Crawling on simple models of web graphs

Colin Cooper
Department of Computer Science,
King's College,
University of London,
London WC2R 2LS, UK
ccooper@dcs.kcl.ac.uk

Alan Frieze*
Department of Mathematical Sciences,
Carnegie Mellon University,
Pittsburgh PA15213.
alan@random.math.cmu.edu

January 17, 2003

1 Introduction

At the present moment there is considerable on going research into the structure of large-scale real networks, and in modeling these networks as the outcomes of discrete random processes. A general introduction to this topic can be found in Hayes [24] or Watts [29]. In particular, there is a strong interest in the structure of the Internet and World Wide Web (**www**). Experimental studies by, Albert, Barabasi and Jeong [5], Broder et al [13] and Faloutsos, Faloutsos and Faloutsos [22] of the structure of the **www** have demonstrated an inverse power law for the proportion of vertices with a given degree.

To model such structures, we require a graph process which (a) evolves randomly by the addition of new vertices and/or edges at each time step t and (b) whose proportional degree sequence follows a power law. Thus the proportion d_k of vertices of degree satisfies $d_k \sim Ck^{-x}$, as $k \to \infty$, where x is a constant. Such random graph process are often referred to as web graphs, or scale-free graphs.

We can also define graph processes, which we refer to as random graph processes, in which the proportional degree sequence is similar to the degree sequence of the traditional models of random graphs introduced by Erdős and Rényi [19], [20]. Thus, in a random graph process, the proportion of vertices of a given degree is Poisson distributed, and the degree sequence drops off exponentially fast in the upper tail.

There are many models of graph processes designed to capture various aspects of the structure of the **www** found in the studies given above. See for example [1], [2], [3], [4],[5], [6], [7], [8], [9], [10], [11], [14], [17], [15], [16], [18], [23], [25], [26] and [27]. A recent survey by Bollobás and Riordan [10] gives an excellent overview of the topic and many detailed structural results.

*Supported in part by NSF grant CCR-9818411.

We consider a simple model of search, in which a particle (which we call a spider) makes a random walk on the nodes of an undirected graph process. It is presumed that the spider examines the data content of the nodes for some specific topic. As the spider is walking the graph is growing, and the spider makes a random transition to whatever neighbours are available at the time. For simplicity, we assume that the growth rate of the process and the transition rate of the random walk are similar, so that the spider has at least a chance of crawling a constant proportion of the process.

We study the success of the spider's search on comparable graph processes of two distinct types: a random graph process and a web graph process. In the simple process we consider, each new vertex directs m edges towards existing vertices, either choosing vertices randomly (giving a random graph process) or copying according to vertex degree (giving a web graph process). Once a vertex has been added the direction of the edges is ignored.

Our results are given in Theorems 1, 2 below. Our main result is the following: For the random graph process the expected proportion of unvisited vertices tends to 0.57. For the comparable web graph process, the expected proportion of unvisited vertices tends to 0.59.

There are several types of search which are appropriate to the **www**. Complete searches of the web, usually in a breadth first manner, are carried out by search engines such as Google [12]. Link and page data for visited pages is stored, and from the link data an undirected model of the **www** can be constructed. This model may be entirely replaced when a new search is made at a future time period or may be continously updated by a continuously ongoing search. Such processes require considerable on-line and off-line memory.

Another possibility, using a single processor, is a search by an agent which examines the semantic content of nodes for some specific topic. This type of search can be made directly on the **www** or on the model of the **www** stored by a search engine. Typical search strategies might include: moving to a random neighbour (sampling pages for content), selecting a random neighbour of large degree (locating the hub/authority vertices of the search topic) or selecting a random neighbour of low degree (favouring the discovery of newer vertices during the search).

Although the edges of a typical **www** graph are directed, the idea of evaluating models of search on an undirected process has many attractions, not least its simplicity. Examples which support the use of an undirected model are as follows.

The Google search engine [12] holds a partial model of the **www** which it is continuously updating. Once a node is added to the search engine database, a list is maintained of pages in the database which point to this node. For a given node with url *PageUrl*, these links can be found by entering the **link:***PageUrl* query to Google. Thus the model of the **www** held by this search engine is equivalent to an undirected web graph. This information is used by Google to compute the page rank of a vertex. The page ranking is based on a weighted union of all directed paths leading to the vertex.

Finally we remark that unless the web graph is strongly connected, a random walk on the directed edges soon becomes stuck at a vertex of out-degree zero. Modifications of the random walk approach which do not suffer from this defect have been studied by [21].

We next give a precise definition of the abstract scenario we consider in this paper.

We have a sequence $(G(t), t = 1, 2, ...)$ of connected random graphs. The graph $G(t)$ is constructed from $G(t-1)$ by adding the vertex t, and m random edges from vertex t to $G(t-1)$. We refer to such graphs as web graphs.

There is also a spider S walking randomly from vertex to vertex on the evolving graph $G(t)$.

The parameter ν_t we estimate is the expected number of vertices which have not been visited by the

spider at step t, when t is large. This process is intended to model the success of a search-engine spider which is randomly crawling the world wide web looking for new web-pages.

We consider the following models for the graph process $G(t)$. Let $m \geq 1$ be a fixed integer. Let $[t] = \{1, ..., t\}$ and let $G(1) \subset G(2) \subset \cdots \subset G(t)$. Initially $G(1)$ consists of a single vertex 1 plus m loops. For $t \geq 2$, $G(t)$ is obtained from $G(t-1)$ by adding the vertex t and m randomly chosen edges $\{t, v_i\}, i = 1, 2, \ldots, m$, where

Model 1: The vertices $v_1, v_2, \ldots, v_m$ are chosen independently and uniformly with replacement from $[t-1]$.

Model 2 The vertices $v_1, v_2, \ldots, v_m$ are chosen proportional to their degree after step $t-1$. Thus if $d(v, \tau)$ denotes the degree of vertex v in $G(\tau)$ then for $v \in [t-1]$ and $i = 1, 2, \ldots, m$,

$$\mathbf{Pr}(v_i = v) = \frac{d(v, t-1)}{2m(t-1)}.$$

While vertex t is being added, the spider S is sitting at some vertex X_{t-1} of $G(t-1)$. After the addition of vertex t, and before the beginning of step $t+1$, the spider now makes a random walk of length ℓ, where ℓ is a fixed positive integer independent of t.

It seems unlikely that at time t, S will have visited every vertex. Let $\nu_{\ell,m}(t)$ denote the expected number of vertices not visited by S at the end of step t.

We will prove the following theorem:

Theorem 1. *In either model, if m is sufficiently large then, as $t \to \infty$,*

$$\nu_{\ell,m}(t) \sim \mathbf{E} \sum_{s=1}^{t} \prod_{\tau=s}^{t} \left(1 - \frac{d(s,\tau)}{2m\tau} \left(1 + O\left(\tfrac{1}{m}\right)\right)\right)^{\ell}. \tag{1}$$

□

We have said that m is fixed. We however have to accept errors of order $1/m$ and so in our asympotics we let $t \to \infty$ first and then take m large.

Let

$$\eta_{\ell,m} = \lim_{t\to\infty} \frac{\nu_{\ell,m}(t)}{t}.$$

We will show that this gives the following limiting results for the models we consider.

Theorem 2. *Let $\eta_\ell = \lim_{m\to\infty} \eta_{\ell,m}$, then*

(a) *For Model 1,*

$$\eta_\ell = \sqrt{\frac{2}{\ell}} e^{(\ell+2)^2/(4\ell)} \int_{(\ell+2)/\sqrt{2\ell}}^{\infty} e^{-y^2/2}\, dy.$$

In particular, $\eta_1 = 0.57\cdots$, and $\eta_\ell \sim 2/\ell$ as $\ell \to \infty$.

(b) *For Model 2*

$$\eta_\ell = e^\ell 2\ell^2 \int_{\ell}^{\infty} y^{-3} e^{-y}\, dy.$$

In particular, $\eta_1 = 0.59\cdots$, and $\eta_\ell \sim 2/\ell$ as $\ell \to \infty$.

□

Thus for large m, t and $\ell = 1$ it is slightly more difficult for the spider to crawl on a web-graph whose edges are generated by a copying process (Model 2) than on a uniform choice random graph (Model 1).

2 Proof of Theorem 1: The main ideas

We first consider the case where $\ell = 1$ and then generalise this case. When $\ell = 1$ the spider makes a random move to an adjacent vertex after vertex t has been added. The construction of $G(t)$ is really the construction of a digraph $D(t)$ where the direction of the arcs (x, y) satisfies $x > y$. The space $\mathcal{G}(t)$ of graphs $G(t)$ induces its measure from this space of digraphs.

Let $\Omega(t)$ denote the set of pairs $(G(t), W(t))$ where $G(t) \in \mathcal{G}(t)$ and $W(t)$ belongs to the set $\mathcal{W}_G(t)$ of t-step walks taken by the spider S which are compatible with the construction of G. Among other things, this means that the τ-th vertex of $G(t)$ visited by the walk must be in $[\tau]$.

The main idea of the proof is as follows. We fix a vertex s and estimate the probability that it is not visited by the end of step t. Thus for $s \le \tau \le t$ we define the events

$$\mathcal{A}_s(\tau) = \{\omega \in \Omega(t) : \text{ Vertex } s \text{ is not visited by } \mathsf{S} \text{ during the time interval } [s, \tau]\}.$$

Let

$$t_0 = t - 100(\ln t)^3.$$

It is convenient to condition on the sequence $d(s, \tau)$ for $\tau = s, s+1, \ldots, t_0$. Let $\boldsymbol{\theta} = (\theta_\tau : 1 \le \tau \le t_0)$ be integers satisfying

$$\theta_1 = \cdots = \theta_s = m \le \theta_\tau \le \theta_t \le \Delta_t^* = 10(\ln t)^5 \text{ and } \theta_{\tau+1} \le \theta_\tau + 5 \text{ for } \tau \le t_0 \tag{2}$$

and let $\boldsymbol{\Theta} = \{\boldsymbol{\theta} : (2) \text{ holds}\}$.

Let

$$\mathcal{D}(\boldsymbol{\theta}) = \{(G(t), W(t)) \in \Omega(t) : d(s, \tau) = \theta_\tau,\ s \le \tau \le t\},$$

and for some event $\boldsymbol{C}$ let $\mathbf{Pr}_{\boldsymbol{\theta}}(\boldsymbol{C}) = \mathbf{Pr}(\boldsymbol{C} \mid \mathcal{D}(\boldsymbol{\theta}))$ be the probability of the corresponding conditional event.

We will show

Lemma 1. *In both Model 1 and Model 2,*

$$\mathbf{Pr}(\bigcup_{\boldsymbol{\theta} \in \boldsymbol{\Theta}} \mathcal{D}(\boldsymbol{\theta})) = 1 - \widetilde{O}(t^{-3})^1.$$

□

Let

$$\sigma_0 = \frac{\ln t}{100 \ln \ln t}$$

and let

$$B_t = \{s \in [t/\ln t, t] : \ s \text{ is within distance } \sigma_0 \text{ of a cycle of length at most } \sigma_0\}.$$

Let

$$\mathcal{G}_1(t) = \{G(t) : \ |B_t| \le t^{7/8}\}.$$

We will prove that

Lemma 2. *If $\boldsymbol{\theta} \in \boldsymbol{\Theta}$ then, in both Model 1 and Model 2,*

$$\mathbf{Pr}_{\boldsymbol{\theta}}(G(t) \notin \mathcal{G}_1(t)) = o(t^{-3}).$$

[1] The $\widetilde{O}$ notation ignores polylog factors.

□

We then prove

Lemma 3. *If $s \in [t/\ln t, t)$, $s \notin B_t$ and $\boldsymbol{\theta} \in \boldsymbol{\Theta}$, then*

(a)

$$\mathbf{Pr}_{\boldsymbol{\theta}}(\overline{\mathcal{A}}_s(t) \mid \mathcal{A}_s(t-1)) = \frac{\theta_{t_0}}{2mt_0}\left(1+O\left(\tfrac{1}{m}\right)\right) + O(t^{-3})\mathbf{Pr}_{\boldsymbol{\theta}}(\mathcal{A}_s(t-1))^{-1}.$$

(b)

$$\mathbf{Pr}_{\boldsymbol{\theta}}(\mathcal{A}_s(s)) = 1 - O(s^{-1}).$$

□

(We condition on $\boldsymbol{\theta}$ in order to avoid some conditioning of the degree $d(s,t_0)$ due to assuming $\mathcal{A}_s(t_0)$.)

