The cover time of the preferential attachment graph

Colin Cooper* Alan Frieze†

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Abstract

The preferential attachment graph $G_m(n)$ is a random graph formed by adding a new vertex at each time step, with m edges which point to vertices selected at random with probability proportional to their degree. Thus at time n there are n vertices and mn edges. This process yields a graph which has been proposed as a simple model of the world wide web [2]. In this paper we show that if $m \geq 2$ then **whp** the cover time of a simple random walk on $G_m(n)$ is asymptotic to $\frac{2m}{m-1} n \log n$.

1 Introduction

Let $G = (V, E)$ be a connected graph, let $|V| = n$, and $|E| = m$. A *random walk* $\mathcal{W}_u$, $u \in V$ on the undirected graph $G = (V, E)$ is a Markov chain $X_0 = u, X_1, \ldots, X_t, \ldots \in V$ associated to a particle that moves from vertex to vertex according to the following rule: the probability of a transition from vertex i, of degree $d(i)$, to vertex j is $1/d(i)$ if $\{i, j\} \in E$, and 0 otherwise. For $u \in V$ let C_u be the expected time taken for $\mathcal{W}_u$ to visit every vertex of G. The *cover time* C_G of G is defined as $C_G = \max_{u \in V} C_u$. The cover time of connected graphs has been extensively studied. It is a classic result of Aleliunas, Karp, Lipton, Lovász and Rackoff [1] that $C_G \leq 2m(n-1)$. It was shown by Feige [8], [9], that for any connected graph G

$$(1 - o(1)) n \log n \leq C_G \leq (1 + o(1)) \frac{4}{27} n^3.$$

The lower bound is achieved by (for example) the complete graph K_n, whose cover time is determined by the Coupon Collector problem.

*Department of Computer Science, King's College, University of London, London WC2R 2LS, UK
†Department of Mathematical Sciences, Carnegie Mellon University, Pittsburgh PA15213. Supported in part by NSF grant CCR-0200945.

In a previous paper [6] we studied the cover time of random graphs $G_{n,p}$ when $np = c\log n$ where $c = O(1)$ and $(c-1)\log n \to \infty$. This extended a result of Jonasson, who proved in [12] that when the expected average degree $(n-1)p$ grows faster than $\log n$, **whp** a random graph has the same cover time (asymptoticaly) as the complete graph K_n, whereas, when $np = \Omega(\log n)$ this is not the case.

Theorem 1. *[6] Suppose that $np = c\log n = \log n + \omega$ where $\omega = (c-1)\log n \to \infty$ and $c \geq 1$. If $G \in G_{n,p}$, then* **whp**[1]

$$C_G \sim c\log\left(\frac{c}{c-1}\right) n\log n.$$

The notation $A_n \sim B_n$ means that $\lim_{n\to\infty} A_n/B_n = 1$.

In another paper [7] we used a different technique to study the cover time of random regular graphs. We proved the following:

Theorem 2. *Let $r \geq 3$ be constant. Let $\mathcal{G}_r$ denote the set of r-regular graphs with vertex set $V = \{1, 2, \ldots, n\}$. If G is chosen randomly from $\mathcal{G}_r$, then* **whp**

$$C_G \sim \frac{r-1}{r-2}\, n\log n.$$

In this paper we turn our attention to the preferential attachment graph $G_m(n)$ introduced by Barabási and Albert [2] as a simplified model of the WWW. The preferential attachment graph $G_m(n)$ is a random graph formed by adding a new vertex at each time step, with m edges which point to vertices selected at random with probability proportional to their degree. Thus at time n there are n vertices and mn edges. We use the generative model of [3] (see also [4]) and build a graph sequentially as follows:

- At each time step t, we add a vertex v_t, and we add an edge from v_t to some vertex u, where u is chosen at random according to the distribution:

$$\mathbf{Pr}(u = v_i) = \begin{cases} \frac{d_{t-1}(v_i)}{2t-1}, & \text{if } v_i \neq v_t; \\ \frac{1}{2t-1}, & \text{if } v_i = v_t; \end{cases}$$

  where $d_{t-1}(v)$ denotes the degree of vertex v at the end of time step $t-1$.

- For some constant m, every m steps we contract the most recently added m vertices to form a single vertex.

Let $G_m(n)$ denote the random graph at time step mn after n contractions of size m. Thus $G_m(n)$ has n vertices and mn edges and may be a multi-graph. It should be noted that without the vertex contractions, we will be generating $G_1(n)$.

[1]A sequence of events $\mathcal{E}_n$ occurs *with high probability* **whp** if $\lim_{n\to\infty} \mathbf{Pr}(\mathcal{E}_n) = 1$.

This is a very nice clean model, but we warn the reader that it allows loops and multiple edges, although **whp** there will be relatively few of them.

We prove

Theorem 3. *If $m \geq 2$ then* **whp** *the preferential attachment graph $G = G_m(n)$ satisfies*

$$C_G \sim \frac{2m}{m-1}\, n \log n.$$

2 The first visit time lemma.

3 The random graph $G_m(n)$

In this section we prove some properties of $G_m(n)$. We first derive crude bounds on degrees. We write $d(j)$ in place of $d_n(j)$.

Lemma 4.
(a)

$$\mathbf{Pr}(\exists k, n:\ d_n(k) \geq (n/k)^{1/2}(\log n)^2) = O(n^{-3}).$$

(b)

$$\mathbf{Pr}(\exists k \leq n^{1/8}:\ d_n(k) \leq n^{1/4}) = O(n^{-1/4}).$$

Proof We consider the model $G_1(N)$, $N = mn$. As discussed in [4], $d_n(s)$ has the same distribution as $d_N(m(s-1)+1) + \cdots + d_N(ms)$. We extract what we need from the results of this paper.

