The cores of random hypergraphs with a given degree sequence. *

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August 11, 2004

Abstract

We study random r-uniform n vertex hypergraphs with fixed degree sequence $\boldsymbol{d} = (d_1, ..., d_n)$, maximum degree $\Delta = o(n^{1/24})$ and total degree θn, where θ is bounded. We give the size, number of edges and degree sequence of the κ-cores ($\kappa \geq 2$) up to a **whp** error of $O(n^{2/3}\Delta^{4/3}\log n)$. In the case of graphs ($r = 2$) we give further structural details such as the number of tree components and, for the case of smooth degree sequences, the size of the mantle. We give various examples, such as the cores of r-uniform hypergraphs with a near Poisson degree sequence, and an improved upper bound for the first linear dependence among the columns in the independent column model of random Boolean matrices.

1 Introduction

An interesting question in the theory of random graphs concerns the size of the largest component of such a graph. In their formative paper [8], Erdős and Rényi proved a strong dichotomy for the size C_1 of the largest component of the random graph $G_{n,m}$ when $m = cn/2$, c constant. Erdős and Rényi showed that, in $G_{n,m}$, if $c < 1$ then **whp**[1] $C_1 = O(\log n)$ and that if $c > 1$ then $C_1 \sim G(c)n$ for some function $G(c) > 0$. A component of order n is called a *giant component*. When $c = 1$ the situation is more complicated and much effort has gone into an analysis of this case. See e.g. Bollobás [3], Łuczak [16], Łuczak, Pittel and Wierman [17], Janson, Knuth, Łuczak and Pittel [12] and the books by Bollobás [4] and by Janson, Łuczak and Ruciński [13].

The κ-core of a hypergraph G is the largest vertex induced subgraph of minimum degree κ. It can be obtained by repeatedly removing edges incident with vertices

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[1]An event $\mathcal{E}$ occurs with high probability (**whp**) if $\mathbf{Pr}(\mathcal{E})$ tends to 1 as $n \to \infty$

of degree less than κ until no such vertices remain. Quite possibly the κ-core is empty.

Let $V_\kappa(G)$ denote the number of vertices in the κ-core of G. Given a space of random graphs, we can ask if a non-empty κ-core exists **whp**. A closely related question concerns the **whp** emergence of a giant κ-core ($V_\kappa(G) = \Theta(n)$) in $G_{n,m}$ or $G_{n,p}$ or other suitably parameterized sequences of spaces. See Pittel, Spencer and Wormald [22] for a substantial treatment of this problem. A further discussion of the method is given by Wormald in [23]. The paper by Kim [14] gives results for κ-cores of r-uniform hypergraphs $G_{n,p}$ using a Poisson cloning technique. More recently, there is also the work of Fernholz and Ramachandran [9] on fixed degree sequence random graphs and of Fountoulakis [11] for $G_{n,m}$.

Let $\boldsymbol{d} = (d_1, d_2, \ldots, d_n)$ be a sequence of non-negative integers and let $\theta n = d_1 + d_2 + \cdots + d_n$. Let $r \geq 2$ be integer, and let $\theta n = mr$ for some integer m. Let $\boldsymbol{G}(\boldsymbol{d})$ be the set of simple r-uniform hypergraphs G with vertex set $V = [n]$ and m edges where for $i = 1, ..., n$ the degree of vertex i is d_i. Let $\Delta = \max_i d_i$ be the maximum degree. Let L_i be the number of vertices of degree i. The total degree of G is $\theta n = \sum iL_i$, and θ is the average degree. We assume $L_0 = 0$ unless stated otherwise. Let $E = E(G)$ denote the edge set of G, then $E = \{e_1, ..., e_t, ..., e_m\}$, where $|e_t| = r,\ t = 1, ..., m$. Let G denote a hypergraph chosen uniformly at random (uar) from $\boldsymbol{G}(\boldsymbol{d})$.

This paper studies the properties of κ-cores ($\kappa \geq 2$) of simple random r-uniform n vertex hypergraphs G with degree sequence $\boldsymbol{d}$. The size of the κ-core depends primarily on a single parameter $\widehat{\sigma} = \widehat{\sigma}(\kappa)$. The parameter $\widehat{\sigma}$ is the smallest solution of

$$q(t) = \theta\left(1 - \frac{rt}{\theta n}\right) \tag{1}$$

for $t \in [0, \theta n/r]$ and the function $q(t)$ is given by

$$q(t) = \sum_{i \geq \kappa} \frac{L_i}{n} \sum_{j=\kappa}^{i} j \binom{i}{j} \left(\left(1 - \tfrac{rt}{\theta n}\right)^{\frac{r-1}{r}}\right)^j \left(1 - \left(1 - \tfrac{rt}{\theta n}\right)^{\frac{r-1}{r}}\right)^{i-j}. \tag{2}$$

We note that $q(\theta n/r) = 0$, so equation (1) always has a solution $\widehat{\sigma}$.

Let $a(t) = \theta(1 - rt/\theta n) - q(t)$. Thus $\widehat{\sigma}$ is the smallest non-negative solution of $a(t) = 0$. The function $a(t)$ comes from the analysis of the algorithm which we use to generate a random multi-hypergraph with degree sequence $\boldsymbol{d}$. The algorithm generates the edges outside the κ-core before generating the (possibly empty) κ-core. The quantity $na(t)$ will be seen to be (asymptotic to) the expected total remaining degree of those vertices of remaining degree less than κ after t edges have been generated and removed.

To simplify notation we measure all errors in the analysis of the algorithm in terms of a single parameter τ where

$$\tau = n^{2/3}\Delta^{4/3}\log n.$$

In order to indicate that various quantities are bounded for all n we use a sequence of absolute positive constants $A_1, A_2, \ldots$.