From this we prove Theorem 1 as follows: If $\boldsymbol{\theta} \in \boldsymbol{\Theta}$ and $s \in [t/\ln t, t]$, $s \notin B_t$ then

$$\mathbf{Pr}_{\boldsymbol{\theta}}(\mathcal{A}_s(t)) = \left(1 - \frac{\theta_{t_0}}{2mt}\left(1+O\left(\tfrac{1}{m}\right)\right)\right)\mathbf{Pr}_{\boldsymbol{\theta}}(\mathcal{A}_s(t-1)) + \widetilde{O}(t^{-3}).$$

We see then that if $\boldsymbol{\theta} \in \boldsymbol{\Theta}$ and $s \in [t/\ln t, t)$, $s \notin B_t$ and if $\tau_0 = \tau - 100(\ln t)^3$,

$$\begin{aligned}
\mathbf{Pr}_{\boldsymbol{\theta}}(\mathcal{A}_s(t)) &= \prod_{\tau=s+1}^{t}\left(1 - \frac{\theta_{\tau_0}}{2m\tau}\left(1+O\left(\tfrac{1}{m}\right)\right)\right) &(3)\\
&= \prod_{\tau=s+1}^{t}\left(1 - \frac{\theta_{\tau}}{2m\tau}\left(1+O\left(\tfrac{1}{m}\right)\right)\right) &(4)
\end{aligned}$$

Note that we can go from (3) to (4) because $\theta_\tau = \theta_{\tau_0}$ except for at most $100(\ln t)^3\Delta_t^*$ instances.

Thus absorbing the cases where $\boldsymbol{\theta} \notin \boldsymbol{\Theta}$ into the error term, (see Lemma 1), summing out the conditional probabilities over degree sequences, we get that for $s \in [t/\ln t, t)$, $s \notin B_t$

$$\begin{aligned}
\mathbf{Pr}(\mathcal{A}_s(t)) &= \sum_{\boldsymbol{\theta}} \mathbf{Pr}(\mathcal{D}(\boldsymbol{\theta})) \prod_{\tau=s+1}^{t}\left(1 - \frac{\theta_{\tau}}{2m\tau}\left(1+O\left(\tfrac{1}{m}\right)\right)\right)\\
&= \mathbf{E}\prod_{\tau=s+1}^{t}\left(1 - \frac{d(s,\tau)}{2m\tau}\left(1+O\left(\tfrac{1}{m}\right)\right)\right).
\end{aligned}$$

Note that the contribution of $s \in [1, t/\ln t] \cup B_t$ to the expectation $\nu_{\ell,m}(t)$ can only be $o(t)$ and (1) follows. □

3 Proof of Theorem 1: The details

We emphasise that $s \geq t/\log t$ throughout and that m is a sufficiently large constant.

3.1 Proof of Lemma 1: Model 1

The degree $d(s,t)$ of vertex s in $G(t)$ is distributed as

$$m + B(m,(s+1)^{-1}) + \cdots + B(m,t^{-1}) \tag{5}$$

where the binomials $B(m,\cdot)$ are independent.

Lemma 4.

(a) $\mathbf{Pr}(\Delta(G(t)) \geq 2m\ln t) = \widetilde{O}(t^{-3})$
where $\Delta(G(t))$ is the maximum degree in $G(t)$.

(b) $\mathbf{Pr}(\exists\tau:\ d(s,\tau+1) - d(s,\tau) > 5) = \widetilde{O}(t^{-4})$.

Proof $\mathbf{E}\,(d(s,t)) = m(1+H_t-H_s) \leq m(2+\ln t/s)$ where $H_k = 1+\frac{1}{2}+\cdots+\frac{1}{k}$. (a) now follows from Theorem 1 of Hoeffding [24]. (b) is easy, since $d(s,\tau+1)-d(s,\tau) = B(m,\tau^{-1})$. □

3.2 Proof of Lemma 1: Model 2

Lemma 5.

(a) $\mathbf{Pr}(d(s,t) \geq 10(\ln t)^5) = \widetilde{O}(t^{-3})$.

(b) $\mathbf{Pr}(\exists\tau:\ d(s,\tau+1) - d(s,\tau) > 5) = \widetilde{O}(t^{-3})$.

Proof (a) In order to get a crude upper bound on $d(s,t)$, we divide the interval $[s,t]$ into sub-intervals using the points (nearest to) $s, se^{1/8}, ...se^{r/8}, ..., se^{k/8}$. Here $se^{(k-1)/8} < t \leq \lceil se^{k/8}\rceil$, so that $k \leq 8\ln\ln t$, as $s \geq t/\ln t$.

Suppose that, at the start of $I_r = (\lceil se^{r/8}\rceil, \lceil se^{(r+1)/8}\rceil]$ we have an upper bound $d(r)$ on the degree of vertex s. We prove that if $d(r) \geq 10\ln t$ then $d(r+1) \leq 2d(r)$ with probability $1-o(t^{-3})$.

Now as long as the degree of s is $\leq 2d(r)$, the number X_τ of edges acquired at step $\tau \in I_r$ is dominated by $B(m, d(r)/(m(\tau-1)))$, so that the number of edges gained during this time has expected value

$$\leq 2d(r)\ln e^{1/8} = \frac{d(r)}{4}.$$

Thus, by Chernoff bounds, provided $d(r) \geq 10\ln t$,

$$\mathbf{Pr}(d(r+1) \geq 2d(r)) \leq \mathbf{Pr}\left(\sum_{\tau\in I_r} B(m,d(r)/(m(\tau-1))) \geq d(r)\right) \leq \left(\frac{e}{4}\right)^{d(r)} = o(t^{-3})$$

and thus $d(r+1) < 2d(r)$ with probability $1-o(t^{-3})$. Choosing $d(0) = 10\ln t$, we see that

$$d(s,t) < d(0)2^k \leq d(0)(\ln t)^4 = 10(\ln t)^5.$$

This proves (a). For (b) we use (a) and the fact that $d(s,\tau+1)-d(s,\tau)$ can then be dominated by $B(10(\ln t)^5, (2m\tau)^{-1})$. □

3.3 Proof of Lemma 2: Model 1

Fix $t/\ln t \le i_1 < \cdots < i_5 \le t$ and let $I = \{i_1, \ldots, i_5\}$. We estimate $\mathbf{Pr}(I \subseteq B_t)$.

For each partition $\mathcal{P}$ of I into parts $A_1, \ldots, A_k$ we consider the event

$\mathcal{E}_{\mathcal{P}} = \{\exists$ small cycles $C_1, \ldots, C_k$ and paths $P_v, v \in I$ such that
(i) $|C_i|, |P_v| \le \sigma_0$ for all i, v.
(ii) If $v \in A_i$ then P_v joins v to $C_i \cup \bigcup_{w \in A_i, w<v} P_w$,
(iii) P_v is edge disjoint from and shares one (endpoint) vertex with $C_i \cup \bigcup_{w \in A_i, w<v} P_w$,
(iv) The k collections $C_i, P_v, v \in A_i$ are pair-wise vertex disjoint. $\}$

Thus $\{I \subseteq B_t\} \subseteq \bigcup_{\mathcal{P}} \mathcal{E}_{\mathcal{P}}$ and

$$\mathbf{Pr}(\mathcal{E}_{\mathcal{P}}) \le \sum_{\substack{C_1,\ldots,C_k \\ P_v, v \in I}} \prod_{(x,y) \in F^*} \frac{m}{\max\{x,y\}} \tag{6}$$

where F^* denotes the edge set of $\bigcup C_i \cup \bigcup P_v$. The term $\frac{m}{\max\{x,y\}}$ is a bound on the probability of the existence of edge (x,y) given the appearance or absence of other edges, not incident with $\max\{x,y\}$.

Thus

$$\mathbf{Pr}(\mathcal{E}_{\mathcal{P}}) \le \prod_{r=1}^{5} \frac{m}{i_r} \sum_{\substack{3 \le |C_i| \le \sigma_0 \\ i=1,\ldots,k}} \sum_{\substack{0 \le |P_v| \le \sigma_0 \\ v \in I}} \prod_{v \in V^*} \frac{m}{v}$$

where V^* denotes the vertex set of $\bigcup C_i \cup \bigcup P_v$, less I,

$$\le \left(\frac{m \ln t}{t}\right)^5 \sum_{\ell=1}^{10\sigma_0} \left(\sum_{v=1}^{t} \frac{m}{v}\right)^{\ell}$$
$$= o(t^{-4}).$$

Thus

$$\mathbf{E}\left(\binom{|B_t|}{5}\right) = o(t)$$

and

$$\mathbf{Pr}(|B_t| \ge t^{7/8}) \le \frac{\mathbf{E}\left(\binom{|B_t|}{5}\right)}{\binom{t^{7/8}}{5}} = o(t^{-3}).$$

□

3.4 Proof of Lemma 2: Model 2

For this model we replace 5 by 10 and let $I = \{i_1, i_2, \ldots, i_{10}\}$. Let

$$\mathcal{G}_2(t) = \{G(t) : \ d(s,t) \le 2m\sqrt{t/s}(\ln t)^2 \text{ for all } 1 \le s \le t\}.$$

It is shown below, see Lemma 11a, that

$$\mathbf{Pr}(G(t) \in \mathcal{G}_2(t)) = 1 - \widetilde{O}(t^{-10}).$$

Therefore, we replace (6) by

$$\begin{aligned}
\mathbf{Pr}(\mathcal{E}_{\mathcal{P}} \mid G(t) \in \mathcal{G}_2(t)) &\leq \sum_{\substack{C_1,\ldots,C_k \\ P_v, v \in I}} \prod_{(x,y)\in F^*} \frac{2m(\ln t)^2}{\max\{x,y\}} \sqrt{\frac{\max\{x,y\}}{\min\{x,y\}}} \frac{1}{\mathbf{Pr}(G(t) \in \mathcal{G}_2(t))} \\
&\leq 2 \sum_{\substack{C_1,\ldots,C_k \\ P_v, v \in I}} \prod_{(x,y)\in F^*} \frac{2m(\ln t)^2}{x^{1/2} y^{1/2}}. \\
&\leq \left(\frac{2m(\ln t)^2}{t^{1/2}}\right)^{10} \sum_{\ell=1}^{20\sigma_0} \left(\sum_{v=1}^{t} \frac{2m(\ln t)^2}{v}\right)^{\ell} \\
&= \widetilde{O}(t^{-23/5}).
\end{aligned}$$

Thus we have

$$\mathbf{E}\left(\binom{|B_t|}{10}\right) = \widetilde{O}(t^{-10}) + \widetilde{O}(t^{10-23/5}) = \widetilde{O}(t^{27/5})$$

and

$$\mathbf{Pr}(|B_t| \geq t^{7/8}) \leq \frac{\mathbf{E}\left(\binom{|B_t|}{10}\right)}{\binom{t^{7/8}}{10}} = o(t^{-3}).$$

□

3.5 Proof of Lemma 3

3.5.1 Rapidly mixing walks

We now consider the random walk made by the spider S. A *random walk* on an fixed undirected graph G is a Markov chain (X_t), $X_t \in V$ associated to a particle that moves from vertex to vertex according to the following rule: The probability of a transition from vertex v, of degree d, to vertex w is $1/d$ if v is adjacent to w, and 0 otherwise. Let π denote the steady state distribution of the random walk. The steady state probability $\pi_G(v)$ of the walk being at a vertex v is,

$$\pi_G(v) = \frac{d(v)}{d(G)}, \tag{7}$$

where $d(v)$ is the degree of v and $d(G)$ is the total degree (i.e. sum of the degrees) of the graph G.