Let $D_k \geq 2k$ be the sum of the degrees of the vertices in $\{v_1, ..., v_k\}$. The following is a slight extension of (3) of [4]: Assume $A \geq 1, k \geq 1$, then

$$\mathbf{Pr}(|D_k - 2\sqrt{kN}| \geq 3A\sqrt{N \log N}) \leq n^{-2A}. \tag{1}$$

We will also need (4) from the same paper: Assume $0 \leq d \leq N - k - s$, then

$$\mathbf{Pr}(d_N(k+1) = d+1 \mid D_k - 2k = s) = (s+d)2^d \frac{(N-k-s)_d}{(2N-2k-s)_{d+1}}. \tag{2}$$

(a): We bound the RHS of (2) from above and simplify to

$$\mathbf{Pr}(d_N(k+1) = d+1 \mid D_k - 2k = s) \leq \exp\left\{-\frac{d(s+(d-1)/2)}{2N}\right\} \tag{3}$$

For $1 \leq k < 100 \log N$ we take $s = 0$ in (3) and write

$$\mathbf{Pr}(d_N(k) \geq 40(N/k)^{1/2} \log N) \leq N \exp\left\{-\frac{800(\log N)^2}{k}\right\} \leq N^{-8}. \tag{4}$$

When $100 \log N \leq k \leq k_0 = N/\log N$ we use (1) with $A = 3$ to argue that

$$\mathbf{Pr}(D_k - 2k \leq 11\sqrt{kN}/10) \leq N^{-6}.$$

Thus from (3) we deduce that

$$\mathbf{Pr}(d_N(k) \geq 10\sqrt{N/k}\log N) \leq N^{-6} + N\exp\left\{-\frac{(10\sqrt{N/k}\log N)(11\sqrt{kN}/10)}{2N}\right\} \leq 2N^{-5}. \tag{5}$$

When $k > k_0$ we use the fact that $d_N(k)$ is stochastically dominated by $d_{2N\log N}(k)$ and so (a) follows from (4), (5) and the relationship between $G_m(n)$ and $G_1(N)$.

(b): Using (1) with $A = 2$ we have

$$\mathbf{Pr}(D_k - 2k \geq 8\sqrt{kN\log N}) \leq N^{-4}$$

where $k \leq mN^{1/8}$.

We then use (2) to write

$$\mathbf{Pr}(d_N(k) \leq N^{1/4}) \leq N^{-4} + \sum_{d=0}^{N^{1/4}} \frac{d + 8\sqrt{kN\log N}}{2N - 2k - 8\sqrt{kN\log N} - d} = O(N^{-1/2}).$$

Multiplying the RHS of the above inequality by $mN^{1/8}$ accounts for the possible values of k and completes the proofof the lemma. □

Let

$$\omega = (\log n)^{1/3}. \tag{6}$$

Let a cycle C be small if $|C| \leq 2\omega + 1$. Let a vertex v be *locally-tree-like* if the sub-graph G_v induced by the vertices at distance 2ω or less is a tree. Thus a locally-tree-like vertex is at distance at least 2ω from any small cycle.

Lemma 5. Whp *$G_m(n)$ does not contain a set of vertices S such that (i) $|S| \leq 100\omega$, (ii) the sub-graph H induced by S has minimum degree at least 2 and (iii) H contains a vertex $v \geq n^{1/10}$ of degree at least 3 in H.*

Proof Let Z_1 denote the number of sets described in the lemma. Then

$$\begin{aligned}
\mathbf{E}(Z_1) &\leq o(1) + \sum_{3\leq s=|S|\leq 100\omega} \sum_{H} \prod_{(v,w)\in E(H)} \frac{(\log n)^2}{(vw)^{1/2}} \qquad (7)\\
&\leq o(1) + \sum_{3\leq s=|S|\leq 100\omega} \sum_{H} (\log n)^{3|E(H)|} \prod_{v\in S} v^{-d_H(v)/2}\\
&\leq o(1) + \sum_{3\leq s=|S|\leq 100\omega} (1+(\log n)^2)^{\binom{s}{2}} n^{-1/20} \prod_{v\in S} \frac{1}{v}\\
&\leq o(1) + \sum_{3\leq s=|S|\leq 100\omega} (1+(\log n)^2)^{\binom{s}{2}} n^{-1/20} \left(\sum_{v=1}^{n} \frac{1}{v}\right)^s\\
&\leq o(1) + 100\omega(\log n)^{10001(\log n)^{2/3}} n^{-1/20}\\
&= o(1).
\end{aligned}$$

Explanation of (7)**:** Suppose that $1 \leq \alpha < \beta \leq n$. Then $\mathbf{Pr}(G_m(n)$ contains edge $(\alpha,\beta))$ is at most $\frac{(\log n)^2}{(\alpha\beta)^{1/2}}$. This is because when β chooses its neighbours, the probability it chooses α is at most $\frac{m(\log n)^2(\beta/\alpha)^{1/2}}{2m(\beta-1)}$. Here the numerator is a bound on the degree of α in $G_m(\beta-1)$. We are using Lemma 4 here and the $o(1)$ term accounts for the failure of this bound. Furthermore, this remains an upper bound if we condition on the existence of some of the other edges of H. □

This lemma is used to justify the following corollary: A small cycle is *light* if it contains no vertex $v \leq n^{1/10}$ (it has no "heavy" vertices), otherwise it is *heavy*.

Corollary 6. Whp $G_m(n)$ *does not contain a small cycle within* 10ω *of a light cycle.*

□

We need to deal with the possibility that $G_m(n)$ contains many cycles.

Lemma 7. Whp $G_m(n)$ *contains at most* $(\log n)^{10\omega}$ *vertices on small cycles.*

Proof Let Z be the number of vertices on small cycles in $G_m(n)$ (including parallel edges). Then

$$\mathbf{E}(Z) \leq \sum_{k=2}^{2\omega+1} k \sum_{a_1,\ldots,a_k} \prod_{i=1}^{k} \frac{(\log n)^2}{(a_i a_{i+1})^{1/2}} = O((\log n)^{9\omega})$$

and the result follows from the Markov inequality. □

We estimate the number of non-locally-tree-like vertices.

Lemma 8. Whp *there are at most* $O(n^{1/2+o(1)})$ *non-locally-tree-like vertices.*

Proof A non-locally-tree-like vertex is within ω of a small cycle. So the expectation of the number Z of such vertices satisfies

$$\begin{aligned}\mathbf{E}\,(Z) &\leq o(1)+\sum_{\substack{0\leq r\leq\omega\\ 3\leq s\leq 2\omega+1\\ 1\leq i\leq s}}\sum_{\substack{a_0,\dots,a_r\\ b_1\dots,b_s}}\frac{(\log n)^3}{(a_0b_i)^{1/2}}\prod_{k=1}^{r-1}\frac{(\log n)^3}{(a_ka_{k+1})^{1/2}}\prod_{l=1}^{s}\frac{(\log n)^2}{(b_lb_{l+1})^{1/2}}\\ &= O(n^{1/2+o(1)}).\end{aligned}$$

The result follows from the Markov inequality. □

Lemma 9. Whp *there are at most* $n(\log n)^{-\omega}$ *vertices* $v \geq n/2$ *which have more than* $(\log n)^{11\omega}$ *vertices at distance* 3ω *or less from them.*

Proof For a fixed vertex v, the expected number of paths of length $\leq 3\omega$ and endpoint v is bounded by

$$\sum_{1\leq r\leq 3\omega}\sum_{a_1,\dots,a_r}\frac{(\log n)^3}{a_r^{1/2}v^{1/2}}\prod_{k=1}^{r-1}\frac{(\log n)^3}{(a_ka_{k+1})^{1/2}}\leq(\log n)^{10\omega}.$$

The result now follows from the Markov inequality. □

Let

$$\omega_0=\log\log\log n. \tag{8}$$

We say that v is *locally regular* if it is locally tree-like and the first $2\omega_0$ levels of G_v form a tree of depth $2\omega_0$, rooted at v, in which every non-leaf has branching factor m.

