ON p-ADIC L-SERIES, p-ADIC COHOMOLOGY AND CLASS FIELD THEORY

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Abstract. We establish several close links between the Galois structures of a range of arithmetic modules including certain natural families of ray class groups, the values at strictly positive integers of p-adic Artin L-series, the Shafarevic-Weil Theorem and the conjectural surjectivity of certain norm maps in cyclotomic $\mathbb{Z}_p$-extensions. Non-commutative Iwasawa theory and the theory of organising matrices play a key role in our approach.

1. Introduction

In a recent article [10] we extended and refined the theory of 'organising modules' introduced by Mazur and Rubin to construct a canonical class of matrices that encodes a range of detailed information about certain natural families of complexes in arithmetic. We then described several concrete applications of the resulting theory of 'organising matrices' including the proof of new results on the explicit structures of Galois groups, ideal class groups and wild kernels in higher algebraic K-theory and the formulation of a range of explicit conjectures concerning both the ranks and Galois structures of the Selmer groups of abelian varieties over finite (non-abelian) Galois extensions of number fields.

The main purpose of this supplementary article is now threefold. Firstly, we will use techniques of non-commutative Iwasawa theory to explain how the apparent dependence of one of the main results in [10] on the validity of both Leopoldt's Conjecture and of a natural analogue of this conjecture due to Schneider can be removed; then we will combine similar Iwasawa-theoretic methods with a p-adic interpretation of the Shafarevic-Weil Theorem to show how the same result from [10] can be strongly refined under a natural hypothesis on the operating Galois group; finally we will show how the latter refinement leads to an explicit, and as far as we are aware new, interpretation of Leopoldt's Conjecture in terms of the cohomological-triviality (as Galois module) of a natural family of ray class groups and then also allows us to both prove several new results and to make further precise conjectures concerning the explicit structure of such groups.

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2. p-adic L-series and Galois structures

We write $\zeta(R)$ for the centre of a ring R. For any finite group Δ we write $\mathrm{Ir}_p(\Delta)$ for the set of irreducible $\mathbb{Q}_p^c$-valued characters of Δ. For any such character ρ we define a primitive idempotent of $\zeta(\mathbb{Q}_p^c[\Delta])$ by setting $e_\rho := (\rho(1)/|\Delta|)\sum_{\delta\in\Delta}\rho(\delta^{-1})\delta$.

For any Galois extension of fields L/K we often write $G_{L/K}$ in place of $\mathrm{Gal}(L/K)$. For any CM-field E we write E^+ for its maximal (totally) real subfield.

2.1. **Statement of the results.** In this section we fix a totally real field k and a finite CM Galois extension F of k inside k^c, set $G := G_{F/k}$ and write τ for the unique non-trivial element of G_{F/F^+}. We fix an odd prime p, write $\mu_p(F)$ for the Iwasawa-theoretic (p-adic) μ-invariant of F and recall that Iwasawa has conjectured that $\mu_p(F) = 0$. We fix a finite set of places Σ of k containing all archimedean places, all which ramify in F/k and all above p. By abuse of notation, for any field L that contains k, we also write Σ for the set of places of L above places in Σ.

For each natural number a we write $\mu_{F,p}^{\otimes a}$ for the a-fold tensor product of the group $\mu_{F,p}$ of p-power order roots of unity in F and $K_{2a}^w(\mathcal{O}_F)$ for the 'wild kernel' of higher algebraic K-theory, as defined by Banaszak in [2]. We write ω_k for the Teichmüller character $G_k \to \mathbb{Z}_p^\times$ and, for any representation $\rho : G \to \mathrm{GL}_m(\mathbb{Q}_p^c)$, we write $\tilde{\rho}$ for the map on $\mathbb{Q}_p[G]$ obtained by extending ρ by $\mathbb{Q}_p$-linearity. For any extension E of k in k^c we denote by $M_{E,\Sigma}$ the maximal abelian pro-p extension of E inside k^c that is unramified outside Σ.

For each integer m write $\mathrm{Ir}_p^{(m)}(G)$ for the subset of $\mathrm{Ir}_p(G)$ comprising representations $\rho : G \to \mathrm{GL}_{\rho(1)}(\mathbb{Q}_p^c)$ with the property that $\ker(\tilde{\rho})$ contains $(-1)^{m+1}\tau$. Then we obtain a well-defined element of $\zeta(\mathbb{Q}_p[G])$ by setting

$$\mathcal{L}_{F/k,\Sigma,m} := \sum_{\rho\in\mathrm{Ir}_p^{(m)}(G)} e_\rho \cdot \lim_{s\to m}(s-1)^{n_{\rho,m}} L_{p,\Sigma}(s, \rho\otimes\omega_k^{1-m}),$$

where $L_{p,\Sigma}(s, \rho\otimes\omega_k^{1-m})$ denotes the Σ-truncated p-adic Artin L-series of the representation $\rho\otimes\omega_k^{1-m}$ and for each representation $\pi : G_k \to \mathrm{GL}_{\pi(1)}(\mathbb{Q}_p^c)$ we define an integer $n_{\pi,m}$ by setting

$$n_{\pi,m} := \begin{cases} 1, & \text{if } \pi \text{ is trivial and } m = 1, \\ 0, & \text{otherwise.} \end{cases}$$

For every matrix H in $\mathrm{M}_n(\mathbb{Z}_p[G])$ we also note that there is a unique matrix H' in $\mathrm{M}_n(\mathbb{Q}_p[G])$ with $HH' = H'H = \mathrm{nr}_{\mathbb{Q}_p[G]}(H)I_n$ and such that for every primitive central idempotent e of $\mathbb{Q}_p[G]$ the matrix $H'e$ is invertible if and only if the product $\mathrm{nr}_{\mathbb{Q}_p[G]}(H)e$ is non-zero. One then obtains an ideal of $\zeta(\mathbb{Z}_p[G])$ by setting

$$\mathcal{A}_p(G) := \{x \in \zeta(\mathbb{Q}_p[G]) : \text{ if } n > 0 \text{ and } H \in \mathrm{M}_n(\mathbb{Z}_p[G]) \text{ then } xH' \in \mathrm{M}_n(\mathbb{Z}_p[G])\}.$$

Such 'denominator ideals' were first introduced by Nickel and have been computed explicitly in many cases by Johnston and Nickel in [16]. (For a convenient discussion of relevant facts see [10, Rem. 2.4(ii)].)

Finally we set $\zeta(I_{G,p}) := \zeta(\mathbb{Z}_p[G]) \cap I_{G,p}$, where $I_{G,p}$ denotes the kernel of the natural augmentation map $\mathbb{Z}_p[G] \to \mathbb{Z}_p$, and write $\mathrm{Ann}_{\zeta(\mathbb{Z}_p[G])}(N)$ for the annihilator in $\zeta(\mathbb{Z}_p[G])$ of each $\mathbb{Z}_p[G]$-module N.

Theorem 2.1. *We assume that if p divides $|G|$, then $\mu_p(F)$ vanishes. Then one has*

$$\mathcal{L}_{F/k,\Sigma,1} \cdot \mathcal{A}_p(G) \cdot \zeta(I_{G,p})^2 \subseteq \mathrm{Ann}_{\zeta(\mathbb{Z}_p[G])}(\mathrm{Gal}(M_{k,\Sigma}/F))$$

and for each integer $m > 1$ also

$$\mathcal{L}_{F/k,\Sigma,m} \cdot \mathcal{A}_p(G) \cdot \mathrm{Ann}_{\zeta(\mathbb{Z}_p[G])}(\mu_{F,p}^{\otimes(m-1)}) \subseteq \mathrm{Ann}_{\zeta(\mathbb{Z}_p[G])}(K^w_{2m-2}(\mathcal{O}_F)_p).$$

Corollary 2.2. *If $k = \mathbb{Q}$, G has a normal Sylow p-subgroup P and the quotient group G/P is abelian, then the inclusions of Theorem 2.1 are valid unconditionally.*

Proof. In view of Theorem 2.1, it suffices to show $\mu_p(F)$ vanishes under the stated hypotheses. But, since $F^P/\mathbb{Q}$ is assumed to be abelian, the vanishing of $\mu_p(F^P)$ is proved by Ferrero and Washington in [12]. The required vanishing of $\mu_p(F)$ therefore follows from the well-known fact that if $\mu_p(E)$ vanishes for any number field E, then Nakayama's Lemma implies that $\mu_p(E')$ also vanishes for any finite p-power degree Galois extension E' of E. □

Remark 2.3. There are interesting classes of non-abelian extensions to which Corollary 2.2 applies. For example, let K be a quadratic field and F a CM field which is abelian over K of exponent dividing $2p^n$ for some natural number n. Then if F is (generalised) dihedral over $\mathbb{Q}$ the Sylow p-subgroup P of $G_{F/\mathbb{Q}}$ is normal and the quotient $G_{F/\mathbb{Q}}/P$ is easily checked to be abelian.

2.2. **Proof of Theorem 2.1.** The starting point is to note that [10, Th. 4.1(iii)] (which is proved by using the theory of organising matrices) asserts that the inclusions in Theorem 2.1 are valid after the respective elements $\mathcal{L}_{F/k,\Sigma,m}$ are replaced by the sum

$$\mathcal{L}'_{F/k,\Sigma,m} := \sum_{\rho \in \mathrm{Ir}_p^m(G)} e_\rho L_{p,\Sigma}(m, \rho \otimes \omega_k^{1-m}),$$

where $\mathrm{Ir}_p^m(G)$ denotes the set of representations ρ in $\mathrm{Ir}_p^{(m)}(G)$ with the property that the $\mathbb{Q}_p^c$-module

$$W(m,\rho) := V_\rho \otimes_{\mathbb{Q}_p[G]} H^2_c(\mathcal{O}_{F,\Sigma}, \mathbb{Q}_p(m))$$

vanishes, where V_ρ is a choice of $\mathbb{Q}_p^c[G]$-module with character ρ. Following [10, Rem. 4.3(ii)], we also note that if $m = 0$, resp. $m > 0$, then to obtain an explicit description of the set $\mathrm{Ir}_p^m(G)$ one must assume the validity of Leopoldt's Conjecture for F at p , resp. that the group $H^2(\mathcal{O}_{F,\Sigma}, \mathbb{Q}_p/\mathbb{Z}_p(1-m))$ vanishes, as is conjectured by Schneider in [27, p. 192].

To prove the stated result it is thus enough to show that both

$$\mathcal{L}_{F/k,\Sigma,1} \cdot \zeta(I_{G,p}) = \mathcal{L}'_{F/k,\Sigma,1} \cdot \zeta(I_{G,p})$$

and for each $m > 1$ that

$$\mathcal{L}_{F/k,\Sigma,m} = \mathcal{L}'_{F/k,\Sigma,m}.$$

However, these equalities are themselves equivalent to asserting that for every strictly positive integer m and every representation ρ in $\mathrm{Ir}_p^{(m)}(G)$ the series $L_{p,\Sigma}(s, \rho \otimes \omega_k^{1-m})$ vanishes at $s = m$ whenever both $n_{m,\rho} = 0$ and $\rho \notin \mathrm{Ir}_p^m(G)$, and this fact follows directly from the next stated result.

For each number field L we write

$$\lambda_{L,p} : \mathcal{O}_L^\times \otimes_{\mathbb{Z}} \mathbb{Z}_p \to \prod_w L_w^\times \tag{1}$$

for the natural diagonal homomorphism where the product runs over all p-adic places w of L. Then Leopoldt's Conjecture predicts that this map is injective and for each character ρ in $\mathrm{Ir}_p(G)$ we shall say that 'Leopoldt's Conjecture for ρ is valid' if the space $V_\rho \otimes_{\mathbb{Z}_p[G]} \ker(\lambda_{F,p})$ vanishes.

In a similar way, for each character ρ in $\mathrm{Ir}_p(G)$ and each integer $m > 1$ we shall say that 'Schneider's Conjecture for the pair (ρ, m) is valid' if the tensor product $V_\rho \otimes_{\mathbb{Z}_p[G]} H^2(\mathcal{O}_{F,\Sigma}, \mathbb{Q}_p/\mathbb{Z}_p(1-m))^\vee$ vanishes where we write $N^\vee$ for the Pontryagin dual of any $\mathbb{Z}_p$-module.

Theorem 2.4. *For each strictly positive integer m and character ρ in $\mathrm{Ir}_p^{(m)}(G)$ we write $v_{\rho,m}$ for the order of vanishing of the function $(s-1)^{n_{\rho,m}} L_{p,\Sigma}(s, \rho \otimes \omega_k^{1-m})$ at $s = m$. Then one has*

$$v_{\rho,m} \geq \dim_{\mathbb{Q}_p^c}(W(m,\rho)) - n_{\rho,m},$$

with equality if either $m = 1$ and Leopoldt's Conjecture for ρ is valid or $m > 1$ and Schneider's Conjecture for (ρ, m) is valid.

The proof of this result is an adaptation of the proof of [8, Th. 4.1] and will occupy the next three subsections. (In the course of this proof we will also find an explicit criterion for equality in the above formula for $v_{\rho,m}$ in terms of the 'semisimplicity' of a natural Iwasawa module - see Proposition 2.5).

2.3. **Iwasawa theory and complexes.** We first discuss some useful preliminaries.

In the sequel for any noetherian ring R we write $D(R)$ for the derived category of complexes of (left) R-modules and $D^{\mathrm{p}}(R)$ for the full triangulated subcategory of $D(R)$ comprising complexes that are perfect.

We now fix a compact p-adic Lie group $\mathcal{G}$ which contains a closed normal subgroup $\mathcal{H}$ such that the quotient group $\Gamma := \mathcal{G}/\mathcal{H}$ is topologically isomorphic to $\mathbb{Z}_p$. We also fix a topological generator γ of Γ and write $\Lambda(\mathcal{G})$ for the p-adic Iwasawa algebra of $\mathcal{G}$.

For each continuous homomorphism $\rho : \mathcal{G} \to \mathrm{Aut}_{\mathcal{O}_\rho}(T_\rho)$, where $\mathcal{O}_\rho$ is a finite extension of $\mathbb{Z}_p$ and T_ρ is a finitely generated free $\mathcal{O}_\rho$-module, we write $\Lambda_{\mathcal{O}_\rho}(\Gamma)$ for the $\mathcal{O}_\rho$-Iwasawa algebra of Γ and consider the tensor product $\Lambda_{\mathcal{O}_\rho}(\Gamma) \otimes_{\mathcal{O}_\rho} T_\rho$ as an $(\Lambda_{\mathcal{O}_\rho}(\Gamma), \Lambda(\mathcal{G}))$-bimodule where $\Lambda_{\mathcal{O}_\rho}(\Gamma)$ acts by multiplication on the left and $\Lambda(\mathcal{G})$ acts on the right via the rule $(\lambda \otimes_{\mathcal{O}_\rho} t)g := \lambda\bar{g} \otimes_{\mathcal{O}_\rho} g^{-1}(t)$ for each $g \in \mathcal{G}$ with image $\bar{g}$ in Γ, $t \in T_\rho$ and $\lambda \in \Lambda_{\mathcal{O}_\rho}(\Gamma)$.