We will need a finite time approximation of the probability distribution π_H pertaining to a random walk on a subgraph $H = G(t) - s$ of $G(t)$. We obtain this by considering the *mixing time* of the walk based on a conductance bound (11) of Jerrum and Sinclair [28].

Let s, t be fixed with $s \in \left[\frac{t}{\ln t}, t\right) \setminus B_t$. Let P denote the transition matrix of the random walk on H. Let $P^{i,\tau}$ denote the distribution of the τth step of a random walk on H which starts at vertex i. For $K \subset V(H) = [t] \setminus \{s\}$ let $\overline{K} = V(H) \setminus K$ and

$$\Phi_K = \frac{\sum_{i\in K, j\in\overline{K}} \pi_H(i) P(i,j)}{\pi_H(K)}.$$

It follows from (7) that

$$\Phi_K = \frac{e(K:\overline{K})}{d(K)}$$

where $e(K:\overline{K})$ is the number of edges from K to $\overline{K}$, and $d(K)$ is the total degree of vertices in set K.

The conductance of the walk is defined by

$$\Phi(s,t) = \min_{\pi_H(K)\le 1/2} \Phi_K.$$

Let

$$\mathcal{G}_3(t) = \left\{ G(t) :\ \Phi(s,t) > \frac{1}{\ln t} \forall s \in [t/\ln t, t] \right\}.$$

Lemma 6. *If $\boldsymbol{\theta} \in \boldsymbol{\Theta}$ then, in both Model 1 and Model 2,*

$$\mathbf{Pr}_{\boldsymbol{\theta}}(G(t) \notin \mathcal{G}_3(t)) = o(t^{-3}).$$

□

3.5.2 Proof of Lemma 6: Model 1

Since $d(K) \le 2m|K| + e(K:\overline{K})$ it suffices to prove a high probability lower bound on $e(K:\overline{K})$, in both models.

Lemma 7.

$$\mathbf{Pr}_{\boldsymbol{\theta}}\left(\Phi(s,t) \le \frac{1}{200}\right) = o(t^{-3}).$$

Proof For $K \subseteq [t]$ let $d(K,t) = \sum_{s\in K} d(s,t)$. Then

$$\mathbf{Pr}(\exists K \subseteq [t] :\ |K| \ge 3t/4 \text{ and } d(K,t) \le (1.1)mt) = o(e^{-cmt}) \tag{8}$$

for some absolute constant $c > 0$.

To see this let $K \subseteq [t]$ with $|K| = k = 3t/4$. Then

$$\begin{aligned}\mathbf{E}\left(d(K,t)\right) \ge \mathbf{E}\left(d([t-k+1,t],t)\right) &= mk + m\sum_{s=t-k+1}^{t} \frac{s-(t-k)}{s} \\ &\ge 2mk - m(t-k)\ln(t/(t-k)) = \left(\frac{3}{2} - \frac{1}{4}\ln 4\right) mt \ge (1.15)mt.\end{aligned}$$

Applying Theorem 1 of Hoeffding we see that

$$\mathbf{Pr}(\exists K \subseteq [t] :\ |K| \ge 3t/4 \text{ and } d(K,t) \le (1.1)mt) \le \binom{t}{3t/4} e^{-c'mt}$$

for some absolute constant $c' > 0$. This completes the proof of (8)

Now for $K, L \subset [t] \setminus \{s\}$, let $e(K:L)$ denote the number of edges of $G(t)$ which have one end in K and the other end in L (we only use this definition for $L = \overline{K} = [t] \setminus (K \cup \{s\})$ and $L = K$).

It follows from (8) that with probability $1 - o(t^{-3})$

$$\Phi(s,t) \ge \min_{\pi(K)\le 1/2} \frac{e(K:\overline{K}) - 5|K|}{m|K| + e(K:\overline{K})} \ge \min_{|K|\le 3t/4} \frac{e(K:\overline{K}) - 5|K|}{m|K| + e(K:\overline{K})}. \tag{9}$$

($e(K:\overline{K})-5|K|$ bounds the number of $K:\overline{K}$ edges in $H_s(t)$ and then observe that the degree sum of K is at most $m|K|+e(K:\overline{K})$.)

We prove the following high probability lower bound on $e(K:\overline{K})$. Together with (9) this proves the lemma.

$$\mathbf{Pr}_{\boldsymbol{\theta}}(\exists K:\ e(K:\overline{K})\le m|K|/150)=o(t^{-3}). \tag{10}$$

Suppose $K\subset[t]$, $k=|K|$ and $Y_K=e(K:\overline{K})$. Let $\kappa=\frac{1}{2}\sqrt{kt}$ and $K_-=K\cap[\kappa]$ and $K_+=K\setminus K_-$.

Case 1: $|K_-|\ge 3k/7$.

$$\mathbf{E}_{\boldsymbol{\theta}}(Y_K)\ge\sum_{\tau=\kappa}^{t-4k/7-1}\frac{3(m-5)k/7}{\tau+4k/7}\ge\frac{3(m-5)k}{7}\ln\left(\frac{t-1}{\kappa+4k/7}\right).$$

Explanation: Consider the $\ge t-\kappa-4k/7-1$ vertices of $[t]-[\kappa]-\{s\}-K$. Each chooses at least $m-5$ random neighbours from lower numbered neighbours (plus themselves) and the sum minimises the expected number of these choices in K_-. The 5 comes from $\theta_{\tau+1}-\theta_t\le 5$ for $\boldsymbol{\theta}\in\boldsymbol{\Theta}$.

Applying Theorem 1 of [24] we obtain

$$\mathbf{Pr}_{\boldsymbol{\theta}}(Y_K\le\mathbf{E}_{\boldsymbol{\theta}}(Y_K)/2)\le\exp\left\{-\frac{1}{8}\frac{3mk}{7}\ln\left(\frac{t-1}{\kappa+4k/7}\right)\right\}=\left(\frac{\kappa+4k/7}{t-1}\right)^{3mk/56}.$$

So,

$$\begin{aligned}\mathbf{Pr}_{\boldsymbol{\theta}}(\exists K:\ |K_-|\ge 3k/7,\ |K|\le 3t/4\text{ and }Y_K\le\mathbf{E}(Y_K)/2)\le\\ \sum_{k=1}^{3t/4}\binom{t}{k}\left(\frac{\kappa+4k/7}{t-1}\right)^{3mk/56}\le\sum_{k=1}^{3t/4}\left(\frac{te}{k}\left(\frac{\kappa+4k/7}{t-1}\right)^{3m/56}\right)^k\le\\ \sum_{k=1}^{3t/4}\left(\frac{3t}{k}\left(\sqrt{\frac{k}{t}}\left(\frac{1}{2}+\frac{4}{7}\sqrt{\frac{3}{4}}\right)\right)^{3m/56}\right)^k=o(t^{-3}).\end{aligned}$$

This yields (10) for this case.

Case 2: $|K_-|\le 3k/7$.
Assume first that $k\ge 1000$. Now let Z_K denote the number of edges from the set W of $\lceil k/15\rceil$ lowest numbered vertices of K_+ which have their lower numbered endpoints also in K. Z_K is dominated by $B(m\lceil k/15\rceil,\sqrt{k/t})$ since there are at most $3k/7+\lceil k/15\rceil\le k/2$ vertices of K below any vertex w of W and there are at least κ vertices in all below such a w. We use $Y_K=e(K:\overline{K})\ge M-Z_K$ where $M=(m-5)\lceil k/15\rceil$. For $|K|\le t/1000$ we write

$$\begin{aligned}\mathbf{Pr}_{\boldsymbol{\theta}}(\exists K:\ 1000\le|K|\le t/1000,\ Z_K\ge M/2)\le\sum_{k=1000}^{t/1000}\binom{t}{k}2^M\left(\frac{k}{t}\right)^{M/2}\le\\ \sum_{k=1000}^{t/1000}\left(\frac{te}{k}\left(\frac{4k}{t}\right)^{(m-5)/30}\right)^k=o(t^{-3}).\end{aligned}$$

For $|K|>t/1000$ we use Chernoff bounds and write, for some absolute positive constant $c>0$

$$\mathbf{Pr}_{\boldsymbol{\theta}}(\exists K:\ t/1000\le|K|\le 3t/4,\ Z_K\ge 9M/10)\le\sum_{k=t/1000}^{3t/4}\binom{t}{k}e^{-cM}=o(t^{-3}).$$

For $|K| \leq 1000$ we can write

$$\mathbf{Pr}_{\boldsymbol{\theta}}(\exists K:\ e(K,K) \geq 3mk/4) \leq \sum_{k=1}^{1000} \binom{t}{k}\binom{mk}{3mk/28}\left(\frac{1000}{t^{1/2}}\right)^{3mk/28} = o(t^{-3}).$$

Note that if $e(K,K) \geq 3mk/4$ then at least $3mk/4 - 3mk/7$ of these edges must have one end in K_+.

This completes the proof of (10). □

3.5.3 Proof of Lemma 6: Model 2

Lemma 8. *There is an absolute constant $\xi > 0$ such that*

$$\mathbf{Pr}_{\boldsymbol{\theta}}(\exists K \subseteq [t] - \{s\}, |K| \geq (1-\xi)t:\ d(K,t) \leq (1+\xi)mt) = o(t^{-3}).$$

Proof Let ζ be a small positive constant and divide $[t]$ into approximately $1/\zeta$ consecutive intervals $I_1, I_2, \ldots$ of size $\lceil \zeta t \rceil$ plus an interval of $t - \lfloor 1/\zeta \rfloor \lceil \zeta t \rceil$. We put a high probability bound on the total degree $d(I_1, t)$. Now consider the random variables $\beta_k, k = 1, 2, \ldots$ where $\beta_k = d(I_1, k\lceil \zeta t \rceil)/(m\lceil \zeta t \rceil)$. Now $\beta_1 = 2$ and conditional on the value of β_k

$$(\beta_{k+1} - \beta_k)m\lceil \zeta t \rceil \text{ is dominated by } B\left(m\lceil \zeta t \rceil, \frac{\beta_k + 1}{2k}\right)$$

It follows that we can find an absolute constant $c > 0$ such that

$$\mathbf{Pr}_{\boldsymbol{\theta}}\left(\beta_{k+1} \leq \beta_k\left(1 + \frac{3}{4k}\right)\right) \leq e^{-cm\zeta t}.$$

So, with probability $1 - O(e^{-cm\zeta t})$ we find that

$$d(I_1, t) \leq 2m\lceil \zeta t \rceil \prod_{k=1}^{\lceil 1/\zeta \rceil}\left(1 + \frac{3}{4k}\right) \leq 2m\lceil \zeta t \rceil \times e^{3/4}\lceil 1/\zeta \rceil^{3/4} \leq 6m\zeta^{1/4}t,$$

for small enough ζ.

Now $d(\lceil \lceil \zeta t \rceil \rceil, t)$ dominates $d(L,t)$ for any set L of size $\lceil \zeta t \rceil$. So, if $m > 1/(c\zeta)$ then the probability there is a set of size $\lceil \zeta t \rceil$ which has degree exceeding $6m\zeta^{1/4}$ is exponentially small ($\leq \binom{t}{\lceil \zeta t \rceil}e^{-t}$)). In which case, every set K of size at least $t - \lceil \zeta t \rceil$ has total degree $d(K,t) \geq 2mt - 6m\zeta^{1/4}t$ and the lemma follows by taking ζ sufficiently small. □

Lemma 9. *If m is sufficiently large then*

$$\mathbf{Pr}_{\boldsymbol{\theta}}\left(\Phi(t) < \frac{1}{\ln t}\right) = O(t^{-3}).$$

Proof For $K \subseteq [t]$, $|K| = k$ we say K is *small* if $\ln t \leq k \leq ct$ and K is large otherwise, where $c = e^{-8}$.

3.5.4 Case of K small

Let $K_- = K \cap [\sqrt{kt}]$ and let $K_+ = K \setminus K_-$.