For $j\in[n]$ we let $X(j)$ denote the set of neighbours of j in $[j-1]$ i.e. the vertices "chosen" by j (although not including j, loops are allowed in this construction).

Lemma 10. Whp, $G_m(n)$ *contains at least* $ne^{-e^{O(\omega_0)}}=n^{1-o(1)}$ *locally regular vertices* $v\geq n/2$.

Proof Let $I_k=\left[n\left(1-\frac{1}{2^k}\right),n\left(1-\frac{1}{2^{k+1}}\right)\right)$ for $1\leq k\leq\omega_0$. Let

$$J_2=\{j\in I_2:\ X(j)\subseteq I_1,\ |X(j)|=m,\ X(j')\cap X(j)=\emptyset,\ \text{for } j\neq j'\}.$$

($|X(j)=m|$ so that there are no parallel edges emanating from j.)

Then for $2<k\leq\omega_0$ we let

$$J_k=\{j\in I_k:\ X(j)\subseteq J_{k-1},\ |X(j)|=m,\ X(j')\cap X(j)=\emptyset,\ \text{for } j\neq j'\}.$$

For $j \in I_2$, $i_{m+1} = j-1$,

$$
\begin{aligned}
&\mathbf{Pr}(j \in J_2) = \\
&\sum_{\{i_1<\cdots<i_m\}\subseteq I_1} \prod_{k=1}^{m} \prod_{\tau=mi_k+1}^{mi_{k+1}} \left(1-\frac{km}{2\tau-1}\right) \cdot m! \prod_{i=1}^{m} \frac{m}{2mj+2i-1} \cdot \prod_{\tau=mj+1}^{mn} \left(1-\frac{m^2}{2\tau-1}\right) \\
&\sim \sum_{\{i_1<\cdots<i_m\}\subseteq I_1} \left(\prod_{k=1}^{m} \frac{i_k}{j}\right)^{m/2} \cdot \frac{m!}{(2j)^m} \cdot \frac{j^{m^2}}{n^{m^2}} \\
&\sim \left(\frac{j^{m-3/2}}{2n^m}\right)^m m! \sum_{\{i_1<\cdots<i_m\}\subseteq I_1} \prod_{k=1}^{m} i_k^{1/2} \\
&\sim \left(\frac{j^{m-3/2}}{2n^m}\right)^m \left(\sum_{i\in I_1} i^{1/2}\right)^m \\
&\sim \frac{j^{m^2-3m/2}}{n^{m^2}} \left(\frac{\sqrt{3}-\sqrt{2}}{3}\right)^m .
\end{aligned}
$$

So we can write, for some A_m, B_m which depend only on m,

$$\mathbf{E}\left(|J_2|\right) \sim \frac{A_m}{n^{m^2}} \sum_{j=3n/4+1}^{7n/8} j^{m^2} \sim B_m n.$$

We argue next that $|J_2|$ is concentrated around its mean. Let $Y_1, Y_2, \ldots, Y_{mn}$ denote the sequence of single choices of edges added. Here when vertex i is choosing one of its m neighbours, we consider each edge $\{u,v\}$ of $G_m(i-1)$ as being 2 directed arcs (u,v) and (v,u). When choosing a neighbour, i chooses (x,y) randomly from the $2m(i-1)$ arcs available and adds the edge $\{i,y\}$. In this way, each vertex is chosen proportional to its degree in $G_m(i-1)$. We let

$$Z_i = \mathbf{E}\left(|J_2| \mid Y_1, Y_2, \ldots, Y_t\right) - \mathbf{E}\left(|J_2| \mid Y_1, Y_2, \ldots, Y_{t-1}\right)$$

and prove that

$$|Z_i| \leq 4. \tag{9}$$

The Azuma-Hoeffding martingale inequality then implies that

$$\mathbf{Pr}(||J_2| - \mathbf{E}\left(|J_2|\right)| \geq u) \leq \exp\left\{-\frac{u^2}{8mn}\right\}. \tag{10}$$

It follows that **qs**[2]

$$||J_2| - \mathbf{E}\left(|J_2|\right)| \leq n^{1/2}\log n. \tag{11}$$

[2]A sequence of events $\mathcal{E}_n$ occurs *quite surely* (**qs**) if $\mathbf{Pr}(\mathcal{E}_n) = 1 - O(n^{-K})$ for any constant $K > 0$.

Fix $Y_1, Y_2, \ldots, Y_i$ and let $Y_i = (x, v)$, $\hat{Y}_i = (\hat{x}, \hat{v})$. Of course $\hat{x} = x$ if m does not divide i. Then for each complete outcome $\mathbf{Y} = Y_1, Y_2, \ldots, Y_T$ we define a corresponding outcome $\hat{\mathbf{Y}} = Y_1, Y_2, \ldots, Y_{i-1}, \hat{Y}_i, \ldots, \hat{Y}_T$ where for $j > i$, $\hat{Y}_j$ is obtained from Y_j as follows: If Y_j creates a new edge (w, v) by randomly choosing edge (x, v) arising from Y_i, then in $\hat{Y}_j$, (w, v) is replaced by $(w, \hat{v})$, otherwise $\hat{Y}_j = Y_j$.

The map $\mathbf{Y} \to \hat{\mathbf{Y}}$ is measure preserving and in going from $\mathbf{Y}$ to $\hat{\mathbf{Y}}$ $|J_2|$ changes by at most 4. (9) follows.

Repeating the argument for $\mathbf{Pr}(j \in J_2)$ we see that for $j \in I_3$ and some $B_m < 1$,

$$\mathbf{Pr}(j \in J_3 \mid J_2) \sim \left(\frac{4j^{m-1}}{7n^{m+1}}\right)^m \left(\sum_{i \in J_2} i\right)^m \geq B_m \frac{|J_2|^{2m+1}}{n^{2m}} \sim B_m A_m^{2m+1} n$$

and given J_2 of size $\sim A_m n$, $|J_3|$ will be concentrated around its mean.