We say that the degree sequence $\boldsymbol{d}$ satisfies the Separation Property if the following conditions hold:

S1: $\Delta = o(n^{1/24})$, $\theta \le A_1$ and $\sum i(i-1)L_i/(\theta n) \le A_2$.

S2: If $t < \widehat{\sigma} - K\tau$ then $na(t) = \Omega(\tau)$, where K is a large positive constant.

S3: $-A_3 \le na'(\widehat{\sigma}) < 0$ and $a(\widehat{\sigma}+h) = ha'(\widehat{\sigma})(1+o(1))$, for $|h| = O(\tau)$.

We give a brief informal discussion of our choice of these conditions. Condition S1 is somewhat arbitrary, but ensures that the multi-hypergraph we generate is simple with probability bounded away from 0 by a positive constant A for any n. Conditions S2, S3 are mainly to avoid thresholds. We identify a threshold for a κ-core with a minimum of $a(t)$ touching the t axis at $t < \theta n/r$. Condition S3 requires $a(t)$ to be clearly decreasing at the root $\widehat{\sigma}$. This is to avoid a near minimum, which we identify with being close to a threshold. Also we need (2) to have a smooth Taylor expansion about $\widehat{\sigma}$. Given the functional form of $q(t)$ it seems reasonable that this could be proved directly, but we have not done so. Condition S2 avoids degree sequences very close to a possible secondary κ-core threshold (earlier than $\widehat{\sigma}$). It also avoids degree sequences which do not fit the framework of our analysis. For example, a space of graphs consisting of a central vertex with $O(\tau)$ paths of length $O(n/\tau)$ attached. Looking for a 2-core by removing vertices of degree 1 means that at most $O(\tau)$ vertices of degree 1 are ever exposed at any step.

We aim to answer two questions. Do graphs in $\boldsymbol{G}(\boldsymbol{d})$ have a significant κ-core **whp** and if so, what are its properties? Theorem 1 gives the order of magnitude of the core, and Theorem 2 gives the precise details if $\tau\log n \le \widehat{\sigma} \le \theta n/r - \tau\log n$.

Theorem 1. *Let A_1, A_2 be positive constants. Let $\boldsymbol{G}(\boldsymbol{d})$ be the space of simple r-uniform n vertex hypergraphs with degree sequence $\boldsymbol{d}$ satisfying $\Delta = o(n^{1/24})$, $\theta \le A_1$ and $\sum i(i-1)L_i/(\theta n) \le A_2$. Let*

$$q(t) = \sum_{i\ge\kappa}\frac{L_i}{n}\sum_{j=\kappa}^{i} j\binom{i}{j}\left(\left(1-\tfrac{rt}{\theta n}\right)^{\frac{r-1}{r}}\right)^{j}\left(1-\left(1-\tfrac{rt}{\theta n}\right)^{\frac{r-1}{r}}\right)^{i-j},$$

and let $a(t) = \theta(1 - rt/\theta n) - q(t)$.
Let $\kappa \ge 2$, and let $\widehat{\sigma}(\kappa)$ be the smallest solution in $[0,1]$ to $a(t) = 0$. The following results hold **whp**

(i) If $\widehat{\sigma}(\kappa) \geq \theta n/r - \tau \log n$ and $a(t) = \Omega(\tau)$ for $t \leq \theta n/r - \tau \log n$,

(a) if $r = 2,\ \kappa = 2$ the number of edges in the κ-core is $O(\tau \log n)$,

(b) if $r = 2,\ \kappa \geq 3$ or $r \geq 3,\ \kappa \geq 2$ the κ-core is empty.

(ii) If $\tau \log n \leq \widehat{\sigma}(\kappa) \leq \theta n/r - \tau \log n$ and $\boldsymbol{d}$ satisfies S2, S3, the number of edges in the κ-core is $\Theta\left(\theta n/r - \widehat{\sigma}\right)$.

(iii) If $\widehat{\sigma}(\kappa) < \tau \log n$ and $\boldsymbol{d}$ satisfies S3, the κ-core contains $\theta n/r - O(\tau \log n)$ edges.

Let $V_\kappa(G),\ E_\kappa(G)$ be the number of vertices and edges of the κ-core (respectively). Let $L_{i,j}(G)$ be the number of vertices of degree i in G which have degree j the κ-core. In the case of graphs ($r = 2$), let $R(G)$ be the number of isolated tree components. The following theorem gives structural details of the κ-core.

Theorem 2. *Let $\boldsymbol{G}(\boldsymbol{d})$ be the space of simple r-uniform hypergraphs with degree sequence $\boldsymbol{d}$ satisfying the Separation Property.*
Let $\kappa \geq 2$. Let $\tau \log n \leq \widehat{\sigma}(\kappa) \leq \theta n/r - \tau \log n$, and let $\widehat{x} = (1 - r\widehat{\sigma}/\theta n)^{\frac{r-1}{r}}$. Then **whp**

$$\begin{aligned}
E_\kappa(G) &= \frac{\theta n}{r} - \widehat{\sigma} + O(\tau) \\
V_\kappa(G) &= \sum_{i \geq \kappa} L_i \sum_{j=\kappa}^{i} \binom{i}{j} \widehat{x}^j \left(1 - \widehat{x}\right)^{i-j} + O(\tau) \\
L_{i,j}(G) &= L_i \binom{i}{j} \widehat{x}^j (1 - \widehat{x})^{i-j} + O(\tau).
\end{aligned}$$

Also, for graphs ($r = 2$),

$$R(G) = \sum_{i \geq 0} L_i (1 - \widehat{x})^i - \frac{\theta n}{2} (1 - \widehat{x})^2 + O(\tau).$$

In Section 2 we define an algorithm, CONSTRUCT to generate a random multi-hypergraph. In Section 3 we give the proof of Theorems 1, 2. In Section 4 we look at results relating to 2-cores of graphs. We combine our results with the work of Molloy and Reed [18, 19] on giant components of smooth degree sequences, to give the size of the mantle of such sequences. In Section 5 we review results for κ-cores of hypergraphs $G_{n,p},\ G_{n,m}$ and give an application to upper-bounding the rank of random Boolean matrices.