Case of $|K_-| \geq k/2$.
Let $X_t = X_t(K_-)$ be the number of those edges directed into K_- from vertices created after time $\sqrt{kt}$. The number of such edges generated at step $\tau \geq \sqrt{kt}$ dominates $B(m-5, mq/(2m\tau))$, independently of any previous step. (Here $q = |K_-|$). Thus

$$\mathbf{E}\,(X_t) \geq \sum_{\tau > \sqrt{kt}}^{t} \frac{(m-5)q}{2\tau} = \frac{(m-5)q}{4} \ln \frac{t}{k}(1+o(1)).$$

Hence

$$\mathbf{Pr}\left(X_t \leq \frac{mq}{6} \ln \frac{t}{k}\right) \leq \exp\left(-\frac{mq}{73} \ln \frac{t}{k}\right).$$

Thus

$$\begin{aligned}
\mathbf{Pr}\left(\exists K_- :\ X_t(K_-) \leq \frac{mq}{6} \ln \frac{t}{k}\right) &\leq \binom{\sqrt{kt}}{q} \exp\left(-\frac{mq}{73} \ln \frac{t}{k}\right) \\
&\leq \exp\left(-q\left(\frac{m}{73} \ln \frac{t}{k} - \ln\left(2e\sqrt{\frac{t}{k}}\right)\right)\right) \\
&\leq t^{-4}
\end{aligned}$$

provided m is sufficiently large. Thus **whp** the set K_- has at least $\frac{mk}{12} \ln t/k$ edges directed into it, of which at most $mk/2$ are incident with K_+. This completes the analysis of this case.

Case of $|K_+| \geq k/2$. We consider the evolution of the set $K_+ = \{u_1, u_2, \ldots, u_r\}$ from step $T = \sqrt{kt}$ onwards. Assume that at the final step t there are δk edges directed into K from $\overline{K}$. We can assume w.l.o.g. that $\delta \leq m/10$, for otherwise there is nothing to prove.

The number Y_{j+1} of $K : K$ edges generated by vertex u_{j+1} is a Binomial random variable with expectation at most

$$\mu_{j+1} = m\frac{2mk + \delta k}{2mt_{j+1}}.$$

The numerator in the above fraction is a bound on the total degree of K.

If $Z = Z(K_+) = \sum_{j=1}^{r} Y_j$ then

$$\begin{aligned}
\mathbf{E}\,(Z) &\leq \frac{2mk+\delta k}{2}\left(\frac{1}{t_1} + \cdots + \frac{1}{t_r}\right) \\
&\leq \frac{2mk+\delta k}{2}\frac{r}{\sqrt{kt}} \\
&\leq 1.05\,\frac{mkr}{\sqrt{kt}}.
\end{aligned}$$

Thus, for $\alpha > 0$,

$$\begin{aligned}
\mathbf{Pr}\,(\exists K_+ :\ Z(K_+) \geq \alpha k) &\leq \sum_{r=k/2}^{k} \binom{t}{r}\left(\frac{e \times 1.05 \times kmr}{\sqrt{kt} \times \alpha k}\right)^{\alpha k} \\
&\leq k\left(\left(\frac{3mk^{1/2}}{\alpha t^{1/2}}\right)^{\alpha} \frac{te}{k}\right)^{k} \\
&\leq t^{-4}
\end{aligned}$$

if $\alpha = m/4$, $k \leq ct$ and m is sufficiently large. We have therefore proved that for small values of k there are at least $mk/2 - mk/4$ out-edges generated by K_+ not incident with K on the condition that $\delta \leq m/10$, completing the analysis of this case.

3.5.5 Case of K large

Let $T = t/2$ and let $ct \leq |K|, |\overline{K}| \leq (1-\xi)t$ where ξ is as in Lemma 8. Let $M = [T]$ and $N = [T+1, t]$. Let $K_- = K \cap M$, $K_+ = K \cap N$, $q = |K_-|$ and $r = |K_+|$. We calculate the expected number of edges $\mu(K_-, K_+)$ of $L = (K_+ \times (M \setminus K_-)) \cup ((N \setminus K_+) \times K_-)$ generated at steps τ, $T \leq \tau \leq t$ which are directed into K. At step τ the number of such edges falling in L is an independent random variable with distribution dominating

$$1_{\tau \in N \setminus K_+} B\left(m-5, \frac{mq}{2m\tau}\right) + 1_{\tau \in K_+} B\left(m-5, \frac{(T-q)m}{2m\tau}\right).$$

Thus

$$\begin{aligned}
\mu(K_-, K_+) &\geq \frac{(m-5)q}{2} \sum_{\tau \in N \setminus K_+} \frac{1}{\tau} + \frac{(m-5)(T-q)}{2} \sum_{\tau \in K_+} \frac{1}{\tau} \\
&= \frac{m-5}{2} \left((k-r) \sum_{\tau \in N \setminus K_+} \frac{1}{\tau} + (T-(k-r)) \sum_{\tau \in K_+} \frac{1}{\tau} \right).
\end{aligned}$$

Let $\mu(k) = \min_{K_-, K_+} \mu(K_-, K_+)$. Then 'somewhat crudely'

$$\begin{aligned}
\sum_{\tau \in N \setminus K_+} \frac{1}{\tau} &\geq \ln \frac{t}{T+r} \\
\sum_{\tau \in K_+} \frac{1}{\tau} &\geq \ln \frac{t}{t-r}.
\end{aligned}$$

Thus

$$\mu(k) \geq \frac{m-5}{2} \left((k-r) \ln \frac{2t}{t+2r} + \left(\frac{t}{2} - (k-r) \right) \ln \frac{t}{t-r} \right).$$

Putting $k = \kappa t$ and $r = \rho t$ we see that

$$\mu(k) \geq \frac{(m-5)t}{2} g(\kappa, \rho)$$

where

$$g(\kappa, \rho) = (\kappa - \rho) \ln \frac{2}{1+2\rho} + \left(\tfrac{1}{2} - \kappa + \rho\right) \ln \frac{1}{1-\rho}.$$

We put a lower bound on g:

$$\rho \leq \frac{\xi}{2} \text{ implies } \kappa - \rho \geq \frac{\xi}{2} \text{ and so } g(\kappa, \rho) \geq \frac{\xi}{2} \ln \frac{2}{1+\xi}.$$

So we can assume that $\rho \geq \xi/2$. Then

$$\begin{aligned}
\kappa - \rho \leq \frac{1-\xi}{2} \quad &\text{implies} \quad g(\kappa, \rho) \geq \frac{\xi}{2} \ln \frac{2}{2-\xi}. \\
\kappa - \rho > \frac{1-\xi}{2} \text{ and } \rho \leq \frac{1-\xi}{2} \quad &\text{implies} \quad g(\kappa, \rho) \geq \frac{1-\xi}{2} \ln \frac{2}{2-\xi}. \\
\kappa - \rho > \frac{1-\xi}{2} \text{ and } \rho > \frac{1-\xi}{2} \quad &\text{implies} \quad \kappa > 1 - \xi.
\end{aligned}$$

We deduce that within our range of interest,

$$\mu(k) \geq \eta m t$$

for some absolute constant η.

Let Z be the number of edges generated within L, so that Z counts a subset of the edges between K and $\overline{K}$. Then

$$\begin{aligned}\mathbf{Pr}\left(\exists K_-, K_+ \subseteq N:\ Z \leq \frac{1}{2}\eta m t\right) &\leq 2^t e^{-\eta m t/8}\\ &\leq e^{-\eta m t/10}.\end{aligned}$$

Recall that m is sufficiently large. This completes the proof of the lemma, except for very small sets K.

For sets K of size $s \leq \ln t$ we note that, as $G(t)$ is connected, the conductance Φ_K is always $\Omega(1/|K|)$. □

3.6 Proof of Lemma 3: Continued

Define

$$\mathcal{G}(t) = \begin{cases}\mathcal{G}_1(t) \cap \mathcal{G}_3(t) & \text{Model 1}\\ \mathcal{G}_1(t) \cap \mathcal{G}_2(t) \cap \mathcal{G}_3(t) & \text{Model 2}\end{cases}.$$

We apply the main result of [28].

$$|P^{i,\tau}(v) - \pi_H(v)| \leq \left(1 - \frac{\Phi^2}{2}\right)^{\tau} \frac{\pi_H(v)}{\pi_{\min}} \tag{11}$$

where $\pi_{\min} = \min_v \pi_H(v)$.

Using (11) and Lemma 6 we see that with $\mu_0 = 10(\ln t)^3$, **whp**

$$|P^{i,\mu_0}(v) - \pi_H(v)| = O(t^{-4}) \qquad \forall v \in [t] \setminus \{s\}. \tag{12}$$

We are glossing over one technical point here. Strictly speaking, (11) only holds for Markov chains in which $P(x,x) \geq 1/2$ for all states x. To get round this one usually makes the walk flip a fair coin and stay put if the coin comes up heads. In our case we also omit to add a new vertex if the coin is heads. So what we have been describing is the outcome, ignoring those times when the coin flip is heads.

For the moment, we fix some $\boldsymbol{\theta} \in \boldsymbol{\Theta}$ and assume that $t/\ln t \leq s < t$.

Now by definition $t_0 = t - 10\mu_0$ and we define

$$\begin{aligned}I &= [t_0+1, t-1]\\ J &= \{\sigma \in I:\ \exists \tau \in I \text{ such that } X_\tau = \sigma\}\\ \mathcal{E}_0 &= \{X_\tau \neq s,\ \tau \in I\}\\ \mathcal{E}_1 &= \{\exists i \in I, j \in J:\ X_i = j \text{ and } j \text{ has a neighbour in } \{X_\sigma:\ \sigma \in [t_0, i-2]\}\}\\ \mathcal{F}_k &= \{|J| = k\} \qquad k \geq 0\\ \mathcal{F}_{\geq k} &= \{|J| \geq k\} \qquad k \geq 0\end{aligned}$$

and write

$$\mathbf{Pr}_{\boldsymbol{\theta}}(X_t = s \mid \mathcal{A}_s(t-1)) =$$
$$\sum_{\substack{G\in\mathcal{G}(t_0)\\ w\in[t_0]\setminus\{s\}}} \mathbf{Pr}_{\boldsymbol{\theta}}(X_t = s \mid X_{t_0} = w, G(t_0) = G, \mathcal{E}_0, \mathcal{A}_s(t_0))\mathbf{Pr}_{\boldsymbol{\theta}}(X_{t_0} = w, G(t_0) = G \mid \mathcal{A}_s(t-1)) +$$
$$\mathbf{Pr}_{\boldsymbol{\theta}}(X_t = s, G(t_0) \notin \mathcal{G}(t_0) \mid \mathcal{A}_s(t-1)). \quad (13)$$

It follows from Lemma 6 that

$$\mathbf{Pr}_{\boldsymbol{\theta}}(X_t = s, G(t_0) \notin \mathcal{G}(t_0) \mid \mathcal{A}_s(t-1)) = o(t^{-3}\mathbf{Pr}_{\boldsymbol{\theta}}(\mathcal{A}_s(t-1))^{-1}). \quad (14)$$

To deal with the rest of (13) we write

$$\begin{aligned}\mathbf{Pr}_{\boldsymbol{\theta}}(X_t = s \mid X_{t_0} = w, G(t_0) = G, \mathcal{E}_0, \mathcal{A}_s(t_0)) &= \mathbf{Pr}_{\boldsymbol{\theta}}(X_t = s \mid X_{t_0} = w, G(t_0) = G, \mathcal{E}_0)\\ &= \sum_{k=0}^{1} \mathbf{Pr}_{\boldsymbol{\theta}}(X_t = s \mid X_{t_0} = w, G(t_0) = G, \mathcal{E}_0, \mathcal{F}_k)\mathbf{Pr}_{\boldsymbol{\theta}}(\mathcal{F}_k \mid X_{t_0} = w, G(t_0) = G, \mathcal{E}_0)\\ &\quad + \mathbf{Pr}_{\boldsymbol{\theta}}(X_t = s \mid X_{t_0} = w, G(t_0) = G, \mathcal{E}_0, \mathcal{F}_{\geq 2})\mathbf{Pr}_{\boldsymbol{\theta}}(\mathcal{F}_{\geq 2} \mid X_{t_0} = w, G(t_0) = G, \mathcal{E}_0). \quad (15)\end{aligned}$$

Given $w, G(t_0)$, conditioning on $\mathcal{B} \stackrel{def}{=} \mathcal{E}_0 \cap \mathcal{F}_0$ is "almost" equivalent to S doing a random walk on $G(t_0) - \{s\}$ starting at w. In fact we get

Lemma 10.

$$\mathbf{Pr}_{\boldsymbol{\theta}}(X_t = s \mid X_{t_0} = w, G(t_0) = G, \mathcal{B}) =$$
$$\frac{\theta_{t_0}}{2mt_0}\left(1 + \frac{1}{\theta_{t_0}}\sum_{y\in N(s,t_0)} \mathbf{E}_{\boldsymbol{\theta}}\left(\frac{d(y,t_0) - d(y,t)}{d(y,t)}\right)\right)\left(1 + O\left(\frac{1}{m}\right)\right). \quad (16)$$

where $N(s,t_0)$ denotes the set of neighbours of s in $G(t_0)$.