Proceeding in this way we find that for $2 \leq k < \omega_0$ we have **qs**

$$|J_{k+1}| \geq C_k \frac{|J_k|^{2m+1}}{n^{2m}}$$

where we can choose $C_k < 1$, depending only on m.

From this we get

$$|J_{\omega_0}| \geq n \prod_{k=0}^{\omega_0 - 1} C_{\omega_0 - k}^{(2m+1)^k} \cdot A_m^{(2m+1)^{\omega_0 - 1}} = n^{1-o(1)}.$$

By construction, any locally tree-like vertex of J_{ω_0} is locally regular. The lemma follows from the bound on on the number of non locally tree-like vertices in Lemma 8. □

3.1 Mixing time

The *conductance* Φ of the walk $\mathcal{W}_u$ is defined by

$$\Phi = \min_{\pi(S) \leq 1/2} \frac{e(S : \overline{S})}{d(S)}.$$

Mihail, Papadimitriou and Saberi [13] proved that the *conductance* Φ of the walks $\mathcal{W}$ are bounded below by some absolute constant. Now it follows from Jerrum and Sinclair [10] that

$$|P_u^{(t)}(x) - \pi_x| \leq (\pi_x / \pi_u)^{1/2} (1 - \Phi^2/2)^t. \tag{12}$$

For sufficiently large t, the RHS above will be $O(n^{-10})$ at τ_0. We remark that there is a technical point here. The result of [10] assumes that the walk is *lazy*, and only makes a move to a neighbour with probability 1/2 at any step. This halves the conductance but we still have

$$T = O(\log n) \tag{13}$$

in (??). It doubles the covertime. It also asymptotically doubles the values R_v. Otherwise, it has a negligible effect on the analysis and we will ignore this for the rest of the paper and continue as though there are no lazy steps.

Notice that Lemma 4 implies $\pi_v = O((\log n)^2 n^{-1/2})$ and so together with (13) we see that

$$T\pi_v = o(1) \tag{14}$$

for all $v \in V$, as required by Lemma ??.

4 Cover time of $G_m(n)$

4.1 Parameters

Recall that the values of ω, ω_0 are given by (6), (8) respectively.

Assume now that $G_m(n)$ (i) has $n^{1-o(1)}$ locally regular vertices, (ii) $d(s) \geq n^{1/4}$ for $s \leq n^{1/10}$, (iii) no small cycle within distance 10ω of a light cycle, (iv) at most $(\log n)^{10\omega}$ vertices on small cycles and (v) at most $n(\log n)^{-\omega}$ vertices $v \geq n/2$ which have more than $(\log n)^{11\omega}$ vertices at distance 3ω or less from them.

Consider first a locally regular vertex v. It was shown in [7] (Lemma 6) that $R_v = \frac{r-1}{r-2} + o(\omega^{-1})$ for a locally-tree-like vertex w of an r-regular graph. We obtain the same result for v by putting $r = m+1$. Note that the degree of v is irrelevant here. It is the branching factor of the rest of the tree G_v that matters.

Lemma 11. *Suppose that v is locally-tree-like. Then*

(a) $R_v \leq \frac{d(v)}{m-1} + o(1)$.

(b) $d(v) \geq m+1$ *implies* $R_v \leq \frac{d(v)(m+m^{-1}-1)}{d(v)(m+m^{-1}-2)-m^{-1}+1} + o(1)$

(c) *If v is locally regular then* $R_v = \frac{m}{m-1} + o(1)$.

Proof We first define an infinite tree T_v^* by taking the tree T_v' defined by the first $\omega + 1$ levels of G_v and then rooting a copy of the infinite tree T_m^∞ which has branching factor m from each leaf of T_v'. This construction is modified in the case that v is locally regular. We now let T_v' be made up from the first ω_0 levels. Thus if v is locally regular, T_v itself is an infinite tree with branching factor m, rooted at v.

Let R_v^* be the expected number of visits to v for an infinite random walk $\mathcal{W}_v^*$ on T_v^*, started at v. We argue first that

$$|R_v - R_v^*| = o(1). \tag{15}$$

Let $r_t^* = \mathbf{Pr}(\mathcal{W}_v^*(t) = v)$. Then

$$\begin{aligned} |R_v - R_v^*| &\leq \sum_{t=\omega+1}^{T} r_t + \sum_{t=\omega+1}^{\infty} r_t^* \\ &\leq o(1) + \sum_{t=\omega+1}^{\infty} e^{-\alpha t} \qquad \text{for some constant } \alpha > 0 \qquad (16) \\ &= o(1). \end{aligned}$$

(When v is locally regular, the sums are from $\omega_0 + 1$.)

Explanation of (16): The term $o(1)$ comes from (12), replace r_t by $\pi_v + O(\zeta^t)$ for some constant $\zeta < 1$. For the second term we project the walk $\mathcal{W}_v^*$ onto $\{0, 1, 2, \ldots, \}$ by letting $\mathcal{X}(t)$ be the distance of $\mathcal{W}_v^*(t)$ from v. The degree of every vertex in T_v^* is at least m and if a vertex has degree exactly m then its immediate descendants have degree at least $m+1$ and so we see that for any positve $\lambda < 1/2$ and $t \geq 0$ we have

$$\begin{aligned} \mathbf{E}\left(e^{-\lambda(\mathcal{X}(2t+2)-\mathcal{X}(2t))} \mid \mathcal{X}(2t)\right) &\leq \frac{1}{3}e^{-2\lambda} + \frac{1}{2} + \frac{1}{6}e^{2\lambda} \qquad (17) \\ &\leq \frac{1}{3}(1 - 2\lambda + 4\lambda^2) + \frac{1}{2} + \frac{1}{6}(1 + 2\lambda + 4\lambda^2) \\ &\leq e^{-\lambda(1-6\lambda)/3}. \qquad (18) \end{aligned}$$

We take $\lambda = 1/12$ and $\alpha = \lambda(1-6\lambda)/3 = 1/72$.

Explanation of (17) We assume that $m = 2$, since the "drift" away from v is greater as m grows. If $\mathcal{W}_v^*(t) = w$ and the degree of w is 2 then all of w's neighbours in T_v^* have degree at least 3. The random variable $\mathcal{X}(t+2) - \mathcal{X}(2)$ is stochastically minimised by assuming that all neighbours of w have degree exactly 3. The expression in (17) gives the expected change in $e^{-\lambda(\mathcal{X}(2t+2)-\mathcal{X}(2t))}$ in this case. When the degree of w is 3, we reduce to the case where al neighbours of w have degree 2.

