ON GLOBAL UNITS, IDEAL CLASS GROUPS AND LOOP OPERATORS

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ABSTRACT. We use the Tate-sequences of Ritter and Weiss to establish both non-trivial bounds on the number of independent Minkowski units in Galois extensions of number fields and also concrete links between the Galois structures of ideal class groups and the Krull-Schmidt decompositions of global unit groups.

1. INTRODUCTION

We fix a prime p and a finite Galois extension of number fields L/K of p-power degree. We set $G := \mathrm{Gal}(L/K)$ and, for a finite set of places S of K that contains the set S_K^∞ of all archimedean places, we write $\mathcal{O}_{L,S}$ for the ring of algebraic S-integers of L. We also write $\mathbb{F}_p$ for the finite field of cardinality p and, with μ_L denoting the group of roots of unity in L, define a $\mathbb{Z}_p[G]$-lattice by setting $U_{L,S} := \mathbb{Z}_p \otimes_{\mathbb{Z}} (\mathcal{O}_{L,S}^\times/\mu_L)$.

If the G-module $U_{L,S}/U_{L,S}^p$ has a direct summand isomorphic to $\mathbb{F}_p[G]^m$ for a natural number m, one says that L/K has a family of 'm independent Minkowski S-units'. In particular, by the Krull-Schmidt Theorem, the maximum size $m_{L/K,S}$ of a family of independent Minkowski S-units for L/K is well-defined. Following beautiful work of Ozaki [20] and of Hajir, Maire and Ramakrishna [12, 13], it is known that $m_{L/K,S}$ plays an important role in relation to studies of the tame Fontaine-Mazur Conjecture, tamely ramified pro-p extensions, deficiencies of p-class tower groups and the inverse Galois problem for the p-class field tower.

The explicit determination of $m_{L/K,S}$ is, however, a difficult problem and, aside from a few special cases that are dealt with by ad hoc techniques (see, for example, [11], [4] and [16]), there is still little known. With this in mind, our primary aim is to show that results of Ritter and Weiss [24] and Gruenberg and Weiss [8] can be combined to establish relatively explicit, non-trivial, upper and lower bounds for $m_{L/K,S}$ in the general case.

To state a representative result, we introduce some general notation. For any abelian group N we set $N/p := N/pN$, $N[p] := \{n \in N : p \cdot n = 0\}$ and $N_p := \mathbb{Z}_p \otimes_{\mathbb{Z}} N$. If Γ is a topologically finitely generated pro-p group, then we respectively write $\mathrm{d}(\Gamma)$ and $\mathrm{r}(\Gamma)$ for the cardinalities of a minimal set of (topological) generators and relations for Γ, and we denote the (topological) deficiency $\mathrm{d}(\Gamma) - \mathrm{r}(\Gamma)$ of Γ by $\mathrm{D}(\Gamma)$. In particular, for any such Γ the quotient Γ^{ab}/p is finite and one has $\mathrm{d}(\Gamma) = \mathrm{d}(\Gamma^{\mathrm{ab}}) = \dim_{\mathbb{F}_p}(\Gamma^{\mathrm{ab}}/p)$ (cf. [19, Prop. (3.9.1)]). Assume now Γ is finite. Then for a (left) $\mathbb{Z}_p[\Gamma]$-lattice N, we set $\mathrm{rk}(N) := \dim_{\mathbb{Q}_p}(\mathbb{Q}_p \otimes_{\mathbb{Z}_p} N)$ and write $\mathrm{cor}_\Gamma(N)$ for the direct sum of all non-projective indecomposable summands in a Krull-Schmidt decomposition of N over $\mathbb{Z}_p[\Gamma]$. (The module $\mathrm{cor}_\Gamma(N)$ is referred to as the 'core' of N (cf. [7, p. 26]) and is independent, up to isomorphism, of the Krull-Schmidt decomposition.) For a $\mathbb{Z}_p[\Gamma]$-module N, we write N^Γ and N_Γ for its maximal submodule and quotient module upon which Γ acts trivially and $\hat{H}^i(\Gamma, N)$ for its Tate-cohomology in degree i.

We write $R_{E/K}$ for the set of places of K that ramify in a field extension E of K and set

$$R_{E/K}^\infty := R_{E/K} \cap S_K^\infty \quad \text{and} \quad R_{E/K}^{\mathrm{f}} := R_{E/K} \setminus S_K^\infty = R_{E/K} \setminus R_{E/K}^\infty.$$

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If E/K is finite, we also set

$$\rho_{E/K} := |R_{E/K}|,\ \ \rho^{\infty}_{E/K} := |R^{\infty}_{E/K}| \quad \text{and} \quad \rho^{\mathrm{f}}_{E/K} := |R^{\mathrm{f}}_{E/K}| = \rho_{E/K} - \rho^{\infty}_{E/K},$$

and write $A_{E,S}$ for the p-part of the ideal class group $\mathrm{Cl}(\mathcal{O}_{E,S})$. Finally, for each place v of K we fix a place w of L above v, write G_v and I_v for the decomposition and inertia subgroups of w in G and set

$$\delta_{L/K} := \sum\nolimits_{v \in R^{\mathrm{f}}_{L/K}} \mathrm{d}\big(G_v\big).$$

In the sequel we will often omit the subscript S in the case it is equal to S^{∞}_K, thereby writing $\mathcal{O}_L$, U_L, A_L and $m_{L/K}$ in place of $\mathcal{O}_{L,S^{\infty}_K}$, U_{L,S^{∞}_K}, A_{L,S^{∞}_K} and $m_{L/K,S^{\infty}_K}$ etc.

We can now state an example of the sort of results proved in §2.2 (and see also Remark 2.5).

Theorem 1.1. *Write L' for the maximal extension of $K(\mu_{L,p})$ in L that has exponent dividing p and is abelian over K. Then one has*

$$m_{L/K} \geq \mathrm{rk}(U_K) - (\delta_{L/K} + \rho^{\infty}_{L/K}) + \mathrm{D}(G) - \mathrm{d}(\hat{H}^{-2}(G, A_L)) - \mathrm{d}(G_{L'/K(\mu_{L,p})}), \tag{1}$$

$$m_{L/K} \leq \mathrm{rk}(U_K) - (\delta_{L/K} + \rho^{\infty}_{L/K}) + \mathrm{d}(G) + \mathrm{d}(A_{L,G}) + \rho^{\mathrm{f}}_{L/K} + 1 \tag{2}$$

and also

$$\mathrm{rk}(\mathrm{cor}_G(U_L)) < |G|(\delta_{L/K} + \mathrm{d}(\hat{H}^{-2}(G, A_L)) + \rho^{\infty}_{L/K}/2 + \mathrm{d}(G_{L'/K(\mu_{L,p})}) - \mathrm{D}(G)).$$

Since $m_{L/K}$ is obviously at most $\mathrm{rk}(U_K)$, the inequality (2) is of interest only if $\delta_{L/K} + \rho^{\infty}_{L/K}$ is strictly greater than $\mathrm{d}(G) + \mathrm{d}(A_{L,G}) + \rho^{\mathrm{f}}_{L/K} + 1$. However, in §2.4 we explain how, in special cases, a closer analysis of the proof of Theorem 1.1 can give stronger bounds. For example, for finite Hilbert p-classfield towers one obtains in this way a formula for $m_{L/K}$ that clarifies aspects of the main results of Hajir at el [13] and for (possibly ramified) cyclic extensions a formula that both complements, and extends, the main result of Lim, Maire and the present author in [4].

To state a general consequence of Theorem 1.1, we fix an abstract finite p-group Γ and write $\mathfrak{F}(\Gamma)$ for the family of Galois extensions of number fields F/E with $\mathrm{Gal}(F/E)$ isomorphic to Γ. We also write $\mathfrak{F}^t(\Gamma)$ for the subset of $\mathfrak{F}(\Gamma)$ comprising tamely ramified extensions F/E (that is, extensions that are unramified at all p-adic places).

Corollary 1.2. *There exist constants c_1 and c_2 that depend only on Γ and are such that*

$$\mathrm{rk}(\mathrm{cor}_{\Gamma}(U_F)) \leq (\mathrm{d}(A_F) + \rho_{F/E})c_1 + c_2 \quad \textit{for all } F/E \in \mathfrak{F}(\Gamma) \tag{3}$$

and

$$\mathrm{rk}(\mathrm{cor}_{\Gamma}(U_F)) \leq (\mathrm{d}(A_E) + \rho_{F/E})2|\Gamma|c_1 + c_2 + (1 - |\Gamma|) \quad \textit{for all } F/E \in \mathfrak{F}^t(\Gamma).$$

By optimising the choice of constants c_1 and c_2 that arise in the proof of this result (given in §2.3) one obtains a strong improvement of the recent result [16, Th. A] of Kumon and Lim. For brevity, however, we prefer to leave the derivation of explicit such formulas for c_1 and c_2 to the reader.

We next recall that, whilst there are infinitely many isomorphism classes of indecomposable $\mathbb{Z}_p[\Gamma]$-lattices unless Γ is cyclic and $|\Gamma| \leq p^2$ (cf. [15]), in all cases the Jordan-Zassenhaus Theorem [5, Th. (24.2)] implies there are only finitely many isomorphism classes of $\mathbb{Z}_p[\Gamma]$-lattices of any fixed rank. In particular, if for $m > 0$ we write $\mathfrak{F}_m(\Gamma)$ for the subset of $\mathfrak{F}(\Gamma)$ comprising extensions F/E for which $\mathrm{d}(A_F) + \rho_{F/E} \leq m$, then the bound (3) implies that only finitely many isomorphism classes of $\mathbb{Z}_p[\Gamma]$-lattices arise from $\mathrm{cor}_{\Gamma}(U_F)$ as F/E ranges over $\mathfrak{F}_m(\Gamma)$.

An effective version of the Jordan-Zassenhaus Theorem therefore leads, via (3), to an effective bound in terms of $\mathrm{d}(A_F) + \rho_{F/E}$ for the number of possible isomorphism classes, as a $\mathbb{Z}_p[\Gamma]$-lattice, of $\mathrm{cor}_{\Gamma}(U_F)$. Even in such cases, however, it can still be very difficult to

determine the decomposition of $\mathrm{cor}_\Gamma(U_F)$ as a direct sum of indecomposable lattices. In this direction, in §3 we use Heller's theory of loop operators to show that, under certain conditions, the argument proving Theorem 1.1 can be finessed to give explicit structural information about both $\mathrm{cor}_\Gamma(U_F)$ and A_F. For interesting examples of such results, see Proposition 3.4 and Remark 3.6.

We note finally that the methods we use to prove the above results can be extended in a straightforward way to establish analogous results in the case of arbitrary finite Galois extensions (rather than just p-extensions). However, since no essentially new ideas are involved, we again prefer to leave the derivation of such results to the reader.

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2. The proofs of Theorem 1.1 and Corollary 1.2

2.1. **Preliminary results.** We fix a finite set S of places of K that contains S_K^∞. We then recall that the argument of Ritter and Weiss in [24, §4] implies the existence of an exact sequence of finitely generated G-modules

$$0 \to \mathcal{O}_{L,S}^\times \to M_0 \to M_1 \to \nabla_{L,S} \to 0 \tag{4}$$

in which M_0 is cohomologically-trivial, M_1 is free and $\nabla_{L,S}$ lies in a short exact sequence

$$0 \to \mathrm{Cl}(\mathcal{O}_{L,S}) \to \nabla_{L,S} \to \big(\bigoplus\nolimits_{v\in S} \mathrm{i}_{G_v}^G\mathbb{Z}\big) \oplus \big(\bigoplus\nolimits_{v\in R_{L/K}\setminus S} \mathrm{i}_{G_v}^G W_v^*\big) \xrightarrow{\epsilon_{L,S}} \mathbb{Z} \to 0. \tag{5}$$

Here, for each place $v \in R_{L/K}$, we have abbreviated the induction functor $\mathrm{Ind}_{G_v}^G$ to $\mathrm{i}_{G_v}^G$ and written W_v^* for the $\mathbb{Z}$-linear dual of the 'inertial' $\mathbb{Z}[G_v]$-lattice $W_v = W(L_w/K_v)$ defined by Gruenberg and Weiss in [8, §11] (for more details see Remark 3.7).

