# Multi-coloured Hamilton cycles in random edge-coloured graphs

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**Abstract**

We define a space of random edge-coloured graphs $\mathcal{G}_{n,m,\kappa}$ which correspond naturally to edge $\kappa$-colourings of $G_{n,m}$. We show that there exist constants $K_0,\ K_1 \geq 10$ such that provided $m \geq K_0 n \log n$ and $\kappa \geq K_1 n$ then a random edge coloured graph contains a multi-coloured Hamilton cycle with probability tending to 1, as the number of vertices $n$ tends to infinity.

# 1 Introduction

Let $\mathcal{G}_{n,m,\kappa}$ denote the space of random edge-coloured graphs, defined as follows: Each $G$ in $\mathcal{G}_{n,m,\kappa}$ has vertex set $[n]$, edge set $E_m(G)$ of size $m$, and each edge is coloured with a label from $[\kappa]$. Thus $|\mathcal{G}_{n,m,\kappa}| = \binom{N}{m}\kappa^m$ where $N = \binom{n}{2}$. The elements of $\mathcal{G}_{n,m,\kappa}$ are given the uniform measure. A random edge-coloured graph $G_{n,m,\kappa}$ is a graph $G$ sampled uniformly at random from $\mathcal{G}_{n,m,\kappa}$.

Given $G \in \mathcal{G}_{n,m,\kappa}$, a subset $S$ of the edges of $G$ is *multi-coloured*, if no two edges of $S$ have the same colour. We are interested in conditions on $n, m, \kappa$ which imply that **whp**[1] $G_{n,m,\kappa}$ contains a multi-coloured Hamilton cycle. We also consider the corresponding randomly arc-coloured random digraph $D_{n,m,\kappa}$ which is defined in the analogous way. We prove

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[1]A sequence of events $\mathcal{E}_n$ is said to occur *with high probability* (**whp**) if $\lim_{n\to\infty} \mathbf{Pr}(\mathcal{E}_n) = 1$.

**Theorem 1** *There exist constants $K_0, K_1 > 0$ such that if $m \geq K_0 n \log n$ and $\kappa \geq K_1 n$ then* **whp** *$G_{n,m,\kappa}$ (resp. $D_{n,m,\kappa}$) contains a multi-coloured Hamilton cycle.*

We note that Theorem 1 is best possible up to a constant factor, for we need at least $n$ distinct colours and the underlying graph (resp. digraph) does not contain a Hamilton cycle **whp** until $m > n \log n/2$. The values of the constants, $(K_0 = 9, K_1 = 10)$, we use in the proof of Theorem 1 are, however, not the best possible values.

There are two general types of results on multi-coloured structures: **whp** existence under random colouring and guaranteed existence under adversarial colouring. When considering adversarial (worst-case) colouring, the guaranteed existence of multi-coloured structures, is called an Anti-Ramsey property.

Erdös, Nesetril and Rödl [4], Hahn and Thomassen [6] and Albert, Frieze and Reed [1] (correction in Rue [8]) considered colourings of the edges of the complete graph $K_n$ where no colour is used more than $k$ times. It was shown in [1] that if $k \leq n/64$, then there must be a multi-coloured Hamilton cycle. Cooper and Frieze [2] proved a random graph threshold for this property to hold in almost every graph in the space studied.

With respect to random colouring, Janson and Wormald [7] gave conditions for the existence of a multi-coloured Hamilton cycle in a random regular graph. We also mention that Frieze and Mckay [5] found a tight threshold for the existence of a multi-coloured spanning tree.

## 2 A sequence of random graphs

Because we are concerned with monotone properties, we can work entirely with the independent model $G_{n,p,\kappa}$ where $p = m/N$ and the underlying uncoloured graph is $G_{n,p}$. Let $p_1$ satisfy $1 - p = (1 - p_1)^2$. Let $D_{n,p_1}$ be the random digraph where each arc occurs independently with probability $p_1$. Suppose now that we randomly colour the arcs of $D_{n,p_1}$ with $\kappa$ colours to obtain the random coloured graph $D_{n,p_1,\kappa}$. Ignoring orientation gives us the random graph $G_{n,p,\kappa}$, provided we make a random choice from the two possible colours when coalescing the edges of directed 2-cycles.

Next let $\mathcal{D}_{d-out,\kappa}$ denote the following set of arc-coloured *digraphs*: Each $D \in \mathcal{D}_{d-out,\kappa}$ has vertex set $[n]$, each vertex has out-degree $d$ and the arcs of $D$ are multi-coloured by $[\kappa]$ i.e. no colour is used more than once. Thus $|\mathcal{D}_{d-out,\kappa}| = \binom{n-1}{d}^n \binom{\kappa}{dn}(dn)!$. The arc-coloured digraph $D_{d-out,\kappa}$ is chosen almost uniformly at random from $\mathcal{D}_{d-out,\kappa}$ (there is a small technical point associated with the colour of coalesced edges). In this paper we will be concerned with $d = O(1)$, in particular we assume that $d = 4$ from now on.