It follows from (18) that

$$\begin{aligned} \mathbf{E}\left(e^{-\lambda\mathcal{X}(2t)}\right) &= \mathbf{E}\left(\prod_{\tau=0}^{t-1} e^{-\lambda(\mathcal{X}(2\tau+2)-\mathcal{X}(2\tau))}\right) \\ &= \mathbf{E}\left(\mathbf{E}\left(e^{-\lambda(\mathcal{X}(2t)-\mathcal{X}(2t-2))} \mid \mathcal{X}(2t-2)\right) \prod_{\tau=0}^{t-2} e^{-\lambda(\mathcal{X}(2\tau+2)-\mathcal{X}(2\tau))}\right) \\ &\leq e^{-\alpha}\mathbf{E}\left(\prod_{\tau=0}^{t-2} e^{-\lambda(\mathcal{X}(2\tau+2)-\mathcal{X}(2\tau))}\right) \\ &\leq e^{-\alpha t}. \end{aligned}$$

Thus

$$r_{2t}^* = \mathbf{Pr}(\mathcal{X}(2t) = 0) = \mathbf{Pr}(e^{-\mathcal{X}(2t)} \geq 1) \leq \mathbf{E}\left(e^{-\mathcal{X}(2t)}\right) \leq e^{-\alpha t}$$

and (16) follows.

Let b_w, $w \in T_v^*$ be the branching factor at w i.e. $b_v = d_v$ and $b_w = d_w - 1$ if w is not the root. Further, if w is in the first ω levels let $b_w = b_w^+ + b_w^-$ where b_w^+ is the number of descendants w' of w with $w > w'$ i.e. w chose w' in the construction of $G_m(n)$. If w is at a higher level, we take $b_w = b_w^+ = m$ and $b_w^- = 0$.

Let $\widehat{T}_w$ be the sub-tree of T_v^* rooted at vertex w. (Thus $\widehat{T}_v = T_v^*$). Let ρ_w denote the probability that a random walk on $\widehat{T}_w$ which starts at w ever returns to w. Our aim is to estimate ρ_v and use

$$R_v^* = \frac{1}{1-\rho_v}. \tag{19}$$

Let $C(w)$ denote the children of w in T_v^*. We use the following recurrence: The parameter k counts the number of returns to x.

$$\begin{aligned}
\rho_w &= 1 - \frac{1}{b_w} \sum_{x \in C(w)} \sum_{k \geq 0} \left(1 - \frac{1}{d_x}\right) \left(\rho_x \left(1 - \frac{1}{d_x}\right)\right)^k (1 - \rho_x) \\
&= 1 - \frac{1}{b_w} \sum_{x \in C(w)} \frac{\left(1 - \frac{1}{d_x}\right)(1 - \rho_x)}{1 - \rho_x \left(1 - \frac{1}{d_x}\right)} \\
&= \frac{1}{b_w} \sum_{x \in C(w)} \frac{1}{b_x + 1 - \rho_x b_x}. \qquad (20)
\end{aligned}$$

We see immediately that if T_v^* is a regular tree with branching factor $m \geq 2$ then, with $\rho_w = \rho$ for all w,

$$\rho = \frac{1}{m + 1 - \rho m} \text{ and hence } \rho = \frac{1}{m}$$

and this deals with the locally regular case.

Now define b_w^+ to be the number of children x of w with $x < w$. These are the children chosen by w. Let $b_w^- = b_w - b_w^+$.

We will now prove the following by induction on $\omega + 1 - \ell_w$, where $\ell_w \leq \omega + 1$ is the level of w in the tree.:

(a) $b_w = m - 1$ implies $\rho_w \leq \frac{1}{m}$.

(b) $b_w^+ = m, b_w^- \geq 1$ implies $\rho_w \leq \frac{1}{b_w}\left(1 + \frac{b_w - m}{m + m^{-1} - 1}\right)$.

(c) $b_w = b_w^+ = m$ implies $\rho_w \leq \frac{1}{m}$.

(d) $b_w^+ = m - 1, b_w^- \geq 1$ implies $\rho_w \leq \frac{1}{b_w}\left(\frac{m-1}{m} + \frac{b_w^-}{m + m^{-1} - 1}\right)$

The base case will be $\ell_w = \omega + 1$. For which, Case (c) applies and the induction hypothesis holds from the locally regular case. (The solution $\rho = 1$, which implies $R_v^* = \infty$ is ruled out by (16) which implies $R_v^* < \infty$).

The lemma follows from this since only cases (b),(c) can apply to the root v, in which case $b_v = d(v)$.

Let us now go through the inductive step. Let us assume these conditions apply to $x \in C(w)$ and then we find that in these cases:

(a) $b_x + 1 - b_x\rho_x \geq m + \frac{1}{m} - 1$.

(b) $b_x + 1 - b_x\rho_x \geq m + (b_x - m)\left(1 - \frac{1}{m+m^{-1}-1}\right) \geq m$.

(c) $b_x + 1 - b_x\rho_x \geq m$.

(d) $b_x + 1 - b_x\rho_x \geq m + \frac{1}{m} - 1 + b_x^-\left(1 - \frac{1}{m+m^{-1}-1}\right) \geq m + \frac{1}{m} - 1$.

Case (a): In this case $b_w = b_w^+$ and only cases (b),(c) are possible for $x \in C(w)$. In which case $b_x + 1 - b_x\rho_x \geq m$ for $x \in C(w)$.

Case (b): In $C(w)$ we have $b_w^+ = m$ cases of (b) or (c) and b_w^- cases of (a) or (d). In the first case we have $b_x + 1 - b_x\rho_x \geq m$. In the second case we have $b_x + 1 - b_x\rho_x \geq m + m^{-1} - 1$. Thus

$$\rho_w \leq \frac{1}{b_w}\left(1 + \frac{b_w - m}{m + m^{-1} - 1}\right).$$

Case (c): This follows as in Case (a).

Case (d): In $C(w)$ we have $m - 1$ cases of (b) or (c) and b_w^- cases of (a) or (d). Thus

$$\rho_w \leq \frac{1}{b_w}\left(\frac{m-1}{m} + \frac{b_w^-}{m + m^{-1} - 1}\right)$$

as is to be shown. □

We deal with non-locally-tree like vertices in a somewhat piece-meal fashion.