As far as we are aware, the sequences (4) and (5) were first used to study an aspect of the Krull-Schmidt decompositions of unit lattices (in unramified Galois extensions) by Ozaki in [21]. In the next two results, we derive some further consequences of these sequences that will be useful for us. In particular, the first of these results reduces the computation of $m_{L/K,S}$ to an analysis of the Tate cohomology of $\nabla_{L,S}$.

For a G-module N, integer a and place v of K we use the respective Tate cohomology groups

$$\hat{H}^a(N) := \hat{H}^a(G,N) \quad \text{and} \quad \hat{H}_v^a(N) := \hat{H}^a(G_v,N)$$

of N. We also set

$$\beta_{L/K,S} := \mathrm{d}(\hat{H}^0(\mathcal{O}_{L,S}^\times)) - \mathrm{d}(\hat{H}^0(U_{L,S})).$$

Proposition 2.1.

(i) $m_{L/K,S} = \mathrm{rk}(U_{K,S}) - \mathrm{d}(\hat{H}^{-2}(\nabla_{L,S})) + \beta_{L/K,S}$.

(ii) If $\mu_{L,p} = (0)$, *then* $\beta_{L/K,S} = 0$. *If* $\mu_{L,p} \neq (0)$, *then*

$$-\mathrm{d}(G_{L'/K(\mu_{L,p})}) \leq \beta_{L/K,S} \leq 1,$$

where L' is as in the statement of Theorem 1.1.

Proof. In this, and subsequent, arguments we repeatedly use the following easy fact: for any exact sequence of finitely generated $\mathbb{Z}_p$-modules $N_1 \xrightarrow{\theta_1} N_2 \xrightarrow{\theta_2} N_3$ one has

$$\max\big\{\mathrm{d}(\mathrm{im}(\theta_1)), \mathrm{d}(\mathrm{im}(\theta_2))\big\} \leq \mathrm{d}(N_2) \leq \mathrm{d}(N_1) + \mathrm{d}(N_3). \tag{6}$$

At the outset we note $U_{K,S}$ is isomorphic to a submodule of $U_{L,S}^G$ of finite index and hence $\mathrm{rk}(U_{K,S}) = \mathrm{rk}(U_{L,S}^G)$. In addition, since the G-modules M_0 and M_1 are both cohomologically-trivial, the double-coboundary map $\hat{H}^{-2}(\nabla_{L,S}) \to \hat{H}^0(\mathcal{O}_{L,S}^\times)$ induced by the exact sequence (4) is bijective. These observations imply that

$$\begin{aligned}\mathrm{rk}(U_{K,S}) - \mathrm{d}(\hat{H}^{-2}(\nabla_{L,S})) + \beta_{L/K,S} &= \mathrm{rk}(U_{L,S}^G) - \mathrm{d}(\hat{H}^0(\mathcal{O}_{L,S}^\times)) + \beta_{L/K,S} \\ &= \mathrm{rk}(U_{L,S}^G) - \mathrm{d}(\hat{H}^0(U_{L,S})).\end{aligned} \tag{7}$$

To prove (i), we are thereby reduced to proving that

$$m_{L/K,S} = \mathrm{rk}(U_{L,S}^G) - \mathrm{d}(\hat{H}^0(U_{L,S})). \tag{8}$$

To do this, we set $M := U_{L,S}$, write T for the map $M \to M^G \subseteq M$ sending each m to $\sum_{g\in G} g(m)$, and then T/p and $\overline{T}$ for the respective maps $M/p \to M/p$ and $M \to M^G/p$ induced by T. As M is torsion-free, the natural map $M^G/p \to M/p$ is injective and we use it to identify $\mathrm{im}(\overline{T})$ with $\mathrm{im}(T/p)$. Setting $d := \dim_{\mathbb{F}_p}(\mathrm{im}(T/p))$, one thus has

$$\begin{aligned}\mathrm{d}\big(\hat{H}^0(M)\big) = \dim_{\mathbb{F}_p}\big(M^G/(\mathrm{im}(T) + pM^G)\big) &= \dim_{\mathbb{F}_p}\big(\big(M^G/p\big)/\mathrm{im}(\overline{T})\big) \\ &= \dim_{\mathbb{F}_p}\big(M^G/p\big) - \dim_{\mathbb{F}_p}(\mathrm{im}(\overline{T})) \\ &= \mathrm{rk}(M^G) - d.\end{aligned}$$

To prove (8) we are therefore reduced to proving $m_{L/K,S} = d$. In addition, if $\mathcal{M}$ is any free $\mathbb{F}_p[G]$-direct summand of M/p of rank $m_{L/K,S}$, then $(T/p)(\mathcal{M})$ is a subspace of $\mathrm{im}(T/p)$ of dimension $m_{L/K,S}$ and so $m_{L/K,S} \leq d$. It is thus enough to show that, if $\{m_i\}_{1\leq i\leq d}$ is any subset of M such that $\{(T/p)(\overline{m_i})\}_{1\leq i\leq d}$ is a basis of $\mathrm{im}(T/p)$, where $\overline{m_i}$ denotes the image of m_i in M/p, then the $\mathbb{Z}_p[G]$-submodule M' of M generated by $\{m_i\}_{1\leq i\leq d}$ is a free direct summand of rank d.

To prove this we use the map of $\mathbb{Z}_p[G]$-modules $\psi : \mathbb{Z}_p[G]^d \twoheadrightarrow M' \subseteq M$ that sends the i-th element in the standard basis of $\mathbb{Z}_p[G]^d$ to m_i. Then the induced map $\psi/p : \mathbb{F}_p[G]^d \to M/p$ is injective. Indeed, since G is a p-group (and $\ker(\psi/p)$ is finite), this follows from the fact that the linear independence of the elements $\{(T/p)(\overline{m_i})\}_{1\leq i\leq d}$ implies the map $(\mathbb{F}_p[G]^d)^G \to (M/p)^G$ obtained by restriction of ψ/p is injective. Now, since ψ/p is injective, one has $\ker(\psi) \subseteq p{\cdot}\mathbb{Z}_p[G]^t$ and hence, as M is torsion-free, $\ker(\psi) = p{\cdot}\ker(\psi)$ so that, by Nakayama's Lemma, ψ is injective. It follows that M' is a free $\mathbb{Z}_p[G]$-module of rank d (as required) and hence also that $H^0(G, M') = T(M')$ and $H^1(G, M') = (0)$. Writing ι for the inclusion $M' \subseteq M$, the long exact sequence of G-cohomology associated to the tautological sequence

$$0 \to M' \xrightarrow{\iota} M \to \mathrm{cok}(\iota) \to 0 \tag{9}$$

therefore gives an exact sequence of $\mathbb{Z}_p$-modules

$$0 \to T(M') \to M^G \to \mathrm{cok}(\iota)^G \to 0. \tag{10}$$

Now, as the elements $\{(T/p)(\overline{m_i})\}_{1\leq i\leq d}$ are linearly independent in M^G/p, Nakayama's Lemma implies $\{T(m_i)\}_{1\leq i\leq d}$ can be extended to give a basis of the $\mathbb{Z}_p$-module M^G and so (10) implies $\mathrm{cok}(\iota)^G$ is torsion-free. It follows that the (finite) torsion-subgroup $\mathrm{cok}(\iota)_{\mathrm{tor}}$ of $\mathrm{cok}(\iota)$ satisfies $(\mathrm{cok}(\iota)_{\mathrm{tor}})^G = (\mathrm{cok}(\iota)^G)_{\mathrm{tor}} = (0)$, and hence, as G is a p-group, that $\mathrm{cok}(\iota)_{\mathrm{tor}} = (0)$. Thus $\mathrm{cok}(\iota)$ is free over $\mathbb{Z}_p$ and so, since the ring $\mathbb{Z}_p[G]$ is Gorenstein (and M' is free), the short exact sequence (9) splits over $\mathbb{Z}_p[G]$, as required to complete the proof of (i).

If $\mu_{L,p} = (0)$, then the natural map $\hat{H}^0(\mathcal{O}_{L,S}^\times) \to \hat{H}^0(U_{L,S})$ is bijective and so claim (ii) is clear. We therefore assume in the sequel that $\mu_{L,p} \neq (0)$. We set $E := K(\mu_{L,p})$, $H := G_{L/E}$ and $\Gamma := G_{E/K} = G/H$ and recall $\mu_{L,p}$ is a cohomomologically-trivial Γ-module (cf. [4, Rem. 2.2]). In particular, from the exact sequence of low-degree terms

$$0 \to H^1(\Gamma, \mu_{L,p}) \to H^1(G, \mu_{L,p}) \to H^1(H, \mu_{L,p})^\Gamma \to H^2(\Gamma, \mu_{L,p})$$

that is induced by the inflation-restriction spectral sequence, one obtains an identification

$$H^1(G,\mu_{L,p})[p] \cong H^1(H,\mu_{L,p})^\Gamma[p] = \mathrm{Hom}(H^{\mathrm{ab}},\mu_L[p])^\Gamma = \mathrm{Hom}(H^{\mathrm{ab}}/p,\mathbb{Z}/p)^\Gamma,$$

where the second equality is valid since $\mu_L[p] = \mu_K[p]$ (as L/K is a p-extension and p is odd). In addition, by Pontryagin duality, there are identifications

$$\mathrm{Hom}(H^{\mathrm{ab}}/p,\mathbb{Z}/p)^\Gamma \cong \mathrm{Hom}((H^{\mathrm{ab}}/p)_\Gamma,\mathbb{Z}/p) = \mathrm{Hom}(G_{L'/E},\mathbb{Z}/p)$$

and so $\mathrm{d}(H^1(G,\mu_{L,p})) = \mathrm{d}(\mathrm{Hom}(G_{L'/E},\mathbb{Z}/p)) = \mathrm{d}(G_{L'/E})$. Thus, from the canonical long exact sequence $\hat{H}^0(\mu_{L,p}) \to \hat{H}^0(\mathcal{O}_{L,S}^\times) \to \hat{H}^0(U_{L,S}) \to H^1(\mu_{L,p})$, we can therefore deduce (by a suitable application of (6)) that

$$1 \geq \mathrm{d}(\hat{H}^0(\mu_{L,p})) \geq \beta_{L/K,S} \geq -\mathrm{d}(H^1(\mu_{L,p})) = -\mathrm{d}(G_{L'/E}),$$

as required. □

Remark 2.2. The above argument also has the following useful consequence: if S' is any finite set of places of K containing S, then $m_{L/K,S} \leq m_{L/K,S'}$. To see this, note the cokernel of the inclusion $U_{L,S} \to U_{L,S'}$ is torsion-free and hence that the induced map $U_{L,S}/p \to U_{L,S'}/p$ is injective. In particular, since $\mathbb{F}_p[G]$ is self-injective, any free direct summand of the $\mathbb{F}_p[G]$-module $U_{L,S}/p$ is a free direct summand of $U_{L,S'}/p$. By the argument of Proposition 2.1(i), this fact implies the claimed inequality $m_{L/K,S} \leq m_{L/K,S'}$.

An alternative to the formula for $m_{L/K,S}$ given in Proposition 2.1(i) will be established (under the assumption that $\mu_L[p] = (0)$) in Proposition 3.3 below. However, to demonstrate the usefulness of the above formula, we next provide explicit bounds for $\mathrm{d}\big(\hat{H}^{-2}(\nabla_{L,S})\big)$.

Lemma 2.3. *Set $S^* := \{v \in S : G_v \neq \{1\}\}$. Then the following claims are valid.*

(i) $\mathrm{d}\big(\hat{H}^{-2}(\nabla_{L,S})\big) \leq \delta_{L/K} + \big(\sum_{v\in S^*\setminus R^{\mathrm{f}}_{L/K}} \mathrm{d}(G_v)\big) + \mathrm{d}(\hat{H}^{-2}(A_{L,S})) + \mathrm{d}(\hat{H}^{-3}(\mathbb{Z}))$.