The rest of this section contains an informal discussion of results, and examples.

Recall that the size of the κ-core depends on a single parameter $\widehat{\sigma}$, which is the smallest solution for $t \in [0, \theta n/r]$ of $q(t) = \theta\left(1 - rt/\theta n\right)$. We define the following variables

$$l_i = L_i/n, \qquad p_i = il_i/\theta, \qquad \rho = \tfrac{r-1}{r}, \qquad x = \left(1 - \tfrac{rt}{\theta n}\right)^{\rho}. \tag{3}$$

Using this notation $q(t)$ can we written as $q(x(t))$ where

$$q(x) = \sum_{i \geq \kappa} l_i \sum_{j \geq \kappa} j \tbinom{i}{j} x^j (1-x)^{i-j}, \tag{4}$$

and $a(t)$ as $a(x(t)) = \theta x^{1/\rho} - q(x)$. We see that $q(x) = \theta x f(x)$ where

$$f(x) = \sum_{i \geq \kappa} p_i \sum_{j \geq \kappa - 1} \tbinom{i-1}{j} x^j (1-x)^{i-1-j}. \tag{5}$$

Let $\widehat{x}$ be defined by $\widehat{x} = \left(1 - \frac{r\widehat{\sigma}}{\theta n}\right)^{\rho}$. Thus $a(\widehat{x}) = 0$ and $q(\widehat{x})$ satisfies

$$q(\widehat{x}) = \theta \widehat{x}^{1/\rho} = \theta \widehat{x} f(\widehat{x}). \tag{6}$$

The solutions to this are $\widehat{x} = 0$ and

$$\begin{aligned} \widehat{x} &= \left(\sum_{i \geq \kappa} p_i \sum_{j \geq \kappa-1} \tbinom{i-1}{j} \widehat{x}^j (1-\widehat{x})^{i-1-j} \right)^{r-1} \\ &= (f(\widehat{x}))^{r-1}. \end{aligned} \tag{7}$$

Informally, the probability $\widehat{x}$ that an edge in general position survives in the κ-core is the probability that each of its $r-1$ terminal vertices survives by having at least $j \geq \kappa - 1$ surviving descendants. This correspondence with a branching process was noted in [22].

The results of this paper are for a fixed degree sequence $\boldsymbol{d}$. If the degree sequence $\boldsymbol{d} = \boldsymbol{d}(c)$ is parameterized by c (e.g. $l_i(c) \sim c^i e^{-c}/i!$), then it may be possible to use the results of this paper to prove that the κ-core exhibits threshold behaviour as c varies.

We note that $a(\theta n/r) = 0$ always, and presume $a(0) > 0$. We say the parameterization c of $\boldsymbol{d}(c)$ is *critical* at c^* if, (i) for $c < c^*$, $a(t,c) > 0$ for $t < \theta n/r$, (ii) at c^* a minimum of $a(t, c^*)$ touches the t-axis at $\widehat{\sigma}(c^*) < \theta n/r$, (iii) for $c > c^*$, $a(t,c)$ crosses the t-axis at some $\widehat{\sigma}(c) < \theta n/r$.

It seems natural to interpret a point c^* as a threshold for a κ-core. As $\partial x/\partial t < 0$ the critical condition (ii) that $a(t) = a'(t) = 0$ at some $t < \theta n/r$ holds iff $a(x) = a'(x) = 0$ for some $x > 0$. Let $G(x) = x^{1/(r-1)} - f(x)$. As $a(x) = cx^{1/\rho} - cxf(x)$ then $a'(x) = 0$ iff $G'(x) = -G(x)/x$. Thus for $x > 0$, $a(x) = a'(x) = 0$ iff $G(x) = G'(x) = 0$ ie. *at a threshold the curves $x^{1/(r-1)}$ and $f(x)$ are tangent.* This can be written as

$$x^{\frac{1}{r-1}} = f(x) = (r-1)xf'(x), \tag{8}$$

a convenient set of equations for the critical parameters $(c^*, \widehat{x}(c^*))$.

Assuming $\boldsymbol{d}(c)$ satisfies S1 and S2, to prove threshold behaviour at c^* the following approach can be adopted.

(i) Derive the critical value c^* using (8).

(ii) Find an $\epsilon \to 0$ such that if $c = c^* - \epsilon$ then $\widehat{\sigma} > \theta n/r - \tau \log n$, and if $c = c^* + \epsilon$ then $\tau \log n < \widehat{\sigma} < \theta n/r - \tau \log n$.

(iii) Prove the degree sequence $\boldsymbol{d}(c)$ satisfies S3 for $c \geq c^* + \epsilon$.

(iv) Apply Theorem 1. For $c \leq c^* - \epsilon$ the (possibly empty) core is of size $O(\tau \log n)$ **whp**. For $c \geq c^* + \epsilon$, the size of the core is $\Theta(\theta n/r - \widehat{\sigma})$ **whp**. The **whp** properties of the core for $c \geq c^* + \epsilon$ are given by Theorem 2.