For each bounded complex of finitely generated projective $\Lambda(\mathcal{G})$-modules $A^\bullet$ we then set

$$A_\rho^\bullet := (\Lambda_{\mathcal{O}_\rho}(\Gamma) \otimes_{\mathcal{O}_\rho} T_\rho) \otimes_{\Lambda(\mathcal{G})} A^\bullet \in D^{\mathrm{p}}(\Lambda_{\mathcal{O}_\rho}(\Gamma))$$

and, for each open normal subgroup U of $\mathcal{G}$, also

$$A_U^\bullet := \mathbb{Z}_p[\mathcal{G}/U] \otimes_{\Lambda(\mathcal{G})} A^\bullet.$$

We note, in particular, that if $\ker(\rho)$ contains U, then there are natural isomorphisms in $D(\mathcal{O}_\rho)$ of the form

$$\mathbb{Z}_p \otimes_{\Lambda(\Gamma)} A_\rho^\bullet \cong T_\rho \otimes_{\Lambda(\mathcal{G})} A^\bullet \cong T_\rho \otimes_{\mathbb{Z}_p[G]} A_U^\bullet.$$

These isomorphisms give rise to a canonical exact triangle in $D(\mathcal{O}_\rho)$

$$A_\rho^\bullet \xrightarrow{\gamma-1} A_\rho^\bullet \to T_\rho \otimes_{\mathbb{Z}_p[G]} A_U^\bullet \to A_\rho^\bullet[1]$$

and hence also, by passing to cohomology, an associated short exact sequence

$$0 \to H^i(A_\rho^\bullet)_\Gamma \to H^i(T_\rho \otimes_{\mathbb{Z}_p[G]} A_U^\bullet) \to H^{i+1}(A_\rho^\bullet)^\Gamma \to 0 \tag{2}$$

in each degree i.

2.4. **Étale cohomology and twisting.** In the sequel we write E^∞ for the cyclotomic $\mathbb{Z}_p$-extension of a number field E.

In particular, in the context of Theorem 2.4 we set $\mathcal{G} := G_{F^\infty/k}$ and we note that this Galois group has the structure described in §2.3 with $\mathcal{H} := G_{F^\infty/k^\infty}$ and with Γ naturally isomorphic to $G_{k^\infty/k}$.

In this case we also regard τ as the generator of the subgroup $G_{F^\infty/(F^\infty)^+}$ of $\mathcal{G}$ and for each integer m set $e_m := (1+(-1)^{1+m}\tau)/2$, regarded as a central idempotent of both $\Lambda(\mathcal{G})$ and $\mathbb{Z}_p[G]$.

We write $\Lambda(\mathcal{G})^\#(m)$ for the (left) $\Lambda(\mathcal{G})$-module $\Lambda(\mathcal{G})$ endowed with the following action of G_k: each σ in G_k acts on $\Lambda(\mathcal{G})^\#(m)$ as right multiplication by $\chi_k(\sigma)^m(\overline{\sigma})^{-1}$, where $\overline{\sigma}$ is the image of σ in $\mathcal{G}$ and $\chi_k : G_k \to \mathbb{Z}_p^\times$ denotes the cyclotomic character. We then recall that the associated compactly supported étale cohomology complex

$$C_m(F^\infty/k) := R\Gamma_{c,\text{ét}}(\mathcal{O}_{k,\Sigma}, e_m\Lambda(\mathcal{G})^\#(m))$$

belongs to $D^{\mathrm{p}}(\Lambda(\mathcal{G}))$.

We may and will assume in the sequel that F^∞ contains all p-power roots of unity in $\mathbb{Q}^c$ (this can be achieved by replacing F by $F(\zeta_p)$ for a primitive p-th root of unity ζ_p and does not effect the validity of Theorem 2.4). The cyclotomic character χ_k then factors through the restriction map $G_k \to \mathcal{G}$.

In this case there is an explicit link between the complexes $C_m(F^\infty/k)$ and $C_1(F^\infty/k)$ and to describe this we write tw_{1-m} for the automorphism of $\Lambda(\mathcal{G})$ sending each g in $\mathcal{G}$ to $\chi_k^{1-m}(g)g$ and consider the module $\Lambda(\mathcal{G}) \otimes_{\Lambda(\mathcal{G}),\mathrm{tw}_{1-m}} \Lambda(\mathcal{G})^\#(1)$. Here the tensor product indicates that the first term $\Lambda(\mathcal{G})$ is regarded as a right $\Lambda(\mathcal{G})$-module via the homomorphism tw_{1-m}. This tensor product is endowed with commuting left actions of $\Lambda(\mathcal{G})$ (via left multiplication on the first factor) and of G_k (via the action on the second factor specified above) and, with respect to these actions, the map $1 \otimes g \mapsto \chi_k(g)^{1-m}g$ induces an isomorphism of $\Lambda(\mathcal{G}) \times \mathbb{Z}_p[G_k]$-modules $\Lambda(\mathcal{G}) \otimes_{\Lambda(\mathcal{G}),\mathrm{tw}_{1-m}} \Lambda(\mathcal{G})^\#(1) \cong \Lambda(\mathcal{G})^\#(m)$. This isomorphism restricts to give an isomorphism of $\Lambda(\mathcal{G}) \times \mathbb{Z}_p[G_k]$-modules

$$\Lambda(\mathcal{G}) \otimes_{\Lambda(\mathcal{G}),\mathrm{tw}_{1-m}} e_1\Lambda(\mathcal{G})^\#(1) = e_m(\Lambda(\mathcal{G}) \otimes_{\Lambda(\mathcal{G}),\mathrm{tw}_{1-m}} \Lambda(\mathcal{G})^\#(1)) \cong e_m\Lambda(\mathcal{G})^\#(m)$$

and hence also an isomorphism in $D^{\mathrm{p}}(\Lambda(\mathcal{G}))$

$$\Lambda(\mathcal{G}) \otimes_{\Lambda(\mathcal{G}),\mathrm{tw}_{1-m}} C_1(F^\infty/k) \cong C_m(F^\infty/k). \tag{3}$$

2.5. **Completion of the proof of Theorem 2.4.** For any profinite group Δ we write $\mathbf{1}_\Delta$ for the trivial (p-adic) character of Δ.

For each integer a we write $A_a(\mathcal{G})$ for the set of continuous representations $\rho : \mathcal{G} \to \mathrm{Aut}_{\mathcal{O}_\rho}(T_\rho)$ for which $\ker(\rho)$ is open and $\rho(\tau) = (-1)^{1+a} \cdot \mathrm{id}_{T_\rho}$. We also write κ_k for the homomorphism $\Gamma := G_{k^\infty/k} \to G_{k^\infty(\zeta_p)/k(\zeta_p)} \to \mathbb{Z}_p^\times$ where the first arrow is the natural identification resulting from the fact that $k(\zeta_p) \cap k^\infty = k$ and the second is induced by restriction of the cyclotomic character χ_k.

For any extension R of $\mathbb{Z}_p$ we write $Q(R[[u]])$ for the total quotient ring of the ring of power series $R[[u]]$ over R in the formal variable u.

Then, for each representation ρ in $A_1(\mathcal{G})$, Deligne and Ribet have shown that there exists a unique element $f_{\Sigma,\rho}(u)$ of $\mathbb{Q}_p^c \otimes_{\mathbb{Z}_p} Q(\mathbb{Z}_p[[u]])$ with the property that

$$L_{p,\Sigma}(1-s,\rho) = f_{\Sigma,\rho}(\kappa_k(\gamma)^s - 1). \tag{4}$$

(For details of the construction and properties of $f_{\Sigma,\rho}(u)$ in the case that ρ is not linear see, for example, Greenberg [13]).

The following observation concerning the function $f_{\Sigma,\rho}(u)$ is key to the proof of Theorem 2.4.

Proposition 2.5. *For each integer m and each representation ρ in $A_m(\mathcal{G})$ write $w_{\rho,m}$ for the order of vanishing of the function $f_{\Sigma,\rho\otimes\omega_k^{1-m}}(\kappa_k(\gamma)^{1-m}(1+u)-1)$ at $u=0$. Then one has*

$$w_{\rho,m} \geq \dim_{\mathbb{Q}_p^c}(\mathbb{Q}_p^c \otimes_{\mathcal{O}_\rho} H^2(C_m(F^\infty/k)_\rho)_\Gamma) - n_{\rho,m},$$

with equality if and only if $\gamma - 1$ acts semisimply on $H^2(C_m(F^\infty/k)_\rho)$.

Proof. For each ρ in $A_m(\mathcal{G})$ we set $f_{\Sigma,\rho,m}(u) := f_{\Sigma,\rho\otimes\omega_k^{1-m}}(\kappa_k(\gamma)^{1-m}(1+u)-1)$.

We note first that Wiles' proof of the main conjecture for totally real fields in [32] implies that for each such ρ the series $f_{\Sigma,\rho,m}(u)$ is a characteristic element in $Q(\mathcal{O}_\rho[[u]])^\times$ for $C_m(F^\infty/k)_\rho$, regarded as a complex of $\mathcal{O}_\rho[[u]]$-modules via the correspondence $u \leftrightarrow \gamma - 1$. In fact, whilst the possibility of this sort of generalisation to non-linear representations of Wiles' results was first observed by Ritter and Weiss in the special case $m=1$ in the course of the proof of [25, Th. 16], this case is also proved more directly by the first author in [8] where, to be precise, it follows by combining the property of [8, §5.4, (P7)] with the argument made just after the proof of [8, Lem. 5.4]. The case of general m is then deduced from the case $m=1$ by the following standard 'twisting' argument.

We fix ρ in $A_m(\mathcal{G})$. Then $\rho\otimes\omega_k^{1-m}$ belongs to $A_1(\mathcal{G})$ and so the above argument implies $f_{\Sigma,\rho\otimes\omega_k^{1-m}}(u) = f_{\Sigma,\rho\otimes\omega_k^{1-m},1}(u)$ is a characteristic element for the complex $C_1(F^\infty/k)_{\rho\otimes\omega_k^{1-m}}$. In addition, the isomorphism (3) combines with the factorisation $\chi_k = \omega_k \times \kappa_k$ to induce an isomorphism in $D^{\mathrm{p}}(\Lambda_{\mathcal{O}_\rho}(\Gamma))$

$$\Lambda_{\mathcal{O}_\rho}(\Gamma) \otimes_{\Lambda_{\mathcal{O}_\rho}(\Gamma),\mathrm{tw}'_{1-m}} C_1(F^\infty/k)_{\rho\otimes\omega_k^{1-m}} \cong C_m(F^\infty/k)_\rho,$$

where tw'_{1-m} is the automorphism of $\Lambda_{\mathcal{O}_\rho}(\Gamma)$ sending each δ in Γ to $\kappa_k^{1-m}(\delta)\delta$.

Thus, writing ι_{1-m} for the automorphism of the group $Q(\mathcal{O}_\rho[[u]])^\times$ induced by sending u to $\kappa_k(\gamma)^{1-m}(1+u)-1$, the naturality of connecting homomorphisms in relative K-theory implies that $\iota_{1-m}(f_{\Sigma,\rho\otimes\omega_k^{1-m}}(u))$ is a characteristic element for $C_m(F^\infty/k)_\rho$. The claimed result is therefore true because $\iota_{1-m}(f_{\Sigma,\rho\otimes\omega_k^{1-m}}(u)) = f_{\Sigma,\rho,m}(u)$.

Given this fact, the description of $C_m(F^\infty/k)_\rho$ given in Lemma 2.6 below implies that

$$\begin{aligned}\mathcal{O}_\rho[[u]] \cdot f_{\Sigma,\rho,m}(u) &= \mathrm{char}_{\mathcal{O}_\rho[[u]]}(H^2_{m,\rho}) \cdot \mathrm{char}_{\mathcal{O}_\rho[[u]]}(H^3_{m,\rho})^{-1} \\ &= \mathrm{char}_{\mathcal{O}_\rho[[u]]}(H^2_{m,\rho}) \cdot u^{-n_{\rho,m}}.\end{aligned} \tag{5}$$

Here in each degree i we set $H^i_{m,\rho} := H^i(C_m(F^\infty/k)_\rho)$, for any finitely generated torsion $\mathcal{O}_\rho[[u]]$-module N we write $\mathrm{char}_{\mathcal{O}_\rho[[u]]}(N)$ for its characteristic ideal and the last displayed equality is valid because the $\mathcal{O}_\rho[[u]]$-module $H^3_{m,\rho} \cong T_{\rho,\mathcal{H}}(m-1)$ identifies with $\mathbb{Z}_p$ if both $\rho = \mathbf{1}_{\mathcal{G}}$ and $m=1$ and is finite in all other cases.

Next we note the tautological exact sequence

$$0 \to u \cdot H^2_{m,\rho} \to H^2_{m,\rho} \to (H^2_{m,\rho})_\Gamma \to 0$$

implies that

$$\mathrm{char}_{\mathcal{O}_\rho[[u]]}(H^2_{m,\rho}) = \mathrm{char}_{\mathcal{O}_\rho[[u]]}(u \cdot H^2_{m,\rho}) \cdot \mathrm{char}_{\mathcal{O}_\rho[[u]]}((H^2_{m,\rho})_\Gamma). \tag{6}$$

Since $(H^2_{m,\rho})_\Gamma$ is annihilated by u one also has $\mathrm{char}_{\mathcal{O}_\rho[[u]]}((H^2_{m,\rho})_\Gamma) = \mathcal{O}_\rho[[u]] \cdot u^{d_m}$ with $d_m = \dim_{\mathbb{Q}_p^c}(\mathbb{Q}_p^c \otimes_{\mathcal{O}_\rho} (H^2_{m,\rho})_\Gamma)$. In addition, it is clear that $\mathrm{char}_{\mathcal{O}_\rho[[u]]}(u \cdot H^2_{m,\rho})$ is not divisible by u if and only if u acts invertibly on the space $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} u \cdot H^2_{m,\rho}$ (or equivalently, $\gamma - 1$ acts semisimply on $H^2_{m,\rho}$).

The equalities (5) and (6) therefore combine to imply that $u^{n_{\rho,m} - d_m} \cdot f_{\Sigma,\rho,m}(u)$ is a generator of the ideal $\mathrm{char}_{\mathcal{O}_\rho[[u]]}(u \cdot H^2_{m,\rho})$ and is therefore divisible by u if and only if $\gamma - 1$ acts semisimply on $H^2_{m,\rho}$. This immediately implies the claimed result. □

For each finite Galois extension L of k that is unramified outside Σ we set $C(L/k) := R\Gamma_{c,\text{ét}}(\mathcal{O}_{L,\Sigma}, \mathbb{Z}_p(1))$. Then, before stating the next result, it is convenient to recall that $C(L/k)$ belongs to $D^{\mathrm{p}}(\mathbb{Z}_p[G_{L/k}])$ and that an explicit computation of its cohomology shows that it is acyclic outside degrees one, two and three, that there is a natural short exact sequence

$$0 \to (\prod_{L \to \mathbb{C}} \mathbb{Z}_p(1))^{G_{\mathbb{C}/\mathbb{R}}} \to H^1(C(L/k)) \to \ker(\lambda_{L,p}) \to 0 \tag{7}$$

where the product runs over all embeddings of L into $\mathbb{C}$, the invariants are taken with respect to the natural diagonal action of $G_{\mathbb{C}/\mathbb{R}}$ and $\lambda_{L,p}$ is the homomorphism (1), and that there are natural identifications of $H^2(C(L/k))$ and $H^3(C(L/k))$ with $G_{M_{L,\Sigma}/L}$ and $\mathbb{Z}_p$ (endowed with the trivial action of $G_{L/k}$) respectively. (For more details of these standard computations see, for example, the proof of [6, Prop. 4.1].)