Lemma 12. *Suppose that either*
(i) G_v contains a unique light cycle C_v, that $v \notin C_v$ and that the shortest path $P = (w_0 = v, w_1, \ldots, w_k)$ from v to C_v is such that $\max\{d(w_1), \ldots, d(w_k)\} \geq \omega^3$, or
(ii) that G_v contains only heavy cycles. Then

(a) $R_v \leq \frac{d(v)}{m-1} + o(1))$.

(b) $d(v) \geq m + 1$ *implies* $R_v \leq \frac{d(v)(m+m^{-1}-1)}{d(v)(m+m^{-1}-2)+m^{-1}-1} + o(1)$

Proof
(a) Let w be the first vertex on the path from v to C_v which has degree at least ω^3. Let G'_v be obtained from G_v by deleting those vertices, other than w, all of whose paths to v in G_v go through w.(By assumption there are one or two paths). Let R'_v be the expected number of returns to v in a random walk of length ω on G'_v where w is an absorbing state. We claim that

$$R_v \leq R'_v + O(\omega^{-2}). \tag{21}$$

Once we verify this, the proof of (a) follows from the proof of Lemma 11 i.e. embed the tree $H'v$ in an infinite tree by rooting a copy of T_m^∞ at each leaf. To verify (21) we couple random walks on G_v, G'_v until w is visited. In the latter the process stops. In the former, we find that when at w, the probability we get closer to v in the next step is at most ω^{-3} and so the expected number of returns from now on is at most $\omega \times \omega^{-3}$ and (21) follows.

(b) Now consider the case where G_v contains only heavy cycles. We argue first that a random walk of length ω that starts at v might as well terminate if it reaches a vertex $w \leq n^{1/10}$, $w \neq v$. We can assume $d(w) \geq n^{1/4}$. Now we can assume from Lemma 7 at least $n_0 = n^{1/4} - (\log n)^{10\omega}$ of the edges incident with w are not in cycles contained in G_v. But then if a walk arrives at w has a more than $\frac{n_0}{n^{1/4}}$ chance of entering a sub-tree T_w of G_v rooted at w for which every vertex is separated from v by w. But then the probability of leaving T_w in ω steps is $O(\omega(\log n)^{10\omega}/n^{1/4})$ and so once a walk has reached w, the expected number of further returns to v is $o(\omega^{-1})$. We can therefore remove T_w from G_v and then replace an edge (x, w) by an edge (x, w_x) and make all the vertices w_x absorbing. Repeating this argument, we are left with a tree to which we can apply the argument of Lemma 11. □

Note that if $v \in V_B$ then no bound on R_v has been established:

$$V_B = \{v : \ G_v \text{ contains a unique light cycle } C_v \text{ and the path from } v \text{ to } C_v \text{ contains no vertex of degree at least } \omega^3\}$$

However, for these it suffices to prove

Lemma 13. *If $v \in V_B$ then $R_v \leq 2\omega$.*

Proof We write, for some constant $\zeta < 1$,

$$\begin{aligned} R_v &= \sum_{t=1}^{\omega} r_t + \sum_{t=\omega+1}^{T} (\pi_v + O(\zeta^t)) \\ &\leq \omega + o(\omega) \end{aligned}$$

and the lemma follows. □

Lemma 14. *There exists a constant $0 < \theta < 1$ such that if $v \in V$ then $|R_T(s)| \geq \theta$ for $|s| \leq 1 + \lambda$.*

Proof Assume first that v is locally tree-like. We write

$$\begin{aligned} R_T(s) &= A(s) + Q(s) \\ &= \frac{1}{1 - B(s)} + Q(s). \end{aligned} \tag{22}$$

Here $A(s) = \sum a_t s^t$ where $a_t = r_t^*$ is the probability that the random walk $\mathcal{W}_v^*$ is at v at time t (see Lemma 11 for the definition of $\mathcal{W}_v^*$). $B(s) = \sum b_t s^t$ where b_t is the probability of a first return at time t. Then $Q(s) = Q_1(s) + Q_2(s)$ where

$$\begin{aligned} Q_1(s) &= \sum_{t=\omega+1}^{T} (r_t - a_t) s^t \\ Q_2(s) &= -\sum_{t=T+1}^{\infty} a_t s^t. \end{aligned}$$

Here we have used the fact that $a_t = r_t$ for $0 \leq t \leq \omega$.

We now justify equation (22). For this we need to show that

$$|B(s)| < 1 \qquad \text{for } |s| \leq 1 + \lambda. \tag{23}$$

We note first that, in the notation of Lemma 11, $B(1) = \rho_v < 1$. Then observe that $b_t \leq a_t \leq e^{-\alpha t}$. The latter inequality is proved in Lemma 11, see (16). Thus the radius of convergence ρ_B of $B(s)$ is at least e^{α}, $B(s)$ is continuous for $0 \leq |s| < \rho_B$, $|B(s)| \leq B(|s|)$ and $B(1) < 1$. Thus there exists a constant $\epsilon > 0$ such that $B(s) < 1$ for $|s| \leq 1 + \epsilon$. We can assume that $\lambda < \epsilon$ and (23) follows. We will use

$$|R_T(s)| \geq \frac{1}{1 + B(|s|)} - |Q(s)| \geq \frac{1}{1 + B(1+\lambda)} - |Q(s)| \geq \frac{1}{2} - |Q(s)|.$$

The lemma for locally tree-like vertices will follow once we show that $|Q(s)| = o(1)$. But

$$\begin{aligned} |Q_1(s)| &\leq (1+\lambda)^T \sum_{t=\omega+1}^{T} (\pi_v + e^{-\Phi^2 t/2} + e^{-\alpha t}) = o(1) \\ |Q_2(s)| &\leq \sum_{t=T+1}^{\infty} (e^{-\alpha}(1+\lambda))^t = o(1). \end{aligned}$$

For non tree-like vertices we proceed more or less as in Lemma 12. If $v \notin V_B$ then we truncate G_v at vertices of large degree, add copies of T_m at leaves and then proceed as above.

If $v \in V_B$ let T_v^* be the graph obtained by adding T_m^∞ to all the leaves of G_v. Thus T_v^* contains a unique cycle $C = (x_1, x_2, \ldots, x_k, x_1)$. We can write an expression equivalent to (22) and then argument rests on showing that $B(1) < 1$ and $a_s \leq \zeta^s$ for some $\zeta < 1$. The latter condition can be relaxed to $a_s \leq e^{o(s)} \zeta^s$, allowing us to take less care with small s.