(ii) If $A_{L,S} = (0)$, then

$$\mathrm{d}\big(\hat{H}^{-2}(\nabla_{L,S})\big) \geq (\delta_{L/K} - \rho^{\mathrm{f}}_{L/K}) + |S \cap R^{\mathrm{f}}_{L/K}| + \big(\sum\nolimits_{v\in S^*\setminus R^{\mathrm{f}}_{L/K}} \mathrm{d}(G_v)\big) - \mathrm{d}(G).$$

Proof. As a first step, we claim that, for each $v \in R^{\mathrm{f}}_{L/K}$, one has

$$\mathrm{d}(G_v) - 1 \leq \mathrm{d}\big(\hat{H}^{-2}(\mathrm{i}^G_{G_v} W_v^*)\big) \leq \mathrm{d}(G_v). \tag{11}$$

To show this, we first take the $\mathbb{Z}$-linear dual of the exact sequence

$$0 \to \mathbb{Z} \xrightarrow{\alpha_v} W_v \xrightarrow{\beta_v} I(G_v) \to 0 \tag{12}$$

of [24, Lem. 5(b)] to obtain an exact sequence of G_v-modules

$$0 \to I(G_v)^* \to W_v^* \to \mathbb{Z} \to 0. \tag{13}$$

We also note the $\mathbb{Z}$-dual of the tautological exact sequence $0 \to I(G_v) \to \mathbb{Z}[G_v] \to \mathbb{Z} \to 0$ gives an exact sequence

$$0 \to \mathbb{Z} \to \mathbb{Z}[G_v] \to I(G_v)^* \to 0 \tag{14}$$

that induces an isomorphism $\hat{H}^a_v(I(G_v)^*) \cong \hat{H}^{a+1}_v(\mathbb{Z})$ in each degree a. Hence, as $\hat{H}^{-1}_v(\mathbb{Z})$ vanishes, $\hat{H}^{-2}_v(W_v^*) \cong \hat{H}^{-2}(\mathrm{i}^G_{G_v} W_v^*)$ and $\hat{H}^0_v(\mathbb{Z}) \cong \mathbb{Z}/|G_v|\mathbb{Z}$, the short exact sequence (13) induces an exact sequence $0 \to \hat{H}^{-2}(\mathrm{i}^G_{G_v} W_v^*) \to \hat{H}^{-2}_v(\mathbb{Z}) \to \mathbb{Z}/|G_v|\mathbb{Z}$. This implies (via (6)) the claimed inequalities (11) since, for any finite group Γ, the group $\hat{H}^{-2}(\Gamma,\mathbb{Z}) \cong H_1(\Gamma,\mathbb{Z})$ is isomorphic to Γ^{ab}.

Next we note that the exact sequence (5) induces long exact sequences

$$\hat{H}^{-2}(A_{L,S}) \to \hat{H}^{-2}(\nabla_{L,S}) \to \hat{H}^{-2}(\ker(\epsilon_{L,S})) \to \hat{H}^{-1}(A_{L,S}) \tag{15}$$

$$\big(\bigoplus\nolimits_{v\in S^*}\hat{H}^{-3}(\mathrm{i}_{G_v}^{G}\mathbb{Z})\big)\oplus\big(\bigoplus\nolimits_{v\in R_{L/K}\setminus S}\hat{H}^{-3}(\mathrm{i}_{G_v}^{G}W_v^*)\big)\to\hat{H}^{-3}(\mathbb{Z})\to\hat{H}^{-2}(\ker(\epsilon_{L,S}))$$
$$\to\big(\bigoplus\nolimits_{v\in S^*}G_v^{\mathrm{ab}}\big)\oplus\big(\bigoplus\nolimits_{v\in R_{L/K}\setminus S}\hat{H}^{-2}(\mathrm{i}_{G_v}^{G}W_v^*)\big)\to G^{\mathrm{ab}}. \quad (16)$$

Then, by using these sequences, and (11), one verifies claim (i) via the following computation

$$\begin{aligned}\mathrm{d}\big(\hat{H}^{-2}(\nabla_{L,S})\big)&\leq\mathrm{d}\big(\hat{H}^{-2}(A_{L,S})\big)+\mathrm{d}\big((\hat{H}^{-2}(\ker(\epsilon_{L,S}))\big)\\&\leq\mathrm{d}\big(\hat{H}^{-2}(A_{L,S})\big)+\mathrm{d}\big(\hat{H}^{-3}(\mathbb{Z})\big)+\big(\textstyle\sum_{v\in S^*}\mathrm{d}(G_v)\big)+\big(\sum_{v\in R_{L/K}\setminus S}\mathrm{d}(G_v)\big)\\&=\delta_{L/K}+\big(\textstyle\sum_{v\in S^*\setminus R^{\mathrm{f}}_{L/K}}\mathrm{d}(G_v)\big)+\mathrm{d}\big(\hat{H}^{-2}(A_{L,S})\big)+\mathrm{d}\big(\hat{H}^{-3}(\mathbb{Z})\big)\end{aligned}$$

(in which (6) has been applied several times).

In addition, if $A_{L,S}=(0)$, then (15) and (16) combine with (11) to imply that

$$\begin{aligned}\mathrm{d}\big(\hat{H}^{-2}(\nabla_{L,S})\big)&=\mathrm{d}\big(\hat{H}^{-2}(\ker(\epsilon_{L,S}))\big)\\&\geq\big(\textstyle\sum_{v\in S^*}\mathrm{d}(G_v)\big)+\big(\sum_{v\in R_{L/K}\setminus S}(\mathrm{d}(G_v)-1)\big)-\mathrm{d}(G)\\&=(\delta_{L/K}-\rho^{\mathrm{f}}_{L/K})+|S\cap R^{\mathrm{f}}_{L/K}|+\big(\textstyle\sum_{v\in S^*\setminus R^{\mathrm{f}}_{L/K}}\mathrm{d}(G_v)\big)-\mathrm{d}(G),\end{aligned}$$

as required to prove claim (ii). □

We finally give a conceptual interpretation of the quantity $\mathrm{d}(\hat{H}^{-3}(\mathbb{Z}))$ in Lemma 2.3(i).

Lemma 2.4. *For a finite p-group Γ, there exists a non-canonical short exact sequence of abelian groups*

$$0\to H^1(\Gamma,\mathbb{Z}/p)\to H^2(\Gamma,\mathbb{Z}/p)\to H^2(\Gamma,\mathbb{Q}/\mathbb{Z})[p]\to 0.$$

In particular, one has $\mathrm{D}(\Gamma)=-\mathrm{d}(\hat{H}^{-3}(\Gamma,\mathbb{Z}))$.

Proof. In each degree a, the short exact sequence $0\to\mathbb{Z}/p\to\mathbb{Q}/\mathbb{Z}\xrightarrow{\times p}\mathbb{Q}/\mathbb{Z}\to 0$ induces a canonical short exact sequence of $\mathbb{F}_p$-modules

$$0\to H^a(\Gamma,\mathbb{Q}/\mathbb{Z})/p\to H^{a+1}(\Gamma,\mathbb{Z}/p)\to H^{a+1}(\Gamma,\mathbb{Q}/\mathbb{Z})[p]\to 0.$$

In particular, since $H^0(\Gamma,\mathbb{Q}/\mathbb{Z})=\mathbb{Q}/\mathbb{Z}$ is divisible, the group $H^1(\Gamma,\mathbb{Z}/p)$ is canonically isomorphic to $H^1(\Gamma,\mathbb{Q}/\mathbb{Z})[p]$, and hence (since $H^1(\Gamma,\mathbb{Q}/\mathbb{Z})$ is finite) non-canonically isomorphic to $H^1(\Gamma,\mathbb{Q}/\mathbb{Z})/p$. The claimed short exact sequence is therefore obtained by combining this isomorphism with the above displayed sequence with $a=1$.

We note next that $\mathrm{d}(\hat{H}^{-3}(\Gamma,\mathbb{Z}))=\dim_{\mathbb{F}_p}\big(H^2(\Gamma,\mathbb{Q}/\mathbb{Z})[p]\big)$ as a consequence of the canonical isomorphism $\hat{H}^{-3}(\Gamma,\mathbb{Z})=H_2(\Gamma,\mathbb{Z})\cong\mathrm{Hom}\big(H^2(\Gamma,\mathbb{Q}/\mathbb{Z}),\mathbb{Q}/\mathbb{Z}\big)$. Given the displayed exact sequence in the statement, the final assertion therefore follows directly from the fact that $\dim_{\mathbb{F}_p}(H^1(\Gamma,\mathbb{Z}/p))=\mathrm{d}(\Gamma)$ and $\dim_{\mathbb{F}_p}(H^2(\Gamma,\mathbb{Z}/p))=\mathrm{r}(\Gamma)$ (see, for example, [19, Prop. (3.9.1) and Cor. (3.9.5)]). □

2.2. **The proof of Theorem 1.1.** The inequality (1) is true since Proposition 2.1(i) (with $S=S_K^\infty$) implies that $m_{L/K}$ is equal to

$$\begin{aligned}&\mathrm{rk}(U_K)-\mathrm{d}(\hat{H}^{-2}(\nabla_{L,S_K^\infty}))+\beta_{L/K,S_K^\infty}\\&\geq\mathrm{rk}(U_K)-\delta_{L/K}-|R^\infty_{L/K}|-\mathrm{d}(\hat{H}^{-2}(A_L))-\mathrm{d}(\hat{H}^{-3}(\mathbb{Z}))+\beta_{L/K,S_K^\infty}\\&\geq\mathrm{rk}(U_K)-(\delta_{L/K}+\rho^\infty_{L/K})+\mathrm{D}(G)-\mathrm{d}(\hat{H}^{-2}(A_L))-\mathrm{d}(G_{L'/K(\mu_{L,p})}).\end{aligned}$$

Here the first inequality follows from Lemma 2.3(i) (with $S=S_K^\infty$, so $S^*\setminus R^{\mathrm{f}}_{L/K}=R^\infty_{L/K}$) and the second from Lemma 2.4 and the result of Proposition 2.1(ii).

To prove the inequality (2) we note that, by Nakayama's Lemma, the minimal number of generators of A_L as a G-module is $\mathrm{d}(A_{L,G})$. By Chebotarev's density theorem, we can therefore fix a set S' of $\mathrm{d}(A_{L,G})$ non-archimedean places of K that split completely in L and are such that $A_{L,S} = (0)$, with $S := S_K^\infty \cup S'$. One then has $S \cap R^{\mathrm{f}}_{L/K} = \emptyset$ and $S^* \setminus R^{\mathrm{f}}_{L/K} = R^\infty_{L/K}$ (in the notation of Lemma 2.3) and also $m_{L/K} \leq m_{L/K,S}$ by Remark 2.2. Upon combining Proposition 2.1(i) with Lemma 2.3(ii) one therefore has

$$\begin{aligned} m_{L/K} \leq m_{L/K,S} &= \mathrm{rk}(U_{K,S}) - \mathrm{d}\big(\hat{H}^{-2}(\nabla_{L,S})\big) + \beta_{L/K,S} \\ &= \mathrm{rk}(U_K) + \mathrm{d}(A_{L,G}) - \mathrm{d}\big(\hat{H}^{-2}(\nabla_{L,S})\big) + \beta_{L/K,S} \\ &\leq \mathrm{rk}(U_K) + \mathrm{d}(A_{L,G}) - (\delta_{L/K} - \rho^{\mathrm{f}}_{L/K} + \rho^\infty_{L/K} - \mathrm{d}(G)) + 1 \\ &= \mathrm{rk}(U_K) - (\delta_{L/K} + \rho^\infty_{L/K}) + \mathrm{d}(G) + \mathrm{d}(A_{L,G}) + \rho^{\mathrm{f}}_{L/K} + 1. \end{aligned}$$

This completes the proof of (2).

