}, \mathbb{Q}^{c\times})$ and $G_{F/\mathbb{Q}}^\wedge := \mathrm{Hom}(G_{F/\mathbb{Q}}, \mathbb{Q}_p^{c\times})$. We fix an isomorphism β as in [14, 4.2.24] and then define $L_{\Sigma,\beta}(A_{/F}, 1)$ to be the unique element of $\mathbb{C}_p[G_{F/\mathbb{Q}}]$ such that for every ρ in $G_{F/\mathbb{Q}}^\wedge$ one has

$$\tilde{\rho}(L_{\Sigma,\beta}(A_{/F},1)) = \frac{L_\Sigma(A,\check{\rho},1)}{\Omega_\infty(M(\check{\rho}))} \cdot \Omega_{p,\beta}(M(\check{\rho})) \cdot \Gamma_{\mathbb{Q}_p}(V_F^0)^{-1} \cdot \frac{P_{L,p}((V_{\check{\rho}}^0)^*(1),1)}{P_{L,p}(V_{\check{\rho}}^0,1)}. \tag{27}$$

Here $L_\Sigma(A,\check{\rho},1)$ denotes the value at $s=1$ of the Σ-truncated Hasse-Weil L-function of A twisted by the contragredient representation $\check{\rho}$, $M(\check{\rho})$ is the tensor product of $h^1(A)(1)$ with the Artin motive associated to $\check{\rho}$, $V_{\check{\rho}}^0$ is the representation $V^0 \otimes V_{\check{\rho}}$ where $V_{\check{\rho}}$ is a representation of character $\check{\rho}$ and the archimedean and p-adic periods $\Omega_\infty(M(\check{\rho}))$ and $\Omega_{p,\beta}(M(\check{\rho}))$, non-zero rational number $\Gamma_{\mathbb{Q}_p}(V_F^0)$ and Euler factors $P_{L,p}(-,s)$ are all as defined by Fukaya and Kato in [14, 4.1.11, 3.3.6, Lem. 4.1.7]. (For a more explicit description of the formula (27) in the case that A is an elliptic curve see §4.4.)
If B denotes either A or A^t, then we write $Ш(B_{/F})$ and $\mathrm{Sel}_p(B_{/F})$ for the (classical) Tate-Shafarevic and pro-p Selmer groups of B over F respectively.

Proposition 4.3. *Let A, p, F and Σ be as above. We set $G = G_{F/\mathbb{Q}}$ and write $I_p(A_{/F})$ for the ideal $\mathbb{Z}_p[G] \cap \bigcap_\rho \ker(\tilde{\rho})$ where ρ runs over all elements of $\mathrm{Hom}(G, \mathbb{Q}_p^{c\times})$ for which $L_\Sigma(A,\check{\rho},1)$ vanishes. We assume that p does not divide $|A(F)_{\mathrm{tor}}|$, that $Ш(A^t_{/F})$ is finite and that the relevant special cases of all of the conjectures discussed by Fukaya and Kato in* [14, §2 and §3] *are valid (see Remark 4.4(i) for more details about these conjectures).*

(i) *If g is the minimal number of generators of the $\mathbb{Z}_p[G]$-module $(A^t(F)_{\mathrm{tor},p})^\vee$, then both*

$$I_p(A_{/F})^{g+2}\mathrm{Ann}_{\mathbb{Z}_p[G]}(A^t(F)_{\mathrm{tor},p})^g \cdot L_{\Sigma,\beta}(A_{/F},1) \subseteq \mathrm{Ann}_{\mathbb{Z}_p[G]}(Ш(A_{/F})_p) \tag{28}$$

and

$$I_p(A_{/F})^{g+3}\mathrm{Ann}_{\mathbb{Z}_p[G]}(A^t(F)_{\mathrm{tor},p})^g \cdot L_{\Sigma,\beta}(A_{/F},1) \subseteq \mathrm{Ann}_{\mathbb{Z}_p[G]}(\mathrm{Sel}_p(A_{/F})). \tag{29}$$