The central idea of this paper is to use a network flow algorithm to take $D_{n,p_1,\kappa}$ and, conditional on an event of probability 1-o(1), return as output, a multi-coloured subdigraph $D$. The distribution of $D$ will be that of $D_{d-out,\kappa}$. If we ignore orientation in $D_{d-out,\kappa}$ and we make a random choice from the two possible colours when coalescing 2-cycles, we obtain the random multi-coloured graph $G_{d-out,\kappa}$. Ignoring colours now gives us the random graph $G_{d-out}$. If it is known **whp** that $G_{d-out}$ has the Hamilton property, then we will have proved that **whp** $G_{n,p,\kappa}$ contains the required multicoloured $H$. To prove Theorem 1 for $G_{n,m,\kappa}$ we only have to do this for $d \geq 4$ and then apply the result of Cooper and Frieze [3] which states that such a graph is Hamiltonian **whp**.

## 2.1 Network Flow Construction

We define a flow network $\mathcal{N}$ as follows. $\mathcal{N}$ has source $s$ and sink $t$. The vertex set $W$ consists of $s, t$, the set of colours $C = [\kappa]$ and the set $V = [n]$ of vertices of the $D_{n,p_1,\kappa}$ under consideration. For each colour $x \in C$ there is an arc $(s, x)$ in $\mathcal{N}$ of capacity 1. There is an arc $(x, v)$ in $\mathcal{N}$ of infinite capacity for every $v \in V$ for which there is an arc $(v, w)$ in $D_{n,p_1,\kappa}$ with tail $v$ and colour $x$. Finally, for each vertex $v \in V$ there is an arc $(v, t)$ of capacity $d$.

For $S \subseteq C$, let $N(S) = \{v : \ x \in S, \ v \in V, \ (x, v) \in \mathcal{N}\}$ be the out-neighbour set of $S$ in $\mathcal{N}$. A cut of finite capacity can be obtained from a set $S \subseteq C$ and $N(S) \subseteq V$. Let $T = N(S)$, $W = \{s\} \cup S \cup T$, and let $\overline{W} = (C \setminus S) \cup (V \setminus T) \cup \{t\}$. The capacity of the cut $(W : \overline{W})$ is $\kappa - |S| + d|T|$. Applying the max-flow min-cut theorem we see that $\mathcal{N}$ admits a flow of value $dn$ if and only if, for all $S \subseteq C$,

$$\kappa - |S| + d|N(S)| \geq dn. \tag{1}$$

We estimate the probability that (1) is not true because, for some set $S$, $|N(S)| < n - (\kappa - |S|)/d$. I.e. there exists a set of colours $S$ of size $s$ and a set of vertices $\overline{T}$ of size $|\overline{T}| > (\kappa - s)/d$ such that every arc of $D$ whose tail is in $\overline{T}$ has a colour in $C \setminus S$.

$p_1$ satisfies $1 - p = (1 - p_1)^2$ and so $p_1 \geq p/2$, for $1 - \sqrt{1-p} \geq p/2$ for $p \geq 0$. We see therefore that $np_1 \geq K_0 \log n$.

Let $\mathcal{E}$ denote the subset of $D_{n,p_1}$ for which $\delta^+(D_{n,p_1}) > np_1/2$.

We first estimate $\mathbf{Pr}(\overline{\mathcal{E}})$. By the Chernoff inequality,

$$\mathbf{Pr}(\delta^+(D_{n,p_1}) \leq np_1/2) \leq ne^{-np_1/8} = O(n^{1-K_0/8}), \tag{2}$$

which is $O(n^{-1/8})$ for $K_0 \geq 9$. Thus $\mathbf{Pr}(\overline{\mathcal{E}}) = o(1)$

Let

$$L(s) = 2\binom{\kappa}{s}\binom{n}{(\kappa - s)/d}\left(\frac{\kappa - s}{\kappa}\right)^{(\kappa-s)np_1/(2d)}$$

be an upper bound on the probability that some set of size $s$ does not satisfy (1) conditional on $\mathcal{E}$. The range of $s$ we need to consider is between $\kappa - dn + 1$ and $\kappa - 1$. For, if $|S| < \kappa - dn$ then (1) is true with $N(S) = \emptyset$, and if $s = \kappa$ then as $\delta^+(D) \geq np_1/2$, $\overline{T} = \emptyset$.

The probability that (1) is not satisfied is bounded by $\Theta$ where

$$\Theta = \mathbf{Pr}(\overline{\mathcal{E}}) + \sum_{s=\kappa-dn+1}^{\kappa-1} L(s). \tag{3}$$

As $\mathbf{Pr}(\overline{\mathcal{E}}) = o(1)$, we can concentrate on the summation term in (3).