Next we note that, since G is a p-group, Swan's Theorem implies a finitely generated $\mathbb{Z}_p[G]$-module is projective if and only if it is free (cf. [5, Th. (32.11)]). Given this fact, the definition of $m_{L/K}$ combines with the argument of Proposition 2.1(i) to imply that U_L is isomorphic, as a $\mathbb{Z}_p[G]$-module, to a direct sum $\mathbb{Z}_p[G]^{m_{L/K}} \oplus \mathrm{cor}_G(U_L)$. In addition, an easy exercise shows that $\mathrm{rk}(U_L)$ is equal to $|G|(\mathrm{rk}(U_K) + 1 - \rho^\infty_{L/K}/2) - 1$, and hence that

$$\begin{aligned} \mathrm{rk}(\mathrm{cor}_G(U_L)) &= \mathrm{rk}(U_L) - \mathrm{rk}(\mathbb{Z}_p[G])m_{L/K} \\ &= |G|(\mathrm{rk}(U_K) + 1 - \rho^\infty_{L/K}/2) - 1 - |G|m_{L/K} \\ &< |G|(\mathrm{rk}(U_K) + 1 - \rho^\infty_{L/K}/2 - m_{L/K}). \end{aligned}$$

The final inequality of Theorem 1.1 now follows as a direct consequence of (1).

2.3. **The proof of Corollary 1.2.** We first bound each of the terms $\delta_{L/K}$, $\mathrm{d}(G_{L'/K(\mu_{L,p})})$ and $\mathrm{d}(\hat{H}^{-2}(\Gamma, M))$ that occur in the final inequality of Theorem 1.1.

If c is the maximal possible value of $\mathrm{d}(\Delta)$ as Δ runs over subgroups of Γ, then it is clear that $\delta_{L/K} \leq \rho^{\mathrm{f}}_{L/K} \cdot c$. Since L'/K is abelian, it is also clear that $\mathrm{d}(G_{L'/K(\mu_{L,p})}) \leq \mathrm{d}(G^{\mathrm{ab}}) = \mathrm{d}(G)$.

Next we note that, for any finitely generated $\mathbb{Z}_p[\Gamma]$-module M, the Γ-module $\mathbb{Z}[\Gamma] \otimes_{\mathbb{Z}} M$ (with diagonal action) is cohomologically-trivial. Thus, writing $I(\Gamma)$ for the augmentation ideal of $\mathbb{Z}[\Gamma]$, the natural short exact sequence of $\mathbb{Z}_p[\Gamma]$-modules

$$0 \to I(\Gamma) \otimes_{\mathbb{Z}} M \to \mathbb{Z}[\Gamma] \otimes_{\mathbb{Z}} M \to M \to 0$$

induces an identification of $\hat{H}^{-2}(\Gamma, M)$ with the subquotient $\hat{H}^{-1}\big(\Gamma, I(\Gamma) \otimes_{\mathbb{Z}} M\big)$ of $I(\Gamma) \otimes_{\mathbb{Z}} M$ and thereby implies $\mathrm{d}(\hat{H}^{-2}(\Gamma, M)) \leq \mathrm{d}(I(\Gamma) \otimes_{\mathbb{Z}} M) = (|\Gamma| - 1)\mathrm{d}(M)$.

Upon combining these bounds with the final assertion of Theorem 1.1, one can directly obtain constants c_1 and c_2 that depend only on Γ and are such that the inequality (3) is valid.

For a finite (possibly empty) set Σ of places of a number field E, we now write $M_{E,\Sigma}$ for the maximal pro-p extension of E that is unramified outside Σ and set $G_{E,\Sigma} := \mathrm{Gal}(M_{E,\Sigma}/E)$. We then fix L/K in $\mathfrak{F}(\Gamma)$ and set $R := R_{L/K}$. Then one has

$$\mathrm{d}(A_L) \leq \mathrm{d}(G_{L,R}) \leq |\Gamma|(\mathrm{d}(G_{K,R}) - 1) + 1 = |\Gamma| \cdot \mathrm{d}(G_{K,R}) + (1 - |\Gamma|),$$

where the first inequality is true since A_L is isomorphic to a quotient of $G_{L,R}$ and the second follows from a direct application of Schreier's Inequality to the open subgroup $G_{L,R}$ of $G_{K,R}$ (cf. [23, Cor. 3.6.3]). In addition, if L/K belongs to $\mathfrak{F}^t(\Gamma)$, then every place in R is tamely ramified in $M_{K,R}$ and hence has cyclic inertia group in $G_{K,R}$. In particular, by applying the exact sequence of [19, Lem. 10.7.4(i)] (with $S = R$ and $T = \emptyset$) and the general result of [19, Prop. (3.9.1)] in this case, one finds that $\mathrm{d}(G_{K,R}) \leq \mathrm{d}(G_{K,\emptyset}) + |R| = \mathrm{d}(A_K) + \rho_{L/K}$ and hence, from the above displayed inequality, that

$$\begin{aligned}\mathrm{d}(A_L)+\rho_{L/K} &\leq |\Gamma|(\mathrm{d}(A_K)+\rho_{L/K})+(1-|\Gamma|)+\rho_{L/K}\\ &\leq 2|\Gamma|(\mathrm{d}(A_K)+\rho_{L/K})+(1-|\Gamma|).\end{aligned}$$

Upon substituting this inequality into (3) one obtains the second claimed inequality in Corollary 1.2. This therefore completes the proof of Corollary 1.2.

Remark 2.5. It will be clear that, by the same arguments, the results of §2.1 actually imply completely analogous versions of Theorem 1.1 and Corollary 1.2 for the quantities $m_{L/K,S}$ and $\mathrm{rk}(\mathrm{cor}_G(U_{L,S}))$ for any given finite set of places S of K containing S_K^∞.

2.4. **Special cases.** To illustrate Theorem 1.1, and the general approach of §2.1, we discuss some concrete applications. As a first step, however, we record another useful general result.

Lemma 2.6. *Assume there exists a place $v_0 \notin S$ with $G_{v_0} = G$ and set $S' := S \cup \{v_0\}$. Then there exists an exact sequence of $\mathbb{Z}_p[G]$-modules $0 \to U_{L,S} \to U_{L,S'} \to \mathbb{Z}_p \to 0$ and inequalities $m_{L/K,S} \leq m_{L/K,S'} \leq m_{L/K,S}+1$.*

Proof. Write $I_{L,S}$ for the group of fractional ideals of $\mathcal{O}_{L,S}$ and θ for the map $U_{L,S'} \to I_{L,S,p}$ that sends each x to $x\mathcal{O}_{L,S,p}$. Then $U_{L,S} = \ker(\theta)$ and, since $G_{v_0} = G$, the image of θ is isomorphic, as a $\mathbb{Z}_p[G]$-module, to $\mathbb{Z}_p$. The claimed exact sequence is thus clear.

Set $m := m_{L/K,S}$ and $m' := m_{L/K,S'}$. Then $m \leq m'$ by Remark 2.2 and to prove $m' \leq m+1$ we set $U := U_{L,S}, U' := U_{L,S'}$ and $X := \mathrm{cor}_G(U')$. Then U' decomposes as $X \oplus \mathbb{Z}_p[G]^{m'}$ and so the exact sequence $0 \to U \to U' \to \mathbb{Z}_p \to 0$ induces a map $\theta : \mathbb{Z}_p[G]^{m'} \to \mathbb{Z}_p$ and hence an exact commutative diagram

$$\begin{array}{ccccc}\ker(\theta)/p & \hookrightarrow & \mathbb{F}_p[G]^{m'} & \xrightarrow{\theta/p} & \mathbb{F}_p\\ \downarrow{\scriptstyle\phi} & & \downarrow & & \downarrow{\scriptstyle 0}\\ \ker(\theta)/p & \hookrightarrow & \mathbb{F}_p[G]^{m'} & \xrightarrow{\theta/p} & \mathbb{F}_p\end{array}$$

in which all vertical arrows send x to $T(x) := \sum_{g\in G} g(x)$. Hence, by the Snake Lemma, one has $\dim_{\mathbb{F}_p}(\mathrm{im}(\phi)) \geq m'-1$ and so, by the argument of Proposition 2.1(i), the submodule $\ker(\theta)$ of U has a direct summand isomorphic to $\mathbb{Z}_p[G]^{m'-1}$. Since $U/\ker(\theta)$ is isomorphic to a submodule of X, and hence torsion-free, and $\mathbb{Z}_p[G]$ is Gorenstein, it then follows U has a direct summand isomorphic to $\mathbb{Z}_p[G]^{m'-1}$. This implies $m \geq m'-1$, as required. □

2.4.1. *Locally cyclic extensions.* If G_v is cyclic for all $v \in R_{L/K}$, then $\delta_{L/K} = \rho^{\mathrm{f}}_{L/K}$.

In particular, if L/K is unramified, then Theorem 1.1 and the argument of Corollary 1.2 combine to imply the existence of a natural number $c(G) := \mathrm{d}(G)+\mathrm{D}(G)$ that depends only on the abstract group G and is such that

$$m_{L/K} > \mathrm{rk}(U_K) - \mathrm{d}(\hat{H}^{-2}(A_L)) - c(G)$$

and

$$\mathrm{rk}(\mathrm{cor}_G(U_L)) < |G|(\mathrm{d}(\hat{H}^{-2}(A_L)) + c(G)+1).$$

On the other hand, if G is itself cyclic (and L/K possibly ramified), then, by Chebotarev's Density Theorem, we can fix a place v_0 of K with $G_{v_0} = G$ and set $S := S_K^\infty \cup \{v_0\}$. In this case the definition of $\epsilon_{L,S}$ ensures (even if G is non-cyclic) that there is an isomorphism of $\mathbb{Z}_p[G]$-modules

$$\ker(\epsilon_{L,S}) \cong \big(\bigoplus\nolimits_{v\in S_K^\infty} \mathrm{i}^G_{G_v}\mathbb{Z}\big) \oplus \big(\bigoplus\nolimits_{v\in R^{\mathrm{f}}_{L/K}} \mathrm{i}^G_{G_v} W_v^*\big). \tag{17}$$

In particular, if we also assume $R^{\infty}_{L/K} = \emptyset$, there is an induced isomorphism of Tate cohomology groups

$$\hat{H}^{-2}(\ker(\epsilon_{L,S})) \cong \bigoplus\nolimits_{v \in R_{L/K}} \hat{H}^{-2}_v(W^*_v) \cong \bigoplus\nolimits_{v \in R_{L/K}} \hat{H}^0_v(W^*_v),$$

where the second isomorphism follows from the cyclicity of each G_v. In addition, for each place $v \in R^{\mathrm{f}}_{L/K}$, the exact sequence (13) implies W^*_v spans $\mathbb{Q}[G_v]$, whilst results of Gruenberg and Weiss [9] imply W^*_v is not cohomologically-trivial over G_v (see the proof of Proposition 3.4 below). By a Herbrand quotient argument, these facts imply $\hat{H}^0_v(W^*_v)$ is a non-trivial cyclic group and hence $\mathrm{d}(\hat{H}^0_v(W^*_v)) = 1$. The above isomorphism therefore implies $\mathrm{d}(\hat{H}^{-2}(\ker(\epsilon_{L,S}))) = \rho_{L/K} (= \rho^{\mathrm{f}}_{L/K})$. By the periodicity of Tate cohomology over G, one has

$$\mathrm{d}(\hat{H}^{-2}(A_{L,S})) = \mathrm{d}(\hat{H}^0(A_{L,S})) \leq \mathrm{d}(A^G_{L,S}).$$

In addition, writing $M^\vee$ for the Pontryagin dual of a finite G-module M (endowed with contragredient G-action), the duality isomorphism $\hat{H}^{-1}(A_{L,S}) \cong \hat{H}^0(A^\vee_{L,S})^\vee$ in Tate cohomology (cf. [3, Chap. VI, Prop. (7.1)]) implies that

$$\mathrm{d}(\hat{H}^{-1}(A_{L,S})) = \mathrm{d}(\hat{H}^0(A^\vee_{L,S})) \leq \mathrm{d}((A^\vee_{L,S})^G) = \mathrm{d}((A_{L,S,G})^\vee) = \mathrm{d}(A_{L,S,G}).$$

Setting $\varepsilon(v_0) := \rho^{\mathrm{f}}_{L/K} - \mathrm{d}(\hat{H}^{-2}(\nabla_{L,S}))$, the exact sequence (15) therefore implies that

$$-\mathrm{d}(A^G_{L,S}) \leq -\mathrm{d}(\hat{H}^{-2}(A_{L,S})) \leq \varepsilon(v_0) \leq \mathrm{d}(\hat{H}^{-1}(A_{L,S})) \leq \mathrm{d}(A_{L,S,G}).$$

Now since $\mathrm{rk}(U_{K,S}) = \mathrm{rk}(U_K) + 1$, Proposition 2.1(i) implies that

$$m_{L/K,S} = (\mathrm{rk}(U_K) + 1) + (\varepsilon(v_0) - \rho^{\mathrm{f}}_{L/K}) + \beta_{L/K,S}.$$

By Lemma 2.6, this quantity is equal to either $m_{L/K}$ or $m_{L/K}+1$. In addition, since $m_{L/K,S} \geq 0$, the formula combines with the upper bound for $\varepsilon(v_0)$ established above to imply that

$$\mathrm{d}(A_{L,S,G}) \geq \rho^{\mathrm{f}}_{L/K} - (\mathrm{rk}(U_K) + 1) - \beta_{L/K,S}.$$

Proposition 2.1(ii) implies $|\beta_{L/K,S}| \leq 1$ (as G is cyclic) and so, for large $\rho^{\mathrm{f}}_{L/K}$, this inequality gives a non-trivial restriction on the G-structure of $A_{L,S}$. For example, if $\mu_K[p] = (0)$ (so $\mu_L[p] = (0)$ and hence $\beta_{L/K,S} = 0$), then it implies $A_{L,S,G}$, and therefore also A_L, is non-trivial whenever $\rho^{\mathrm{f}}_{L/K}$ is greater than $\mathrm{rk}(U_K) + 1$.