We next give two examples of thresholds in r-uniform hypergraphs obtained in this way. The first example is the well known case of random hypergraphs $G_{n,c/n}$. These results have been given by several authors (eg. [22, 20]). For such hypergraphs, the degree sequence is approximately $nPo(c)$. This approximation is close enough to produce accurate results, as is shown in Section 5. For the sequence $(l_i = c^i e^{-c}/i!,\ i \geq 0)$, we find

$$q(x) = cx\left(1 - e^{-cx}\left(1 + cx + \cdots + \frac{(cx)^{\kappa-2}}{(\kappa-2)!}\right)\right). \tag{9}$$

Let $y = cx$ so that $f(y) = 1 - e^{-y}\left(1 + \cdots + \frac{y^{\kappa-2}}{(\kappa-2)!}\right)$. The equation (8) becomes

$$\left(\frac{y}{c}\right)^{\frac{1}{r-1}} = f(y) = (r-1)\frac{y^{\kappa-1}}{(\kappa-2)!}e^{-y}. \tag{10}$$

As c increases the curve $h(y,c) = (y/c)^{1/(r-1)}$ drops until at some c it is tangent to $f(y)$. The conditions $h(y,c) = f(y)$ and $h'(y,c) = f'(y)$ give the derived condition that $h(y,c) = (r-1)(y^{\kappa-1}/(\kappa-2)!)e^{-y}$ and hence the critical values $(c^*, \widehat{y})$ given by the solutions of

$$\begin{aligned} (r-1)(\kappa-1) - 1 &= \frac{y}{\kappa} + \frac{y^2}{\kappa(\kappa+1)} + \cdots + \frac{y^j}{\kappa^{(j)}} + \cdots \\ c &= y\left(\frac{(\kappa-2)!e^y}{y^{\kappa-1}(r-1)}\right)^{r-1}. \end{aligned}$$

The second example is the emergence of $(d+1)$-cores in graphs with a mixture of vertices of degree d and $d+2$ in the proportions $1-c$ and c respectively[2]. For the sequence $(l_d = 1-c,\ l_{d+2} = c)$, then $\theta = d + 2c$ and

$$q(x) = (d+2)c\left((d+1)x^{d+1}(1-x) + x^{d+2}\right).$$

[2]This nice example was suggested by Andrzej Rucinski

Using $q(x) = \theta x f(x)$ and applying (8) gives the critical values for graphs as

$$\widehat{x} = \frac{d^2 - 1}{d^2} \qquad\qquad c^* = \frac{d^{2d}}{(d+1)(d+2)(d^2-1)^{d-1} - 2d^{2d-1}}.$$

For example, if $d = 1$, the threshold for a 2-core occurs when the proportion of vertices of degree 3 is 1/4.

2 Generating hypergraphs with a fixed degree sequence.

We say a hypergraph $G = (V, E)$ is simple if no edge $e \in E$ contains a repeated vertex, and there are no repeated edges in E (ie: E is not a multi-set). We remark that as far as the proofs in this paper are concerned there is nothing to prevent a more restrictive definition of simple, for example that no pair of edges share a common pair of vertices (the edges are of co-degree 1).

Let $\boldsymbol{G}(\boldsymbol{d})$ be the set of hypergraphs with vertex set $V = [n]$ and degree sequence $\boldsymbol{d}$. The standard method for generating uniformly distributed simple graphs (and hypergraphs) with a given degree sequence is to use the configuration model of Bollobás [2] which is a probabilistic interpretation of the counting formula of Bender and Canfield [1].

Let $W = [\theta n]$. Let $\boldsymbol{d} = (d_1, d_2, ..., d_n)$ and let $P(\boldsymbol{d})$ be the ordered partition $W_1, W_2, ..., W_n$ of W into sets of size $|W_i| = d_i$. Thus $(\max x \in W_i) + 1 = \min y \in W_{i+1}$. Define $\phi_{\boldsymbol{d}} : W \to [n]$ by $\phi(w) = i$ if $w \in W_i$. We refer to the elements of W as (*configuration*) *points*, and the sets W_i $i \in [n]$ as *vertices*. Relative to the partition $P(\boldsymbol{d})$, which remains fixed in all future discussions, the *degree sequence* of W is $\boldsymbol{d}$.

Let F be a partition of W into $m = \theta n/r$ sets of size r. Let Ω be the set of such partitions F of W. Thus $|\Omega| = (\theta n)!/((\theta n/r)!(r!)^{\theta n/r})$. We refer to the elements F of Ω as *configurations*, the elements of F as *edges*.

Let $\gamma(F)$ denote the multi-hypergraph with vertex set $[n]$ and edge multiset $E_F = \{\phi(e) = \{\phi(x_1), \phi(x_2), ..., \phi(x_r)\} : e = \{x_1, x_2, ..., x_r\} \in F\}$. Relative to the fixed partition $P(\boldsymbol{d})$ the *degree sequence* of $\gamma(F)$ is $\boldsymbol{d}$. Let $\boldsymbol{M}(\boldsymbol{d}) = \{\gamma(F) : F \in \Omega\}$ be the set of underlying multi-hypergraphs and $\boldsymbol{G}(\boldsymbol{d})$ the subset of simple hypergraphs. It is a standard result that under the mapping γ the subset $\boldsymbol{G}(\boldsymbol{d})$ has uniform measure if Ω does.

In the case of graphs, Frieze and Łuczak [10] describe a simple algorithm for obtaining a uar multi-graph configuration F from Ω. The algorithm which was used in [7] under the name CONSTRUCT, generalizes naturally to the uar generation of multi-hypergraph configurations:

Given a hypergraph G, the obvious way to find the κ-core is to repeatedly delete an edge incident with a vertex of degree less than κ in the remaining subgraph H, until either H is empty or all vertices have degree at least κ. We use CONSTRUCT to generate uniformly at random a configuration multi-hypergraph F with degree sequence $\boldsymbol{d}$. In the version of CONSTRUCT we use the edges $e_1, e_2, ..., e_{\theta n/r}$ of F are generated in an order suitable for deletion by the process given above to obtain the κ-core.