Now, choosing $\kappa \geq 10n$,,

$$\begin{aligned}
\sum_{s=\kappa-dn+1}^{\kappa-1} L(s) &\leq 2 \sum_{s=\kappa-dn+1}^{\kappa-1} \left( \frac{\kappa e}{\kappa - s} \left( \frac{ned}{\kappa - s} \right)^{1/d} \left( \frac{\kappa - s}{\kappa} \right)^{(K_0 \log n)/(2d)} \right)^{\kappa - s} \\
&\leq 2 \sum_{s=\kappa-dn+1}^{\kappa-1} \left( 3 \left( \frac{\kappa - s}{\kappa} \right)^{K_0 \log n/(2d) - 1/d - 1} \right)^{\kappa - s} \\
&\leq 2 \sum_{s=\kappa-dn+1}^{\kappa-1} \left( 3 \left( \frac{dn-1}{edn} \right)^{(K_0 \log n)/(2d) - 1/d - 1} \right)^{\kappa - s} \qquad (4) \\
&\leq 2K_1 n \exp\left\{ -\frac{K_0 \log n}{2d} + \frac{1}{d} + 3 \right\}, \qquad (5)
\end{aligned}$$

which is $O(n^{-1/8})$ for $K_0 \geq 9$ when $d = 4$.

Thus **whp** $\mathcal{N}$ contains a flow of value $nd$. The capacities of $\mathcal{N}$ are integral and so we can assume this flow is integral. It decomposes into $nd$ $(s,t)$-paths, each of which assigns a colour $x$ to a vertex $v$. By construction a colour can be assigned at most once to an edge and each vertex is assigned $d$ colours. For each assignment of a colour $x$ to a vertex $v$ we choose (randomly from $D$) an arc of colour $x$ which has tail $v$. We thus obtain a multi-coloured member of $\mathcal{D}_{d-out,\kappa}$. It is easy to argue that the underlying uncoloured digraph is distributed as $D_{d-out,\kappa}$. Indeed we could start with $D_{n,p_1,\kappa}$ and then replace each arc $(v,w)$ by $(v, \pi_v(w))$ where the $\pi_v$, $v \in V$ are independent permutations of $V \setminus \{v\}$. After this transformation the digraph is still distributed as $D_{n,p_1,\kappa}$. We run the network flow algorithm and **whp** we obtain a multi-coloured member of $\mathcal{D}_{d-out,\kappa}$. By replacing each arc $(v,w)$ by $(v, \pi_v^{-1}(w))$ we obtain a subgraph of the original $D_{n,p_1,\kappa}$ which is distributed as $D_{d-out,\kappa}$. For those cases where both $(v,w)$ and $(w,v)$ are selected by the algorithm to be edges of $D_{n,p_1,\kappa}$ and have to be coalesced, we can assume that the colour choice is the same as that made when coalescing these same edges in going from $D_{n,p_1,\kappa}$ to $G_{n,p,\kappa}$.

In summary, **whp** $D_{n,p_1,\kappa}$ contains a multi-coloured subgraph which is distributed as

$D_{d-out,\kappa}$. Ignoring orientation we obtain a graph which **whp** contains a Hamilton cycle. This verifies Theorem 1 for the case of undirected graphs.

Consider now the directed case i.e. we start with $D_{n,p,\kappa}$. We first split this into two independent copies $D_1, D_2$ of $D_{n,p_1,\kappa}$. We then use a slightly modified network. Now we have vertices $s, t, C$ and two copies $V_1, V_2$ of $V$. The $s, C$ edges are as before and there are $V_1, t$ and $V_2, t$ edges of capacity $d$ (now we can take $d = 2$). We join $x \in C$ to $v \in V_1$ by an infinite capacity arc if $V_1$ contains an arc of colour $x$ and tail $v$. We join $x \in C$ to $v \in V_2$ by an infinite capacity arc if $V_1$ contains an arc of colour $x$ and head $v$. The network flow algorithm constructs a random multi-coloured 2-in,2-out digraph, which **whp** has a Hamilton cycle, by the result of [3].

As a final remark, we did not really make arguments about Hamiltonicity only about constructing a random subgraph which is distributed as $D_{d-out,\kappa}$. Clearly, other monotone graph properties can be treated in this manner.

Finally we mention two natural related problems: Suppose we fix $K_0$ at the threshold value $\frac{1}{2} + o(1)$. What is the least value of $K_1 = K_1(n)$ for which $G_{n,p,\kappa}$ contains a multi-coloured Hamilton cycle **whp**? Similarly, if we fix $K_1 = 1$, what is the least value of $K_0 = K_0(n)$ for which $G_{n,p,\kappa}$ contains a multi-coloured Hamilton cycle **whp**?

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