This non-vanishing criterion is of the same form as those obtained by using genus theory (see, for example, the result [18, Cor. 3.6] of Maire) and also complements recent results of Lim, Maire and the present author. Specifically, in [4, Lem. 2.3, Exam. 2.4] it is shown that if A_K is cyclic, then there are infinitely many cyclic p-extensions L/K for which $\rho^{\mathrm{f}}_{L/K} = \mathrm{rk}(U_K) + 1$ and, for some $v \in R^{\mathrm{f}}_{L/K}$, one has $A_{L,S'} = (0)$ with $S' = S^\infty_K \cup \{v\}$. Finally, we note that the latter article also provides (in the case that G is a cyclic p-group) an alternative approach to studying $m_{L/K,S}$ for certain sets S.

2.4.2. *p-Hilbert classfield towers.* Fix a number field E with $\mu_E[p] \neq (0)$ and the p-Hilbert classfield tower L^E/E of finite degree, and set $G^E := \mathrm{Gal}(L^E/E)$.

Then L^E/E is unramified and A_{L^E} vanishes, and so we can combine Proposition 2.1(i) with the exact sequences (15) and (16) with $S = S^\infty_E$ and $S^* = R_{L^E/E} = \emptyset$. In this way, we deduce

$$\begin{aligned} \mathrm{rk}(U_E) - m_{L^E/E} &= \mathrm{d}(\hat{H}^{-2}(G^E, \nabla_{L^E,S^\infty_E})) - \beta_{L^E/E,S^\infty_E} \\ &= \mathrm{d}(\hat{H}^{-3}(G^E, \mathbb{Z})) - \beta_{L^E/E,S^\infty_E} \\ &= -\mathrm{D}(G^E) - \beta_{L^E/E,S^\infty_E}, \end{aligned}$$

where the last equality follows from Lemma 2.4. The resulting formula for $m_{L^E/E}$ is a refinement (in the case of finite p-Hilbert classfield towers) of the inequalities for $m_{L^E/E}$ that are

established by the main result (Theorem A) of Hajir et al [13]. (Note here that our definition of the deficiency of a group is the *negative* of that used in loc. cit.)

In addition, for any finite p-group Γ and large enough integer k, the result [12, §1, Th.] of Hajir et al implies there are infinitely many extensions K/E of degree p^k with $\mathrm{Gal}(L^K/K) \cong \Gamma$. For such fields L^K, Corollary 1.2 gives an upper bound for $\mathrm{rk}(\mathrm{cor}_\Gamma(U_{L^K}))$ that depends only on Γ, and hence implies $m_{L^K/K} \to \infty$ as $[K : E] \to \infty$.

2.4.3. *Uniform pro-p extensions.* Fix an abelian number field K, with cyclotomic $\mathbb{Z}_p$-extension K_∞, and a finite Galois p-extension L of K with $L \cap K_\infty = K$. Set $\Gamma := \mathrm{Gal}(L/K) \cong \mathrm{Gal}(LK_n/K_n)$, where K_n is such that $K \subseteq K_n \subset K_\infty$ and $[K_n : K] = p^n$. Then the Iwasawa μ-invariant of LK_∞/L vanishes (as K is abelian and L/K of p-power degree) and so $\mathrm{d}(A_{LK_n})$ is bounded independently of n (by [25, Prop. 13.23]). The result of Corollary 1.2 therefore implies the existence of an upper bound for $\mathrm{rk}(\mathrm{cor}_\Gamma(U_{LK_n}))$ that is independent of n.

To generalise this example, we fix a Galois extension K'_∞/K of number fields with $R_{K'_\infty/K}$ finite and $\mathcal{G} := \mathrm{Gal}(L/K)$ a uniform pro-p group. We write $(\mathcal{G}_n)_n$ for the p-central descending series of $\mathcal{G}$ and, for each n, set $K'_n := (K'_\infty)^{\mathcal{G}_n}$. We then fix a finite Galois extension L/K with $L \cap K'_\infty = K$, write $\mathcal{X}_\infty$ for the Galois group of the maximal unramified p-extension of LK'_∞ and assume $\mathcal{X}_\infty[p]$ is torsion over the Iwasawa algebra $\mathbb{F}_p[[\mathcal{G}]]$.

In this situation, results of Perbet [22] imply that

$$\mathrm{d}(A_{LK'_n}) = O(p^{-n}[K'_n : K])$$

and so, if no place in $R_{L/K}$ has finite decomposition group in K'_∞/K, the first inequality of Theorem 1.1 combines with the argument of Corollary 1.2 to imply $m_{LK'_n/K'_n} \to \infty$ as $n \to \infty$. However, if $\mathcal{X}_\infty[p]$ is not torsion over $\mathbb{F}_p[[\mathcal{G}]]$ (which, by Hajir et al [10, Th. 1.3], should happen often), then a more delicate analysis is required to extract information about the asymptotic behaviour of $m_{LK'_n/K'_n}$ from Theorem 1.1.

3. U_L AND LOOP OPERATORS

In this section we show that, under suitable hypotheses, the general approach of §2.1 combine with Heller's algebraic theory of loop operators to derive concrete information about the $\mathbb{Z}_p[G]$-structures of both $\mathrm{cor}_G(U_L)$ and A_L (see, in particular, Proposition 3.4 and Remark 3.6).

3.1. **Loop operators.** Fix a finite group Γ. For a finitely generated $\mathbb{Z}_p[\Gamma]$-lattice M, we write $\mathrm{d}_\Gamma(M)$ and $\mathrm{r}_\Gamma(M)$ for the minimal numbers of its generators and relations (as a $\mathbb{Z}_p[\Gamma]$-module) and denote its 'deficiency' $\mathrm{d}_\Gamma(M) - \mathrm{r}_\Gamma(M)$ by $\mathrm{D}_\Gamma(M)$. We also write $\mathrm{m}_\Gamma(M)$ for the maximal rank of a free direct $\mathbb{Z}_p[\Gamma]$-summand of M (as determined by its Krull-Schmidt decomposition). In particular, if M is free, then it is isomorphic to $\mathbb{Z}_p[\Gamma]^{\mathrm{m}_\Gamma(M)}$.

We next recall that $\mathbb{Z}_p[\Gamma]$ is semi-perfect and hence that every finitely generated $\mathbb{Z}_p[\Gamma]$-module M has a projective cover $\theta : P \to M$ that is unique up to isomorphism (cf. [5, Prop. (6.20), Th. (6.23)]). The $\mathbb{Z}_p[\Gamma]$-module

$$\Omega_\Gamma(M) = \Omega^1_\Gamma(M) := \ker(\theta)$$

is then independent, up to isomorphism, of θ and, for each integer $n > 1$, we inductively set

$$\Omega^n_\Gamma(M) := \Omega_\Gamma(\Omega^{n-1}_\Gamma(M)).$$

These modules form part of the theory of 'loop-operators' introduced by Heller [14] and their properties are essentially well-understood. In particular, the following result records several properties relevant to our study.

Lemma 3.1. *Let M_1, M_2 and M_3 be $\mathbb{Z}_p[\Gamma]$-lattices, and N a $\mathbb{Z}_p[\Delta]$-lattice for some subgroup Δ of Γ.*

(i) $\Omega_\Gamma(M_1) = (0) \iff M_1$ *is free;* $\Omega_\Gamma(M_1)$ *is non-free indecomposable* $\iff \mathrm{cor}_\Gamma(M_1)$ *is non-zero indecomposable;* $\Omega_\Gamma(M_1 \oplus M_2)$ *is isomorphic to* $\Omega_\Gamma(M_1) \oplus \Omega_\Gamma(M_2)$.

(ii) If $M_1 \hookrightarrow M_2 \twoheadrightarrow M_3$ *is a short exact sequence, then* $\mathrm{m}_\Gamma(M_2) \geq \mathrm{m}_\Gamma(M_1) + \mathrm{m}_\Gamma(M_3)$ *and there exists an exact sequence of* $\mathbb{Z}_p[\Gamma]$*-lattices*

$$0 \to \mathrm{cor}_\Gamma(M_1) \to \mathrm{cor}_\Gamma(M_2) \oplus \mathbb{Z}_p[\Gamma]^{\mathrm{m}_\Gamma(M_2)-\mathrm{m}_\Gamma(M_1)-\mathrm{m}_\Gamma(M_3)} \to \mathrm{cor}_\Gamma(M_3) \to 0.$$

(iii) If $N_1 \hookrightarrow N_2 \twoheadrightarrow N_3$ *is a short exact sequence of finitely generated* $\mathbb{Z}_p[\Gamma]$*-modules, then* $\mathrm{D}_\Gamma(N_2) \geq \mathrm{D}_\Gamma(N_1) + \mathrm{D}_\Gamma(N_3)$ *and there exists an exact sequence of* $\mathbb{Z}_p[\Gamma]$*-lattices*

$$0 \to \Omega^2_\Gamma(N_1) \to \Omega^2_\Gamma(N_2) \oplus \mathbb{Z}_p[\Gamma]^{\mathrm{D}_\Gamma(N_2)-\mathrm{D}_\Gamma(N_1)-\mathrm{D}_\Gamma(N_3)} \to \Omega^2_\Gamma(N_3) \to 0.$$

Further, the element of $\mathrm{Ext}^1_{\mathbb{Z}_p[\Gamma]}(\Omega^2_\Gamma(N_3), \Omega^2_\Gamma(N_1))$ *that corresponds to this sequence is uniquely determined by the element of* $\mathrm{Ext}^1_{\mathbb{Z}_p[\Gamma]}(N_3, N_1)$ *that corresponds to the original sequence.*

(iv) $\Omega_\Gamma(M_1) = \Omega_\Gamma(\mathrm{cor}_\Gamma(M_1)) = \mathrm{cor}_\Gamma(\Omega_\Gamma(M_1))$ *and* $\Omega_\Gamma(\mathrm{i}^\Gamma_\Delta N) = \mathrm{i}^\Gamma_\Delta(\Omega_\Delta(N))$.

(v) $\mathrm{cor}_\Gamma(\mathrm{i}^\Gamma_\Delta(N))$ *is a direct summand of* $\mathrm{i}^\Gamma_\Delta(\mathrm{cor}_\Delta(N))$*, with equality if* Δ *is normal in* Γ.

Proof. The first assertion of (i) is clear since Γ is a p-group and so a finitely generated projective $\mathbb{Z}_p[\Gamma]$-module is free (by Swan's Theorem). The second assertion of (i) follows, for example, from [9, Lem. 5.1] (and is also straightforward to prove directly) and the third is true since the direct sum of projective covers of M_1 and M_2 is a projective cover of $M_1 \oplus M_2$.