(ii) *If p does not divide $|A^t(F)_{\rm tor}|$, then*

$$\mathbb{Z}_p[G]\cdot L_{\Sigma,\beta}(A_{/F},1) = \mathrm{Fit}_{\mathbb{Z}_p[G]}(H^2(\mathrm{SC}(U_\Sigma,T_F,T_F^0))) \subseteq \mathrm{Ann}_{\mathbb{Z}_p[G]}(\mathrm{Sel}_p(A_{/F})).$$

Proof. At the outset we recall that if $Ш(A_{/F})$ is finite, then [5, Prop. 5.4] implies that $\mathrm{Sel}(T_F^*(1))$ and $Ш(T_F^*(1))$ are canonically isomorphic to $\mathrm{Sel}_p(A_{/F})$ and $Ш(A_{/F})_p$ respectively and that the space $H^1_f(\mathbb{Q},V_F)\cong H^1_f(F,V)$ identifies with $\mathbb{Q}_p\otimes A^t(F) = \mathbb{Q}_p\otimes_{\mathbb{Q}} H^1_f(M_F)$. We also recall that the assumption $(*)$ of Proposition 4.1 is satisfied in this case (cf. [9, Exam. 6.3]).
We set $C := \mathrm{SC}(U_\Sigma,T_F,T_F^0)[-1]$ and $C^* := R\,\mathrm{Hom}_{\mathbb{Z}_p}(C,\mathbb{Z}_p[-4])$ and also $\mathfrak{A} := \mathbb{Z}_p[G]$, $e_0 := e_0(C)$ and $\mathfrak{A}_0 := \mathfrak{A}_0(C)$. The key point in this argument is that if the conjectures discussed in [14, §2 and §3] are valid in the relevant cases, then one has

$$\mathrm{Det}_{\mathfrak{A}_0}(C_0) = \mathfrak{A}_0\cdot L_{\Sigma,\beta}(A_{/F},1) = \mathfrak{A}\cdot L_{\Sigma,\beta}(A_{/F},1) \tag{30}$$

where $L_{\Sigma,\beta}(A_{/F},1)$ is the element of $\mathbb{C}_p[G]$ that is defined via the interpolation property (27). The first equality in (30) is a direct consequence of [14, Th. 4.1.12] and the explicit definition of $L_{\Sigma,\beta}(A_{/F},1)$ via (27). To verify the second equality of (30) we note first that if the Deligne-Beilinson Conjecture (in the form of [14, 2.2.8(1)]) is valid with $M = M_F$ and $K = \mathbb{Q}[G]$, then a homomorphism $\rho\in G^\wedge$ belongs to $G_0^\wedge := \{\rho\in G^\wedge : L_\Sigma(A,\check\rho,1)\neq 0\}$ if and only if $e_\rho(\mathbb{Q}_p^c\otimes_{\mathbb{Q}_p} H^1_f(\mathbb{Q},V_F)) = 0$, or equivalently (by Proposition 4.1(iv)) $e_0e_\rho = e_\rho$. One therefore has

$$\begin{aligned}
L_{\Sigma,\beta}(A_{/F},1) &= \sum_{\rho\in G^\wedge}\rho(L_{\Sigma,\beta}(A_{/F},1))e_\rho\\
&= \sum_{\rho\in G_0^\wedge}\rho(L_{\Sigma,\beta}(A_{/F},1))e_\rho\\
&= \sum_{\rho\in G_0^\wedge}\rho(L_{\Sigma,\beta}(A_{/F},1))e_0e_\rho\\
&= e_0\sum_{\rho\in G_0^\wedge}\rho(L_{\Sigma,\beta}(A_{/F},1))e_\rho\\
&= e_0L_{\Sigma,\beta}(A_{/F},1)
\end{aligned}$$