Proof We emphasize that throughout the proof of this lemma, a graph $G \in \mathcal{G}(t_0)$ is fixed as well as $X_{t_0} = w$. All probabilities are conditional on this, even if not stated explicitly. The only randomness in the graph $G(t)$ itself is due to new vertices.

Let

$$\mathcal{M} = \{\nexists v \in [t_0], v \neq s : \quad v \text{ has more than five neighbours in } I\}.$$

Then in both models

$$\mathbf{Pr}_{\boldsymbol{\theta}}(\overline{\mathcal{M}} \mid G(t_0) = G) \leq |I|^6\left(\frac{2mt_0^{1/2}(\ln t_0)^2}{mt_0}\right)^6 = \widetilde{O}(t^{-3}). \quad (17)$$

Fix $y \in N(s,t_0)$ and let $\mathcal{W}_k(y)$ denote the set of walks in $H = G(t_0) - s$ which start at w, finish at y, are of length $\lambda_0 = t - t_0 = 100(\ln t)^3$ and which *leave* N^* exactly k times where N^* is the (random) set of neighbours of $I \cup \{s\}$ in $G(t_0)$. Let $\mathcal{W}_k = \bigcup_y \mathcal{W}_k(y)$ and let $W = (w_0, w_1, \ldots, w_{\lambda_0}) \in \mathcal{W}_k(y)$. Let

$$\rho_W = \frac{\mathbf{Pr}_{\boldsymbol{\theta}}(X_G(i) = w_i, i = 0,1,\ldots,\lambda_0 \mid \mathcal{M})}{\mathbf{Pr}_{\boldsymbol{\theta}}(X_H(i) = w_i, i = 0,1,\ldots,\lambda_0)}. \quad (18)$$

Here $X_G(i), i = 0,1,\ldots,\lambda_0$ is the sequence of vertices visited by S at times $t_0, t_0+1, \ldots, t$ and we will use $W_G = W_{w,G}$ to denote this walk. We let $X_H(i), i = 0,1,\ldots,\lambda_0$ is the set of vertices of H visited by a random walk $W_H = W_{w,H}$ on H with start vertex w.

Then

$$1 \geq \rho_W \geq \left(\frac{m-5}{m}\right)^k.$$

This is because a vertex can have at most 5 edges joining it to s and then

$$\frac{\mathbf{Pr}_{\boldsymbol{\theta}}(X_G(i) = w_i \mid X_G(i-1) = w_{i-1}, \mathcal{M})}{\mathbf{Pr}_{\boldsymbol{\theta}}(X_H(i) = w_i \mid X_H(i-1) = w_{i-1})} \begin{cases} \geq \frac{d_G(w_{i-1})-5}{d_G(w_{i-1})} & w_{i-1} \in N^*. \\ = 1 & w_{i-1} \notin N^*. \end{cases}$$

Furthermore,

$$\begin{aligned} \mathbf{Pr}_{\boldsymbol{\theta}}(\mathcal{B} \mid \mathcal{M}) &= \sum_{k \geq 0} \sum_{W \in \mathcal{W}_k} \mathbf{Pr}_{\boldsymbol{\theta}}(W_{w,G}(\lambda_0) = W \mid \mathcal{M}) \\ &= \sum_{k \geq 0} \sum_{W \in \mathcal{W}_k} \rho_W \mathbf{Pr}_{\boldsymbol{\theta}}(W_{w,H}(\lambda_0) = W) \\ &\geq \sum_{k \geq 0} p_k \left(\frac{m-5}{m}\right)^k \end{aligned}$$

where

$$p_k = \sum_{W \in \mathcal{W}_k} \mathbf{Pr}_{\boldsymbol{\theta}}(W_H(\lambda_0) = W) = \mathbf{Pr}_{\boldsymbol{\theta}}(W_H(\lambda_0) \in \mathcal{W}_k).$$

We will show later that

$$p_0 + p_1 + p_2 \geq 1 - O(m^{-1}) \tag{19}$$

which immediately implies that

$$\mathbf{Pr}_{\boldsymbol{\theta}}(\mathcal{B} \mid \mathcal{M}) \geq p_0 + p_1 \left(1 - \frac{5}{m}\right) + p_2 \left(1 - \frac{5}{m}\right)^2 \geq 1 - O(m^{-1}).$$

Now write

$$\begin{aligned} \mathbf{Pr}_{\boldsymbol{\theta}}(X_G(\lambda_0) = y \mid \mathcal{B}, \mathcal{M}) &= \sum_{k \geq 0} \sum_{W \in \mathcal{W}_k(y)} \mathbf{Pr}_{\boldsymbol{\theta}}(W_G(\lambda_0) = W \mid \mathcal{M}) \mathbf{Pr}_{\boldsymbol{\theta}}(\mathcal{B} \mid \mathcal{M})^{-1} \\ &= \sum_{k \geq 0} \sum_{W \in \mathcal{W}_k(y)} \rho_W \mathbf{Pr}_{\boldsymbol{\theta}}(W_H(\lambda_0) = W) \mathbf{Pr}_{\boldsymbol{\theta}}(\mathcal{B} \mid \mathcal{M})^{-1}. \end{aligned}$$

Now if

$$\begin{aligned} p_{k,y} &= \frac{\mathbf{Pr}_{\boldsymbol{\theta}}(W_H \in \mathcal{W}_k(y))}{\mathbf{Pr}_{\boldsymbol{\theta}}(X_H(\lambda_0) = y)} \\ &= \mathbf{Pr}_{\boldsymbol{\theta}}(W_H(\lambda_0) \text{ leaves } N^* \text{ exactly } k \text{ times} \mid X_H(\lambda_0) = y) \end{aligned}$$

then we have

$$\sum_{k \geq 0} p_{k,y} \left(\frac{m-5}{m}\right)^k \leq \frac{\mathbf{Pr}_{\boldsymbol{\theta}}(X_G(\lambda_0) = y \mid \mathcal{B}, \mathcal{M})}{\mathbf{Pr}_{\boldsymbol{\theta}}(X_H(\lambda_0) = y)} \leq \mathbf{Pr}_{\boldsymbol{\theta}}(\mathcal{B} \mid \mathcal{M})^{-1}. \tag{20}$$

We need to be careful about probability spaces here. Let $N^\#$ denote the the set of neighbours of I in $G(t_0)$. In our conditional probability space, the $p_{k,y}$ are now to be thought of as random variables dependent on $N^\#$.

We will show later that

$$\mathbf{Pr}_{\boldsymbol{\theta},\#}(p_{0,y}+p_{1,y}+p_{2,y}\geq 1-O(m^{-1}))=1-\widetilde{O}(t^{-1/2}). \tag{21}$$

where $\mathbf{Pr}_{\boldsymbol{\theta},\#}$ stresses that $N^{\#}$ is randomly chosen.

So, from (20), we obtain

$$\mathbf{Pr}_{\boldsymbol{\theta}}(X_H(\lambda_0)=y)\left(1-\frac{5}{m}\right)^2\mathbf{Pr}_{\boldsymbol{\theta},\#}(p_{0,y}+p_{1,y}+p_{2,y}\geq 1-O(m^{-1}))\leq \mathbf{Pr}_{\boldsymbol{\theta},\#}(X_G(\lambda_0)=y\mid\mathcal{B},\mathcal{M})\leq \mathbf{Pr}_{\boldsymbol{\theta}}(X_H(\lambda_0)=y)(1+O(m^{-1}))$$

or using (17)

$$\left|\frac{\mathbf{Pr}_{\boldsymbol{\theta}}(X_G(\lambda_0)=y\mid\mathcal{B},\mathcal{M})}{\mathbf{Pr}_{\boldsymbol{\theta}}(X_H(\lambda_0)=y)}-1\right|=O\left(\frac{1}{m}\right).$$

Therefore

$$\begin{aligned}\mathbf{Pr}_{\boldsymbol{\theta}}(X_G(\lambda_0)=y\mid\mathcal{B},\mathcal{M}) &= (1+O(m^{-1}))\mathbf{Pr}_{\boldsymbol{\theta}}(X_H(\lambda_0)=y)\\ \mathbf{Pr}_{\boldsymbol{\theta}}(X_G(\lambda_0)=y,\mathcal{B},\mathcal{M}) &= (1+O(m^{-1}))\mathbf{Pr}_{\boldsymbol{\theta}}(X_H(\lambda_0)=y)\mathbf{Pr}_{\boldsymbol{\theta}}(\mathcal{B},\mathcal{M})\end{aligned}$$

or

$$\mathbf{Pr}_{\boldsymbol{\theta}}(X_G(\lambda_0)=y,\mathcal{B})+\tilde{O}(t^{-3}) = (1+O(m^{-1}))\mathbf{Pr}_{\boldsymbol{\theta}}(X_H(\lambda_0)=y)(\mathbf{Pr}_{\boldsymbol{\theta}}(\mathcal{B})+\tilde{O}(t^{-3})) \tag{22}$$

Now we will show later that

$$\mathbf{Pr}_{\boldsymbol{\theta}}(\mathcal{B})\geq\frac{1}{2} \tag{23}$$

and (11) implies

$$\mathbf{Pr}_{\boldsymbol{\theta}}(X_H(\lambda_0)=y)=\frac{d(y,t_0)-\delta_y}{2mt_0}+O(t^{-3})=\Omega(t^{-1})$$

where $1\leq\delta_y\leq 5$ is the number of (s,y) edges in $G(t_0)$. So from (22) we have

$$\mathbf{Pr}_{\boldsymbol{\theta}}(X_G(\lambda_0)=y\mid X_{t_0}=w,G(t_0)=G,\mathcal{B})=(1+O(m^{-1}))\frac{d(y,t_0)}{2mt_0}.$$

Thus,

$$\mathbf{Pr}_{\boldsymbol{\theta}}(X_t=s\mid X_{t_0}=w,G(t_0)=G,\mathcal{B})=(1+O(m^{-1}))\mathbf{E}_{\boldsymbol{\theta}}\left(\sum_{y\in N(s,t_0)}\frac{d(y,t_0)}{2mt_0}\frac{1}{d(y,t)}\right)+O(m^{-1})\mathbf{Pr}_{\boldsymbol{\theta}}(X_{t-1}\in N(s,t-1)\setminus N(s,t_0)).$$

And then Lemma 10 follows from

$$\mathbf{Pr}_{\boldsymbol{\theta}}(X_{t-1}\in N(s,t-1)\setminus N(s,t_0))=\widetilde{O}(t^{-2}). \tag{24}$$

To verify this we observe that

$$\mathbf{Pr}_{\boldsymbol{\theta}}(|N(s,t-1)\setminus N(s,t_0)|\geq 2)=\widetilde{O}(t^{-2})$$

and so we only need to consider the case $N(s,t-1)\setminus N(s,t_0)=\{y_1\}$ where $y_1\in I$.

If $y_1-t_0\leq 5\mu_0$ then we can prove $\mathbf{Pr}_{\boldsymbol{\theta}}(X_{t-1}=y_1\mid X_{y_1}=w',\ldots)=\widetilde{O}(t^{-1})$ essentially by replacing t_0 by y_1. If $y_1-t_0>5\mu_0$ then either (i) there exists $u\in I$ such that $X_u=y_1$ or (ii) we can condition on $X_u\neq y_1, u\in I$ and proceed as for (16). Finally note that the probability of (i) is $\widetilde{O}(t^{-1})$ by the argument for (16). This completes the proof of (24) and hence Lemma 10 (modulo the proofs below).