Lemma 2.6. *For each representation ρ in $A_m(\mathcal{G})$ the complex $C_m(F^\infty/k)_\rho$ is acyclic outside degrees two and three. In addition $H^2(C_m(F^\infty/k)_\rho)$ is a finitely generated torsion $\Lambda_{\mathcal{O}_\rho}(\Gamma)$-module whilst $H^3(C_m(F^\infty/k)_\rho)$ is isomorphic as a $\Lambda_{\mathcal{O}_\rho}(\Gamma)$-module to $T_{\rho,\mathcal{H}}(m-1)$ with $\mathcal{H} = G_{F^\infty/k^\infty}$.*

Proof. For each non-negative integer n write F^n for the subextension of F^∞ that has degree p^n over F and set $\mathcal{G}_n := G_{F^n/k}$.

Then, since the group $H^1(C(F^n/k))$ described by (7) (with $L = F^n$) is $\mathbb{Z}_p$-free and $C(F^n/k)$ is acyclic outside degrees one, two and three, it can be represented by a complex of finitely generated projective $\mathbb{Z}_p[\mathcal{G}_n]$-modules of the form $P^1_n \to P^2_n \to P^3_n$, where the first module is placed in degree one.

By passing to the inverse limit over n, this implies that $C_1(F^\infty/k)$ is represented by a complex of finitely generated projective $\Lambda(\mathcal{G})$-modules $P^\bullet$ of the form $P^1 \to P^2 \to P^3$, where again the first module occurs in degree one, that $H^1(C_1(F^\infty/k))$ vanishes (since e_1 annihilates each module $(\prod_{F^n \to \mathbb{C}} \mathbb{Z}_p(1))^{G_{\mathbb{C}/\mathbb{R}}}$ and Iwasawa's proof of the 'weak Leopoldt Conjecture' in [15] implies the vanishing of the inverse limit of the groups $\ker(\lambda_{F^n,\Sigma})$ under the natural field-theoretic norm maps), that $H^2(C_1(F^\infty/k))$ identifies with $e_1 G_{M_{F^\infty,\Sigma}/F^\infty}$ and so is a finitely generated torsion $\Lambda(\mathcal{G})$-module (by Iwasawa's theorem on the growth of class numbers) and that $H^3(C_1(F^\infty/k))$ identifies with the trivial $\Lambda(\mathcal{G})$-module $\mathbb{Z}_p$.

Now for each integer m the isomorphism (3) implies that $C_m(F^\infty/k)_\rho$ is represented by the complex

$$P^\bullet_{(m,\rho)} := (\Lambda_{\mathcal{O}_\rho}(\Gamma) \otimes_{\mathcal{O}_\rho} T_\rho) \otimes_{\Lambda(\mathcal{G})} (\Lambda(\mathcal{G}) \otimes_{\Lambda(\mathcal{G}),\mathrm{tw}_{1-m}} P^\bullet)$$

and so is acyclic outside degrees one, two and three. In addition, the above description of $H^3(C_1(F^\infty/k))$ implies that for each ρ in $A_m(\mathcal{G})$ the module $H^3(C_m(F^\infty/k)_\rho)$ is canonically isomorphic to $(\Lambda_{\mathcal{O}_\rho}(\Gamma) \otimes_{\mathcal{O}_\rho} T_\rho) \otimes_{\Lambda(\mathcal{G})} \mathbb{Z}_p(m-1)$ which is easily seen to be isomorphic to the given module $T_{\rho,\mathcal{H}}(m-1)$. This proves the final assertion of the lemma.

At this stage it suffices to show that $C_m(F^\infty/k)_\rho$ is acyclic in degree one and that $H^2(C_m(F^\infty/k)_\rho)$ is a torsion $\Lambda_{\mathcal{O}_\rho}(\Gamma)$-module. Hence, since $H^1(C_m(F^\infty/k)_\rho)$ is isomorphic to a submodule of the free $\Lambda_{\mathcal{O}_\rho}(\Gamma)$-module $P^1_{(m,\rho)}$, it will suffice to show that $H^1(C_m(F^\infty/k)_\rho)$ and $H^2(C_m(F^\infty/k)_\rho)$ are both torsion $\Lambda_{\mathcal{O}_\rho}(\Gamma)$-modules. Then, as the hypertor-spectral sequence computes these groups in terms of the $\Lambda_{\mathcal{O}_\rho}(\Gamma)$-modules $\mathrm{Tor}_p^{\Lambda(\mathcal{G})}(\Lambda_{\mathcal{O}_\rho}(\Gamma) \otimes_{\mathcal{O}_\rho} T_\rho, H^q(C_m(F^\infty/k)))$, it is enough to observe that each of the latter modules is torsion, as follows easily from the fact that the above descriptions imply each group $H^q(C_m(F^\infty/k))$ is a torsion module over $\Lambda(G_{F^\infty/F^n})$ for a natural number n large enough to ensure that G_{F^∞/F^n} is contained in $\ker(\rho)$. □

We can now prove Theorem 2.4. To do this we set

$$C_m(F/k) := \mathbb{Z}_p[G]e_m \otimes_{\mathbb{Z}_p[G]} R\Gamma_{c,\text{ét}}(\mathcal{O}_{F,\Sigma}, \mathbb{Z}_p(m)) \in D^{\mathrm{p}}(\mathbb{Z}_p[G]).$$

Then for each character ρ in $\mathrm{Ir}_p^{(m)}(G)$ the interpolation property (4) combines with Proposition 2.5 to imply that

$$\begin{aligned}
(8)\quad v_{\rho,m} &= n_{\rho,m} + \mathrm{ord}_{u=0} f_{\Sigma,\rho\otimes\omega_k^{1-m}}(\kappa_k(\gamma)^{1-m}(1+u)-1) \\
&\geq n_{\rho,m} + (\dim_{\mathbb{Q}_p^c}(\mathbb{Q}_p^c \otimes_{\mathcal{O}_\rho} H^2(C_m(F^\infty/k)_\rho)_\Gamma) - n_{\rho,m}) \\
&= \dim_{\mathbb{Q}_p^c}(V_\rho \otimes_{\mathbb{Z}_p[G]} H^2(C_m(F/k))) - \dim_{\mathbb{Q}_p^c}(\mathbb{Q}_p^c \otimes_{\mathcal{O}_\rho} H^3(C_m(F^\infty/k)_\rho)^\Gamma) \\
&= \dim_{\mathbb{Q}_p^c}(W(m,\rho)) - n_{\rho,m}.
\end{aligned}$$

Here the second equality follows from the natural 'descent' isomorphism

$$\mathbb{Z}_p[G] \otimes^{\mathbb{L}}_{\Lambda(\mathcal{G})} C_m(F^\infty/k) \cong C_m(F/k)$$

in $D^{\mathrm{p}}(\mathbb{Z}_p[G])$ and the appropriate case of the short exact sequence (2) and the last from the definition of $W(m,\rho)$ and from the explicit description of $H^3(C_m(F^\infty/k)_\rho)$ that is given in Lemma 2.6.

We note also that if $m=1$ and Leopoldt's Conjecture for ρ is valid, then the exact sequence (7) with $L=F$ implies that the space $V_\rho \otimes_{\mathbb{Z}_p[G]} H^1(C_1(F/k))$ vanishes. The appropriate case of (2) then implies that $\mathbb{Q}_p^c \otimes_{\mathcal{O}_\rho} H^2(C_1(F^\infty/k)_\rho)^\Gamma$ and hence also its subspace $\mathbb{Q}_p^c \otimes_{\mathcal{O}_\rho} (u \cdot H^2(C_1(F^\infty/k)_\rho))^\Gamma$ vanishes. This fact in turn implies that $\gamma - 1$ acts semisimply on $H^2(C_1(F^\infty/k)_\rho)$ and hence, via Proposition 2.5, that there is equality in the formula (8).

In a similar way, one finds that if $m>1$ and Schneider's Conjecture is valid for the pair (m,ρ), then $V_\rho \otimes_{\mathbb{Z}_p[G]} H^1(C_m(F/k))$ vanishes and so Proposition 2.5 implies that there is equality in (8).

At this stage we have completed the proof of Theorem 2.4 and hence also that of Theorem 2.1.

3. The Shafarevich-Weil Theorem

In the sequel we shall use the p-adic interpetation of the Shafarevich-Weil Theorem that is provided by Theorem 3.1 below.

For any noetherian ring R, any object C of $D(R)$ and any integer n we write $\tau^{\leq n}C$ and $\tau^{\geq n}C$ for the truncations of C in degrees less than or equal to n and greater than or equal to n respectively.

For any abelian group A we write $A_p^\wedge$ for its pro-p-completion $\varprojlim_n A/p^nA$. For any group Δ we write I_Δ for the kernel of the augmentation map $\mathbb{Z}[\Delta] \to \mathbb{Z}$.

3.1. **Statement of the result.** For any number field E and any finite set of places Σ of E that contains all archimedean places we write $R\Gamma_{c,\text{ét}}(\mathcal{O}_{E,\Sigma}, \mathbb{G}_m)$ for the cohomology with compact support of the étale sheaf $\mathbb{G}_m$ on $\mathrm{Spec}(\mathcal{O}_{E,\Sigma})$.

We also write $C_\Sigma(E)$ for the group belonging to the Σ-class formation described by Milne in [21, Chap. I, §4].

Theorem 3.1. *Let L/K be a finite Galois extension of number fields and Σ a finite set of places of K which contains all archimedean places and also all those which ramify in L/K.*

(i) *Fix a representative*

$$0 \to C_\Sigma(L) \to \mathcal{E}_\Sigma(L/K) \xrightarrow{\theta} G_{L/K} \to 0$$

of the fundamental class in $H^2(G_{L/K}, C_\Sigma(L))$. Then $R\Gamma_{c,\text{ét}}(\mathcal{O}_{L,\Sigma}, \mathbb{G}_m)$ is canonically isomorphic in $D(\mathbb{Z}[G_{L/K}])$ to the complex of $G_{L/K}$-modules

$$\mathbb{Z}[G_{L/K}] \otimes_{\mathbb{Z}[\mathcal{E}_\Sigma(L/K)]} I_{\mathcal{E}_\Sigma(L/K)} \xrightarrow{d} \mathbb{Z}[G_{L/K}] \to \mathbb{Q} \tag{9}$$

where the first term is placed in degree one, the homomorphism d sends $x \otimes (\varepsilon - 1)$ to $x(\theta(\varepsilon) - 1)$ and the second arrow sends each element of $G_{L/K}$ to 1.

(ii) *We now fix a prime number p and assume further that the given set Σ also contains all p-adic places of K. Then the complex $\tau^{\geq 2}R\Gamma_{c,\text{ét}}(\mathcal{O}_{L,\Sigma}, \mathbb{Z}_p(1))$ is canonically isomorphic in $D(\mathbb{Z}_p[G_{L/K}])$ to the complex*

$$\mathbb{Z}_p[G_{L/K}] \otimes_{\mathbb{Z}[G_{M_{L,\Sigma}/K}]_p^\wedge} (I_{G_{M_{L,\Sigma}/K}})_p^\wedge \xrightarrow{\delta_p} \mathbb{Z}_p[G_{L/K}], \tag{10}$$

where the first term is placed in degree two and δ_p is induced by the natural projection $G_{M_{L,\Sigma}/K} \to G_{L/K}$.

If one identifies the non-zero cohomology groups of the complex (10) with $G_{M_{L,\Sigma}/L}$ and $\mathbb{Z}_p$ via the homomorphisms which send each a in $G_{M_{L,\Sigma}/L}$ to $1 \otimes (a-1) \in \ker(\delta_p)$ and each g in $G_{L/K}$ to 1, then the above isomorphism induces the identity map on cohomology in all degrees i.

3.2. **The proof.** The very definition of $R\Gamma_{c,\text{ét}}(\mathcal{O}_{L,\Sigma}, \mathbb{G}_m)$ ensures that it lies in an exact triangle in $D(\mathbb{Z}[G_{L/K}])$ of the form

$$R\Gamma_{c,\text{ét}}(\mathcal{O}_{L,\Sigma}, \mathbb{G}_m) \to R\Gamma(\mathcal{O}_{L,\Sigma}, \mathbb{G}_m) \to \bigoplus_{w \in \Sigma_L} R\Gamma(L_w, \mathbb{G}_m) \to R\Gamma_{c,\text{ét}}(\mathcal{O}_{L,\Sigma}, \mathbb{G}_m)[1]$$

where the second arrow denotes the natural restriction map.

By analysing the long exact cohomology sequence of this triangle one finds that $R\Gamma_{c,\text{ét}}(\mathcal{O}_{L,\Sigma}, \mathbb{G}_m)$ is acyclic outside degrees one and three and that there are canonical identifications $H^1_{c,\text{ét}}(\mathcal{O}_{L,\Sigma}, \mathbb{G}_m) = C_\Sigma(L)$ and $H^3_{c,\text{ét}}(\mathcal{O}_{L,\Sigma}, \mathbb{G}_m) = \mathbb{Q}/\mathbb{Z}$. The

complex $\tau^{\geq 1}\tau^{\leq 3}R\Gamma_{c,\text{ét}}(\mathcal{O}_{L,\Sigma},\mathbb{G}_m)$ therefore represents an element of the Yoneda Ext-group $\text{Ext}^3_{G_{L/K}}(\mathbb{Q}/\mathbb{Z},C_\Sigma(L))$ and it is shown in [9, Prop. 3.5(b)] that the image of this element under the composite isomorphism

$$\text{Ext}^3_{G_{L/K}}(\mathbb{Q}/\mathbb{Z},C_\Sigma(L)) \cong \text{Ext}^2_{G_{L/K}}(\mathbb{Z},C_\Sigma(L)) \cong H^2(G_{L/K},C_\Sigma(L))$$

is equal to the fundamental class defined by global class field theory. (The first displayed isomorphism here is induced by the exact sequence of Ext-groups associated to the short exact sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ (and the vanishing of $\text{Ext}^i_{G_{L/K}}(\mathbb{Q},C_\Sigma(L))$ for $i = 2,3$) and the second by the Universal Coefficient Spectral Sequence.)

By a standard argument of homological algebra (see, for example, [14, p. 210]) it follows that $\tau^{\geq 1}\tau^{\leq 3}R\Gamma_{c,\text{ét}}(\mathcal{O}_{L,\Sigma},\mathbb{G}_m)$ is represented by the explicit complex (9). More precisely, if one identifies the cohomology group in degree one of this complex with $C_\Sigma(L)$ via the map which sends each c in $C_\Sigma(L)$ to $1\otimes(c-1)\in\ker(d)$, then there exists an isomorphism between $\tau^{\geq 1}\tau^{\leq 3}R\Gamma_{c,\text{ét}}(\mathcal{O}_{L,\Sigma},\mathbb{G}_m)$, and hence also $R\Gamma_{c,\text{ét}}(\mathcal{O}_{L,\Sigma},\mathbb{G}_m)$, and the complex (9) which induces the identity map on cohomology in all degrees.