and hence $\mathfrak{A}\cdot L_{\Sigma,\beta}(A_{/F},1) = \mathfrak{A}\cdot e_0L_{\Sigma,\beta}(A_{/F},1) = \mathfrak{A}_0\cdot L_{\Sigma,\beta}(A_{/F},1)$, as claimed in (30). The fact that $G_0^\wedge$ is equal to $\{\rho\in G^\wedge : e_0e_\rho = e_\rho\}$ also implies $I_p(A_{/F})$ is equal to the ideal $I_{\mathfrak{A}_0}$ that occurs in Theorem 2.1.
Next we recall that the modules $H^0(\mathbb{Z}_\Sigma, W_F^*(1))$ and $H^0(\mathbb{Z}_\Sigma, W_F)$ identify with $A(F)_{{\rm tor},p}$ and $A^t(F)_{{\rm tor},p}$ respectively. In particular, since p is assumed not to divide $|A(F)_{\rm tor}|$, the hypothesis of Proposition 4.1(i) is satisfied in this case. Also, since C is concentrated in degrees 2 and 3, the discussion just prior to Lemma 2.8 implies that $H^3(C^*)$ is isomorphic to $(H^2(C)_{\rm tor})^\vee$. From the injection (26) we therefore obtain a surjection $(A^t(F)_{{\rm tor},p})^\vee\cong H^0(\mathbb{Z}_\Sigma,W_F)^\vee \twoheadrightarrow (H^2(C)_{\rm tor})^\vee\cong H^3(C^*)$ and hence an inequality $g\geq g(C^*)$ where $g(C^*)$ is the integer defined in Theorem 2.1. Applying

both (2) and Proposition 4.1(ii) and (iii) in this context therefore gives inclusions

$$\begin{aligned} & I_p(A_{/F})^{g+2}\mathrm{Ann}_{\mathbb{Z}_p[G]}(A^t(F)_{\mathrm{tor},p})^g \cdot \mathrm{Det}_{\mathfrak{A}_0}(C_0) \\ \subseteq\, & (I_{\mathfrak{A}_0})^{g+2}\mathrm{Ann}_{\mathfrak{A}}(H^2(C)_{\mathrm{tor}})^g \cdot \mathrm{Det}_{\mathfrak{A}_0}(C_0) \\ \subseteq\, & (I_{\mathfrak{A}_0})^{g(C^*)+n(C^*)}\mathrm{Ann}_{\mathfrak{A}}(H^2(C)_{\mathrm{tor}})^{g(C^*)} \cdot \mathrm{Det}_{\mathfrak{A}_0}(C_0) \\ \subseteq\, & \mathrm{Ann}_{\mathfrak{A}}(H^3(C)_{\mathrm{tor}}) \\ \subseteq\, & \mathrm{Ann}_{\mathfrak{A}}(\text{Ш}(T^*_F(1))) \\ =\, & \mathrm{Ann}_{\mathfrak{A}}(\text{Ш}(A_{/F})_p). \end{aligned}$$