3.6.1 Proof of (19,21)

Clearly (21) implies (19). We therefore start with (21). Observe first that $N^{\#}$ the set of neighbours of I in $G(t_0)$ satisfies

$$\mathbf{Pr}(W_H(\lambda_0) \cap N^{\#} \neq \emptyset) = \widetilde{O}(t^{-1/2}). \tag{25}$$

To see this we use the fact that the walk is defined on H and $N^{\#}$ is independent of H. We also need an upper bound of $\widetilde{O}(t^{1/2})$ for maximum degree in $G(t_0)$ and this follows from Lemmas 4(a) and 11(a).

Since $N^* = N^{\#} \cup N(s,t_0)$ we only have to show now that the probability that W_H leaves a vertex of $N(s,t_0)$ three times or more is $O(m^{-1})$. Furthermore, this event depends on $G(t_0)$, which is fixed and not on $G(t)$.

Given $G \in \mathcal{G}(t_0)$ and $W = W_H(\lambda_0)$, the total degree of the vertices of W is $\widetilde{O}(t^{1/2})$ in Model 2. In Model 1 we would have $\widetilde{O}(t^{-1})$ for the RHS of (25).

Now let $\mathcal{W}(a,b,\gamma)$ denote the set of walks in H from a to b of length γ and for $W \in \mathcal{W}(a,b,\gamma)$ let $\mathbf{Pr}(W) = \mathbf{Pr}(W_{a,H}(\gamma) = W)$. Then for $x \in V(H)$ we have

$$\begin{aligned}
\mathbf{Pr}(X_{w,H}(\lambda_0/2) = x \mid X_{w,H}(\lambda_0) = y) &= \sum_{\substack{W_1 \in \mathcal{W}(w,x,\lambda_0/2) \\ W_2 \in \mathcal{W}(x,y,\lambda_0/2)}} \frac{\mathbf{Pr}(W_1)\mathbf{Pr}(W_2)}{\mathbf{Pr}(\mathcal{W}(w,y,\lambda_0))} \\
&= \pi_{x,H}^{-1} \sum_{\substack{W_1 \in \mathcal{W}(w,x,\lambda_0/2) \\ W_2 \in \mathcal{W}(x,y,\lambda_0/2)}} \frac{\mathbf{Pr}(W_1)\pi_{x,H}\mathbf{Pr}(W_2)}{\mathbf{Pr}(\mathcal{W}(w,y,\lambda_0))}
\end{aligned}$$

and with W_3 equal to the reversal of W_2,

$$\begin{aligned}
&= \pi_{x,H}^{-1}\pi_{y,H} \sum_{\substack{W_1 \in \mathcal{W}(w,x,\lambda_0/2) \\ W_3 \in \mathcal{W}(y,x,\lambda_0/2)}} \frac{\mathbf{Pr}(W_1)\mathbf{Pr}(W_3)}{\mathbf{Pr}(\mathcal{W}(w,y,\lambda_0))} \\
&= \frac{\pi_{x,H}^{-1}\pi_{y,H}}{\mathbf{Pr}(\mathcal{W}(w,y,\lambda_0))}\mathbf{Pr}(\mathcal{W}(w,x,\lambda_0/2))\mathbf{Pr}(\mathcal{W}(y,x,\lambda_0/2)) \\
&= \frac{\pi_{x,H}^{-1}\pi_{y,H}}{\mathbf{Pr}(\mathcal{W}(w,y,\lambda_0))}(\pi_{x,H} - O(t^{-24}))^2 \\
&= \pi_{x,H} - O(t^{-23}).
\end{aligned}$$

It follows that the variation distance between the distribution of a random walk of length λ_0 from w to y and that of $W_1, W_3^{reversed}$ is $O(t^{-22})$ where W_1, W_3 are obtained by (i) choosing x from the steady state distribution and then (ii) choosing a random walk W_1 from w to x and a random walk W_3 from y to x. Furthermore, the variation distance between distribution of W_1 and a random walk of length $\lambda_0/2$ from w is $O(t^{-24})$. Similarly, the variation distance between distribution of W_3 and a random walk of length λ_0 from y is $O(t^{-24})$.

Now consider W_1 and let Z_i be the distance of $X_H(i)$ from s. We observe that since $s \notin B_{t_0} \subseteq B_t$, while the walk is within σ_0 of s the distance to s must go up or down in one step and that

$$\mathbf{Pr}\left(Z_{i+1} = Z_i + 1 \mid Z_i < \sigma_0\right) \geq 1 - \frac{1}{m-5}.$$

We will deduce from this that, where $N_H(s)$ is the set of $G(t_0)$ neighbours of s,

$$\mathbf{Pr}(W_1 \text{ or } W_3 \text{ make a } \textit{return} \text{ to } N_H(s)) = O(1/m) \tag{26}$$

and this together with (25) implies (21).

To verify (26) we first see that

$$\begin{aligned}\mathbf{Pr}_{\boldsymbol{\theta}}(\exists 1 \le i \le \lambda_0/2 : Z_i = 1 \mid Z_0 = \sigma_0) &\le \lambda_0 \sum_{k=0}^{\lambda_0/2} \binom{\sigma_0+2k}{\sigma_0+k} \left(\frac{1}{m-5}\right)^{\sigma_0+k} \\ &\le \lambda_0 \sum_{k=0}^{\lambda_0/2} \left(\frac{(\sigma_0+2k)e}{(m-5)(\sigma_0+k)}\right)^{\sigma_0+k} \\ &\le \lambda_0^2 (2e/(m-5))^{\sigma_0} \qquad (27)\end{aligned}$$

Then we have

$$\mathbf{Pr}_{\boldsymbol{\theta}}(\exists i > 0 :\ Z_{2i} = 1, 1 < Z_1, Z_2, \ldots, Z_{2i-1} \le \sigma_0 \mid Z_0 = 1) \le \sum_{i>0} \binom{2i}{i} \left(\frac{1}{m-5}\right)^i < \sum_{i>0} \left(\frac{2}{m-5}\right)^i = \frac{2}{m-7}. \qquad (28)$$

Equation (26) follows from (27) and (28).

3.6.2 Proof of (23)

This follows from $\mathbf{Pr}(\mathcal{E}_0 \mid X_{t_0} = w, G(t_0) = G, \mathcal{F}_0) = 1 - O(m^{-1})$, and this follows much as in the proof of (21). In particular, we see that if a walk in G starts at $w \neq s \notin B_t$ then the probability it visits s in λ_0 steps is $O(m^{-1})$. Then we will see that $\mathbf{Pr}_{\boldsymbol{\theta}}(\mathcal{F}_0 \mid X_{t_0} = w, G(t_0) = G) = 1 - o(1)$ (see (29) below). In the proof below we condition on $\mathcal{E}_0$ but the proof is valid without this conditioning.

This completes the proof of Lemma 10. □

We will next argue that

$$\begin{aligned}\mathbf{Pr}_{\boldsymbol{\theta}}(\mathcal{F}_{\ge k} \mid X_{t_0} = w, G(t_0) = G, \mathcal{E}_0) &= \widetilde{O}(t^{-k}) \qquad k = 1, 2. \qquad (29)\\ \mathbf{Pr}_{\boldsymbol{\theta}}(X_t = s \mid X_{t_0} = w, G(t_0) = G, \mathcal{E}_0, \mathcal{F}_1) &= \widetilde{O}(t^{-1/2}). \qquad (30)\\ \mathbf{E}_{\boldsymbol{\theta}}\left(d(y,t) - d(y,t_0)\right) &= \widetilde{O}(t^{-1/2}). \qquad (31)\end{aligned}$$

It follows from (13)–(16) and (29)–(31) that

$$\mathbf{Pr}_{\boldsymbol{\theta}}(X_t = s \mid X_{t_0} = w, G(t_0) = G, \mathcal{E}_0) = \frac{\theta_{t_0}}{2mt}\left(1 + O\left(\frac{1}{m}\right)\right) + O(t^{-3}\mathbf{Pr}_{\boldsymbol{\theta}}(\mathcal{A}_s(t-1))^{-1}). \qquad (32)$$

and removing the conditioning on $X_{t_0} = w, G(t_0) = G$ yields Lemma 3a.

For part (b) we see that $X_s = s$ if and only if [i] s chooses X_{s-1} as one of its m neighbours and then [ii] S moves to s. If we condition on $X_{s-1} = x$ and $d(x, s-1) = d$ then $\mathbf{Pr}_{\boldsymbol{\theta}}([i]) \le \frac{md}{2m(s-1)}$ and $\mathbf{Pr}_{\boldsymbol{\theta}}([ii] \mid [i]) \le \frac{2}{d}$ (we write $\le \frac{2}{d}$ in place of the more natural $\frac{1}{d+1}$ to account for x being chosen more than once). This proves part (b).

3.6.3 Proof of (29)

Let us generate $X_i, i \in I$ using as little information about the edges incident with I as possible. Thus, at step i we first establish whether any of $t_0 + 1, \ldots, i$ are neighbours of X_{i-1}. If the answer

is NO, we do not determine these neighbours. Thus up to the first time we get the answer YES, the conditional distribution of the neighbours of $t_0, t_0+1, \ldots, i$ is that they are chosen from a set of size $t - o(t)$ either randomly (Model 1) or from the same set with probabilities proportional to degree (Model 2). Let $\mathcal{Y}_i = \{\text{YES at } i \text{ and } X_i \in \{t_0+1, \ldots, i\}\}$. If $d(X_{i-1}, i) = d$ then

$$\mathbf{Pr}(\mathcal{Y}_i \mid d) = O\left(|I| \cdot \frac{d}{t} \cdot \frac{1}{d}\right) = O\left(\frac{|I|}{t}\right). \tag{33}$$

Since $\mathcal{F}_{\geq 1} \subseteq \bigcup_{i \in I} \mathcal{Y}_i$ we have (29) for $k = 1$.

Now assume that i_1 is the first i for which $\mathcal{Y}_i$ occurs and that $X_{i_1-1} = j_1$. Arguing as in the first paragraph of this subsection, we see that the conditional probability that $\mathcal{Y}_i$ occurs for $i_2 > i_1$, with $X_{i_2-1} = j_2 \neq j_1$ is also $\widetilde{O}(t^{-1}|I|)$ and this completes the proof of (29).

3.6.4 Proof of (30)

Let $J = \{j_1\}$ and let j_1 be visited for the first at time t_1 and let $i_1 = X_{t_1-1}$. We condition now on i_1, j_1, t_1 a well. We write

$$\begin{aligned}
&\mathbf{Pr}_{\boldsymbol{\theta}}(X_t = s \mid X_{t_0} = w, G(t_0) = G, \mathcal{E}_0, \mathcal{F}_1) = \\
&\quad \mathbf{Pr}_{\boldsymbol{\theta}}(X_t = s \mid X_{t_0} = w, G(t_0) = G, \mathcal{E}_0, \overline{\mathcal{E}}_1, \mathcal{F}_1)\mathbf{Pr}(\overline{\mathcal{E}}_1 \mid X_{t_0} = w, G(t_0) = G, \mathcal{E}_0, \mathcal{F}_1) + \\
&\quad \mathbf{Pr}_{\boldsymbol{\theta}}(X_t = s \mid X_{t_0} = w, G(t_0) = G, \mathcal{E}_0, \mathcal{E}_1, \mathcal{F}_1)\mathbf{Pr}(\mathcal{E}_1 \mid X_{t_0} = w, G(t_0) = G, \mathcal{E}_0, \mathcal{F}_1).
\end{aligned}$$

Observe that

$$\mathbf{Pr}_{\boldsymbol{\theta}}(\mathcal{E}_1, X_{t_0} = w, G(t_0) = G, \mathcal{E}_0, \mathcal{F}_1) = \widetilde{O}(t^{-3/2}).$$

Use (33) plus an extra $\widetilde{O}(t^{-1/2})$ factor for the extra neighbour(s). Furthermore, it is easily seen that

$$\mathbf{Pr}_{\boldsymbol{\theta}}(\mathcal{F}_{\geq 1} \mid X_{t_0} = w, G(t_0) = G, \mathcal{E}_0,) = \tilde{\Omega}(t^{-1}).$$

Combining these two statements with (29) for $k = 2$ we see that

$$\mathbf{Pr}_{\boldsymbol{\theta}}(\mathcal{E}_1 \mid X_{t_0} = w, G(t_0) = G, \mathcal{E}_0, \mathcal{F}_1) = \widetilde{O}(t^{-1/2}).$$

So we can assume that $\mathcal{E}_1$ does not hold.