$\boldsymbol{B(1) < 1}$: If $m \geq 3$ there is a $\geq 1 - \frac{2}{m}$ probability of the first move of $\mathcal{W}_v^*$ going into an infinite tree rooted at a neighbour of v and then the probability of return to v is bounded below by a positive constant. The same argument is valid for $m = 2$ when $v \notin C$. So assume that $v \in C$ and that T_v^* consists of C plus a tree T_i attached to x_i for $i = 1, 2, \ldots, k$. Here T_i is empty (if degree of x_i is 2) or infinite. Furthermore, T_i empty, implies that T_{i-1}, T_{i+1} are both infinite. Thus the walk $\mathcal{W}_v^*$ has a constant positive probability of moving into an infinite tree within 2 steps and then never returning to v.

$\boldsymbol{a_s \leq e^{o(s)}\zeta^s}$: If C is an even cycle then we can couple the distance X_t of $W_v^*(t)$ to v with a random walk on $\{0, 1, 2, \ldots, \}$ as we did in Lemma 11. If C is an odd cycle let w_1, w_2 be the vertices of C which are furthest from v in T_v^*. If $W_v^*(t) \neq w_1, w_2$ then $\mathbf{E}\,(X_{t+2} - X_t) \geq 1/6$ and otherwise $\mathbf{E}\,(X_{t+2} - X_t) \geq 0$. Thus $\mathbf{E}\,(X_{t+4} - X_t) \geq 1/6$ always and we can use Hoeffding's theorem. □

Lemma 15. *If $v \in V$ and its degree $d_n(v) \leq (\log n)^2$ then $H_v < CR_v + o(1)$ for some constant $C < 1$.*

Proof As in Section **??** let f_t be the probability that $\mathcal{W}_u$ has a first visit to v at time t. As $H(s) = F(s)R(s)$ we have

$$\begin{aligned} H_v &\leq \mathbf{Pr}(\mathcal{W}_u \text{ visits } v \text{ by time } T-1)R_v \\ &= R_v \sum_{t=1}^{T} f_t. \end{aligned}$$

We now estimate $\sum_{t=1}^{T} f_t$, the probability that $\mathcal{W}_u$ visits v by time T. We first observe that (12) implies

$$\sum_{t=\omega+1}^{T} f_t \leq \sum_{t=\omega+1}^{T} (((\log n)^2/m)^{1/2} e^{-\Phi^2 t/2} + \pi_v) = o(1).$$

Thus it suffices to bound $\sum_{t=1}^{\omega} f_t$, the probability that $\mathcal{W}_u$ visits v by time ω.

Let $v_1, v_2, \ldots, v_k$ be the neighbours of v and let w be the first neighbour of v visited by $\mathcal{W}_u$. Then

$$\begin{aligned} \mathbf{Pr}(\mathcal{W}_u \text{ visits } v \text{ by time } \omega) &= \sum_{i=1}^{k} \mathbf{Pr}(\mathcal{W}_u \text{ visits } v \text{ by time } \omega \mid w = v_i)\mathbf{Pr}(w = v_i) \\ &\leq \sum_{i=1}^{k} \mathbf{Pr}(\mathcal{W}_{v_i} \text{ visits } v \text{ by the time } \omega)\mathbf{Pr}(w = v_i). \end{aligned}$$

So it suffices to prove the lemma when u is a neighbour of v.

Let the neighbours of u be $u_1, u_2, \ldots, u_d$, $d \geq m$ and $v = u_d$. If u is locally tree-like than we can write

$$\mathbf{Pr}(\mathcal{W}_u \text{ does not visit } v \text{ by time } \omega) \geq \rho \frac{d-1}{d} - o(1) > 0. \tag{24}$$

Here ρ is a lower bound on the probability of not returning to u in ω steps, given that $\mathcal{W}_u(1) \neq v$. We have seen in the previous lemma that this is at least some positive constant.

If $u \notin V_B$ then we truncate H_u as we did in Lemma 12 and argue for (24).

If $u \in V_B$ and there exist neighbours $u_1, \ldots, u_k$ say, which are not on the unique cycle C_u of H_u then there is a probability k/d that $\mathcal{W}_u^*(1) = u_i$ for some $i \leq k$ and then the probability that $\mathcal{W}_u$ does not return to u_i in ω steps is bounded below by a constant. The final case is where $m = 2$, $d_n(u) = 2$ and u, u_1, v are part of the unique cycle of H_u. But then with probability 1/2 $\mathcal{W}_u(1) = u_1$ and then with conditional probability at least 1/3 $x = \mathcal{W}_u(2)$ is not on C_u and then the probability that $\mathcal{W}_u$ does not return to x in ω steps is bounded below by a constant. □

4.2 Upper bound on cover time

Let $t_0 = \lceil \frac{2m}{m-1} n \log n \rceil$. We prove that **whp**, for $G_m(n)$, for any vertex $u \in V$, $C_u \leq t_0 + o(t_0)$.

Let $T_G(u)$ be the time taken to visit every vertex of G by the random walk $\mathcal{W}_u$. Let U_t be the number of vertices of G which have not been visited by $\mathcal{W}_u$ at step t. We note the following:

$$C_u = \mathbf{E}\left(T_G(u)\right) = \sum_{t>0} \mathbf{Pr}(T_G(u) \geq t), \tag{25}$$

$$\mathbf{Pr}(T_G(u) \geq t) = \mathbf{Pr}(T_G(u) > t-1) = \mathbf{Pr}(U_{t-1} > 0) \leq \min\{1, \mathbf{E}\, U_{t-1}\}. \tag{26}$$

It follows from (25), (26) that for all t

$$C_u \leq t + 1 + \sum_{s \geq t} \mathbf{E}\left(U_s\right) = t + 1 + \sum_{v \in V} \sum_{s \geq t} \mathbf{Pr}(\boldsymbol{A}_s(v)). \tag{27}$$

For vertices v satisfying Corollary **??** we see that

$$\sum_{s \geq t} \mathbf{Pr}(\boldsymbol{A}_s(v)) \leq (1 + O(T\pi_v)) \frac{R_v}{\pi_v} e^{-(1+O(T\pi_v))t\pi_v/R_v} + O(\lambda^{-2} e^{-\lambda t/2}).$$

Recall that V_B is the set of vertices v such that G_v contains a unique light cycle C_v and the path from v to C_v contains no vertex of degree at least ω^3.

We write $V = V_1 \cup V_2 \cup V_3$ where $V_1 = (V \setminus V_B) \cap \{d_n(v) \leq (\log n)^2\}$, $V_2 = \{d_n(v) \geq (\log n)^2\}$ and $V_3 = V_B \cap \{d_n(v) \leq (\log n)^2\}$.