This observation implies that (28) follows directly upon combining the last displayed inclusion with the (conjectural) equality (30). In a similar way one obtains (29) by multiplying the above displayed inclusions by $I_p(A_{/F}) = I_{\mathfrak{A}_0}$, using (30), noting that $I_{\mathfrak{A}_0}$ annihilates $H^3(C)_{\mathrm{tf}} \subset H^3(\mathbb{Q}_p \otimes_{\mathbb{Z}_p} C)$ so $I_{\mathfrak{A}_0} \cdot \mathrm{Ann}_{\mathfrak{A}}(H^3(C)_{\mathrm{tor}}) \subseteq \mathrm{Ann}_{\mathfrak{A}}(H^3(C))$, and then recalling that $\mathrm{Ann}_{\mathfrak{A}}(H^3(C)) \subseteq \mathrm{Ann}_{\mathfrak{A}}(\mathrm{Sel}(T^*_F(1))) = \mathrm{Ann}_{\mathfrak{A}}(\mathrm{Sel}_p(A_{/F}))$ by the final assertion of Proposition 4.1(ii). This proves claim (i).
If p does not divide $|A^t(F)_{\mathrm{tor}}|$, then $H^0(\mathbb{Z}_\Sigma, W_F)$ vanishes and so the injective homomorphism (26) implies that $H^2(C)_{\mathrm{tor}}$ vanishes. Since C is acyclic outside degrees 2 and 3 (by Proposition 4.1(i)), the equality in claim (ii) is therefore obtained by substituting (30) into (3) and noting that $H^3(C) = H^3(\mathrm{SC}(U_\Sigma, T_F, T_F^0)[-1]) = H^2(\mathrm{SC}(U_\Sigma, T_F, T_F^0))$. The remaining inclusion in claim (ii) is then valid because $\mathrm{Fit}_{\mathfrak{A}}(H^3(C)) \subseteq \mathrm{Ann}_{\mathfrak{A}}(H^3(C)) \subseteq \mathrm{Ann}_{\mathfrak{A}}(\mathrm{Sel}_p(A_{/F}))$ where the last inclusion follows from Proposition 4.1(ii). □

Remarks 4.4.
(i) The conjectures from [14, §2 and §3] that are being assumed in Proposition 4.3 are as follows: [14, 2.2.8] (the Deligne-Beilinson conjecture) for the motive M_F and algebra $K = \mathbb{Q}[G]$; [14, Conj. 2.3.2] (the non-commutative Tamagawa number conjecture), [14, Conj. 3.4.3] (the local non-commutative Tamagawa number conjecture) and [14, Conj. 3.5.5] (compatibility of the above conjectures with the relevant functional equation), in each case for the ring $\Lambda = \mathbb{Z}_p[G]$ and sheaf $T = T_F$.
(ii) In certain special cases the inclusions of Proposition 4.3 become even more explicit. For example, if $A^t(F)$ is finite, then [14, 2.2.8(1)] predicts that $L_\Sigma(A, \check{\rho}, 1) \neq 0$ for all $\rho \in G^\wedge_{F/\mathbb{Q}}$ and hence that $I_p(A_{/F}) = \mathbb{Z}_p[G]$. For a more explicit version of Proposition 4.3(ii) see Proposition 4.5 below.
(iii) The equality of Proposition 4.3(ii) is a 'strong main conjecture' of the kind that Mazur and Tate explicitly ask for in [20, Remark after Conj. 3]. Under certain additional hypotheses on A, F and p it is also possible to precisely relate the ideals $\mathrm{Fit}_{\mathbb{Z}_p[G]}(H^2(\mathrm{SC}(U_\Sigma, T_F, T_F^0)))$ and $\mathrm{Fit}_{\mathbb{Z}_p[G]}(\mathrm{Sel}_p(A_{/F})^\vee)$ (for further details see [6, §12.2]). In particular, it would be interesting to know the precise relation between the equality of Proposition 4.3(ii) and the explicit conjectural formulas for the Fitting ideals of Selmer groups that are formulated (in certain special cases) by Kurihara in [18].
(iv) It can be shown that the observations made in Proposition 4.1(ii) imply that the ideal $\mathrm{Fit}_{\mathbb{Z}_p[G]}(H^2(\mathrm{SC}(U_\Sigma, T_F, T_F^0)))$ which occurs in Proposition 4.3(ii) is a subset of $I_{G,p}^{\mathrm{rk}(A(\mathbb{Q}))}$ where $I_{G,p}$ is the augmentation ideal of $\mathbb{Z}_p[G]$ and $\mathrm{rk}(A(\mathbb{Q}))$ is the rank of the