We consider two possibilities:

(a) There is an $i < t-1$ such that $X_i = j_1$ and $X_{i+1} \neq i_1$.

Model 1: From time $i+1$ on, our walk is conditioned to walk on the graph H' induced by $([t_0] \cup \{j_1\}) \setminus \{s\}$. If $v \in H'$ then its steady state probability $\pi(v)$ for a random walk on H' is at least $1/(2t_0)$ and the probability that $X_{i+1} = v$ is at most twice this. It follows that in any subsequent step of a simple random walk, the probability S is at v is at most $2\pi(v)$. Although the walk is conditioned rather than constrained to walk on H', the arguments similar to those of Lemma 10 show that this only changes such probabilities by a factor $1 + O(m^{-1})$. Thus in this case the probability we arrive at a neighbour of s, at time $t-1$ is also $\widetilde{O}(t^{-1})$ and (30) follows for this case.

Model 2: Now if $v \in [t_0] \setminus \{s\}$ its steady state random walk probability $\pi(v)$ is asymptotically equal to the probability it is chosen as X_{i+1} and we can use the analysis for Model 1.

(b) For all $i < t-1$, $X_i = j_1$ implies $X_{i+1} = i_1$.

Now our after replacing some sequences i_1, j_1, i_1 by just i_1 our walk is conditioned on $\mathcal{F}_0$ and then (30) is implied by Lemma 10.

3.6.5 Proof of (31)

This follows from the fact that in Model 2, the maximum degree in $G(t)$ is $O(t^{1/2})$ **whp**, see e.g. [17]. For Model 1 the maximum degree is $O(\ln t)$ with sufficiently high probability.

3.7 $\ell \geq 1$

We follow the above analysis and note that the degrees do not change during the spider's walk and that error estimates do not increase (no new vertices are added).

4 Proof of Theorem 2

4.1 Model 1

Theorem 3.

$$\eta_{\ell,m}(t) = (1+O(m^{-1}))\int_0^1 \exp\left((m+\frac{\ell}{2})\ln x + \frac{2m^2}{\ell}\left(1-x^{\frac{\ell}{2m}}\right)\right)\,dx.$$

$$\eta_\ell = \sqrt{\frac{2}{\ell}}e^{(\ell+2)^2/(4\ell)}\int_{(\ell+2)/\sqrt{2\ell}}^{\infty} e^{-z^2/2}\,dz.$$

Thus, when $\ell = 1$, $\eta_1 = 2\sqrt{\pi}e^{9/4}(1-\Phi(3/\sqrt{2}))$ where $\Phi(x)$ is the standard normal cumulate. Thus $\eta_1 \sim 0.5699953....$ Furthermore, as $\ell \to \infty$, $\eta_\ell \sim 2/\ell$.

Proof We write $d(s,t)$ as

$$d(s,t) = X_s + X_{s+1} + \cdots + X_\tau + \cdots + X_t,$$

where $X_s = m$ and for $\tau > s$, the $X_\tau = B(m, \frac{1}{\tau-1})$ are independent.

Now

$$\begin{aligned}\sum_{\tau=s}^{t}\frac{d(s,\tau)}{\tau} &= \sum_{\tau=s}^{t}\frac{1}{\tau}\sum_{r=s}^{\tau}X_r\\ &= \sum_{r=s}^{t}X_r\left(\sum_{\tau=r}^{t}\frac{1}{\tau}\right).\end{aligned}$$

$$\sum_{\tau=r}^{t}\frac{1}{\tau} = \ln\frac{t}{r} + O\left(\frac{1}{r}\right).$$

Thus

$$\begin{aligned}\prod_{\tau=s}^{t}\left(1-\frac{d(s,\tau)}{2m\tau}\left(1+O\left(\frac{1}{m}\right)\right)\right)^{\ell} &= \exp\left(-\left(1+O\left(\frac{1}{m}\right)\right)\ell\sum_{\tau=s}^{t}\frac{d(s,\tau)}{2m\tau}\right) \\ &= \exp\left(-\left(1+O\left(\frac{1}{m}\right)\right)\frac{\ell}{2m}\sum_{r=s}^{t}X_r\ln\frac{t}{r}\right) \\ &= \prod_{r=s}^{t}\left(\frac{r}{t}\right)^{\frac{\ell X_r}{2m}\left(1+O\left(\frac{1}{m}\right)\right)}.\end{aligned} \tag{34}$$

Then we can write

$$\prod_{r=s}^{t}\left(\frac{r}{t}\right)^{\frac{\lambda_2 X_r}{2m}} \le \prod_{\tau=s}^{t}\left(1-\frac{d(s,\tau)}{2m\tau}\left(1+O\left(\frac{1}{m}\right)\right)\right)^{\ell} \le \prod_{r=s}^{t}\left(\frac{r}{t}\right)^{\frac{\lambda_1 X_r}{2m}}$$

where $\ell-\frac{A}{m}\le\lambda_1\le\ell\le\lambda_2\le\ell+\frac{A}{m}$ for some constant $A>0$.

Now

$$\begin{aligned}\mathbf{E}\prod_{r=s}^{t}\left(\frac{r}{t}\right)^{\frac{\lambda_1 X_r}{2m}} &= \prod_{r=s}^{t}\mathbf{E}\left(\frac{r}{t}\right)^{\frac{\lambda_1 X_r}{2m}} \\ &= \left(\frac{s}{t}\right)^{\frac{\lambda_1}{2}}\prod_{r=s+1}^{t}\left(1-\frac{1}{r-1}+\frac{1}{r-1}\left(\frac{r}{t}\right)^{\frac{\lambda_1}{2m}}\right)^{m} \\ &= (1+o(1))\left(\frac{s}{t}\right)^{\frac{\lambda_1}{2}}\exp\left\{m\ln\frac{s}{t}+\frac{2m^2}{\lambda_1}\left(1-\left(\frac{s}{t}\right)^{\frac{\lambda_1}{2m}}\right)\right\}.\end{aligned}$$

Thus

$$\begin{aligned}\nu_{\ell,m}(t) &\ge (1+o(1))t\int_0^1\exp\left(\left(m+\frac{\lambda_1}{2}\right)\ln x+\frac{2m^2}{\lambda_1}\left(1-x^{\frac{\lambda_1}{2m}}\right)\right)\,dx \\ &= \left(1+O\left(\frac{1}{m}\right)\right)t\int_0^1\exp\left(\left(m+\frac{\ell}{2}\right)\ln x+\frac{2m^2}{\ell}\left(1-x^{\frac{\ell}{2m}}\right)\right)\,dx.\end{aligned}$$

Replacing λ_1 by λ_2 to get an upper bound, we deduce that

$$\nu_{\ell,m}(t)=\left(1+O\left(\frac{1}{m}\right)\right)t\int_0^1\exp\left(\left(m+\frac{\ell}{2}\right)\ln x+\frac{2m^2}{\ell}\left(1-x^{\frac{\ell}{2m}}\right)\right)\,dx.$$

The values of this integral are easily tabulated. For $\ell=1$ they quickly reach a value of about 0.57 as m grows. The approximation is accurate to the second decimal place for $m\ge 4$.

As $m\to\infty$, by using the transformations $x=e^{-y}$ and $z=\sqrt{\ell/2}y+(l+2)/(\sqrt{2\ell})$ we obtain

$$\begin{aligned}\eta_\ell &= \int_0^\infty\exp\left(-\frac{\ell+2}{2}y-\frac{\ell}{4}y^2\right)\,dy \\ &= \sqrt{\frac{2}{\ell}}e^{(\ell+2)^2/(4\ell^2)}\int_{(\ell+2)/\sqrt{2\ell}}^{\infty}e^{-z^2/2}\,dz \\ &\sim \frac{2}{\ell}\end{aligned}$$

since $e^{x^2/2}\int_x^\infty e^{-y^2/2}dy\sim 1/x$ as $x\to\infty$.

4.2 Model 2

Theorem 4.

$$\eta_\ell = e^\ell 2\ell^2 \int_\ell^\infty y^{-3}e^{-y}\,dy.$$

When $\ell = 1$, $\eta_1 = 0.59634....$ Furthermore, as $\ell \to \infty$, $\eta_\ell \sim 2/\ell$.

Lemma 11.

(a)

$$\mathbf{Pr}(\exists s,t:\ d(s,t) \geq 2m\sqrt{t/s}(\ln t)^2) = \widetilde{O}(t^{-10}).$$

(b) *If $t/\ln t \leq s \leq t$ and $r \leq 2m\sqrt{t/s}(\ln t)^2$ then*

$$\mathbf{Pr}(d(s,t) = m+r) = \binom{m+r-1}{r}\left(\frac{s}{t}\right)^{m/2}\left(1-\left(\frac{s}{t}\right)^{\frac{1}{2}}\right)^r\left(1+O\left(\tfrac{(m+r)^3}{s}\right)+O\left(\tfrac{r}{\sqrt{s}}\right)\right).$$

Proof (a) Fix $s \leq t$ and let $X_\tau = d(s,\tau)$ for $\tau = s, s+1, \ldots, t$. Now conditional on $X_\tau = x$, we have

$$X_{\tau+1} = X_\tau + B\left(m, \frac{x}{2m\tau}\right)$$

and for $\lambda \leq 1$

$$\begin{aligned}\mathbf{E}\left(e^{\lambda X_{\tau+1}} \mid X_\tau = x\right) &= e^{\lambda x}\left(1-\frac{x}{2m\tau}+\frac{x}{2m\tau}e^\lambda\right)^m\\ &\leq \exp\left\{\lambda x - \frac{x}{2\tau}+\frac{x}{2\tau}(1+\lambda+\lambda^2)\right\}\\ &= \exp\left\{\lambda x\left(1+\frac{1+\lambda}{2\tau}\right)\right\}.\end{aligned}$$

Thus

$$\begin{aligned}\mathbf{E}\left(e^{\lambda X_{\tau+1}}\right) &\leq \mathbf{E}\left(\exp\left\{\lambda\left(1+\frac{1+\lambda}{2\tau}X_\tau\right)\right\}\right)\\ &\leq \exp\left\{m\lambda\prod_{\tau=s+1}^{\tau}\left(1+\frac{1+\lambda}{2\tau}\right)\right\}\\ &\leq \exp\{2m\lambda(t/s)^{(1+\lambda)/2}\}\\ &\leq \exp\{2m\lambda(t/s)^{1/2}\ln t\}.\end{aligned}$$

Putting $u = 2m(t/s)^{1/2}(\ln t)^2$, $\lambda = \frac{2\ln\ln t}{\ln t}$ we get

$$\begin{aligned}\mathbf{Pr}(X_t \geq u) &\leq \exp\{\lambda(2m(t/s)^{1/2}\ln t - u)\}\\ &\leq \exp\{-m(t/s)^{1/2}\ln t)^2\}\end{aligned}$$

and part (a) follows.