Algorithm CONSTRUCT
begin
  $W(0) := [\theta n]$
  $F(0) := \emptyset$
  $C(0) = (F(0), W(0))$
  For $s = 1$ **to** $\theta n/r$ **do**
  begin
    Choose a point $x_s \in W(s-1)$ *using a deterministic rule* **(R*)**
    Choose $r-1$ points $\boldsymbol{y}_s = (y_{s,2}, ..., y_{s,r})$
      uniformly at random without replacement from $W(s-1) \setminus \{x_s\}$
    $F(s) := F(s-1) \cup \{\{x_s, y_{s,2}, ..., y_{s,r}\}\}$
    $W(s) := W(s-1) \setminus \{x_s, y_{s,2}, ..., y_{s,r}\}$
    $C(s) = (F(s), W(s))$
  end
  $F := F(\theta n/r)$
end

The set $C(s) = (F(s), W(s))$ describes the output of CONSTRUCT after step s. It consists of a set of edges $F(s)$ and a set of remaining configuration points $W(s)$.

Let v be a vertex. At the end of step s, $W_v(s) = W_v \cap W(s)$ is the set of remaining configuration points of vertex v. Let $d(v, s) = |W_v(s)|$. We call $d(v, s)$ the *residual degree* of vertex v at (the end of) step s.

During step s, the algorithm generates the edge sequence $\boldsymbol{e}_s = (x_s, y_{s,2}, ..., y_{s,r})$. Let $\phi(e_s) = \{u, v_1, ..., v_{r-1}\} = \{\phi(x_s), \phi(y_{s,2}), ..., \phi(y_{s,r})\}$ denote the underlying multi-hypergraph edge generated. We refer to x_s as the *initial point* and to $y_{s,k}$, $k = 2, ..., r$ as the terminal points, and similarly to $u = \phi(x_s)$ as the *initial vertex* of the edge, and to $v_2 = \phi(y_{s,2}), ..., v_r = \phi(y_{s,r})$ as the *terminal vertices.* If the context is clear we suppress the ϕ, thus statements such as $v \in e_s$ mean $z \in e_s,\ v = \phi(z)$ etc.

Let $\boldsymbol{C}(s) = \{C(s)\}$ be the possible outputs of CONSTRUCT after step s. Because of the deterministic choice of x_s by **(R*)** we have $|\boldsymbol{C}(s)| = |\boldsymbol{C}(s-1)|(\theta n - r(s -$

$1) - 1)_{(r-1)}/(r-1)!$ and thus

$$|\boldsymbol{C}(s)| = \frac{(\theta n)_{(rs)}}{(\theta n/r)_{(s)}(r!)^s}. \tag{11}$$

Also, the probability that $C(s)$ is the output at step s is $1/|\boldsymbol{C}(s)|$.

Algorithm CONSTRUCT generates configuration multi-hypergraphs. The proofs in this paper are based on the analysis of CONSTRUCT. In order for the proofs to be valid for simple hypergraphs we require that the subset of configurations induced by $\boldsymbol{G}(\boldsymbol{d})$ has constant measure in Ω.

In general, the expected number of loops (edges with repeated vertices) is $O(\Delta^2)$. The expected number of repeated edges is $O(\Delta^{2r}/n^{r-2})$. This is $o(1/\sqrt{n})$ for $r \geq 3$ and $\Delta = o(n^{1/24})$. By requiring that θ and $\sum i(i-1)L_i/(\theta n)$ are bounded above by an absolute constant (condition S1 of the Separation Property) we can ensure that $\boldsymbol{G}(\boldsymbol{d})$ is large enough.

Lemma 3. *[6] Let θ be constant, let $\sum i(i-1)L_i/(\theta n)$ be finite and let the maximum degree $\Delta = o(n^{1/24})$, then $\mathbf{Pr}(G$ is simple $) > \epsilon(n)$ where $\epsilon(n) \geq A > 0$ and $\lim_{n\to\infty} \epsilon(n) = \epsilon > 0$.*

Finally, we remark that given the sequence of edges $e_1, ..., e_t$ generated at the end of step t of the algorithm, the remaining configuration is generated uar from the set $W(t)$. Thus, conditional on the residual degree sequence and the hypergraph being simple, the κ-core is a random hypergraph with the given degree sequence.

Properties of algorithm CONSTRUCT

We use a version of CONSTRUCT which first generates the edges outside the (possibly empty) κ-core $\kappa = 2, 3, ...$, and then generates the κ-core. It chooses initial point x_{t+1} using the following rule:

(R*): Select x_{t+1} from $W(t)$ as follows.

$$\begin{aligned} \delta(t) &= \min_{u\in V}\{|W_u(t)| : |W_u(t)| > 0\} \\ U(t) &= \{u : |W_u(t)| = \delta(t)\} \\ X(t) &= \cup_{u\in U(t)} W_u(t) \\ x_{t+1} &= \min\{y : y \in X(t)\} \end{aligned}$$

Informally we can write:

> **(R*)**: Select x_{t+1} from $W(t)$ by choosing the smallest point label from the vertices of minimum positive residual degree.

From now on let κ be fixed. For convenience we refer to the version of CONSTRUCT using **(R*)** above as CONSTRUCT(κ). This is to distinguish it from a closely associated version of CONSTRUCT, called CONSTRUCT-$[v]$ uses the following rule, which differs only in that vertex v is never selected as initial.