In order to prove claim (ii), we first note that the proof of claim (i) combines with [14, Ch.V.9] to imply that the exact sequence

$$0 \to C_\Sigma(L) \to \mathbb{Z}[G_{L/K}] \otimes_{\mathbb{Z}[\mathcal{E}_\Sigma(L/K)]} I_{\mathcal{E}_\Sigma(L/K)} \xrightarrow{d} \mathbb{Z}[G_{L/K}] \to \mathbb{Z} \to 0, \tag{11}$$

where the second arrow sends each element c of $C_\Sigma(L)$ to $1\otimes(c-1)$ and the fourth arrow sends each element of $G_{L/K}$ to 1, is a representative of the image in $\text{Ext}^2_{G_{L/K}}(\mathbb{Z},C_\Sigma(L))$ of the fundamental class. We write C for the complex of $\mathbb{Z}[G_{L/K}]$-modules

$$\mathbb{Z}[G_{L/K}] \otimes_{\mathbb{Z}[\mathcal{E}_\Sigma(L/K)]} I_{\mathcal{E}_\Sigma(L/K)} \xrightarrow{d} \mathbb{Z}[G_{L/K}]$$

with the first term placed in degree zero and identify $H^0(C)$ and $H^1(C)$ with $C_\Sigma(L)$ and with $\mathbb{Z}$ respectively via the exact sequence (11).

We next recall (from, for example, [31, Th. 11.5]) that the theorem of Shafarevich-Weil implies the existence of an exact commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C_\Sigma(L) & \longrightarrow & \mathcal{E}_\Sigma(L/K) & \xrightarrow{\theta} & G_{L/K} & \longrightarrow & 0 \\
 & & {\scriptstyle r_{L,\Sigma}}\big\downarrow & & {\scriptstyle r'_{L,\Sigma}}\big\downarrow & & \big\| & & \\
0 & \longrightarrow & G_{M_{L,\Sigma}/L} & \longrightarrow & G_{M_{L,\Sigma}/K} & \longrightarrow & G_{L/K} & \longrightarrow & 0
\end{array}$$

in which the lower sequence is the tautological exact sequence and $r_{L,\Sigma}$ is the global reciprocity map so that $\ker(r_{L,\Sigma}) = \ker(r'_{L,\Sigma})$ is the connected component of $C_\Sigma(L)$ and is hence divisible. The above diagram induces a map of complexes of $\mathbb{Z}[G_{L/K}]$-modules from C to the complex

$$D : \mathbb{Z}[G_{L/K}] \otimes_{\mathbb{Z}[G_{M_{L,\Sigma}/K}]} I_{G_{M_{L,\Sigma}/K}} \to \mathbb{Z}[G_{L/K}],$$

with the first term placed in degree zero and the differential induced by the natural projection $G_{M_{L,\Sigma}/K} \to G_{L/K}$. By passing to pro-$p$-completions, this map in turn induces a map of complexes of $\mathbb{Z}_p[G_{L/K}]$-modules from

$$C_{\lim} := \varprojlim_n C/p^n$$

(where the inverse limit is taken with respect to the natural transition morphisms) to the complex (10) which is in fact a quasi-isomorphism. Indeed, the exact sequence (11) induces an exact sequence

$$0 \to C_\Sigma(L)/p^n \to (\mathbb{Z}[G_{L/K}] \otimes_{\mathbb{Z}[\mathcal{E}_\Sigma(L/K)]} I_{\mathcal{E}_\Sigma(L/K)})/p^n \xrightarrow{d/p^n} \mathbb{Z}[G_{L/K}]/p^n \to \mathbb{Z}/p^n \to 0$$

and hence also canonical identifications $H^0(C/p^n) = C_\Sigma(L)/p^n$ and $H^1(C/p^n) = \mathbb{Z}/p^n$. Similarly $H^0(D/p^n) = G_{M_{L,\Sigma}/L}/p^n$ and $H^1(D/p^n) = \mathbb{Z}/p^n$. The fact that $\ker(r_{L,\Sigma})$ is divisible therefore implies that the map $H^0(C_{\lim}) = \varprojlim_n C_\Sigma(L)/p^n \to G_{M_{L,\Sigma}/L}$ induced on degree zero cohomology is bijective, while [9, Lem. 9] implies that the canonical maps $H^1(C_{\lim}) \to \varprojlim_n H^1(C/p^n) = \mathbb{Z}_p$ and $H^1(D_{\lim}) \to \varprojlim_n H^1(D/p^n) = \mathbb{Z}_p$ are bijective and hence that the map induced on degree one cohomology is bijective too.

We now observe that (the proof of) [6, Lem. 4.3] implies the existence of a complex of $\mathbb{Z}[G_{L/K}]$-modules Q which is acyclic outside degrees zero and one and has all of the following properties: Q is of the form $Q^{-1} \to Q^0 \to Q^1$, with the first term placed in degree -1, and all of the modules Q^i are $\mathbb{Z}$-free; for every natural number n, the quotient complex Q/p^n consists of finite, projective $\mathbb{Z}/p^n[G_{L/K}]$-modules; there exist canonical identifications of $H^0(Q)$ and $H^1(Q)$ with $C_\Sigma(L)$ and $\mathbb{Z}$ respectively and a canonical isomorphism

$$Q \to C \tag{12}$$

in $D(\mathbb{Z}[G_{L/K}])$ which induces the identity map on cohomology in each degree. We define a complex of $\mathbb{Z}_p[G_{L/K}]$-modules by setting

$$Q_{\lim} := \varprojlim_n Q/p^n.$$

Then by using the short exact sequences $0 \to Q \xrightarrow{p^n} Q \to Q/p^n \to 0$ and the above description of the cohomology of Q, one finds that $Q_{\lim}$ is acyclic outside degrees -1, 0 and 1 and that there exist canonical identifications of $H^0(Q_{\lim}) = \varprojlim_n H^0(Q/p^n)$ and $H^1(Q_{\lim}) = \varprojlim_n H^1(Q/p^n)$ with $\varprojlim_n C_\Sigma(L)/p^n$ and $\mathbb{Z}_p$ respectively. Furthermore, the isomorphism (12) induces a canonical exact triangle

$$H^{-1}(Q_{\lim})[1] \to Q_{\lim} \to C_{\lim} \to H^{-1}(Q_{\lim})[2]$$

in $D(\mathbb{Z}_p[G_{L/K}])$ and so we conclude that the complex (10) is canonically isomorphic in $D(\mathbb{Z}_p[G_{L/K}])$ to $\tau^{\geq 0}Q_{\lim}$.

The explicit descriptions given in claim (ii) therefore now follow because the proof of [6, Lem. 4.4] shows that, given our choice of the complex Q, there exists a canonical isomorphism $Q_{\lim} \to R\Gamma_{c,\text{ét}}(\mathcal{O}_{L,\Sigma}, \mathbb{Z}_p(1))[2]$ in $D(\mathbb{Z}_p[G_{L/K}])$ which induces the identity map on cohomology in degrees zero and one.

4. Cyclic Sylow p-subgroups

In this section we fix an odd prime p and a finite Galois extension F/k of number fields and set $G := G_{F/k}$. We will assume further that G has both a cyclic Sylow p-subgroup P and a p-complementary subgroup H. (Note that, by Hall's Theorem, the existence of such a subgroup H is automatic whenever G is solvable.)

We also fix a finite set of places Σ of k containing all archimedean places, all which ramify in F/k and all above p. Under these hypotheses we shall show that Theorem 3.1 implies various explicit restrictions on the structure of the group $G_{M_{F,\Sigma}/F}$.

For each subgroup J of G we write T_J for the element $\sum_{h\in J} h$ of $\mathbb{Z}_p[G]$ and e_J for the associated idempotent $|J|^{-1}T_J$ of $\mathbb{Q}_p[G]$. We observe that T_J and e_J are central in $\mathbb{Q}_p[G]$ if and only if J is normal in G.

At the outset we shall also fix a generator π of P.

4.1. **The element γ_π^*.** For any pre-image γ_π of π under the canonical projection map $G_{M_{F,\Sigma}/k} \to G$ the element $\gamma_\pi^{|P|}$ belongs to $G_{M_{F,\Sigma}/F}$. We therefore obtain an element of $G_{M_{F,\Sigma}/F}$ by setting $\gamma_\pi^* := e_H(\gamma_\pi^{|P|})$.

In the sequel we shall use the following properties of this element.

Lemma 4.1. *Fix G and π as above. Then γ_π^* is central in $G_{M_{F,\Sigma}/k}$, unique up to multiplication by an element of $T_G(G_{M_{F,\Sigma}/F})$ and can be chosen to have infinite order.*

Proof. We use the fact that, by assumption, $G = PH = HP$ with $P \cap H = \{1\}$.

In particular, since $he_H = e_H$ for all h in H the element γ_π^* is central in $G_{M_{F,\Sigma}/k}$ provided that $(\pi e_H)(\gamma_\pi^{|P|}) = e_H(\gamma_\pi^{|P|})$. Thus, since $\gamma_\pi^{|P|}$ is obviously fixed by the natural (conjugation) action of P on $G_{M_{F,\Sigma}/F}$ it suffices to show that πe_H can be written as $|H|^{-1}\sum_{h\in H}(h\cdot\varpi(h))$ with each element $\varpi(h)$ in P. But this is true because if $\pi h_1 = h_0\rho_1$ and $\pi h_2 = h_0\rho_2$ with h_0, h_1 and h_2 in H and ρ_1 and ρ_2 in P, then $h_1^{-1}h_2 = \rho_1^{-1}\rho_2$ belongs to $P\cap H = \{1\}$ and so $h_1 = h_2$.

We now set $A := G_{M_{F,\Sigma}/F}$. Then the second assertion is true because for any a in A one has $(a\gamma_\pi)^{|P|} = T_P(a)\gamma_\pi^{|P|}$ and hence also $e_H((a\gamma_\pi)^{|P|}) = e_H(T_P(a)\gamma_\pi^{|P|}) = T_G(a^{1/|H|})e_H(\gamma_\pi^{|P|})$.

To prove the last assertion it is enough to prove that $T_G(A)$ is infinite. To prove this we note that the Tate cohomology group $\hat{H}^{-1}(G,A)$ is finite (since A is a finitely generated $\mathbb{Z}_p$-module) and hence that the natural short exact sequence

$$0 \to \hat{H}^{-1}(G,A) \to H_0(G,A) \xrightarrow{T_G} T_G(A) \to 0$$

implies $T_G(A)$ is infinite if and only if $H_0(G,A)$ is infinite. It is thus enough to note that the conjugation action of G on $G_{F^\infty/F}$ is trivial and hence that the restriction map $A \to G_{F^\infty/F}$ induces a surjection from $H_0(G,A)$ to the infinite group $H_0(G, G_{F^\infty/F}) = G_{F^\infty/F}$. □

In the sequel we fix elements γ_π and γ_π^* as above and write $M_{F,\Sigma}^{\gamma_\pi^*}$ for the subfield of $M_{F,\Sigma}$ that is fixed by the subgroup of $G_{M_{F,\Sigma}/F}$ generated by γ_π^*.

4.2. **The module Y_{F,Σ,γ_π}.** It is also convenient to introduce an important auxiliary module. To do this we fix, for every element h of H, a pre-image $\widehat{h}$ of h under the canonical projection $G_{M_{F,\Sigma}/k} \to G$ and then define an element $\widehat{t_H} := \sum_{h\in H}\widehat{h}$ of $\mathbb{Z}[G_{M_{F,\Sigma}/k}]$. We then consider the following homomorphisms

$$\mathbb{Z}_p[G]e_H \xrightarrow{(\alpha_\pi,-d_\pi)} (\mathbb{Z}_p[G]\otimes_{\mathbb{Z}[G_{M_{F,\Sigma}/k}]_p^\wedge}(I_{G_{M_{F,\Sigma}/k}})_p^\wedge)\oplus\mathbb{Z}_p[G]e_H \xrightarrow{(\delta_p,\mathrm{id})} \mathbb{Z}_p[G].$$

Here α_π and d_π send e_H to $\frac{1}{|H|}e_H\otimes(\gamma_\pi-1)\widehat{t_H}$ and $e_H(\pi-1)e_H$ respectively and δ_p is the homomorphism that is induced by the projection $G_{M_{F,\Sigma}/k} \to G$ as in

Theorem 3.1 (ii). One checks easily that $\mathrm{im}((\alpha_\pi, -d_\pi)) \subseteq \ker((\delta_p, \mathrm{id}))$ and so we can set

$$Y_{F,\Sigma,\gamma_\pi} := \ker((\delta_p, \mathrm{id}))/\,\mathrm{im}((\alpha_\pi, -d_\pi)).$$

Proposition 4.4 below will show that there is a canonical short exact sequence of the form

$$0 \to G_{M^{\gamma_\pi^*}_{F,\Sigma}/F} \to Y_{F,\Sigma,\gamma_\pi} \to N_\pi \to 0 \tag{13}$$

in which the module N_π is finite.

4.3. **Statement of the result.** If F is a totally real field, we obtain a well-defined element of $\zeta(\mathbb{Q}_p[G])$ by setting

$$\mathcal{L}_{F/k,\Sigma,1} := \sum_{\rho \in \mathrm{Ir}_p(G)} e_\rho \cdot \lim_{s\to 1}(s-1)^{n_{\rho,1}} L_{p,\Sigma}(s,\rho)$$

In this case we also define an element of $\zeta(\mathbb{Q}_p[G])^\times$ by setting

$$\epsilon_{F/k,\gamma_\pi^*} := \mathrm{Nrd}_{\mathbb{Q}_p[G]}(d_\pi) + \log_p(\chi_k(\gamma_\pi^*))e_G$$

where d_π is the endomorphism of $\mathbb{Q}_p[G]e_H$ defined above (and in this regard see also Remark 4.3(i) below).

Theorem 4.2. *We assume (as we may, following Lemma 4.1) that the element γ_π^* of $G_{M_{F,\Sigma}/F}$ has infinite order.*

Then the $\mathbb{Z}_p[G]$-module Y_{F,Σ,γ_π} has a presentation in which the number of generators is greater than or equal to the number of (non-zero) relations.

In addition, if F is totally-real and $\mu_p(F)$ vanishes when p divides $|G|$, then the (non-commutative) Fitting invariant $\mathrm{Fit}_{\mathbb{Z}_p[G]}(Y_{F,\Sigma,\gamma_\pi})$ is generated over $\zeta(\mathbb{Z}_p[G])$ by the set of elements of the form $u \cdot \epsilon_{F/k,\gamma_\pi^*} \cdot \mathcal{L}_{F/k,\Sigma,1}$ with u in $\mathrm{Nrd}_{\mathbb{Q}_p[G]}(K_1(\mathbb{Z}_p[G]))$.