(b) Let $\boldsymbol{\tau} = (\tau_1, ..., \tau_r)$ where τ_j is the step at which the transition from degree $m+j$ to degree $m+j+1$ occurs. Let $\tau_0 = s$ and let $\tau_{r+1} = t$. Let $p(s,t,r:\boldsymbol{\tau}) = \mathbf{Pr}(d(s,t) = m+r \text{ and } \boldsymbol{\tau})$. Then

$$p(s,t,r:\boldsymbol{\tau}) = \prod_{j=0}^{r}\left(\Phi_j(\tau_j)\prod_{\tau_j<T<\tau_{j+1}}\left(1-\frac{m+j}{2mT}\right)^m\right),$$

where $\Phi_0 = 1$ and

$$\Phi_j = \left(1 + O\left(\tfrac{m+j}{\tau_j}\right)\right) \frac{m(m+j-1)}{2m\tau_j} \left(1 - \frac{m+j-1}{2m\tau_j}\right)^{m-1}.$$

In the above and in the following we use the fact that $\tau_j \geq s \gg (m+r)^3$ and $r = o(s^{1/2})$. The events that an edge sends two edges to s only contribute to the error terms here.

Now

$$\begin{aligned}
\prod_{\tau_j < T < \tau_{j+1}} \left(1 - \frac{m+j}{2mT}\right)^m &= \exp\left(-\frac{m+j}{2} \sum_{\tau_j < T < \tau_{j+1}} \left(\frac{1}{T} + O\left(\frac{m+j}{T^2}\right)\right)\right) \\
&= \exp\left(-\frac{m+j}{2}\left(\log\frac{\tau_{j+1}}{\tau_j} + O\left(\frac{m+j}{\tau_j}\right)\right)\right) \\
&= \left(\frac{\tau_j}{\tau_{j+1}}\right)^{\frac{m+j}{2}} \left(1 + O\left(\tfrac{(m+j)^2}{\tau_j}\right)\right).
\end{aligned}$$

Thus

$$\mathbf{Pr}(d(s,t) = m+r) = \sum_{\boldsymbol{\tau}} p(s,t,r:\boldsymbol{\tau})$$

where

$$p(s,t,r:\boldsymbol{\tau}) = \left(1 + O\left(\tfrac{(m+r)^3}{s}\right)\right) \frac{m(m+1)\cdots(m+r-1)}{2^r} \left(\frac{s}{t}\right)^{m/2} \frac{1}{t^{r/2}} \frac{1}{\sqrt{\tau_1}} \ldots \frac{1}{\sqrt{\tau_r}}. \tag{35}$$

Now

$$\begin{aligned}
\sum_{\boldsymbol{\tau}} \frac{1}{\sqrt{\tau_1}} \ldots \frac{1}{\sqrt{\tau_r}} &= \frac{1}{r!}\left(\int_s^t \frac{1}{\sqrt{\tau}}\, d\tau + O\left(\frac{1}{\sqrt{s}}\right)\right)^r \\
&= \left(1 + O\left(\tfrac{r}{\sqrt{s}}\right)\right) \frac{2^r}{r!} \left(\sqrt{t} - \sqrt{s}\right)^r.
\end{aligned}$$

The result follows. □

Assuming the same conditions on r, s as in Lemma 11b, define

$$\rho(s,t) = \prod_{\tau=s}^{t} \exp\left(-\ell \frac{d(s,\tau)}{2m\tau}\right).$$

As in the proof of Lemma 11 let $d(s,t) = m+r$ and let $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_r)$ denote the transition steps of $d(s,t)$ from m to $m+r$. As before, let $\tau_0 = s$ and $\tau_{r+1} = t$. Let $\rho(s,t:\boldsymbol{\tau})$ be the value of ρ given $\boldsymbol{\tau}$.

Then

$$\begin{aligned}
\rho(s,t:\boldsymbol{\tau}) &= \exp\left(-\frac{l}{2m} \sum_{j=0}^{r} \sum_{\tau_j \leq T < \tau_{j+1}} \frac{m+j}{T}\right) \\
&= \exp\left(-\frac{\ell}{2m} \sum_{j=0}^{r} (m+j)\left(\log\frac{\tau_{j+1}}{\tau_j} + O\left(\frac{1}{\tau_j}\right)\right)\right) \\
&= \left(1 + O\left(\tfrac{(m+r)^2}{s}\right)\right) \left(\frac{s}{t}\right)^{\frac{\ell}{2}} t^{-r\ell/2m} {\tau_1}^{\ell/2m} \ldots {\tau_r}^{\ell/2m}.
\end{aligned}$$

Thus, combining $\rho(s,t:\boldsymbol{\tau})$ with $p(s,t,r:\boldsymbol{\tau})$ (as given in (35)) for $\boldsymbol{\tau}$ of length r and summing over $\boldsymbol{\tau}$ we have

$$\begin{aligned}
\mathbf{E}\,\rho(s,t) &= \sum_r \sum_{\boldsymbol{\tau}} \rho(s,t:\boldsymbol{\tau}) p(s,t,r:\boldsymbol{\tau}) \\
&= \sum_r \left(1+O\left(\tfrac{(m+r)^3}{s}\right)\right)\left(\frac{s}{t}\right)^{(m+\ell)/2}\binom{m+r-1}{r}\frac{r!}{2^r}\frac{1}{t^{r(1+\ell/m)/2}}\sum_{\boldsymbol{\tau}}\prod_{j=1}^{r}\frac{1}{\tau_j^{(1-\ell/m)/2}} \\
&= \sum_r \left(1+O\left(\tfrac{(m+r)^3}{s}\right)+O\left(\tfrac{r}{s^{(1-\ell/m)/2}}\right)\right)\binom{m+r-1}{r}\left(\frac{1-\left(\frac{s}{t}\right)^{(1+\ell/m)/2}}{1+\ell/m}\right)^r \\
&= (1+o(1))\left(\frac{1+\frac{\ell}{m}}{1+\frac{\ell}{m}\left(\frac{t}{s}\right)^{(1+\ell/m)/2}}\right)^m.
\end{aligned}$$

Thus, using the transformations, $x=s/t$ and $y=\ell/\sqrt{x}$, we find

$$\begin{aligned}
\lim_{m,t\to\infty}\frac{\mathbf{E}\,\nu_{\ell,m}(t)}{t} &= \lim_{m,t\to\infty}\frac{1}{t}\sum_{s=1}^{t}\left(\frac{1+\frac{\ell}{m}}{1+\frac{\ell}{m}\left(\frac{t}{s}\right)^{(1+\ell/m)/2}}\right)^m \\
&= e^{\ell}\int_0^1 e^{-\ell/\sqrt{x}}\,dx \\
&= e^{\ell}2\ell^2\int_{\ell}^{\infty} y^{-3}e^{-y}\,dy.
\end{aligned}$$

as required.

5 Extensions and further research

There are some natural questions to be explored in the context of the above models.

- It should be possible to extend the analysis to other models of web-graphs e.g. [13], [17]. In principal, one should only have to establish that random walks on these graphs are rapidly mixing.
- One can consider non-uniform random walks. Suppose for example that each $v\in[t]$ is given a weight $\lambda(v)$ and when at a vertex v the spider chooses its next vertex with probability proportional to $\lambda(v)$. If $\Lambda(v)=\sum_{N(v)}\lambda(v)$ ($N(v)$ denotes the neighbours of v) then the steady state probability $\pi(v)$ of being at v in such a walk is proportional to $\Theta(v)=\lambda(v)\Lambda(v)$. Again, once one shows rapid mixing it should be possible to obtain an expression like (1) for the number of unvisited vertices.
- We have only estimated the expectation of the number of unvisited vertices. It would be interesting to establish a concentration result.

References

[1] D. Achlioptas, A. Fiat, A.R. Karlin and F. McSherry, Web search via hub synthesis, *Proceedings of the 42nd Annual IEEE Symposium on Foundations of Computer Science* (2001) 500-509.

[2] M. Adler and M. Mitzenmacher, *Toward Compressing Web Graphs*, *Proceedings of the 2001 Data Compression Conference* (2001) 203-212.

[3] W. Aiello, F. Chung and L. Lu, Random evolution in massive graphs, *Proceedings of the 42nd Annual IEEE Symposium on Foundations of Computer Science* (2001) 510-519.

[4] W. Aiello, F. Chung and L. Lu, A random graph model for power law graphs, *Experiment. Math.* 10 (2001) 53-66.

[5] R. Albert, A. Barabasi and H. Jeong, *Diameter of the world wide web.* Nature 401:103-131 (1999) see also `http://xxx.lanl.gov/abs/cond-mat/9907038`

[6] A. Barabasi, R. Albert and H. Jeong, Scale-free characteristics of random networks: The topology of the world wide web, *Physics A* 272 (1999) 173-187.

[7] A. Barabasi and R. Albert, Emergence of scaling in random networks, *Science* 286 (1999) 509-512).

[8] B. Bollobás, O. Riordan, J. Spencer and G. Tusnády, The degree sequence of a scale free random graph process, Random Structures and Algorithms 18 (2001) 279-290.

[9] B. Bollobás and O. Riordan, *The diameter of a scale free random graph*, to appear.

[10] B. Bollobás, O. Riordan, Mathematical results on scale-free graphs, *Handbook of Graphs and Networks*, S. Bornholdt, H. Schuster (eds), Wiley-VCH, Berlin (2002).

[11] B. Bollobás and O. Riordan, *Robustness and vulnerability of scale-free random graph*, to appear.

[12] S. Brin and L. Page, The anatomy of a large scale hypertextual web search engine, *Computer Networks* 30, (1998), 107-117.

[13] A. Broder, R. Kumar, F.Maghoul, P. Raghavan, S. Rajagopalan, R. Stata, A. Tomkins and J. Wiener. *Graph structure in the web.*
`http://gatekeeper.dec.com/pub/DEC/SRC /publications/stata/www9.htm`

[14] Buckley, P.G. and Osthus, D., Popularity based random graph models leading to a scale-free degree sequence, preprint available from http://www.informatik.hu-berlin.de/~osthus/.

[15] F. Chung and L. Lu, Connected components in a random graph with given degree sequences, *Annals of Combinatorics*, to appear.

[16] F. Chung and L. Lu, The average distances in random graphs with given expected degrees, *PNAS* 99 (2002) 15879-15882.

[17] C. Cooper and A.M. Frieze, A general model of web graphs, *Proceedings of ESA 2001*, 500-511.

[18] E. Drinea, M. Enachescu and M. Mitzenmacher, Variations on random graph models for the web, Harvard Computer Science Technical Report TR-06-01.

[19] P. Erdős and A. Rényi, On random graphs I, *Publicationes Mathematicae Debrecen*, 5, (1959), 290-297.

[20] P. Erdős and A. Rényi, On the evolution of random graphs, *Magyar Tud. Akad. Mat. Kutató Int. Közl.* 5, (1960), 17-61.

[21] R. Fagin, A. Karlin, J. Kleinberg, P. Raghavan, S. Rajagopalan, R. Rubinfeld, M. Sudan and A. Tomkins, Random walks with back buttons, *STOC 2000* 484-493.

[22] M. Faloutsos, P. Faloutsos and C. Faloutsos, On power-law relationships of the internet topology, *SIGCOMM 1999, Comput. Commun. Rev.* 29, (1999), 251.

[23] M.R. Henzinger, A. Heydon, M. Mitzenmacher and M. Najork, Measuring Index Quality Using Random Walks on the Web, *WWW8 / Computer Networks* 31 (1999) 1291-1303.

[24] W. Hoeffding, Probability inequalities for sums of bounded random variables, *Journal of the American Statistical Association* 58 (1963) 13-30.

[25] R. Kumar, P. Raghavan, S. Rajagopalan, D. Sivakumar, A. Tomkins and E. Upfal. The web as a graph, *Proceedings of the 19th Annual ACM Symposium on Principles of Database Systems* (2000) 1-10.

[26] R. Kumar, P. Raghavan, S. Rajagopalan, D. Sivakumar, A. Tomkins and E. Upfal. Stochastic models for the web graph, Proceedings of the 41st Annual IEEE Symposium on Foundations of Computer Science (2000) 57-65.

[27] L. Lu, The diameter of random massive graphs, Proceedings of the 12th ACM-SIAM Symposium on Discrete Algorithms (2001) 912-921.

[28] A. Sinclair and M. Jerrum, Approximate counting, uniform generation, and rapidly mixing Markov chains, *Information and Computation* 82 (1989) 93-133.

[29] D. Watts, *Small Worlds: The Dynamics of Networks between Order and Randomness*, Princeton University Press (1999).