To prove (ii), we note that given sequence combines with the Krull-Schmidt theorem to induce a short exact sequence of $\mathbb{Z}_p[\Gamma]$-modules

$$0 \to \mathrm{cor}_\Gamma(M_1) \oplus \mathbb{Z}_p[\Gamma]^{\mathrm{m}_\Gamma(M_1)} \to \mathrm{cor}_\Gamma(M_2) \oplus \mathbb{Z}_p[\Gamma]^{\mathrm{m}_\Gamma(M_2)} \to \mathrm{cor}_\Gamma(M_3) \oplus \mathbb{Z}_p[\Gamma]^{\mathrm{m}_\Gamma(M_3)} \to 0.$$

This induces the claimed exact sequence since $\mathbb{Z}_p[\Gamma]$ is Gorenstein, and hence then also implies the claimed inequality $\mathrm{m}_\Gamma(M_2) - \mathrm{m}_\Gamma(M_1) - \mathrm{m}_\Gamma(M_3) \geq 0$.

To prove (iii) we fix projective covers $\theta_i : P_i \to N_i$ for $i \in \{1, 3\}$ and consider the exact diagram

$$\begin{array}{ccccc} P_1 & \xhookrightarrow{\phi_1} & P_1 \oplus P_3 & \xrightarrow{\phi_2} & P_3 \\ \downarrow{\scriptstyle \theta_1} & & \downarrow{\scriptstyle \theta_2'} & & \downarrow{\scriptstyle \theta_3} \\ N_1 & \hookrightarrow & N_2 & \twoheadrightarrow & N_3. \end{array} \tag{18}$$

Here the lower row is the given sequence, the maps ϕ_1 and ϕ_2 are the obvious maps and θ_2' is chosen to make the diagram commute. Then θ_2' is surjective, and so, for a projective cover $\theta_2 : P_2 \to N_2$, the module $P_1 \oplus P_3$ is isomorphic to $P_2 \oplus \mathbb{Z}_p[\Gamma]^t$ for some non-negative integer t. Upon applying the Snake lemma to the above diagram, one therefore obtains a short exact sequence of the form

$$\Omega_\Gamma(N_1) \hookrightarrow \Omega_\Gamma(N_2) \oplus \mathbb{Z}_p[\Gamma]^t \twoheadrightarrow \Omega_\Gamma(N_3). \tag{19}$$

Given the universal property of projective covers, it is straightforward to check that the Yoneda extension class of this sequence (regarded as a 1-extension of $\mathbb{Z}_p[\Gamma]$-modules) is uniquely determined by that of the original sequence. In addition, since P_1, P_2 and P_3 are respectively free (over $\mathbb{Z}_p[\Gamma]$) of ranks $\mathrm{d}_\Gamma(N_1)$, $\mathrm{d}_\Gamma(N_2)$ and $\mathrm{d}_\Gamma(N_3)$, the exactness of the upper row of (18) implies $t = \mathrm{d}_\Gamma(N_1) + \mathrm{d}_\Gamma(N_3) - \mathrm{d}_\Gamma(N_2)$. Noting that $\mathrm{r}_\Gamma(N_i) = \mathrm{d}_\Gamma(\Omega_\Gamma(N_i))$ for $i \in \{1, 2, 3\}$, one can therefore obtain the claimed exact sequence in (ii) by repeating the argument after replacing the given sequence by (19). Given the claimed sequence, one then also deduces that the exponent $\mathrm{D}_\Gamma(N_2) - \mathrm{D}_\Gamma(N_1) - \mathrm{D}_\Gamma(N_3)$ must be non-negative, as claimed.

The first equality in (iv) follows directly from the exact sequence (19) with $N_1 \hookrightarrow N_2 \twoheadrightarrow N_3$ taken to be an exact sequence of the form $\mathrm{cor}_\Gamma(M_1) \hookrightarrow M_1 \twoheadrightarrow \mathbb{Z}_p[\Gamma]^{\mathrm{m}_\Gamma(M_1)}$. We next fix a decomposition of M_1 as a direct sum $\bigoplus_{i \in I} M_{1i}$ of indecomposable modules, and for each index i a projective cover θ_i of M_{1i}. Then $\bigoplus_{i \in I} \theta_i$ is a projective cover of M_1 and, given

this, the equality $\Omega_\Gamma(\mathrm{cor}_\Gamma(M_1)) = \mathrm{cor}_\Gamma(\Omega_\Gamma(M_1))$ follows from (i). To prove the remainder of (iv), and also (v), set $M := \mathrm{i}^\Gamma_\Delta N$. Then, by Nakayama's Lemma, one has $\mathrm{d}_\Gamma(M) = \dim_{\mathbb{F}_p}(\mathbb{F}_p \otimes_{\mathbb{Z}_p[\Gamma]} M) = \dim_{\mathbb{F}_p}(\mathbb{F}_p \otimes_{\mathbb{Z}_p[\Delta]} N) = \mathrm{d}_\Delta(N)$. Thus, if θ is a projective cover of the $\mathbb{Z}_p[\Delta]$-module N, then $\mathrm{i}^\Gamma_\Delta\theta$ is a projective cover of the $\mathbb{Z}_p[\Gamma]$-module $\mathrm{i}^\Gamma_\Delta N$ and so the second assertion of (iv) is true since $\Omega_\Gamma(M) = \ker(\mathrm{i}^\Gamma_\Delta\theta) = \mathrm{i}^\Gamma_\Delta(\ker(\theta)) = \mathrm{i}^\Gamma_\Delta(\Omega_\Delta(N))$.

The first assertion of (v) follows easily from the Krull-Schmidt Theorem. To prove the second, write $\mathcal{O}$ for the valuation ring of the completion of the maximal unramified extension $\mathbb{Q}_p^{\mathrm{un}}$ of $\mathbb{Q}_p$ in $\mathbb{Q}_p^c$. Then, for a $\mathbb{Z}_p$-lattice M, the group $\Omega := \mathrm{Gal}(\mathbb{Q}_p^{\mathrm{un}}/\mathbb{Q}_p)$ acts semi-linearly on $M' := \mathcal{O} \otimes_{\mathbb{Z}_p} M$ and $M = H^0(\Omega, M')$. In particular, for a subgroup Υ of Γ, a $\mathbb{Z}_p[\Upsilon]$-lattice L has a free direct summand if and only if the $\mathcal{O}[\Upsilon]$-lattice L' has a free direct summand and so $\mathrm{cor}_\Upsilon(L)' = \mathrm{cor}_{\mathcal{O}[\Upsilon]}(L')$. It follows that $\mathrm{cor}_\Delta(N)'$ is a direct sum of non-free indecomposable $\mathcal{O}[\Delta]$-lattices and hence, by Green's Indecomposability Theorem [6], that $\mathrm{i}^\Gamma_\Delta(\mathrm{cor}_\Delta(N)') = \mathrm{cor}_{\mathcal{O}[\Gamma]}(\mathrm{i}^\Gamma_\Delta(N)')$ provided Δ is normal in Γ. In this case, therefore, one has

$$\begin{aligned}\mathrm{i}^\Gamma_\Delta(\mathrm{cor}_\Delta(N)) &= H^0(\Omega, \mathrm{i}^G_\Delta(\mathrm{cor}_\Delta(N))') = H^0(\Omega, \mathrm{i}^\Gamma_\Delta(\mathrm{cor}_\Delta(N)'))\\ &= H^0(\Omega, \mathrm{cor}_{\mathcal{O}[\Gamma]}(\mathrm{i}^\Gamma_\Delta(N)')) = H^0(\Omega, \mathrm{cor}_\Gamma(\mathrm{i}^\Gamma_\Delta(N))') = \mathrm{cor}_\Gamma(\mathrm{i}^\Gamma_\Delta(N)),\end{aligned}$$

as required. □

Example 3.2. It is clear that $\Omega_\Gamma(\mathbb{Z}_p[\Gamma]) = (0)$, $\Omega_\Gamma(\mathbb{Z}_p) = I(\Gamma)_p$, $\mathrm{cor}_\Gamma(I(\Gamma)_p) = I(\Gamma)_p$, $\mathrm{cor}_\Gamma(\mathbb{Z}_p[\Gamma]) = (0)$ and $\mathrm{cor}_\Gamma(\mathbb{Z}_p) = \mathbb{Z}_p$. In addition, since $\mathbb{Z}_p \otimes_{\mathbb{Z}_p[\Gamma]} I(\Gamma)_p \cong I_p(\Gamma)/I_p(\Gamma)^2 \cong \Gamma^{\mathrm{ab}}$, Nakayama's Lemma implies $\mathrm{d}_\Gamma(I(\Gamma)_p) = \mathrm{d}(\Gamma)$ and so $\Omega_\Gamma(I(\Gamma)_p)$ is the kernel of a surjective map $\theta : \mathbb{Z}_p[\Gamma]^{\mathrm{d}(\Gamma)} \to I(\Gamma)_p$. In particular, for the tautological short exact sequence of $\mathbb{Z}_p[\Gamma]$-modules

$$0 \to I(\Gamma)_p \to \mathbb{Z}_p[\Gamma] \to \mathbb{Z}_p \to 0, \tag{20}$$

the short exact sequences in Lemma 3.1(ii) and (19) are respectively the sequence itself and $0 \to \Omega_\Gamma(I(\Gamma)_p) \to \mathbb{Z}_p[\Gamma]^{\mathrm{d}(\Gamma)} \xrightarrow{\theta} I(\Gamma)_p \to 0$.

3.2. **Module deficiencies and the core of U_L.** We start by describing $m_{L/K,S}$ and $\mathrm{cor}_G(U_{L,S})$ in terms of the module $\nabla_{L,S,p}$.

Proposition 3.3. *Fix a finite set S of places of K that contains S_K^∞, and assume $\mu_L[p] = (0)$. Then $m_{L/K,S} = \mathrm{D}_G(\nabla_{L,S,p}) - |R_{L/K} \setminus S|$ and the $\mathbb{Z}_p[G]$-module $\mathrm{cor}_G(U_{L,S})$ is isomorphic to $\Omega^2_G(\nabla_{L,S,p})$.*

Proof. Since $\mu_L[p] = (0)$ one has $U_{L,S} = \mathcal{O}^\times_{L,S,p}$. Hence, if we fix a projective cover of $\mathbb{Z}_p[G]$-modules $\theta : P \to \nabla_{L,S,p}$, then a standard construction of homological algebra implies the existence of an exact sequence of $\mathbb{Z}_p[G]$-modules

$$0 \to U_{L,S} \to P_1 \xrightarrow{\theta_1} P \xrightarrow{\theta} \nabla_{L,S,p} \to 0 \tag{21}$$

that has the same Yoneda extension class as the pro-p completion of (4). Then, since $U_{L,S}$ and P are finitely generated and torsion-free the same is true of P_1. In addition, since P is projective and the modules M_0 and M_1 in (4) are both cohomologically-trivial (over G), the module P_1 must also be cohomologically-trivial. These facts combine with [1, Th. 8] to imply that the $\mathbb{Z}_p[G]$-module P_1 is projective, and hence free (by Swan's Theorem) of rank $\mathrm{m}_G(P_1)$. In addition, the exact sequences (5) and (13) (for each $v \in R_{L/K} \setminus S$) combine with Dirichlet's Unit Theorem to imply the existence of an isomorphism of $\mathbb{Q}_p[G]$-modules

$$\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \nabla_{L,S,p} = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \ker(\epsilon_{L,S})_p \cong (\mathbb{Q}_p \otimes_{\mathbb{Z}_p} U_{L,S}) \oplus \mathbb{Q}_p[G]^{|R_{L/K} \setminus S|}$$

(see also [24, §1, Cor.]). From the exactness of (21), it therefore follows that

$$\mathrm{m}_G(P_1) = \mathrm{m}_G(P) - |R_{L/K} \setminus S| = \mathrm{d}_G(\nabla_{L,S,p}) - |R_{L/K} \setminus S|. \tag{22}$$