Remark 4.3.
(i) Given the explicit structure of G it is straightforward to compute explicitly both the term $\mathrm{Nrd}_{\mathbb{Q}_p[G]}(d_\pi)$ that occurs in the definition of $\epsilon_{F/k,\gamma_\pi^*}$ and the module N_π that occurs in the exact sequence (13). We only record the following easy cases.
(a) If H is normal in G, then e_H is central in $\mathbb{Q}_p[G]$,

$$\mathrm{Nrd}_{\mathbb{Q}_p[G]}(d_\pi) = 1 - e_H + (1-\pi)e_H$$

and the module N_π vanishes. In this case also $\mathcal{A}_p(G) = \zeta(\mathbb{Z}_p[G])$ (by [16, Prop. 4.8(i), (v)]).
(b) If G is dihedral of order $2p$, then one can compute that

$$\mathrm{Nrd}_{\mathbb{Q}_p[G]}(d_\pi) = -\sum_{i=1}^{i=(p-1)/2} \sum_{g\in P} (\phi(g^i)-1)^2 g,$$

where ϕ is any choice of non-trivial irreducible character of P. In addition, one finds that N_π is equal to the abelian group $\mathbb{Z}/p\mathbb{Z}$ upon which π acts trivially and each element of G of order two acts as multiplication by -1. In this case $\mathcal{A}_p(G)$ is equal to $\mathbb{Z}_p[G] \cdot \{p, \pi-1\} \cap \zeta(\mathbb{Z}_p[G])$.
(ii) The final assertion of Theorem 4.2 refines the first assertion of Theorem 2.1. To see this note that the former assertion implies that

$$\epsilon_{F/k,\gamma_\pi^*} \cdot \mathcal{L}_{F/k,\Sigma,1} \cdot \mathcal{A}_p(G) \subseteq \mathrm{Ann}_{\mathbb{Z}_p[G]}(Y_{F,\Sigma,\gamma_\pi}) \subseteq \mathrm{Ann}_{\mathbb{Z}_p[G]}(G_{M^{\gamma_\pi^*}_{F,\Sigma}/F})$$

where the last inclusion follows from the sequence (13). In addition, if H is normal in G, then the example in (i)(a) above shows that

$$\epsilon_{F/k,\gamma_\pi^*} \cdot \zeta(I_{G,p}) = (1 - e_H + (1-\pi)e_H) \cdot \zeta(I_{G,p}) = \zeta(I_{G,p})^2$$

where the second equality is because $(1-e_H)+(1-\pi)e_H$ is a generator of $\zeta(I_{G,p})$ over $\zeta(\mathbb{Z}_p[G])$. Since the element γ_π^* is central in $G_{M_{F,\Sigma}/k}$ (by Lemma 4.1) and hence annihilated by any element of $\zeta(I_{G,p})$, the tautological exact sequence

$$0 \to \overline{\langle \gamma_\pi^* \rangle} \to G_{M_{F,\Sigma}/F} \to G_{M_{F,\Sigma}^{\gamma_\pi^*}/F} \to 0 \tag{14}$$

then combines with the above equality to imply that

$$\epsilon_{F/k,\gamma_\pi^*} \mathcal{L}_{F/k,\Sigma,1} \cdot \mathcal{A}_p(G) \cdot \zeta(I_{G,p}) = \mathcal{L}_{F/k,\Sigma,1} \cdot \mathcal{A}_p(G) \cdot \zeta(I_{G,p})^2$$

belongs to $\mathbb{Z}_p[G]$ and annihilates $G_{M_{F,\Sigma}/F}$. This statement refines the first assertion of Theorem 2.1.

(iii) The final assertion of Theorem 4.2 can also usefully be compared with the results proved by Barrett and the first author in [3, Cor. 3.4]. To do this assume (as in loc. cit.) that G is cyclic, F is totally real and Leopoldt's Conjecture is valid for F. Then the equality of [3, (17)] asserts $\mathrm{Fit}_{\mathbb{Z}_p[G]}(G_{M_{F,\Sigma}/F^\infty})$ is generated by the element $(\pi-1)\cdot\mathcal{L}_{F/k,\Sigma,1}$. Whilst this assertion is similar in spirit to that of Theorem 4.2 (in the relevant special case) we are aware of no direct link between the structures of the Galois groups of $M_{F,\Sigma}/F^\infty$ and $M_{F,\Sigma}^{\gamma_\pi^*}/F$. In this regard note also that the equality of [3, (17)] both refines and generalises the main results of Oriat in [23] (for more details of this connection see [3, Rem. 3.6(iii)]).

4.4. **Proof of Theorem 4.2.** We start by proving the following useful preliminary result. In this result we write $C_\pi^\bullet$ for the complex of projective $\mathbb{Z}_p[G]$-modules

$$\mathbb{Z}_p[G]e_H \xrightarrow{d_\pi} \mathbb{Z}_p[G]e_H$$

where the first term is placed in degree two. We also identify the cohomology groups of $R\Gamma_{c,\text{ét}}(\mathcal{O}_{F,\Sigma},\mathbb{Z}_p(1))$ as in Theorem 3.1(ii).

Proposition 4.4. *There exist canonical identifications of both $H^2(C_\pi^\bullet)$ and $H^3(C_\pi^\bullet)_{\mathrm{tf}}$ with $\mathbb{Z}_p$ and a canonical exact triangle in $D^{\mathrm{p}}(\mathbb{Z}_p[G])$ of the form*

$$C_\pi^\bullet \xrightarrow{\theta_\pi} R\Gamma_{c,\text{ét}}(\mathcal{O}_{F,\Sigma},\mathbb{Z}_p(1)) \to D_\pi^\bullet \to C_\pi^\bullet[1] \tag{15}$$

in which $H^2(\theta_\pi)(1) = \gamma_\pi^*$ and $H^3(\theta_\pi)$ factors through the map $\mathrm{id}_{\mathbb{Z}_p}$.

The complex $D_\pi^\bullet$ is acyclic outside degrees one and two, there are canonical exact sequences

$$\begin{cases} 0 \to H^1_{c,\text{ét}}(\mathcal{O}_{F,\Sigma},\mathbb{Z}_p(1)) \to H^1(D_\pi^\bullet) \to \mathbb{Z}_p \cdot m_{\gamma_\pi^*} \to 0 \\ 0 \to G_{M_{F,\Sigma}^{\gamma_\pi^*}/F} \to H^2(D_\pi^\bullet) \to \mathrm{cok}(d_\pi)_{\mathrm{tor}} \to 0 \end{cases} \tag{16}$$

where $m_{\gamma_\pi^*}$ is equal to the order of γ_π^* if this is finite and to zero otherwise, and a canonical identification $H^2(D_\pi^\bullet) = Y_{F,\Sigma,\gamma_\pi}$. In addition, if H is normal in G, then $\mathrm{cok}(d_\pi)_{\mathrm{tor}}$ vanishes.

Proof. We claim first that there is an exact commutative diagram of $\mathbb{Z}_p[G]$-modules of the following form

$$\begin{array}{ccccccccc}
 & & 0 & & & & & & 0 \\
 & & \downarrow & & & & & & \downarrow \\
 & & \mathbb{Z}_p \cdot m_{\gamma_\pi^*} & & & & & & \mathrm{cok}(d_\pi)_{\mathrm{tor}} \\
 & & \subseteq\downarrow & & & & & & \downarrow\subseteq \\
0 & \to & \mathbb{Z}_p & \xrightarrow{\iota} & \mathbb{Z}_p[G]e_H & \xrightarrow{d_\pi} & \mathbb{Z}_p[G]e_H & \xrightarrow{\iota'} & \mathrm{cok}(d_\pi) & \to 0 \\
 & & \alpha'\downarrow & & \alpha_\pi\downarrow & & \downarrow\alpha'' & & \downarrow\epsilon' & \\
0 & \to & G_{M_{F,\Sigma}/F} & \xrightarrow{\delta_p'} & \mathbb{Z}_p[G]\otimes_{\mathbb{Z}[G_{M_{F,\Sigma}/k}]_p^\wedge}(I_{G_{M_{F,\Sigma}/k}})_p^\wedge & \xrightarrow{\delta_p} & \mathbb{Z}_p[G] & \xrightarrow{\epsilon} & \mathbb{Z}_p & \to 0 \\
 & & \downarrow & & & & & & \downarrow & \\
 & & G_{M_{F,\Sigma}^{\gamma_\pi^*}/F} & & & & & & 0 & \\
 & & \downarrow & & & & & & & \\
 & & 0. & & & & & & &
\end{array}$$

Here $\iota(1) = T_P \cdot e_H = |P|e_G$, ι' is the tautological projection, $\alpha'(1) = \gamma_\pi^*$, α_π is as defined in §4.2, $\alpha''(e_H) = e_H$, $\epsilon(g) = 1$ for all $g \in G$, ϵ' is induced by the restriction of ϵ, $\delta_p'(a) = 1 \otimes (a-1)$ for all $a \in G_{M_{F,\Sigma}/F}$ and δ_p is as described in Theorem 3.1(ii).

Commutativity of all squares is then easily checked, exactness of the first column is clear and exactness of the second row follows from Theorem 3.1(ii). Exactness of the first row and last column also follows from Lemma 4.5 below.

We now set $E^\bullet := R\Gamma_{c,\text{ét}}(\mathcal{O}_{F,\Sigma}, \mathbb{Z}_p(1))$. Then the natural exact triangle that relates $\tau^{\leq 1}E^\bullet, E^\bullet$ and $\tau^{\geq 2}E^\bullet$ gives rise to an exact sequence

$$\mathrm{Hom}_{\mathcal{D}}(C_\pi^\bullet, \tau^{\leq 1}E^\bullet) \to \mathrm{Hom}_{\mathcal{D}}(C_\pi^\bullet, E^\bullet) \xrightarrow{\beta} \mathrm{Hom}_{\mathcal{D}}(C_\pi^\bullet, \tau^{\geq 2}E^\bullet) \to \mathrm{Hom}_{\mathcal{D}}(C_\pi^\bullet, \tau^{\leq 1}E^\bullet[1])$$

with $\mathcal{D} = D^{\mathrm{p}}(\mathbb{Z}_p[G])$. In addition, the first and last groups here vanish since $C_\pi^\bullet$ is a complex of projective modules concentrated in degrees two and three and the complexes $\tau^{\leq 1}E^\bullet$ and $\tau^{\leq 1}E^\bullet[1]$ are both acyclic in degrees greater than one, and hence the map β is bijective.

This last fact combines with Theorem 3.1(ii) to imply that the morphisms α_π and α'' together constitute both a morphism θ_π' from $C_\pi^\bullet$ to $\tau^{\geq 2}E^\bullet$ and, at the same time, a morphism θ_π from $C_\pi^\bullet$ to $E^\bullet$. One therefore has an exact triangle of the required form (15) together with a natural identification of the truncated complex $\tau^{\geq 2}(D_\pi^\bullet)$ with $\mathrm{Cone}(\theta_\pi')$. By using the exact sequence of cohomology of this triangle one checks easily that $D_\pi^\bullet$ is acyclic outside degrees one and two and that there are exact sequences of the form (16). Further, one has $H^2(D_\pi^\bullet) = H^2(\tau^{\geq 2}D_\pi^\bullet) = H^2(\mathrm{Cone}(\theta_\pi'))$ and, recalling the explicit definition of mapping cone, it is clear that the latter group identifies with the module Y_{F,Σ,γ_π} defined in §4.2. □

Lemma 4.5. *The kernel of d_π is $\iota(\mathbb{Z}_p)$, the map ϵ' is surjective and* $\ker(\epsilon')$ *is finite. If H is normal in G, then* $\ker(\epsilon')$ *vanishes.*

Proof. We first prove that $\iota(\mathbb{Z}_p) = \ker(d_\pi)$. Recall that, by assumption, every element g of G can be written uniquely in the form $g = \underline{g}\overline{g}$ with $\underline{g} \in P$ and $\overline{g} \in H$.

This implies that $ge_H = \underline{g}e_H$ for each such g and hence also

$$(17)\quad |H|\cdot d_\pi(e_H) = |H|\cdot e_H(\pi-1)e_H = \sum_{h\in H} h(\pi-1)e_H$$
$$= \sum_{h\in H}(\pi^h h - h)e_H = \sum_{h\in H}(\pi^h-1)e_H = \sum_{h\in H}(\underline{\pi^h}-1)e_H.$$

This formula makes clear that $\iota(\mathbb{Z}_p)$ is contained in $\ker(d_\pi)$. Thus, since $\iota(\mathbb{Z}_p)$ is also equal to $(\mathbb{Z}_p[G]e_H)^P$, one has $\iota(\mathbb{Z}_p) = \ker(d_\pi)$ if and only if for every non-trivial character ψ of P the map $d_{\pi,\psi}$ that is induced by d_π on $e_\psi(\mathbb{Q}_p^c[G]e_H) = \mathbb{Q}_p^c\cdot e_\psi e_H$ is non-zero (and hence injective). Now the formula (17) implies that $d_{\pi,\psi}(e_\psi e_H)$ is equal to $|H|^{-1}\sum_{h\in H}(\psi(\underline{\pi^h})-1)e_\psi e_H$ and so vanishes if and only if one has $\sum_{h\in H}\psi(\underline{\pi^h}) = |H|$. But the last equality cannot be valid because $\{\psi(\underline{\pi^h}) : h\in H\}$ is a set of roots of unity at least one of which (that with h equal to the identity element of H) is non-trivial because π is a generator of P and the character ψ is assumed to be non-trivial.

The surjectivity of ϵ' follows directly from that of ϵ. The exactness of the first row of the above commutative diagram then implies that $\mathbb{Q}_p\otimes_{\mathbb{Z}_p}\ker(\epsilon')$ vanishes and hence that $\ker(\epsilon')$ is finite.

Finally, we note that if H is normal in G, then e_H is central in $\mathbb{Q}_p[G]$ and so $d_\pi(e_H) = (\pi-1)e_H$. It is then easy to see that, for any element x of $\mathbb{Z}_p[G]e_H$ with $\epsilon(x)=0$, one has $x\in\mathrm{im}(d_\pi)$, and so we conclude that $\ker(\epsilon')$ vanishes in this particular case. □

We now turn to the proof of Theorem 4.2. To prove the first claim note that the descriptions of Proposition 4.4 imply that $D^\bullet_\pi$ is an object of $D^{\mathrm{p}}(\mathbb{Z}_p[G])$ which is acyclic outside degrees one and two, that $H^1(D^\bullet_\pi)$ is $\mathbb{Z}_p$-free and that the Euler characteristic of $\mathbb{Q}_p\otimes_{\mathbb{Z}_p} D^\bullet_\pi$ in $K_0(\mathbb{Q}_p[G])$ vanishes. It follows (from, for example, [10, Prop. 3.2]) that $D^\bullet_\pi$ can be represented by a complex of finitely generated free modules of the form $N\to N$ and hence, since $H^2(D^\bullet_\pi) = Y_{F,\Sigma,\gamma_\pi}$, that there is an exact sequence of $\mathbb{Z}_p[G]$-modules

$$(18)\qquad N\to N\to Y_{F,\Sigma,\gamma_\pi}\to 0.$$

This is a presentation of Y_{F,Σ,γ_π} in which the number of generators is greater than or equal to the number of (non-zero) relations, as claimed.

To proceed we use the notion of characteristic element defined in [10, §1.3.2]. We also assume that F is totally real, write Υ for the subset of $\mathrm{Ir}_p(G)$ comprising characters ρ for which the space

$$W_\rho := V_\rho\otimes_{\mathbb{Z}_p[G]} Y_{F,\Sigma,\gamma_\pi}$$

vanishes and obtain an idempotent of $\zeta(\mathbb{Q}_p[G])$ by setting $e := \sum_{\rho\in\Upsilon} e_\rho$.

Then the presentation (18) combines with the definition of (non-commutative) Fitting invariants to imply that $\mathrm{Fit}_{\mathbb{Z}_p[G]}(Y_{F,\Sigma,\gamma_\pi})$ is generated over $\zeta(\mathbb{Z}_p[G])$ by the set of elements of the form $u\cdot\mathcal{L}e$ where u is in $\mathrm{Nrd}_{\mathbb{Q}_p[G]}(K_1(\mathbb{Z}_p[G]))$ and $\mathcal{L}$ is any characteristic element for the complex $D^\bullet_\pi$.

In addition, the description of Theorem 3.1(ii) combines with the exact sequences (13) and (14) to imply that

$$(19)\qquad \dim_{\mathbb{Q}_p^c}(W_\rho) = \dim_{\mathbb{Q}_p^c}(W(1,\rho)) - n_{1,\rho}$$

for all ρ in $\mathrm{Ir}_p(G)$ and so Theorem 2.4 implies that $\mathcal{L}_{F/k,\Sigma,1} = \mathcal{L}_{F/k,\Sigma,1}\cdot e$.