Let $t_1 = (1 + \epsilon)t_0$ where $\epsilon = n^{-1/3}$ can be assumed by Lemma 4 to satisfy $T\pi_v = o(\epsilon)$ for all $v \in V - V_2$

If $v \notin V_B$ then by Lemmas 11(a) and 12(a),

$$t_1(1+O(T\pi_v))\pi_v/R_v \geq \frac{2m}{m-1}n\log n \cdot \frac{d(v)}{2mn} \cdot \frac{m-1}{d(v)} = \log n. \tag{28}$$

Plugging (28) into (27) and using $R_v \leq 5$ (Lemmas 11 and 12) and $\pi_v \geq \frac{1}{2n}$ for all $v \in V \setminus V_B$ we get

$$\sum_{v \in V_1} \sum_{s \geq t_1} \mathbf{Pr}(\boldsymbol{A}_s(v)) \leq 10n. \tag{29}$$

Suppose now that $v \in V_2$ ie. $d_n(v) \geq (\log n)^2$. After a walk of length T there is an $\Omega((\log n)^2/n)$ chance of being at v. Thus for some constant $c > 0$ and $s \geq t_1$, we have

$$\mathbf{Pr}(\boldsymbol{A}_s(v)) \leq \left(1 - \frac{c(\log n)^2}{n}\right)^{\lfloor s/T \rfloor} \leq \exp\left\{-\frac{cs\log n}{2K_1 n}\right\}.$$

Thus

$$\begin{aligned} \sum_{v \in V_2} \sum_{s \geq t_1} \mathbf{Pr}(\boldsymbol{A}_s(v)) &\leq n \sum_{s \geq t_1} \exp\left\{-\frac{cs\log n}{2K_1 n}\right\} & (30) \\ &\leq \frac{3K_1 n^2}{c\log n} \exp\left\{-\frac{ct_1\log n}{2K_1 n}\right\} = o(1). & (31) \end{aligned}$$

It remains to deal with $v \in V_3$. We first observe that

$$|V_B| \leq (\log n)^{10\omega}\omega^{3\omega} \leq (\omega \log n)^{10\omega} \tag{32}$$

and from Lemmas 13, 15 we have

$$\begin{aligned} \sum_{v \in V_3} \sum_{s \geq t_1} \mathbf{Pr}(\boldsymbol{A}_s(v)) &\leq (\omega\log n)^{10\omega}\left(2n\omega e^{-(1+o(1))t_1\pi_v/(2\omega)} + O(\lambda^{-2}e^{-\lambda t_1/2})\right) & (33) \\ &= o(n). & (34) \end{aligned}$$

Thus combining (29) with (30), (31) and (33), (34) gives

$$C_u \leq t_1 + O(n) = t_0 + o(t_0),$$

completing our proof of the upper bound on cover time.

4.3 Lower bound on cover time

For some vertex u, we can find a set of vertices S such that at time $t_1 = t_0(1-\epsilon)$, $\epsilon \to 0$, the probability the set S is covered by the walk $\mathcal{W}_u$ tends to zero. Hence $T_G(u) > t_1$ **whp** which implies that $C_G \geq t_0 - o(t_0)$.

We construct S as follows. Let S be some maximal set of locally regular vertices all of which are at least distance $2\omega+1$ apart. Thus $|S| \geq ne^{-e^{O(\omega_0)}}(\log n)^{-11\omega} \geq n(\log n)^{-12\omega}$.

Let $S(t)$ denote the subset of S which has not been visited by $\mathcal{W}_u$ after step t. Now, provided $t \geq T$

$$\mathbf{E}\left(|S(t)|\right) \geq (1-o(1)) \sum_{v \in S} \left(\frac{c_{u,v}}{(1+p_v)^t} + o(n^{-2}) \right).$$

Let u be a fixed vertex of S. Let $v \in S$ and let $H_T(1)$ be given by (**??**), then (12) implies that

$$H_T(1) \leq \sum_{t=\omega}^{T-1} (\pi_v + e^{-\Phi^2 t/2}) = o(1). \tag{35}$$

Thus $c_{uv} = 1 - o(1)$. Setting $t = t_1 = (1-\epsilon)t_0$ where $\epsilon = 2\omega^{-1}$, we have

$$\begin{aligned} \mathbf{E}\left(|S(t_1)|\right) &= (1+o(1))|S|e^{-(1-\epsilon)t_0 p_v} \\ &\geq n^{1/\omega}. \end{aligned} \tag{36}$$

Let $Y_{v,t}$ be the indicator for the event that $\mathcal{W}_u$ has not visited vertex v at time t. Let $Z = \{v, w\} \subset S$. We will show (below) that that for $v, w \in S$

$$\mathbf{E}\left(Y_{v,t_1} Y_{w,t_1}\right) = \frac{c_{u,Z}}{(1+p_Z)^{t+2}} + o(n^{-2}), \tag{37}$$

where $c_{u,Z} \sim 1$ and $p_Z \sim (m-1)/(mn)$. Thus

$$\mathbf{E}\left(Y_{v,t_1} Y_{w,t_1}\right) = (1+o(1))\mathbf{E}\left(Y_{v,t_1}\right)\mathbf{E}\left(Y_{w,t_1}\right). \tag{38}$$

It follows from (36) and (38), that

$$\mathbf{Pr}(S(t_1) \neq 0) \geq \frac{\mathbf{E}\left(|S(t_1)|\right)^2}{\mathbf{E}\left(|S(t_1)|^2\right)} = \frac{1}{\frac{\mathbf{E}(|S_{t_1}|(|S_{t_1}|-1))}{\mathbf{E}(|S(t_1)|)^2} + \mathbf{E}\left(|S_{t_1}|\right)^{-1}} = 1 - o(1).$$

Proof of (37). Let Γ be obtained from G by merging v, w into a single node Z. This node has degree $2m$.

There is a natural measure preserving mapping from the set of walks in G which start at u and do not visit v or w, to the corresponding set of walks in Γ which do not visit Z. Thus the probability that $\mathcal{W}_u$ does not visit v or w in the first t steps is equal to the probability that a random walk $\widehat{\mathcal{W}}_u$ in Γ which also starts at u does not visit Z in the first steps.

We apply Lemma **??** to Γ. That $\pi_Z = \frac{1}{n}$ is clear, and $c_{u,Z} = 1 - o(1)$ is argued as in (35). The derivation of R_Z in Lemma 11 is also valid. The vertex Z is tree-like up to distance ω in Γ. The fact that the root vertex of the corresponding infinite tree has degree $2m$ does not affect the calculation of E_0. $\square$

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