On the other hand, if we fix a projective cover $\theta'_1 : P'_1 \to \ker(\theta)$, then there exists an isomorphism of $\mathbb{Z}_p[G]$-modules $\phi : P_1 \cong P'_1 \oplus P''_1$ for some projective, and hence free, $\mathbb{Z}_p[G]$-module P''_1 so that $\theta_1 \circ \phi^{-1} = \theta'_1 \oplus 0_{P''_1}$. There are therefore isomorphisms of $\mathbb{Z}_p[G]$-modules

$$U_{L,S} \cong \ker(\theta_1) \cong \ker(\theta_1 \circ \phi^{-1}) = \ker(\theta'_1) \oplus P''_1 \cong \Omega^2_G(\nabla_{L,S,p}) \oplus P''_1.$$

Taken in conjunction with Lemma 3.1(iv), this implies $\mathrm{cor}_G(U_{L,S})$ is isomorphic to $\Omega^2_G(\nabla_{L,S,p})$ and hence that $m_{L/K,S} = \mathrm{m}_G(P''_1)$. It is therefore enough to note that (22) implies

$$\begin{aligned}\mathrm{m}_G(P''_1) = \mathrm{m}_G(P_1) - \mathrm{m}_G(P'_1) &= (\mathrm{d}_G(\nabla_{L,S,p}) - |R_{L/K} \setminus S|) - \mathrm{d}_G(\ker(\theta)) \\ &= (\mathrm{d}_G(\nabla_{L,S,p}) - |R_{L/K} \setminus S|) - \mathrm{r}_G(\nabla_{L,S,p}) \\ &= \mathrm{D}_G(\nabla_{L,S,p}) - |R_{L/K} \setminus S|.\end{aligned}$$

□

With further effort, the above result can be made much more explicit. To give a concrete example of this, for each $v \in R^{\mathrm{f}}_{L/K}$, we define an integer by setting

$$\varepsilon_v = \varepsilon_{L/K,v} := (\mathrm{d}(G_v) + 1) - \mathrm{d}_{G_v}(W_{v,p}).$$

These integers are considered in detail in Remark 3.7 below. For the moment, however, we note only that $\mathrm{d}(G_v) = \mathrm{d}_{G_v}(I(G_v)_p)$ (cf. Example 3.2) and hence that the sequence (12) implies ε_v is equal to either 0 or 1.

Proposition 3.4. *Assume $R^\infty_{L/K} = \emptyset$ and the existence of $v_0 \in R_{L/K}$ with $I_{v_0} = G$. Write $R'_{L/K}$ and $R''_{L/K}$ for $\{v \in R_{L/K} : I_v \neq G_v\}$ and $R_{L/K} \setminus (R'_{L/K} \cup \{v_0\})$ respectively. Set*

$$t_{L/K} := \mathrm{D}_G(\nabla_{L,S^\infty_K,p}) - \mathrm{D}_G(A_L) - |S^\infty_K| - (\rho_{L/K} - \delta_{L/K}) + (1 - \mathrm{d}(G)) - \sum\nolimits_{v \in R'_{L/K}} \varepsilon_v$$

and define lattices

$$M_{L/K} := \bigoplus\nolimits_{v \in R'_{L/K}} \mathrm{i}^G_{G_v} \Omega_{G_v}(W_{v,p}) \quad \text{and} \quad N_{L/K} := \bigoplus\nolimits_{v \in R''_{L/K}} (\Omega^2_G(\mathrm{i}^G_{G_v}\mathbb{Z}_p) \oplus \Omega_G(\mathrm{i}^G_{G_v}\mathbb{Z}_p)).$$

Then $t_{L/K} \geq 0$ and there exists a short exact sequence of $\mathbb{Z}_p[G]$-lattices

$$0 \to \Omega^2_G(A_L) \to \mathrm{cor}_G(U_L) \oplus \mathbb{Z}_p[G]^{t_{L/K}} \to I(G)_p \oplus M_{L/K} \oplus N_{L/K} \to 0. \tag{23}$$

The $\mathbb{Z}_p[G]$-lattices $I(G)_p$, and both $\Omega^2_G(\mathrm{i}^G_{G_v}\mathbb{Z}_p)$ and $\Omega_G(\mathrm{i}^G_{G_v}\mathbb{Z}_p)$ for each $v \in R''_{L/K}$, are non-free indecomposable. For each $v \in R'_{L/K}$, the $\mathbb{Z}_p[G_v]$-lattice $\Omega_{G_v}(W_{v,p})$ is non-free indecomposable and, if G_v is normal in G, then the $\mathbb{Z}_p[G]$-lattice $\mathrm{i}^G_{G_v}\Omega_{G_v}(W_{v,p}) = \Omega_G(\mathrm{i}^G_{G_v}W_{v,p})$ is also non-free indecomposable.

Proof. We start by noting the argument of Gruenberg and Weiss that proves [9, Th. 6.1] also establishes the following facts about the $\mathbb{Z}_p[G_v]$-lattices $W^*_{v,p}$.

If $v \in R'_{L/K}$, then $W^*_{v,p}$ (and hence also $W_{v,p}$) is non-free indecomposable. In particular, since the sequences (13) and (14) combine to imply $\mathrm{d}_{G_v}(W^*_{v,p}) \leq 2$ and $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} W^*_{v,p} \cong \mathbb{Q}_p[G_v]$, one deduces in this case that

$$\mathrm{d}_{G_v}(W^*_{v,p}) = 2. \tag{24}$$

From the short exact sequence $0 \to W_{v,p} \to \mathbb{Z}_p[G_v] \oplus \mathbb{Z}_p[G_v] \to W^*_{v,p} \to 0$ of [9, Lem. 4.1], we can then deduce

$$\mathrm{r}_{G_v}(W^*_{v,p}) = \mathrm{d}_{G_v}(W_{v,p}) \tag{25}$$

and also that there exists an isomorphism of $\mathbb{Z}_p[G_v]$-modules

$$\Omega^2_{G_v}(W^*_{v,p}) \cong \Omega_{G_v}(W_{v,p}). \tag{26}$$

If now $v \in R_{L/K} \setminus R'_{L/K}$, then the exact sequence of G_v-modules (13) splits and so $W^*_{v,p}$ is isomorphic to the direct sum $\mathbb{Z}_p \oplus I(G_v)^*_p$. In this case, therefore, the exact sequence (14) induces an isomorphism of $\mathbb{Z}_p[G_v]$-modules

$$\Omega^2_{G_v}(W^*_{v,p}) \cong \Omega^2_{G_v}(\mathbb{Z}_p) \oplus \Omega^2_{G_v}(I(G_v)^*_p) \cong \Omega^2_{G_v}(\mathbb{Z}_p) \oplus \Omega_{G_v}(\mathbb{Z}_p) \tag{27}$$

and also combines with the tautological short exact sequence (20) with $\Gamma = G_v$ and equality $\mathrm{d}_G(I(G_v)_p) = \mathrm{d}(G_v)$ to imply

$$\mathrm{D}_{G_v}(W^*_{v,p}) = \mathrm{D}_{G_v}(\mathbb{Z}_p) + \mathrm{D}_{G_v}(I(G_v)^*_p) = (1 - \mathrm{d}(G_v)) + (1 - 1) = 1 - \mathrm{d}(G_v). \tag{28}$$

In addition, since $G_{v_0} = G$, the splitting of (13) for $v = v_0$ implies that the tautological exact sequence of $\mathbb{Z}_p[G]$-modules

$$0 \to \ker(\epsilon_L)_p \to \big(\bigoplus\nolimits_{v \in S^\infty_K} \mathrm{i}^G_{G_v}\mathbb{Z}_p\big) \oplus \big(\bigoplus\nolimits_{v \in R^{\mathrm{f}}_{L/K}} \mathrm{i}^G_{G_v} W^*_{v,p}\big) \xrightarrow{\epsilon_{L,p}} \mathbb{Z}_p \to 0$$

is also split and hence that there is an isomorphism of $\mathbb{Z}_p[G]$-modules

$$\ker(\epsilon_L)_p \cong \big(\bigoplus\nolimits_{v \in S^\infty_K} \mathrm{i}^G_{G_v}\mathbb{Z}_p\big) \oplus I(G)^*_p \oplus \tilde{M}_{L/K} \oplus \tilde{N}_{L/K} \tag{29}$$

with

$$\tilde{M}_{L/K} := \bigoplus\nolimits_{v \in R'_{L/K}} \mathrm{i}^G_{G_v} W^*_{v,p} \quad \text{and} \quad \tilde{N}_{L/K} := \bigoplus\nolimits_{v \in R''_{L/K}} (\mathrm{i}^G_{G_v}\mathbb{Z}_p \oplus \mathrm{i}^G_{G_v} I(G_v)^*_p).$$

Since the $\mathbb{Z}_p[G]$-module $\bigoplus_{v \in S^\infty_K} \mathrm{i}^G_{G_v}\mathbb{Z}$ is a free of rank $|S^\infty_K|$ (as $R^\infty_{L/K} = \emptyset$), the exact sequence (5) (with $S = S^\infty_K$) therefore induces an exact sequence

$$0 \to A_L \to \nabla_{L,p} \to \mathbb{Z}_p[G]^{|S^\infty_K|} \oplus I(G)^*_p \oplus \tilde{M}_{L/K} \oplus \tilde{N}_{L/K} \to 0, \tag{30}$$

with $\nabla_{L,p} := \nabla_{L,S^\infty_K,p}$. Upon applying Lemma 3.1(iii) to this sequence, and recalling $\Omega^2_G(\nabla_{L,p})$ is isomorphic to $\mathrm{cor}_G(U_L)$ (by Proposition 3.3 with $S = S^\infty_K$), we thus derive an exact sequence

$$0 \to \Omega^2_G(A_L) \to \Omega^2_G(\nabla_{L,p}) \oplus \mathbb{Z}_p[G]^{t'_{L/K}} \to \Omega^2_G(I(G)^*_p) \oplus \Omega^2_G(\tilde{M}_{L/K}) \oplus \Omega^2_G(\tilde{N}_{L/K}) \to 0, \tag{31}$$

with

$$t'_{L/K} := \mathrm{D}_G(\nabla_{L,p}) - \mathrm{D}_G(A_L) - \mathrm{D}_G(\mathbb{Z}_p[G]^{|S^\infty_K|}) - \mathrm{D}_G(I(G)^*_p) - \mathrm{D}_G(\tilde{M}_{L/K} \oplus \tilde{N}_{L/K}).$$

It is clear $\mathrm{D}_G(\mathbb{Z}_p[G]^{|S^\infty_K|}) = |S^\infty_K|$ and, as a consequence of the exact sequence (14) with G_v replaced by G, also $\mathrm{D}_G(I(G)^*_p) = 0$. In addition, after taking account of (24), (25) and (28), one computes that the quantity $\mathrm{D}_G(\tilde{M}_{L/K} \oplus \tilde{N}_{L/K})$ is equal to

$$\begin{aligned}
&\big(\textstyle\sum_{v \in R'_{L/K}} \mathrm{D}_{G_v}(W^*_{v,p})\big) + \big(\sum_{v \in R''_{L/K}} (\mathrm{D}_{G_v}(\mathbb{Z}_p) + \mathrm{D}_{G_v}(I(G_v)^*_p)\big)\\
&= \big(\textstyle\sum_{v \in R'_{L/K}} (2 - \mathrm{d}_{G_v}(W_{v,p})\big) + \big(\sum_{v \in R''_{L/K}} (1 - \mathrm{d}(G_v))\big)\\
&= \big(\textstyle\sum_{v \in R'_{L/K}} (1 - \mathrm{d}(G_v) + \varepsilon_v)\big) + \big(\sum_{v \in R''_{L/K}} (1 - \mathrm{d}(G_v))\big)\\
&= \big(\textstyle\sum_{v \in R_{L/K} \setminus \{v_0\}} (1 - \mathrm{d}(G_v))\big) + \sum_{v \in R'_{L/K}} \varepsilon_v\\
&= \big(\textstyle\sum_{v \in R_{L/K}} (1 - \mathrm{d}(G_v))\big) - (1 - \mathrm{d}(G_{v_0})) + \sum_{v \in R'_{L/K}} \varepsilon_v\\
&= (\rho^{\mathrm{f}}_{L/K} - \delta_{L/K}) - (1 - \mathrm{d}(G)) + \textstyle\sum_{v \in R'_{L/K}} \varepsilon_v.
\end{aligned}$$

Taken together, these computations prove $t'_{L/K} = t_{L/K}$. Given this, the claimed exact sequence (23) is then obtained by combining the exact sequence (31) with the isomorphisms (26) and (27) (and Lemma 3.1(iv)) and the fact $\Omega^2_G(I(G)^*_p) \cong \Omega_G(\mathbb{Z}_p) \cong I(G)_p$.