It is thus enough for us to show that if $\mu_p(F)$ vanishes when p divides $|G|$, then there is a characteristic element $\mathcal{L}$ for $D^\bullet_\pi$ with the property that

$$\mathcal{L} \cdot e_\rho = e_\rho \cdot \epsilon_{F/k,\gamma^*_\pi} \lim_{s\to 1}(s-1)^{n_{\rho,1}} L_{p,\Sigma}(s,\rho) \tag{20}$$

for all ρ in Υ. This will be proved in the next subsection.

4.5. **Interpolation data.** Set $\mathfrak{A} := \mathbb{Z}_p[G]e$ and $B^\bullet_0 := \mathfrak{A} \otimes^{\mathbb{L}}_{\mathbb{Z}_p[G]} B^\bullet$ for each object $B^\bullet$ in $D^{\mathrm{p}}(\mathbb{Z}_p[G])$.

Then the exact triangle (15) induces an exact triangle in $D^{\mathrm{p}}(\mathfrak{A})$

$$C^\bullet_{\pi,0} \xrightarrow{\theta_{\pi,0}} E^\bullet_0 \to D^\bullet_{\pi,0} \to C^\bullet_{\pi,0}[1], \tag{21}$$

with $E^\bullet := R\Gamma_{c,\text{ét}}(\mathcal{O}_{F,\Sigma}, \mathbb{Z}_p(1))$ as in the proof of Proposition 4.4. In addition, the equality (19) implies that Leopoldt's Conjecture is valid for all characters ρ in Υ and so there are canonical identifications

$$\mathbb{Q}_p \otimes_{\mathbb{Z}_p} H^i(E^\bullet_0) = \begin{cases} \mathbb{Q}_p \otimes_{\mathbb{Z}_p} G_{F^\infty/F}, & \text{if } i=2 \text{ and } \mathbf{1}_G \in \Upsilon, \\ \mathbb{Q}_p, & \text{if } i=3 \text{ and } \mathbf{1}_G \in \Upsilon, \\ 0, & \text{otherwise.} \end{cases}$$

Using these identifications we write

$$\lambda_0 : \mathbb{Q}_p \otimes_{\mathbb{Z}_p} H^2(E^\bullet_0) \cong \mathbb{Q}_p \otimes_{\mathbb{Z}_p} H^3(E^\bullet_0)$$

for the isomorphism of $\mathbb{Q}_p[G]$-modules which, if $\mathbf{1}_G$ belongs to Υ, maps each a in $G_{F^\infty/F}$ to $\log_p(\chi_k(a))$.

Proposition 4.6.

(i) *If $\mu_p(F)$ vanishes when p divides $|G|$, then there exists a characteristic element $\mathcal{L}_0$ for the pair $(E^\bullet_0, \lambda_0)$ for which one has*

$$e_\rho \mathcal{L}_0 = \lim_{s\to 1}(s-1)^{n_{\rho,1}} L_{p,\Sigma}(s,\rho)$$

at each ρ in Υ.

(ii) *$D^\bullet_{\pi,0}$ has finite cohomology groups and the map $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} H^i(\theta_{\pi,0})$ is bijective in each degree i.*

(iii) *$\epsilon^{-1}_{F/k,\gamma^*_\pi} \cdot e$ is a characteristic element for the pair*

$$(C^\bullet_{\pi,0}, (\mathbb{Q}_p \otimes_{\mathbb{Z}_p} H^3(\theta_{\pi,0}))^{-1} \circ \lambda_0 \circ (\mathbb{Q}_p \otimes_{\mathbb{Z}_p} H^2(\theta_{\pi,0}))).$$

Proof. Claim (i) is a consequence of the main conjecture of non-commutative Iwasawa theory. To be more precise we write F' for the composite of F and the cyclotomic $\mathbb{Z}_p$-extension k^∞ of k, set $\mathcal{G} := G_{F'/k}$ and write $\Lambda(\mathcal{G})$ for the associated (non-commutative) p-adic Iwasawa algebra and S for the Ore set in $\Lambda(\mathcal{G})$ that was introduced by Venjakob. We also fix a topological generator γ_k of $G_{k^\infty/k}$ and set $c_k := \log_p(\chi_k(\gamma_k))$.

We assume that $\mu_p(F)$ vanishes if p divides $|G|$ and recall that in this case the main conjecture of non-commutative Iwasawa theory for totally real fields has been proved by Ritter and Weiss [26] (and then also subsequently by Kakde [17]). In particular, this result implies, via the argument used to prove [7, Th. 9.1] and using the same notation as in loc. cit., the existence of an element ξ of $K_1(\Lambda(\mathcal{G})_S)$ which is both a characteristic element of the complex $E^\bullet_\infty := R\Gamma_{c,\text{ét}}(\mathcal{O}_{F,S}, \Lambda(\mathcal{G})^\#(1))$ of $\Lambda(\mathcal{G})$-modules and also satisfies $\xi^*(\rho) = c_k^{-n_{\rho,1}} \cdot \lim_{s\to 1}(s-1)^{n_{\rho,1}} L_{p,\Sigma}(s,\rho) \neq 0$ for

each Artin representation ρ at which Leopoldt's Conjecture is valid (in the sense discussed just prior to the statement of Theorem 2.4).

One also knows from [11, Lem. 8.2] that at any such representation ρ the complex $E^\bullet_\infty$ is 'semisimple' (in the terminology of loc. cit.), that the induced complex $\mathbb{Z}_p[G]e\otimes^{\mathbb{L}}_{\Lambda(\mathcal{G})}E^\bullet_\infty$ identifies with $E^\bullet_0$ and that, with respect to the latter identification, the relevant Bockstein homomorphism is given by $c_k^{-1}\cdot\lambda_0$ where λ_0 is the map that occurs in the statement of claim (i).

We may therefore apply the descent formalism of [11, Th. 1.2] to the complex $E^\bullet_\infty$, characteristic element ξ and homomorphism $\Lambda(\mathcal{G})\to\mathbb{Z}_p[G]e$ and in so doing deduce the existence of a characteristic element $\mathcal{L}'_0$ for the pair $(E^\bullet_0, c_k^{-1}\cdot\lambda_0)$ which has the property that $e_\rho\mathcal{L}'_0 = c_k^{-n_{\rho,1}}\cdot\lim_{s\to 1}(s-1)^{n_{\rho,1}}L_{p,\Sigma}(s,\rho)$ at each ρ in Υ. It is then straightforward to check that the product

$$\mathcal{L}_0 := ((1-e_G)+c_k\cdot e_G)\cdot\mathcal{L}'_0$$

is a characteristic element for $(E^\bullet_0,\lambda_0)$ and also at each ρ in Υ satisfies

$$e_\rho\mathcal{L}_0 = c_k^{n_{\rho,1}}\cdot e_\rho\mathcal{L}'_0 = \lim_{s\to 1}(s-1)^{n_{\rho,1}}L_{p,\Sigma}(s,\rho),$$

as required to complete the proof of claim (i).

The definition of the idempotent e ensures that $\mathbb{Q}_p^c\otimes_{\mathbb{Z}_p}(\mathfrak{A}\otimes_{\mathbb{Z}_p[G]}Y_{F,\Sigma,\gamma_\pi})$ vanishes and hence also, since $D^\bullet_\pi$ is acyclic in degrees greater that two, that $\mathbb{Q}_p^c\otimes_{\mathbb{Z}_p}H^2(D^\bullet_{\pi,0})$ vanishes. The description of Proposition 4.4 therefore implies that $\mathbb{Q}_p^c\otimes_{\mathbb{Z}_p}H^1(D^\bullet_{\pi,0})$ vanishes too and hence the first assertion of claim (ii) is valid. Given this, the second assertion of claim (ii) follows immediately from the long exact cohomology sequence of the triangle (21).

Claim (iii) follows from an explicit computation. This takes into account the properties of $H^2(\theta_\pi)$ and $H^3(\theta_\pi)$ given in Proposition 4.4 and the explicit description of refined Euler characteristics (and hence of the notion of characteristic element) in the relevant special case that is given, for example, in [10, §3.1]. □

The observations in Proposition 4.6 combine with the exact triangle (21) and the additivity criterion for refined Euler characteristics of [5, Cor. 6.6] to imply that the product $\mathcal{L}_1 := \epsilon_{F/k,\gamma_\pi^*}\mathcal{L}_0$ is a characteristic element for the complex $D^\bullet_{\pi,0}$.

Taking account of Bass's Theorem (cf. [19, Chap. 7, (20.9)]) as in the proof of [8, Lem. 6.2], we may therefore deduce the existence of a characteristic element $\mathcal{L}$ for $D^\bullet_\pi$ which projects to give $\mathcal{L}_1$ under the natural projection $\zeta(\mathbb{Q}_p[G])^\times\to\zeta(\mathbb{Q}_p[G]e)^\times$ and hence has the required interpolation property (20).

This completes the proof of Theorem 4.2.

5. Leopoldt's Conjecture and class field theory

Leopoldt's Conjecture, as originally formulated in [20], predicts that for every number field E the topological closure of the embedding of $\mathcal{O}_E^\times$ in $\prod_{v\in S_p(E)}E_v^\times$ has $\mathbb{Z}_p$-rank equal to the rank of $\mathcal{O}_E^\times$ as an abelian group. This prediction is clearly equivalent to the version of Leopoldt's Conjecture recalled in §2.2 and it is also by now known that there are many other equivalent forms of this important conjecture (see, for example, the extensive discussion in Neukirch et al. [22, Chap. X, §3]). In particular, recent work of Khare and Wintenberger [18], with additional contributions by Sharifi [29], has given a reformulation of the conjecture in terms of

the finite generation of a module which measures the Galois structure of a natural family of (infinite) ray class groups.

In this section we show that the approach of §4 also leads to an explicit interpretation of Leopoldt's Conjecture in terms of the cohomological-triviality as Galois modules of a different family of ray class groups and that this interpretation in turn leads to new results and predictions concerning the explicit structure of such groups.

5.1. **Statement of the main result.** We fix an odd prime p and for any number field E and natural number n we write E_∞ for the cyclotomic $\mathbb{Z}_p$-extension of E and E_n for the unique subfield of E_∞ that has degree p^n over E. We also fix a primitive p-th root of unity ζ_p in $\mathbb{Q}^c$.

In the sequel we shall write $Z(G)$ for the centre of a group G and J_E for the group of ideles of a number field E.

Theorem 5.1. *The following claims are equivalent.*

(i) *Leopoldt's Conjecture is valid at p.*

(ii) *For any number field L with $\zeta_p \in L$, any finite set Σ of places of L which contains all archimedean and p-adic places and also all those which ramify in L_1/L and any $\gamma \in G_{M_{L_1,\Sigma}/L} \backslash G_{M_{L_1,\Sigma}/L_1}$ write $\psi_{L,\Sigma,\gamma} : J_{L_1} \to G_{M^{\gamma^p}_{L_1,\Sigma}/L_1}$ for the Artin reciprocity map. Then the following equivalent conditions are satisfied:*

(a) *The transfer map $G_{M^{\gamma^p}_{L_1,\Sigma}/L} \to G_{M^{\gamma^p}_{L_1,\Sigma}/L_1}$ has image $Z(G_{M^{\gamma^p}_{L_1,\Sigma}/L})$.*

(b) *$\psi_{L,\Sigma,\gamma}(J_L) = \psi_{L,\Sigma,\gamma}(J_{L_1})^{G_{L_1/L}}$.*

(c) *$G_{M^{\gamma^p}_{L_1,\Sigma}/L_1}$ is a cohomologically-trivial $G_{L_1/L}$-module.*

(iii) *Let L/K be any finite p-power degree Galois extension of number fields that is unramified outside a finite set of places Σ of K which contains both all archimedean and all p-adic places of K. Then the homomorphism (of abelian groups)*

$$G_{M_{K,\Sigma}/K} \to (\mathbb{Z}_p[G_{L/K}] \otimes_{\mathbb{Z}[G_{M_{L,\Sigma}/K}]^\wedge_p} (I_{G_{M_{L,\Sigma}/K}})^\wedge_p)^{G_{L/K}}$$

which sends each a in $G_{M_{K,\Sigma}/K}$ to $\sum_{g \in G_{L/K}} g \otimes (\tilde{a} - 1)$ for any lift $\tilde{a}$ of a to $G_{M_{L,\Sigma}/K}$ is bijective.

Remark 5.2. (i) The assertion of Theorem 5.1(ii)(c) is equivalent to the vanishing of the finite module $\hat{H}^0(G_{L_1/L}, G_{M^{\gamma^p}_{L_1,\Sigma}/L_1})$. The assertion of Theorem 5.1(ii)(b) differs slightly from all equivalent versions of Leopoldt's Conjecture of which the authors are aware (such as can be derived by combining [22, Cor. (10.3.9) and Th. (3.6.4)]) in that the role of groups such as $G_{M_{L_1,\Sigma}/L_1}$ in loc. cit. is here played by the smaller group $G_{M^{\gamma^p}_{L_1,\Sigma}/L_1}$.

(ii) Let L/K be any finite Galois extension of number fields that is unramified outside a finite set of places Σ of K which contains both all archimedean and all p-adic places of K. Theorem 3.1(ii) implies that the truncated complex $\tau^{\geq 2} R\Gamma_{c,\text{ét}}(\mathcal{O}_{L,\Sigma}, \mathbb{Z}_p(1))$ corresponds to a canonical element of the Yoneda Ext-group $\text{Ext}^2_{\mathbb{Z}_p[G_{L/K}]}(\mathbb{Z}_p, G_{M_{L,\Sigma}/L})$ and we write this element as $c_{L/K,\Sigma}$. Then the same approach as used below can also be used to show that the condition in Theorem 5.1(iii) is equivalent to asserting that for all such extensions L/K and sets Σ

the pair $(G_{L/K}, G_{M_{L,\Sigma}/L})$ is a class formation with fundamental class $c_{L/K,\Sigma}$ (in the sense of [22, Chap. III, Def. (3.1.8)]).

5.2. **The proof of Theorem 5.1.** We start by making an easy observation.

For any number field L we use the homomorphism $\lambda_{L,p}$ defined in (1).

Lemma 5.3. *The following conditions are equivalent.*

(i) *Leopoldt's Conjecture is valid at p.*

(ii) *For every number field L and any sufficiently large integer n the homomorphism $\ker(\lambda_{L_{n+1},p}) \to \ker(\lambda_{L_n,p})$ that is induced by the field-theoretic norm $L_{n+1} \to L_n$ is surjective.*

Proof. If Leopoldt's Conjecture is valid at p, then the group $\ker(\lambda_{L_n,p})$ vanishes and so the second condition is clearly satisified.

To prove the converse we recall that (as already used in the proof of Lemma 2.6) the known validity of the 'weak Leopoldt Conjecture' in this case implies the vanishing of the inverse limit $\varprojlim_m \ker(\lambda_{L_m,p})$, where the transition morphisms are induced by the field-theoretic norm maps. In particular, if these transition morphisms are surjective for each integer m with $m \geq m_0$, then the projection map $\varprojlim_m \ker(\lambda_{L_m,p}) \to \ker(\lambda_{L_{m_0},p})$ is surjective. It follows that the group $\ker(\lambda_{L_{m_0},p})$, and hence also $\ker(\lambda_{L,p})$, vanishes, as required. □

We next analyse the surjectivity of the maps appearing in Lemma 5.3(ii).