Mordell-Weil group of $A_{/\mathbb{Q}}$. Under the conditions of Proposition 4.3(ii) the element $L_{\Sigma,\beta}(A_{/F},1)$ therefore belongs to $I_{G,p}^{\mathrm{rk}(A(\mathbb{Q}))}$. Further, under these hypotheses, the techniques of [6, §9] allow one to formulate an explicit conjecture for the residue class of $L_{\Sigma,\beta}(A_{/F},1)$ modulo $I_{G,p}^{\mathrm{rk}(A(\mathbb{Q}))+1}$. Such conjectural formulas constitute a natural generalisation of the congruences for modular symbols that are conjectured by Mazur and Tate in [20] and will be discussed elsewhere.

4.4. **Elliptic curves.** We now make the predictions of Proposition 4.3 more explicit in the special case that A is equal to an elliptic curve E. To do this we let u in $\mathbb{Z}_p^\times$ be the unit root of the polynomial $1 - a_pX + pX^2$ where $a_p := p + 1 - |\tilde{E}_p(\mathbb{F}_p)|$ with $\tilde{E}_p$ the reduction of E modulo p. For each $\rho \in G^\wedge_{F/\mathbb{Q}}$ we set $V_\rho := e_\rho\mathbb{Q}_p^c[G]$, regarded as a module over $G_\mathbb{Q}$ via the natural projection $G_\mathbb{Q} \twoheadrightarrow G$, and define a polynomial

$$P_p(\rho, X) := \det_{\mathbb{Q}_p^c}(1 - \varphi_p \cdot X \mid H^0(I_p, V_\rho)) \in \mathbb{Q}_p^c[X]$$

where I_p is the inertia subgroup of p in G and φ_p the geometric frobenius of p in G/I_p. We write p^{f_ρ} for the p-part of the conductor of ρ and $\epsilon_p(\rho)$ for the local ϵ-factor of ρ at the prime p. Regarding $\mathbb{Q}^c$ as a subfield of $\mathbb{C}$ we write τ for the element of $G_\mathbb{Q}$ obtained by restricting complex conjugation and define integers $d_+(\rho)$ and $d_-(\rho)$ by setting $d_\pm(\rho) := \dim_{\mathbb{Q}_p^c}(\{v \in V_\rho : \tau(v) = \pm v\})$. We also define the periods $\Omega_+(E)$ and $\Omega_-(E)$ of E by setting

$$\Omega_\pm(E) := \int_{\gamma^\pm} \omega$$

where ω is the Néron differential and $\gamma^\pm$ is a generating element for the (free rank one $\mathbb{Z}$-)submodule of $H_1(E(\mathbb{C}),\mathbb{Z})$ upon which complex conjugation acts as multiplication by ± 1. Then the same argument as used in the proof of [10, Prop. 7.8] shows that the expression on the right hand side of (27) is equal to

$$\frac{L_\Sigma(E,\check{\rho},1)}{\Omega_+(E)^{d_+(\rho)}\Omega_-(E)^{d_-(\rho)}}\epsilon_p(\rho)u^{-f_\rho}\frac{P_p(\rho,u^{-1})}{P_p(\check{\rho},up^{-1})}. \tag{31}$$

It is important to note at this stage that the argument of [10, Prop. 7.8] uses the explicit computation of [14, Th. 4.2.26] and hence relies upon choosing the isomorphism β (which arises in the definition of the p-adic period $\Omega_{p,\beta}(M(\check{\rho}))$) as in [14, 4.2.24].

Proposition 4.5. *Assume the notation and hypotheses of Proposition 4.3 in the case that A is equal to an elliptic curve E. Let $L_\Sigma(E_{/F},1)$ denote the element of $\mathbb{Q}_p^c[G]$ for which $\tilde{\rho}(L_\Sigma(E_{/F},1))$ is equal to the expression in (31) for every $\rho \in G^\wedge$. Then $L_\Sigma(E_{/F},1)$ belongs to $\mathbb{Z}_p[G]$ and annihilates each of the modules* $\mathrm{Sel}_p(E_{/F})$, $Ш(E_{/F})_p$ *and* $\mathbb{Z}_p \otimes E(F)$.