It is now enough to verify the assertions regarding indecomposability. Firstly, the $\mathbb{Z}_p[G]$-lattice $I(G)^*_p$ is clearly not free and so is indecomposable since $\mathrm{d}_G(I(G)^*_p) = 1$. In addition, for each $v \in R''_{L/K}$, the $\mathbb{Z}_p[G_v]$-lattice $\mathrm{i}^G_{G_v}\mathbb{Z}_p$ is non-free indecomposable and so Lemma 3.1(i) implies the same is true of both $\Omega^2_G(\mathrm{i}^G_{G_v}\mathbb{Z}_p)$ and $\Omega_G(\mathrm{i}^G_{G_v}\mathbb{Z}_p)$. We now fix $v \in R'_{L/K}$ and note that, since the $\mathbb{Z}_p[G_v]$-lattice $W_{v,p}$ is non-free indecomposable, the same is true of $\Omega_{G_v}(W_{v,p})$ (by Lemma 3.1(i)). Further, if G_v is normal in G, then (using the notation of the proof of Lemma 3.1(v)) Green's Theorem combines with Lemma 3.1(i) to imply $\Omega_G(\mathrm{i}^G_{G_v}W_{v,p})$ is non-free indecomposable provided $\mathcal{O} \otimes_{\mathbb{Z}_p} W_{v,p}$ is a non-free indecomposable module over $R := \mathcal{O}[G_v]$. Since this module is clearly not free (since $W_{v,p}$ is not free over $\mathbb{Z}_p[G_v]$), it is enough to show the R-module $\mathcal{O} \otimes_{\mathbb{Z}_p} W^*_{v,p}$ is indecomposable. However, if $\mathcal{O} \otimes_{\mathbb{Z}_p} W^*_{v,p}$ decomposes as a direct sum $M \oplus M'$ of non-zero modules, then precisely one of the summands, M say, has zero isotypic component at the trivial character of G_v and so belongs to the kernel of the map of R-modules $\theta : \mathcal{O} \otimes_{\mathbb{Z}_p} W^*_{v,p} \to \mathcal{O}$ induced by the exact sequence (13). This implies a decomposition $\ker(\theta) = M \oplus (M' \cap \ker(\theta))$ and so, since $\ker(\theta)$ is equal to the indecomposable R-module $N := \mathcal{O} \otimes_{\mathbb{Z}_p} I(G)^*_p$, it follows that $M = N$ and $M' \cap \ker(\theta) = (0)$ so θ induces an isomorphism $M' \cong \mathcal{O}$. This implies that the scalar extension of (13) splits over R and hence that the original sequence splits over $\mathbb{Z}[G_v]$. The latter contradiction therefore implies that the R-module $\mathcal{O} \otimes_{\mathbb{Z}_p} W^*_{v,p}$ is indecomposable, as required. □

Remark 3.5. The existence of a place $v_0 \in R_{L/K}$ with $I_{v_0} = G$ strongly restricts L/K. However, with further effort, the approach of Proposition 3.4 allows one to prove similar results for wider classes of extensions. For example, if L/K is any Galois extension for which there exists a non-archimedean place v_0 of K with $G_{v_0} = G$, and we set $S := S^\infty_K \cup \{v_0\}$, then the above argument can be combined with the pro-p completion of the isomorphism (17) (in place of (29)) to analyse the module $\mathrm{cor}_G(U_{L,S})$ and thereby also, via Lemmas 2.6 and 3.1(ii), the module $\mathrm{cor}_G(U_L)$.

Remark 3.6. Since $t_{L/K} \geq 0$, Propositions 3.3 (with $S = S^\infty_K$) and 3.4 combine to imply that, under the stated conditions, one has

$$\begin{aligned} m_{L/K} + \rho_{L/K} &= \mathrm{D}_G(\nabla_{L,S^\infty_K,p}) \\ &\geq \mathrm{D}_G(A_L) + |S^\infty_K| + (\rho_{L/K} - \delta_{L/K}) - (1 - \mathrm{d}(G)) + \sum\nolimits_{v \in R'_{L/K}} \varepsilon_v \end{aligned}$$

and hence also

$$\mathrm{D}_G(A_L) \leq \kappa_{L/K} := (m_{L/K} - |S^\infty_K|) + (\sum\nolimits_{v \in R_{L/K} \setminus \{v_0\}} \mathrm{d}(G_v)) + (1 - \sum\nolimits_{v \in R'_{L/K}} \varepsilon_v).$$

This gives a concrete link between $m_{L/K}$ and the G-structure of A_L. For instance, if $\kappa_{L/K} < 0$, then it implies $\mathrm{r}_G(A_L) > \mathrm{d}_G(A_L)$ (and so $A_L \neq (0)$). As a specific example, assume $R_{L/K} = \{v_0\}$ and $m_{L/K} < \mathrm{rk}(U_K)$: then $\kappa_{L/K} = m_{L/K} - |S^\infty_K| + 1 = m_{L/K} - \mathrm{rk}(U_K) < 0$ so $A_L \neq (0)$ and hence, by [25, Th. 10.4], also $A_K \neq (0)$. In particular, if K is any number field for which $\mu_K[p]$ and A_K both vanish, then for any finite Galois p-extension L/K that is (totally) ramified at precisely one non-archimedean place one must have $m_{L/K} = \mathrm{rk}(U_K)$. This result was first obtained, by different means and under the additional hypothesis that G is cyclic, in [4, Exam. 5.2(i)], where it is also shown $\mathrm{cor}_G(U_L)$ is isomorphic, as a $\mathbb{Z}_p[G]$-module, to $I(G)^*_p$. However, by using the approach of Bouazzaoui and Lim [2], one finds that the latter isomorphism cannot be valid (under the stated conditions on K and $R_{L/K}$) if G is not cyclic - I am grateful to Donghyeok Lim for pointing this out to me.

Remark 3.7. Fix $v \in R'_{L/K}$ and consider the integer ε_v defined just before Proposition 3.4. To do this, set $f := |G_v/I_v|$, fix a generator ϕ of G_v/I_v and write π for the canonical projection $\mathbb{Z}[G_v] \to \mathbb{Z}[G_v/I_v]$. For a natural number t, set $T_t := \sum_{i=0}^{i=t-1} \phi^i$ and then $z(g) := (g-1, T_{a(g)})$

for $g \in G_v$, with $a(g)$ the unique integer such that $1 \leq a(g) \leq f$ and $\pi(g) = \phi^{a(g)}$. Then, from the discussion in [9, §1], one has

$$W_v = \{(x, y) \in I(G_v) \oplus \mathbb{Z}[G_v/I_v] : \pi(x) = (\phi - 1)y\},$$

(so $z(g) \in W_v$ for each g) and the maps in (12) are such that $\alpha_v(1) = (0, T_f)$ and $\beta_v(z(g)) = g - 1$ for $g \in G$. Hence, if we set $d := \mathrm{d}(G_v)$ and fix a subset $\{g_j\}_{1 \leq j \leq d}$ of G_v so that $\{g_j - 1\}_{1 \leq j \leq d}$ generates the $\mathbb{Z}_p[G_v]$-module $I(G_v)_p$, then the $\mathbb{Z}_p[G_v]$-module $W_{v,p}$ is generated by the union of $\alpha_v(1)$ and $Z := \{z(g_j)\}_{1 \leq j \leq d}$. In particular, if some g_j is such that its order $o(j)$ is equal to the order of $\pi(g_j)$, then $(\sum_{i=0}^{o(j)-1} g_j^i) z(g_j) = \alpha_v(1)$ so that $W_{v,p}$ is generated by Z and hence $\varepsilon_v = 1$. (If G is abelian, one can even check that the existence of such an element g_j is a necessary condition for $\varepsilon_v = 1$.) Now an element g_j with these properties automatically exists, and so $\varepsilon_v = 1$, if G_v is the semi-direct product of I_v and G_v/I_v (which, by [17, Lem. 3.3], one can always assume after composing L_w with the unramified extension of K_v of degree $[L_w : K_v]$). However, it can be shown that such elements can also exist (and hence one has $\varepsilon_v = 1$) if G_v is not a semi-direct product.

Remark 3.8. Since (29) implies $\ker(\epsilon_L)_p$ has a very explicit structure, the proof of Proposition 3.4 reduces the study of $\mathrm{cor}_G(U_L)$ to determining both $\Omega^2_G(A_L)$ and the (Yoneda) extension class of the sequence (30). Whilst the explicit determination (in a general setting) of $\Omega^2_G(A_L)$ is surely difficult, the results of Ritter and Weiss imply the following 'independence' result for the extension class of (30). Fix a finite p-group Γ, a finite set Σ and, for each $\sigma \in \Sigma$, subgroups Γ_σ and Γ^0_σ of Γ, with Γ^0_σ normal in Γ_σ and $\Gamma_\sigma/\Gamma^0_\sigma$ cyclic, and an element γ_σ of Γ_σ that projects to a generator of $\Gamma_\sigma/\Gamma^0_\sigma$. Write (Γ, Σ) for the set of Galois extensions L/K of number fields for which there exists a (fixed) group isomorphism $\iota_{L/K} : \Gamma \to \mathrm{Gal}(L/K)$ and bijection $\kappa_{L/K} : \Sigma \to R_{L/K}$ so that, for each $\sigma \in \Sigma$, the groups $\iota_{L/K}(\Gamma_\sigma)$ and $\iota_{L/K}(\Gamma^0_\sigma)$ are the decomposition and inertia groups in Γ of some place of L above $\kappa_{L/K}(\sigma)$, and $\iota_{L/K}(\gamma_\sigma)$ is an associated frobenius element. For L/K and L'/K' in (Γ, Σ), the maps $\iota_{L/K}$ and $\iota_{L'/K'}$ make A_L and $A_{L'}$ into $\mathbb{Z}_p[\Gamma]$-modules and $\ker(\epsilon_L)_p$ and $\ker(\epsilon_{L'})_p$ into canonically isomorphic $\mathbb{Z}_p[\Gamma]$-modules. In particular, for any isomorphism of Γ-modules $j : A_L \cong A_{L'}$, there exists an induced isomorphism of groups $j_* : \mathrm{Ext}^1_{\mathbb{Z}_p[\Gamma]}(\ker(\epsilon_L)_p, A_L) \cong \mathrm{Ext}^1_{\mathbb{Z}_p[\Gamma]}(\ker(\epsilon_{L'})_p, A_{L'})$. Then the argument used by Ritter and Weiss to prove [24, Th. 2] shows that the extension classes of the respective sequences (30) for L/K and L'/K' correspond under j_*, and hence that the $\mathbb{Z}_p[\Gamma]$-lattices $\mathrm{cor}_\Gamma(U_L)$ and $\mathrm{cor}_\Gamma(U_{L'})$ are isomorphic, provided there exists an exact commutative diagram of group homomorphisms

$$\begin{array}{ccccc} A_L & \hookrightarrow & \mathrm{Gal}(H_L/K) & \twoheadrightarrow & \mathrm{Gal}(L/K) \\ \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle j'} & & \downarrow{\scriptstyle \iota_{L'/K'} \circ \iota_{L/K}^{-1}} \\ A_{L'} & \hookrightarrow & \mathrm{Gal}(H_{L'}/K') & \twoheadrightarrow & \mathrm{Gal}(L'/K'). \end{array}$$

Here H_L and $H_{L'}$ are the p-Hilbert classfields of L and L' and the rows are induced by Galois theory and the identifications $A_L \cong \mathrm{Gal}(H_L/L)$ and $A_{L'} \cong \mathrm{Gal}(H_{L'}/L')$ given by Artin reciprocity.

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