Lemma 5.4. *Let L/K be a finite cyclic extension of number fields of p-power degree with Galois group G. We fix a finite set Σ of places of K which contains all archimedean and p-adic places and also all those which ramify in L/K. We set $n := |G|$ and fix an element γ of $G_{M_{L,\Sigma}/K}$ which has infinite order and projects to give a generator of G. Then the following conditions are equivalent.*

(i) *The map $\ker(\lambda_{L,p}) \to \ker(\lambda_{K,p})$ induced by taking norms with respect to L/K is surjective.*

(ii) *The image of the transfer map $G_{M^{\gamma^n}_{K,\Sigma}/L} \subseteq G_{M^{\gamma^n}_{K,\Sigma}/K} \to G_{M^{\gamma^n}_{L,\Sigma}/L}$ contains $G_{M^{\gamma^n}_{L,\Sigma}/L} \cap Z(G_{M^{\gamma^n}_{L,\Sigma}/K})$.*

Proof. We use the exact sequence of $\mathbb{Z}_p[G]$-modules that is given by (7) with k replaced by K. In particular, since the first module in this sequence is projective, and so cohomologically-trivial, the Tate cohomology group $\hat{H}^0(G, H^1_{c,\text{ét}}(\mathcal{O}_{L,\Sigma}, \mathbb{Z}_p(1)))$ vanishes if and only if $\hat{H}^0(G, \ker(\lambda_{L,p}))$ vanishes. As $H^0(G, \ker(\lambda_{L,p})) = \ker(\lambda_{K,p})$, this last condition is in turn equivalent to the surjectivity of the map that occurs in claim (i).

On the other hand, under the stated hypotheses on G, Proposition 4.4 implies the existence of an exact sequence of $\mathbb{Z}_p[G]$-modules of the form

$$0 \to H^1_{c,\text{ét}}(\mathcal{O}_{L,\Sigma}, \mathbb{Z}_p(1)) \to Q^1 \to Q^2 \to \Gamma_L \to 0 \tag{22}$$

in which Q^1 and Q^2 are both projective and we have set $\Gamma_L := G_{M^{\gamma^n}_{L,\Sigma}/L}$. Thus, since the Tate cohomology of G is periodic of order two, if one splits the above displayed exact sequence into two short exact sequences and considers the associated long exact cohomology sequences one obtains isomorphisms

$$\hat{H}^0(G, H^1_{c,\text{ét}}(\mathcal{O}_{L,\Sigma}, \mathbb{Z}_p(1))) \cong \hat{H}^2(G, H^1_{c,\text{ét}}(\mathcal{O}_{L,\Sigma}, \mathbb{Z}_p(1))) \cong \hat{H}^0(G, \Gamma_L).$$

Taken in conjunction with the observation made above, this shows that the map in claim (i) is surjective if and only if the group $\hat{H}^0(G, \Gamma_L)$ vanishes. But the latter group vanishes if and only if the homomorphism

$$\mathrm{Tr} : H_0(G, \Gamma_L) \to H^0(G, \Gamma_L)$$

that is induced by applying the 'trace element' $\sum_{g \in G} g$ is surjective.

To understand this last condition we set $\Xi := G_{M_{L,\Sigma}^{\gamma^n}/K}, \Delta := G_{M_{K,\Sigma}^{\gamma^n}/L}$ and $\Gamma_K := G_{M_{K,\Sigma}^{\gamma^n}/K}$. Then $H_0(G, \Gamma_L) = \Delta$ since $M_{K,\Sigma}^{\gamma^n}$ is the maximal extension of L in $M_{L,\Sigma}^{\gamma^n}$ that is abelian over K, $H^0(G, \Gamma_L) = \Gamma_L \cap Z(\Xi)$ and the restriction to Δ of the transfer homomorphism $\Gamma_K = \Xi^{\mathrm{ab}} \to \Gamma_L$ coincides with the map Tr defined above. The claimed equivalence is therefore clear. □

We are now ready to prove Theorem 5.1. We note first that the equivalence of the conditions (i) and (ii) in Theorem 5.1 will follow from the results of Lemmas 5.3 and 5.4 if we can show that whenever ζ_p belongs to L, then $Z(G_{M_{L_1,\Sigma}^{\gamma^p}/L})$ is equal to $G_{M_{L_1,\Sigma}^{\gamma^p}/L_1} \cap Z(G_{M_{L_1,\Sigma}^{\gamma^p}/L})$, or equivalently that $Z(G_{M_{L_1,\Sigma}^{\gamma^p}/L}) \subseteq G_{M_{L_1,\Sigma}^{\gamma^p}/L_1}$. But, since L_1/L has degree p, this is clear since the group $G_{M_{L_1,\Sigma}^{\gamma^p}/L}$ is non-abelian (as otherwise, $M_{L_1,\Sigma}^{\gamma^p} = M_{L,\Sigma}^{\gamma^p}$ and hence $M_{L_1,\Sigma} = M_{L,\Sigma}$ which is obviously false).

We next set $G := G_{L/K}$ and claim that the condition in Theorem 5.1(iii) is equivalent to asserting that $\ker(\lambda_{L,p})$ is a cohomologically-trivial G-module. We also note that the equivalence of this latter condition with the validity of Leopoldt's Conjecture at p is an immediate consequence of Lemma 5.3.

To prove the required interpretation of condition (iii) we again use the exact sequence (7) (with k replaced by K), this time to observe that it implies $\ker(\lambda_{L,p})$ is a cohomologically-trivial G-module if and only if $H^1_{c,\text{ét}}(\mathcal{O}_{L,\Sigma}, \mathbb{Z}_p(1))$ is G-cohomologically-trivial, or equivalently that the complex $H^1_{c,\text{ét}}(\mathcal{O}_{L,\Sigma}, \mathbb{Z}_p(1))[-1]$ belongs to $D^{\mathrm{p}}(\mathbb{Z}_p[G])$.

In addition, since $R\Gamma_{c,\text{ét}}(\mathcal{O}_{L,\Sigma}, \mathbb{Z}_p(1))$ belongs to $D^{\mathrm{p}}(\mathbb{Z}_p[G])$ and is acyclic in degrees less than one, the natural exact triangle

$$\begin{aligned} H^1_{c,\text{ét}}(\mathcal{O}_{L,\Sigma}, \mathbb{Z}_p(1))[-1] &\to R\Gamma_{c,\text{ét}}(\mathcal{O}_{L,\Sigma}, \mathbb{Z}_p(1)) \\ &\to \tau^{\geq 2} R\Gamma_{c,\text{ét}}(\mathcal{O}_{L,\Sigma}, \mathbb{Z}_p(1)) \to H^1_{c,\text{ét}}(\mathcal{O}_{L,\Sigma}, \mathbb{Z}_p(1))[0] \end{aligned}$$

implies that this last condition is satisfied if and only if $\tau^{\geq 2} R\Gamma_{c,\text{ét}}(\mathcal{O}_{L,\Sigma}, \mathbb{Z}_p(1))$ belongs to $D^{\mathrm{p}}(\mathbb{Z}_p[G])$. The explicit description of this complex given in Theorem 3.1(i) implies in turn that this is true if and only if the G-module

$$N := \mathbb{Z}_p[G] \otimes_{\mathbb{Z}[G_{M_{L,\Sigma}/K}]_p^\wedge} (I_{G_{M_{L,\Sigma}/K}})_p^\wedge$$

is cohomologically-trivial and the general result of [1, §9, Th. 9] shows that this is true if and only if the natural 'trace' map $H_0(G, N) \to H^0(G, N)$ is bijective. To obtain the required interpretation of condition (iii) it is thus enough to observe that there is an isomorphism of abelian groups $H_0(G, N) \cong G_{M_{K,\Sigma}/K}$ with respect to which the trace map identifies with the displayed map in claim (iii): indeed, this

follows from the isomorphisms

$$H_0(G,N) = \mathbb{Z}_p \otimes_{\mathbb{Z}[G_{M_{L,\Sigma}/K}]_p^\wedge} (I_{G_{M_{L,\Sigma}/K}})_p^\wedge$$
$$\cong H_0(G_{M_{L,\Sigma}/K}, (I_{G_{M_{L,\Sigma}/K}})_p^\wedge) \cong (G_{M_{L,\Sigma}/K})^{\mathrm{ab}} \cong G_{M_{K,\Sigma}/K},$$

where the first isomorphism is clear, the (inverse of) the second is induced by the map which sends each element c of $G_{M_{L,\Sigma}/K}$ to the class of the element $c-1$ of $(I_{G_{M_{L,\Sigma}/K}})_p^\wedge$ and the final isomorphism is induced by the natural restriction map $G_{M_{L,\Sigma}/K} \to G_{M_{K,\Sigma}/K}$.

This completes the proof of Theorem 5.1.

5.3. **Module structures and dévissage.** Theorem 5.1 implies that Leopoldt's Conjecture entails a variety of explicit predictions concerning the structure of the group $G_{M_{L,\Sigma}/L}$. To discuss this aspect of the theory further we will restrict to the case that $G_{L/K}$ has order p and also refrain from making a full analysis of even this case. Nevertheless, we feel that the result of Corollary 5.5 below already makes clear that this approach should in general lead to predictions about the precise structure of $G_{M_{L,\Sigma}/L}$ which it may be interesting to investigate directly.

Conversely, the same analysis also shows that if one can control cohomological aspects of the torsion subgroup of $G_{M_{L,\Sigma}/L}$, then Nakayama's Lemma will allow the cohomological-triviality of the module in Theorem 5.1(ii)(c) to be verified by computing norm maps on the maximal elementary abelian (finite degree) p-extension of L that is unramifed outside Σ.

5.3.1. *Explicit structures.* In the sequel we fix a cyclic extension of number fields L/K of degree p and a finite set of places Σ of K which contains all archimedean places, all which ramify in L/K and all above p. We set $\Gamma := G_{L/K}$, $G := G_{M_{L,\Sigma}/K}$ and $A := G_{M_{L,\Sigma}/L}$ and then fix an element γ of $G \setminus A$. We note that γ^p belongs to $A \cap Z(G)$ and write A_γ for the quotient $A/\overline{\langle \gamma^p \rangle}$ so that there is a tautological exact sequence of $\mathbb{Z}_p[\Gamma]$-modules

$$0 \to \overline{\langle \gamma^p \rangle} \to A \to A_\gamma \to 0. \tag{23}$$

We also write r_2 for the number of complex places of K, T_Γ for the element $\sum_{\gamma\in\Gamma}\gamma$ of $\mathbb{Z}_p[\Gamma]$ and $\mathcal{O}_p$ for the $\mathbb{Z}_p[\Gamma]$-module $\mathbb{Z}_p[\Gamma]/(T_\Gamma)$. Finally we write $\mathbb{F}_p$ for the field with p elements.

Corollary 5.5. *Assume that Leopoldt's Conjecture is valid for L at p.*

(i) *Then the integer $d_A := \dim_{\mathbb{F}_p}(\hat{H}^0(\Gamma, A_{\gamma,\mathrm{tor}}))$ is independent of the choice of γ and satisfies $d_A \le r_2$.*

(ii) *The quotient A/A_{tor} is isomorphic as a $\mathbb{Z}_p[\Gamma]$-module to the direct sum $\mathbb{Z}_p^{d_A+1} \oplus \mathcal{O}_p^{d_A} \oplus \mathbb{Z}_p[\Gamma]^{r_2-d_A}$.*

Proof. Up to isomorphism, the only indecomposable finitely generated torsion-free $\mathbb{Z}_p[\Gamma]$-modules are $\mathbb{Z}_p, \mathcal{O}_p$ and $\mathbb{Z}_p[\Gamma]$. The Krull-Schmidt Theorem therefore implies that for any finitely generated torsion-free $\mathbb{Z}_p[\Gamma]$-module M there are unique non-negative integers $a(M), b(M)$ and $c(M)$ for which there exists an isomorphism of $\mathbb{Z}_p[\Gamma]$-modules

$$M_{\mathrm{tf}} \cong \mathbb{Z}_p^{a(M)} \oplus \mathcal{O}_p^{b(M)} \oplus \mathbb{Z}_p[\Gamma]^{c(M)}.$$

Now, since Leopoldt's Conjecture is valid for L at p, the exact sequences (7) and (22) combine to imply that A_γ spans a free $\mathbb{Q}_p[\Gamma]$-module and hence that $a(A_\gamma) = b(A_\gamma)$.

In addition, since A_γ is cohomologically-trivial (as a consequence of Theorem 5.1) and $\hat{H}^{-1}(\Gamma, \mathbb{Z}_p)$ vanishes, the cohomology sequence associated to the tautological exact sequence $0 \to A_{\gamma,\mathrm{tor}} \to A_\gamma \to A_{\gamma,\mathrm{tf}} \to 0$ induces an isomorphism

$$\hat{H}^0(\Gamma, A_{\gamma,\mathrm{tor}}) \cong \hat{H}^{-1}(\Gamma, A_{\gamma,\mathrm{tf}}) \cong \hat{H}^{-1}(\Gamma, \mathcal{O}_p)^{a(A_\gamma)} \cong (\mathbb{Z}/p)^{a(A_\gamma)}$$

and so implies that $a(A_\gamma) = d_A$.

By considering degrees one then computes that

$$\begin{aligned} p \cdot r_2 = \dim_{\mathbb{Q}_p}(\mathbb{Q}_p \otimes_{\mathbb{Z}_p} A_\gamma) &= a(A_\gamma) + (p-1)b(A_\gamma) + p \cdot c(A_\gamma) \\ &= d_A + (p-1)d_A + p \cdot c(A_\gamma) = p(d_A + c(A_\gamma)) \end{aligned}$$

and hence that $c(A_\gamma) = r_2 - d_A$.

We note finally that the sequence (23) implies $a(A) = a(A_\gamma) + 1 = d_A + 1$, $b(A) = b(A_\gamma) = d_A$ and $c(A) = c(A_\gamma) = r_2 - d_A$.

All claims are now clear. □

Corollary 5.5 leaves one to study the structure of $A_{\gamma,\mathrm{tor}}$. To make a first step in this direction we clarify the connection between $A_{\gamma,\mathrm{tor}}$ and A_{tor}.

To do this we recall that L/K is said to be '$\mathbb{Z}/p^n\mathbb{Z}$-extendable' for some natural number n, resp. to be '$\mathbb{Z}_p$-extendable', if L can be embedded into a Galois extension L' of K that is unramified outside Σ and is such that the group $G_{L'/K}$ is isomorphic to $\mathbb{Z}/p^n\mathbb{Z}$, resp. to $\mathbb{Z}_p$. Such extensions have been characterized by Bertrandias and Payan in [4], resp. by Seo in [28]. We note that, following Theorem 5.1(ii), the case that L/K is $\mathbb{Z}_p$-extendable is of particular interest in our context.

Lemma 5.6. *Let n be a natural number for which L/K is $\mathbb{Z}/p^{n+1}\mathbb{Z}$-extendable. Then $A[p^n] = A_\gamma[p^n]$ and the image of γ^p in A/p^n has order equal to p^n.*

In particular, if L/K is $\mathbb{Z}_p$-extendable, then $A_{\mathrm{tor}} = A_{\gamma,\mathrm{tor}}$ and the exact sequence (23) splits as a sequence of $\mathbb{Z}_p$-modules.