> **(R*)**: Select x_{t+1} from $W(t) - W_v(t)$ by choosing the smallest point label from the vertices of minimum positive residual degree other than v. If $W(t) = W_v(t)$ then stop.

Note that in this version of the algorithm it is possible that $d(v,t) < \kappa$ but x_{t+1} is chosen from a vertex of degree at least κ, so that the algorithm does not find the κ-core. Also the algorithm halts if only v has positive residual degree.

If however $d(v,t) \geq \kappa$ and there are vertices of degree greater than zero but less than κ then both versions of CONSTRUCT behave identically towards v. The rest of the section spells this out in detail.

Let $C(t) = (F(t), W(t))$ be some *fixed* output of CONSTRUCT after t steps, where $F(t)$ is the set of generated edges, and $W(t)$ is the set of unused configuration points. The algorithm has no memory. Given it is in the state $C(t)$ after step t the outputs after step $t+\tau$, which we denote by $\boldsymbol{C}(t+\tau\,; C(t))$, are exactly those obtained by running it for τ steps with initial input $(F(t), W(t))$.

In what follows we calculate the probability $\mathbf{Pr}(d(v,t) = j)$ as a function of v, t, j and other variables, particularly $\boldsymbol{d}, n$. Generally we have $\mathbf{Pr}(d(v,t) = j, d(v,s) = i, \boldsymbol{d}(s), \boldsymbol{d}, n, Alg\ )$, where $i \geq j$, $s \leq t$, $\boldsymbol{d}(s)$ is the residual degree sequence at step s and Alg is the version of CONSTRUCT. In its simplest form it is $\mathbf{Pr}(d(v,t) = j, \boldsymbol{d}, n,$ CONSTRUCT(κ)). In CONSTRUCT(κ), the probability $\mathbf{Pr}(d(v,t) = j)$ is not a function of v, and any other vertex w such that $d(w,0) = d(v,0)$ would give the same value.

From now on let v be a fixed vertex. For CONSTRUCT(κ) let $\boldsymbol{C}^\kappa(j,t)$ be the subset of outputs $C(t) = (F(t), W(t))$ of $\boldsymbol{C}^\kappa(t)$ such that $d(v,t) = j$ (ie: $|W_v(t)| = j$). Let P_κ denote probability over outputs of CONSTRUCT(κ). For CONSTRUCT-$[v]$, we similarly use the notation $\boldsymbol{C}^v(t)$, and P_v respectively.

Let $d(v,0) = i$, and for CONSTRUCT-$[v]$ let $\pi_{i,j}(0,t;v) = P_v(d(v,t) = j)$. Thus

$$\pi_{i,j}(0,t;v) = \frac{|\boldsymbol{C}^v(j,t)|}{|\boldsymbol{C}^v(t)|}.$$

The notation $\pi_{i,j}(0,t) = \pi_{i,j}(t)$ assumes we start at time 0 with $d(v,0) = i$ and $W(0) = \theta n$ configuration points. The notation $\pi_{i,j}(s,t)$ assumes $d(v,s) = i$ and we start at step $s+1$ with $W(s) = \theta n - rs$ points. The values of these probabilities are given by the following lemma, which is proved in the Appendix.

Lemma 4. *Let* $x(s,t) = ((\theta n - rt)/(\theta n - rs))^\rho$ *where* $\rho = (r-1)/r$. *Let*

$$\Delta_{i,j}(s,t) = \binom{i}{j} x^j (1-x)^{i-j}. \tag{12}$$

Let $d(v,s) = i$, *then in* CONSTRUCT-$[v]$

$$\pi_{i,j}(s,t;v) = \Delta_{i,j}(s,t)\left(1 + O\left(\frac{\Delta^3}{\theta n - rt} + \frac{\Delta^3}{t-s}\right)\right).$$

Remark 1. *We note that*

$$\sum_{i=j}^{l} \Delta_{l,i}(0,s)\Delta_{i,j}(s,t) = \Delta_{l,j}(0,t). \tag{13}$$

To see this, let $x = \left(\frac{\theta n - rs}{\theta n}\right)^\rho$, $y = \left(\frac{\theta n - rt}{\theta n - rs}\right)^\rho$ then $xy = \left(\frac{\theta n - rt}{\theta n}\right)^\rho$. The result follows from

$$\sum_i \binom{l}{i} x^i (1-x)^{l-i} \binom{i}{j} y^j (1-y)^{i-j} = \binom{l}{j} (xy)^j (1-xy)^{l-j}.$$

Let $L_j(t) = |\{u : d(u,t) = j\}|$ count the vertices of residual degree j in $W(t)$. Given an output $C(t)$, let $\boldsymbol{L}(t) = (L_0(t), L_1(t), ..., L_\Delta(t))$. Let $A(t) = L_1(t) + 2L_2(t) + \cdots + (\kappa-1)L_{\kappa-1}(t)$, be the number of *available* points.

We say $C(t)$ is τ-*good* if $A(t) \geq r\tau$. The following lemma is pivotal in the proof.

Lemma 5. *Let* $i \geq j \geq \kappa$. *For* $C(t+\tau) \in \boldsymbol{C}(t+\tau\,; C(t))$ *we have in* CONSTRUCT*(*κ*) that*

$$P_\kappa(d(v,t+\tau) = j \mid d(v,t) = i, C(t) \text{ is } \tau\text{-good}) = \pi_{i,j}(t,t+\tau;v).$$

Proof We claim that

$$\begin{aligned} |\boldsymbol{C}^v(t+\tau\,; C(t))| &= |\boldsymbol{C}^\kappa(t+\tau\,; C(t))|, && (14)\\ \boldsymbol{C}^v(j,t+\tau\,; C(t)) &= \boldsymbol{C}^\kappa(j,t+\tau\,; C(t)). && (15) \end{aligned}$$

Because $d(v,t) = i \geq \kappa$ and $C(t)$ is τ-good (ie. $A(t) \geq r\tau$) then for any T, $t+1 \leq T \leq t+\tau$ there is always an $x_T \notin W_v(T-1)$ for CONSTRUCT-$[v]$ to select, and it has not halted by the end of step $t+\tau$.