Proof. Since $L_\Sigma(E_{/F},1)$ is equal to the element $L_{\Sigma,\beta}(E_{/F},1)$ defined via (27) Proposition 4.3(ii) implies that $L_\Sigma(E_{/F},1)$ belongs to $\mathbb{Z}_p[G]$ and annihilates $\mathrm{Sel}_p(E_{/F})$. Now the module $Ш(E_{/F})_p \cong Ш(T_F^*(1))$ is a quotient of $\mathrm{Sel}(T_F^*(1)) \cong \mathrm{Sel}_p(E_{/F})$, whilst the Pontryagin dual of (12) (with $T = T_F$) implies that $(\mathbb{Z}_p \otimes E(F))^* \cong H^1_f(\mathbb{Q}, T_F^*(1))^* \cong (\mathbb{Q}_p/\mathbb{Z}_p \otimes_{\mathbb{Z}_p} H^1_f(\mathbb{Q}, T_F^*(1)))^\vee$ is isomorphic to a quotient of $\mathrm{Sel}_p(E_{/F})^\vee$. This implies that both $Ш(E_{/F})_p$ and $(\mathbb{Z}_p \otimes E(F))^*$ are annihilated by $L_\Sigma(E_{/F},1)$. Finally, since $\mathbb{Z}_p \otimes E(F)_{\mathrm{tor}}$ is assumed to vanish, the module $\mathbb{Z}_p \otimes E(F)$ is isomorphic to $(\mathbb{Z}_p \otimes E(F))^{**}$ and so is also annihilated by $L_\Sigma(E_{/F},1)$. □

Numerical evidence in favour of the containment $L_\Sigma(E_{/F}, 1) \in \mathbb{Z}_p[G]$ predicted by Proposition 4.5 (and also other related predictions) is described by Bley in [4].

References

[1] M. F. Atiyah, C. T. C. Wall, Cohomology of Groups In: 'Algebraic Number Theory' (Ed. J. W. S. Cassels and A. Frohlich), Academic Press, London, (1967) 94-115.

[2] G. Banaszak, Generalisation of the Moore exact sequence and the wild kernel for higher K-groups, Compositio Math. **86** (1993) 281-305.

[3] J. Barrett, Annihilating the Tate-Shafarevic Groups of Tate Motives, Ph.D. Thesis, King's College London, 2009.

[4] W. Bley, Numerical evidence for the equivariant Birch and Swinnerton-Dyer conjecture, preprint, 2009.

[5] S. Bloch, K. Kato, L-functions and Tamagawa numbers of motives, In: 'The Grothendieck Festschrift' vol. 1, Prog. Math. **86**, Birkhäuser, Boston, (1990) 333-400.

[6] D. Burns, Leading terms and values of equivariant motivic L-functions, Pure App. Math. Q. **6** (2010) 83-172.

[7] D. Burns, C. Greither, On the Equivariant Tamagawa Number Conjecture for Tate motives, Invent. math. **153** (2003) 303-359.

[8] D. Burns, C. Greither, Equivariant Weierstrass Preparation and values of L-functions at negative integers, Doc. Math., Extra volume (2003) 157-185.

[9] D. Burns, O. Venjakob, Leading terms of Zeta isomorphisms and p-adic L-functions in non-commutative Iwasawa theory, Doc. Math., Extra Volume (2006) 165-209.

[10] D. Burns, O. Venjakob, On descent theory and main conjectures in non-commutative Iwasawa theory, to appear in J. Inst. Math. Jussieu.

[11] B. Ferrero, L. Washington, The Iwasawa invariant μ_p vanishes for abelian number fields, Ann. Math. **109** (1979), 377-395.