Proof. By considering multiplication by p^n on the exact sequence (23) one obtains an exact sequence

$$0 \to A[p^n] \xrightarrow{\subseteq} A_\gamma[p^n] \xrightarrow{d'} \langle\gamma^p\rangle/p^n \to A/p^n \xrightarrow{d} A_\gamma/p^n \to 0. \tag{24}$$

Now $\ker(d)$ is equal to $\langle\gamma^p, A^{p^n}\rangle/A^{p^n}$ and so is isomorphic to $\langle\gamma^p\rangle/(\langle\gamma^p\rangle \cap A^{p^n})$. On the other hand, since G/A^{p^n} has a quotient that is isomorphic to $\mathbb{Z}/p^{n+1}\mathbb{Z}$ and this quotient is generated by the image of γ, the smallest power of γ^p that belongs to A^{p^n} is $\gamma^{p^{n+1}} = (\gamma^p)^{p^n}$. The order of $\ker(d)$ is therefore p^n and so $\mathrm{im}(d')$ is trivial. This shows that $A[p^n] = A_\gamma[p^n]$ and also that the order of the image of γ^p in A/A^{p^n} is p^n, as required.

The final assertion then follows easily from the fact that if L is $\mathbb{Z}_p$-extendable then it is $\mathbb{Z}/p^n\mathbb{Z}$-extendable for every n. □

Example 5.7. To give a concrete example we assume K is a complex abelian field that is disjoint from the cyclotomic $\mathbb{Z}_p$-extension $\mathbb{Q}^\infty$ of $\mathbb{Q}$. We let L denote the compositum of K and the first layer of $\mathbb{Q}^\infty$ (so L/K is cyclic of degree p), write Σ for the union of the places of K which divide either p or the discriminant of $K/\mathbb{Q}$ and set $A_L := G_{M_{L,\Sigma}/L}$ and $A_K := G_{M_{K,\Sigma}/K}$. Then $L/\mathbb{Q}$ is abelian so Leopoldt's Conjecture is valid for L at p. Further, L/K is $\mathbb{Z}_p$-extendable and so Lemma 5.6 implies that, for any choice of γ in $G_{M_{L,\Sigma}/K} \setminus A_L$, the torsion subgroup of $G_{M_{L,\Sigma}^{\gamma^p}/L}$ is equal to that of A_L. Thus, since in this case $r_2 = [K : \mathbb{Q}]/2$, Corollary 5.5

implies (unconditionally) that the quotient $A_L/A_{L,\mathrm{tor}}$ is isomorphic as a $\mathbb{Z}_p[\Gamma]$-module to $\mathbb{Z}_p^{d+1} \oplus \mathcal{O}_p^d \oplus \mathbb{Z}_p[\Gamma]^{[K:\, QQ]/2-d}$ where the non-negative integer d satisfies $d \leq [K:\mathbb{Q}]/2$ and can be computed via the equality

$$d = \dim_{\mathbb{F}_p}(\hat{H}^0(\Gamma, A_{L,\mathrm{tor}})) = \dim_{\mathbb{F}_p}((A_L \cap Z(G_{M_{L,\Sigma}/K}))_{\mathrm{tor}}/\operatorname{im}(\mathrm{ver}_{\mathrm{tor}})).$$

Here $\mathrm{ver}_{\mathrm{tor}}$ denotes the transfer map $A_{K,\mathrm{tor}} \subseteq A_K \to A_L$ and the last equality follows from the computation of $\hat{H}^0(\Gamma, A_{L,\mathrm{tor}})$ made in the proof of Theorem 5.1.

In the next result we describe explicit connections between the invariant d_A that occurs in Corollary 5.5 and the structures of the torsion subgroups of both A and $Z(G)$. In this result we write N_γ for the (normal) subgroup of G that is generated by A^p and γ^p and $\mathrm{ver}_{G,A}$ for the transfer map $G \to A$. We also write $\mathrm{rk}_p(M)$ for the 'p-rank' $\dim_{\mathbb{F}_p}(M[p]) = \dim_{\mathbb{F}_p}(M/pM)$ of a finite $\mathbb{Z}_p$-module M.

Proposition 5.8. *Assume that Leopoldt's Conjecture is valid for L at p.*

(i) *The quotient G/N_γ has exponent p if and only if* $\operatorname{im}(\mathrm{ver}_{G,A}) \subseteq N_\gamma$.
(ii) *If G/N_γ has exponent p, then $d_A = r_2$, $\mathrm{rk}_p(Z(G)_{\mathrm{tor}}) \geq 2r_2 - 1$ and if also $r_2 \neq 0$, then A_γ has an element of order p^2.*
(iii) *If G/N_γ does not have exponent p, then $\mathrm{rk}_p(Z(G)_{\mathrm{tor}}) \geq d_A$. In this case one can also choose γ so that both $A_\gamma[p] = A[p]$ and the following claims are valid.*
  (a) *A_{tor} vanishes if and only if A_γ is a free $\mathbb{Z}_p[\Gamma]$-module.*
  (b) *d_A vanishes if and only if $A_{\gamma,\mathrm{tor}}$ is a cohomologically-trivial Γ-module. If this is the case, and G is not abelian, then either A_γ has an element of order p^2 (for every choice of γ as above) or A_{tor} is isomorphic to the induced $\mathbb{Z}_p[\Gamma]$-module $\mathbb{Z}_p[\Gamma] \otimes_{\mathbb{Z}_p} Z(G)_{\mathrm{tor}}$.*

Proof. The group G/N_γ has exponent p if and only if for every element a of A one has $(a\gamma)^p \in N_\gamma$. But for each such a one has $(a\gamma)^p = (\prod_{i=0}^{i=p-1} a^{\gamma^i})\gamma^{-p}$ and so the required condition is satisfied if ond only if $\prod_{i=0}^{i=p-1} a^{\gamma^i} \in N_\gamma$ for all $a \in A$. Lastly note that for a in A one has $\prod_{i=0}^{i=p-1} a^{\gamma^i} = \mathrm{ver}_{G,A}(a)$.

To prove claim (ii) we assume that $\operatorname{im}(\mathrm{ver}_{G,A}) \subseteq N_\gamma$. Then the map $\mathrm{tr}_{A_\gamma/p}$ which occurs in Lemma 5.9 below is the zero map and so, since ver_γ/p is assumed to be surjective, that result implies both that $A_\gamma^\Gamma \subseteq p \cdot A_\gamma$ and that $\hat{H}^0(\Gamma, A_\gamma[p])$ is isomorphic to $(A_\gamma/p)^\Gamma$.

The inclusion $A_\gamma^\Gamma \subseteq p \cdot A_\gamma$ implies that the $\mathbb{Z}_p[\Gamma]$-module A_γ can have no non-zero free direct summand and hence, via Corollary 5.5(ii), that $d_A = r_2$. This implies that $A_\gamma/A_{\gamma,\mathrm{tor}}$ is isomorphic to $(\mathbb{Z}_p \oplus \mathcal{O}_p)^{r_2}$ and the isomorphism $\hat{H}^0(\Gamma, A_\gamma[p]) \cong (A_\gamma/p)^\Gamma$ then combines with the natural exact sequence

$$0 \to (A_{\gamma,\mathrm{tor}}/p)^\Gamma \to (A_\gamma/p)^\Gamma \to ((A_\gamma/A_{\gamma,\mathrm{tor}})/p)^\Gamma \to H^1(\Gamma, A_{\gamma,\mathrm{tor}}/p)$$

and the fact that $H^1(\Gamma, A_{\gamma,\mathrm{tor}}/p)$ has the same order as $\hat{H}^0(\Gamma, A_{\gamma,\mathrm{tor}}/p) = (A_{\gamma,\mathrm{tor}}/p)^\Gamma$ to imply that

$$\begin{aligned}\mathrm{rk}_p(\hat{H}^0(\Gamma, A_\gamma[p])) = \mathrm{rk}_p((A_\gamma/p)^\Gamma) &\geq \mathrm{rk}_p(((A_\gamma/A_{\gamma,\mathrm{tor}})/p)^\Gamma) \\ &= \mathrm{rk}_p(((\mathbb{F}_p \oplus \mathcal{O}_p/p)^\Gamma)^{r_2}) = 2r_2.\end{aligned} \tag{25}$$

Now for any natural number m the group $\hat{H}^0(\Gamma, A_\gamma[p^m])$ is in all cases a subquotient of $A^\Gamma_{\gamma,\mathrm{tor}} = (A^\Gamma_\gamma)_{\mathrm{tor}}$ and one has $A^\Gamma_{\mathrm{tor}} \subseteq Z(G)_{\mathrm{tor}}$. Thus in general one has

$$\mathrm{rk}_p(Z(G)_{\mathrm{tor}}) \geq \mathrm{rk}_p(A^\Gamma_{\mathrm{tor}}) \geq \mathrm{rk}_p(A^\Gamma_{\gamma,\mathrm{tor}}) - 1 \geq \mathrm{rk}_p(\hat{H}^0(\Gamma, A_\gamma[p^m])) - 1, \tag{26}$$

where the second inequality follows from (24) with $n = 1$. From the inequality (25) one therefore has $\mathrm{rk}_p(Z(G)_{\mathrm{tor}}) \geq 2r_2 - 1$. In addition, if $r_2 \neq 0$, then (25) also combines with the fact that $d_A = r_2$ to imply $A_\gamma[p] \neq A_{\gamma,\mathrm{tor}}$ and hence that A_γ contains an element of order p^2.

We turn to claim (iii). In this case we can replace γ by $a\gamma$ for a suitable a in A to ensure that both $\mathrm{im}(\mathrm{ver}_{G,A}) \nsubseteq N_\gamma$ and $\gamma^p \notin A^p$. Under this assumption the sequence (24) with $n = 1$ implies $A[p] = A_\gamma[p]$. Then also $A^\Gamma[p] = A[p]^\Gamma = A_\gamma[p]^\Gamma = A^\Gamma_\gamma[p]$ and so $\mathrm{rk}_p(A^\Gamma_{\mathrm{tor}}) = \mathrm{rk}_p(A^\Gamma[p]) = \mathrm{rk}_p(A^\Gamma_\gamma[p]) = \mathrm{rk}_p(A^\Gamma_{\gamma,\mathrm{tor}})$. In this case the inequality (26) (for any sufficiently large m) therefore implies that $\mathrm{rk}_p(Z(G)_{\mathrm{tor}}) \geq \mathrm{rk}_p(\hat{H}^0(\Gamma, A_{\gamma,\mathrm{tor}})) = d_A$, as claimed.

Note also that, in this case, A_{tor} vanishes if and only if $A_{\gamma,\mathrm{tor}}$ vanishes and that then A_γ is a cohomologically-trivial torsion-free $\mathbb{Z}_p[\Gamma]$-module, and hence free. Conversely, if A_γ is a free $\mathbb{Z}_p[\Gamma]$-module, then the exact sequence (23) splits and shows that A_{tor} vanishes. This proves claim (iii)(a).

Regarding (iii)(b), the first assertion is clear because $A_{\gamma,\mathrm{tor}}$ is cohomologically-trivial if and only if $\hat{H}^0(\Gamma, A_{\gamma,\mathrm{tor}})$ vanishes. Also, if $A_{\gamma,\mathrm{tor}}$ has exponent p, then so does A_{tor} and hence $A_{\mathrm{tor}} = A[p] = A_\gamma[p] = A_{\gamma,\mathrm{tor}}$. Given this, and the fact that $Z(G) = A^\Gamma$ if G is not abelian, the second assertion of (iii)(b) follows from fact (due to Nakayama and Rim, and proved, for example, in [1, §9, Th. 6]) that a finite cohomologically-trivial $\mathbb{F}_p[\Gamma]$-module M is free and hence isomorphic to the induced module $\mathbb{F}_p[\Gamma] \otimes_{\mathbb{F}_p} M^\Gamma = \mathbb{Z}_p[\Gamma] \otimes_{\mathbb{Z}_p} M^\Gamma$. □

5.3.2. *Dévissage.* By Nakayama's Lemma, the surjectivity of the transfer map $A_\gamma \to A^\Gamma_\gamma$ (as predicted by Theorem 5.1(ii)) is equivalent to the surjectivity of the induced map $\mathrm{ver}_\gamma/p : (A_\gamma/p)_\Gamma \to A^\Gamma_\gamma/p$.

To analyse this condition one can use the next result. In this result we write $\mathrm{tr}_{A_\gamma/p}$ for the homomorphism $(A_\gamma/p)_\Gamma \to (A_\gamma/p)^\Gamma$ that is induced by the action of $T_\Gamma := \sum_{\gamma\in\Gamma} \gamma$.

Lemma 5.9. *There is a natural exact commutative diagram of finite groups*

$$\begin{array}{ccccccccc} H^1(\Gamma, A_\gamma[p]) & \to & A^\Gamma_\gamma/p & \xrightarrow{\iota} & (A_\gamma/p)^\Gamma & \to & H^1(\Gamma, pA_\gamma) & \to & H^1(\Gamma, A_\gamma) \\ & & {\scriptstyle \mathrm{ver}_\gamma/p}\uparrow & & \uparrow{\scriptstyle \mathrm{tr}_{A_\gamma/p}} & & & & \\ & & (A_\gamma/p)_\Gamma & = & (A_\gamma/p)_\Gamma & & & & \end{array}$$

in which the map ι is induced by the inclusion $A^\Gamma_\gamma \subseteq A_\gamma$.

In particular, if Leopoldt's Conjecture is valid for L, then ver_γ/p is surjective and $\hat{H}^0(\Gamma, A_\gamma[p])$ is isomorphic to $\mathrm{cok}(\mathrm{tr}_{A_\gamma/p})$.

Proof. Commutativity of the square is clear and exactness of the upper row follows by an easy exercise using the long exact sequences of Γ-cohomology associated to the following short exact sequences

$$0 \to A_\gamma[p] \to A_\gamma \xrightarrow{\times p} pA_\gamma \to 0, \quad 0 \to pA_\gamma \to A_\gamma \to A_\gamma/p \to 0.$$

Next, if Leopoldt's Conjecture is valid, then A_γ is a cohomologically-trivial Γ-module (by Theorem 5.1(ii)) and so the exact cohomology sequence of the first displayed sequence above implies that $H^1(\Gamma, pA_\gamma)$ is isomorphic to $\hat{H}^0(\Gamma, A_\gamma[p])$. Since ver_γ/p is surjective in this case the given exact commutative diagram (together with the fact that $H^1(\Gamma, A_\gamma)$ vanishes) then gives an isomorphism of $\hat{H}^0(\Gamma, A_\gamma[p])$ with $\mathrm{cok}(\mathrm{tr}_{A_\gamma/p})$. □

Lemma 5.9 suggests that it should be possible, at least under certain hypotheses, to provide evidence for the surjectivity or otherwise of ver_γ/p (and hence, following Theorem 5.1, for the validity or otherwise of Leopoldt's Conjecture for L) by analysing the action of Γ on the quotient $A_\gamma/p = A/N_\gamma$ of the Galois group over L_1 of the maximal elementary abelian p-extension L_1^{elem} of L_1 that is unramified outside p. For example, if $H^1(\Gamma, A_\gamma[p])$ vanishes (and hence, a fortiori, if A_γ is torsion-free), then it implies that the surjectivity of ver_γ/p would follow from that of the map $\mathrm{tr}_{A_\gamma/p}$ associated to the finite extension L_1^{elem}/L_1.

Conversely, if Leopoldt's Conjecture is valid for L, then the final assertion of Lemma 5.9 implies that analysis of the map $\mathrm{tr}_{A_\gamma/p}$ can be used to gain structural information about the torsion subgroups of both A_γ and A.

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