The outputs obtained are equivalent to running the appropriate version of CONSTRUCT for τ steps, starting with $C(t)$ (ie: $|W(0)| = \theta n - rt = \Theta n$). From (11) a non-halting version of CONSTRUCT starting with Θn points has $|\boldsymbol{C}(\tau)| = (\Theta n)_{(r\tau)}/(\Theta n/r)_{(\tau)}(r!)^\tau$ possible outputs after τ steps. This proves (14).

For (15) we note first that because $A(t) \geq r\tau$ either version of CONSTRUCT always chooses an initial point x_T from a vertex of degree $d(w,T) < \kappa$ at each step $T = t+1, ..., t+\tau$.

It is enough to give a one step induction. Let $t+1 \leq T \leq t+\tau$. Given $C(T-1)$ such that $d(v, T-1) \geq \kappa$ the same choice of $x_T \notin W_v(T-1)$ is made by rule **(R*)** in both algorithms. Thus the sets $\boldsymbol{C}(T\ ; C(T-1))$ of CONSTRUCT-$[v]$ and CONSTRUCT(κ) are the same. □

Lemma 6. *If $C(t)$ is τ-good then for all $j \geq \kappa$*

$$P_\kappa\left(|L_j(t+\tau) - \mathbf{E}L_j(t+\tau)| > \sqrt{8\alpha(r-1)^2\tau\log n}\ \mid\ C(t)\right) = O(n^{-\alpha}).$$

The probability is over outputs $\boldsymbol{C}^\kappa(t+\tau\ ; C(t))$ of CONSTRUCT(κ) *given the input $C(t) = (F(t), W(t))$.*

Proof We use Azuma's martingale inequality. Let $X_0, ..., X_m$ be a martingale with X_0 constant and $|X_{i+1} - X_i| \leq c,\ 0 \leq i \leq m-1$, then

$$\mathbf{Pr}(|X_m - X_0| \geq \omega) \leq 2e^{-\omega^2/2mc^2}.$$

The martingale X_i is the conditional expectation of $L_j(t+\tau)$ given $C(t+i)$, $i = 0, ..., \tau-1$. We prove that $|X_{i+1} - X_i| \leq 2(r-1)$.

All outputs $f \in \boldsymbol{C}^\kappa(t+\tau\ ; C(t))$ are equiprobable. Without loss of generality we establish a bijection $\psi(f) = f'$ between outputs $f = (e_1, ..., e_\tau)$ and any other output $f' = (e'_1, ..., e'_\tau)$ where the difference occurs on the first (ordered) edge e'_1.

Let the digraph Γ have vertex set $W = \cup W_v$ and edge set $f \to \psi(f)$. We can write $f = (z_1, ..., z_{r\tau})$ so that $f(j) = (z_1, ..., z_j)$. Thus for $s = 0, ..., r\tau$, $\Gamma(s)$ has edges $f(s) \to \psi(f(s))$ and $\Gamma = \Gamma(r\tau)$.

Let $e_1 = (x_1, y_{1,2}, ..., y_{1,r})$ and let $e'_1 = (x'_1, y'_{1,2}, ..., y'_{1,r})$. Certainly $x_1 = x'_1$ as it is chosen deterministically from $W(t)$. Let $y_{1,i} \to y'_{1,i},\ i = 2, ..., r$.

We now consider $e_j, j \geq 2$. Let $x_j \to x'_j$ where x'_j is chosen deterministically by **(R*)** given $f'(r(j-1))$. Let $y_{j,i} \to y_{j,i}$ if $y_{j,i} \notin f'(r(j-1)+i-1)$. If $y_{j,i}$ is already used in f' then it has indegree 1 and out degree 0 in the digraph $\Gamma(r(j-1)+i-1)$. Hence it is the terminal vertex of a path in that digraph with initial vertex u of indegree 0 (ie: $u \in f, u \notin f'$). Let $y_{j,i} \to u$ completing a cycle in the digraph Γ.

We now prove that $|L_j(f) - L_j(\psi(f))| \leq 2(r-1)$. Note there are at most $2(r-1)$ vertices of degree j in f (or $\psi(f)$) incident with e_1 or e'_1. We claim that if $v \notin e_1,\ v \notin e'_1$ and $d(v, t+\tau, f) = j \geq \kappa$ then $d(v, t+\tau, \psi(f)) = j$.

If $d(v, t+\tau, f) \geq \kappa$ then $d(v,t) \geq \kappa$. Let $s \geq 2$ be the first step, if any, that a point $z = z_{r(s-1)+i}$ of v is selected in f or f'. z is not in $x_s \to x'_s$ as, by guarantee, only points of degree less than κ are selected as initial, and $d(v, t+s-1, f') = d(v, t+s-1, f) \geq \kappa$.