[12] M. Flach, A generalisation of the Cassels-Tate pairing, J. Reine u. Angew. Math. **412** (1990) 113-127.

[13] J.-M. Fontaine, B. Perrin-Riou, Autour des conjectures de Bloch et Kato: cohomologie galoisienne et valeurs de fonctions L, In: Motives (Seattle), Proc. Symp. Pure Math. **55**, I, (1994) 599-706.

[14] T. Fukaya, K. Kato, A formulation of conjectures on p-adic zeta functions in non-commutative Iwasawa theory, Proc. St. Petersburg Math. Soc. Vol. XII, 1–85, Amer. Math. Soc. Transl. Ser. 2, **219**, Amer. Math. Soc., Providence, RI, 2006.

[15] K. Iwasawa, On the μ-invariants of $\mathbb{Z}_l$-extensions, In: Number Theory, Algebraic Geometry and Commutative Algebra, in honor of Y. Akizuki, Kinokuniya, Tokyo 1973, 1-11.

[16] K. Kato, Lectures on the approach to Iwasawa theory of Hasse-Weil L-functions via B_{dR}, Part I, In: Arithmetical Algebraic Geometry (ed. E. Ballico), Lecture Notes in Math. **1553** (1993) 50-163, Springer, New York, 1993.

[17] F. Knudsen, D. Mumford, The projectivity of the moduli space of stable curves I: Preliminaries on 'det' and 'Div', Math. Scand. **39** (1976) 19-55.

[18] M. Kurihara, Iwasawa theory and Fitting ideals, J. Reine u. Angew. Math. **561** (2003) 39-86.

[19] S. Lichtenbaum, Values of zeta functions, étale cohomology and algebraic K-theory, Lecture Notes in Math. **342** , 489-501, Springer Verlag, 1973.

[20] B. Mazur, J. Tate, Refined Conjectures of the Birch and Swinnerton-Dyer Type, Duke Math. J. **54** (1987) 711-750.

[21] J. S. Milne, Arithmetic Duality Theorems, Academic Press, Boston, 1986.

[22] J. Nekovář, Selmer complexes, Astérisque **310**, S.M.F., Paris, 2006.

[23] J. Neukirch, A. Schmidt, K. Wingberg, Cohomology of number fields, Grundlehren der mathematischen Wissenschaften, vol. 323, Springer, 2000.

[24] B. Oriat, Annulation de groupes de classes reelles, Nagoya Math. J. **81** (1981) 45-56.

[25] A. Parker, Non-commutative Fitting invariants and equivariant Tamagawa numbers, Ph.D. Thesis, King's College London, 2007.

[26] B. Perrin-Riou, Representations p-adiques ordinaires, Astérisque **223** (1994), 185-220.

[27] P. Schneider, Über gewisse Galoiscohomologiegruppen, Math. Zeit. **168** (1979) 181-205.
[28] V. P. Snaith, Relative K_0, annihilators, Fitting ideals and Stickelberger phenomena, Proc. London Math. Soc. **90** (2005) 545-590.
[29] J. Tate, Relations between K_2 and Galois Cohomology, Invent. math. **36** (1976) 257-274.
[30] J. Tate, Les Conjectures de Stark sur les Fonctions L d'Artin en $s = 0$ (notes par D.Bernardi et N. Schappacher), Progress in Math., **47**, Birkhäuser, Boston, 1984.
[31] L. C. Washington, Introduction to Cyclotomic Fields, Graduate Texts in Math. **83**, Springer, 1982.
[32] C. Weibel, The norm residue isomorphism theorem, J. Topology **2** (2009) 346-372.
[33] A. Wiles, The Iwasawa conjecture for totally real fields, Ann. Math. **131** (1990) 493-540.

King's College London,
Dept. of Mathematics,
London WC2R 2LS,
United Kingdom

King's College London,
Dept. of Mathematics,
London WC2R 2LS,
United Kingdom
david.burns@kcl.ac.uk