If z is selected for f then $z \to z$ so the degree of v in f and f' remains the same. What if z is selected for f' but not f? Then $w \to z$ where w is already selected in f' and hence z is the initial vertex of a path in $\Gamma(r(s-1)+i-1)$ terminating at w. Thus z was previously selected in f but not f'. This is a contradiction, as no points of v were selected previously (or, general induction: all previously selected points of v were mapped to themselves). $\square$

3 The size of the κ-core

For the purpose of analysis, we say CONSTRUCT is in *Phase I* at step t, if for all steps $T \leq t$ the point x_T chosen as initial comes from a vertex of degree less than κ (ie: $\forall_{T \leq t}\ d(\phi(x_T), T-1) < \kappa$). If some vertex of degree at least κ has been chosen as initial by the end of step t, we say the algorithm is in *Phase II*.

During Phase I of CONSTRUCT(κ), the initial vertex $v = \phi(x_s)$ of the edge we generate (and remove) at each step s has residual degree at most $\kappa - 1$. Let $\sigma + 1$ be the first step at which the minimum residual degree is at least κ. Thus σ is the stopping time of Phase I. The remaining (ungenerated) multi-hypergraph $H = G - \{e_1, ..., e_\sigma\}$ is the κ-core of the multi-hypergraph G. At the end of Phase I the output is $C(\sigma) = (F(\sigma), W(\sigma))$. We have generated and removed edges $F(\sigma) = \{e_1, ..., e_\sigma\}$. Phase II then generates a random κ-core configuration conditional on the degree sequence of $W(\sigma)$.

Let $L_i(t)$ be the number of vertices of residual degree i at the end of step t of CONSTRUCT(κ), and let $L_i(0) = L_i$. Let $Q(t)$ be the total degree of vertices of residual degree at least κ. Thus

$$Q(t) = \sum_{i \geq \kappa} i L_i(t). \tag{16}$$

At the end of step t, $A(t)$ the total degree of vertices of residual degree less than κ, is related to $Q(t)$ by

$$A(t) + Q(t) + rt = \theta n. \tag{17}$$

The stopping time σ for Phase I is defined as $\sigma = \min\{s : A(s) = 0\}$.

Recall that we have defined $q(t)$ by

$$q(x(t)) = \sum_{i \geq \kappa} l_i \sum_{j=\kappa}^{i} j \binom{i}{j} x^j (1-x)^{i-j},$$

where $l_i = L_i/n$ and $x(t) = (1 - rt/\theta n)^\rho$. Also

$$a(t) = \theta\left(1 - \tfrac{rt}{\theta n}\right) - q(t),$$

and that $\widehat{\sigma}$ is the smallest non-negative solution of $a(t) = 0$. We claim that **whp** $\widehat{\sigma}$ is a good approximation to σ.

Let

$$\tau = \lceil n^{2/3}\Delta^{4/3}\log n\rceil \tag{18}$$

$$\sigma_0 = \widehat{\sigma} - K\tau, \qquad \sigma_1 = \widehat{\sigma} + K\tau, \tag{19}$$

where K is a large positive (generic) constant. We prove that for $t \leq \sigma_0$ **whp** CONSTRUCT(κ) is in Phase I (or $\sigma_0 \leq 0$) and for $t \geq \sigma_1$ **whp** CONSTRUCT(κ) is in Phase II (or $\sigma_1 > \theta n/r$).

To do this we exhibit a sequence of steps $t_k = k\tau, k = 0, ..., \lceil\sigma_0/\tau\rceil$ such that **whp** $A(t_k) \geq r\tau$. Conditional on $A(t_k) \geq r\tau$, we can *guarantee* that CONSTRUCT(κ) proceeds to step t_{k+1} using only initial points x_t from vertices of degree less than κ. Thus we are always removing a non-core edge and the algorithm remains in Phase I.

Lemma 7. *Let σ_0, σ_1 be given by (19). Let $\widehat{\sigma}$ be the smallest solution of $a(t) = 0$, and let $\tau\log n \leq \widehat{\sigma} \leq \theta n/r - \tau\log n$, then* **whp** $\sigma \in [\sigma_0, \sigma_1]$.

Proof **Lower Bound.** Let $S = \lceil\sigma_0/\tau\rceil$ and for $k = 0, 1, ..., S$ let $t_k = k\tau$. Let $\lambda = O(\Delta^3/\tau)$ and let $\xi = O(\sqrt{2(r-1)^2\alpha\tau\log n})$. Recall that $\Delta = o(n^{1/24})$.

The proof is inductive. The induction is from t_k to t_{k+1}, on the proposition that for $\kappa \leq j \leq \Delta$ with probability $1 - O(k\Delta n^{-\alpha})$

$$L_j(t_k) = k\xi + (1 + k\lambda)\sum_{i\geq j} L_i\Delta_{i,j}(0, k\tau),$$

where $\Delta_{i,j}(0, t)$ is given by (12). Given $\boldsymbol{L}(t_k)$ satisfying this, from Lemma's 4, 5 and 6, with probability $1 - O(n^{-\alpha})$

$$\begin{aligned}
L_j(t_{k+1}) &= \mathbf{E}L_j(t_{k+1}) + \xi \\
&= \xi + \sum_{i\geq j} L_i(t_k)\pi_{i,j}(t_k, t_k + \tau) \\
&= \xi + \sum_{i\geq j}\pi_{i,j}(t_k, t_k+\tau)\left(k\xi + (1 + k\lambda)\sum_{l\geq i} L_l\Delta_{l,i}(0, k\tau)\right) \\
&= (k+1)\xi + (1 + (k+1)\lambda)\sum_{l\geq i} L_l\sum_{i\geq j}\Delta_{l,i}(0, k\tau)\Delta_{i,j}(k\tau, (k+1)\tau) \\
&= (k+1)\xi + (1 + (k+1)\lambda)\sum_{l\geq j} L_l\Delta_{l,j}(0, (k+1)\tau).
\end{aligned}$$

Thus

$$\begin{aligned}
Q(t_k) &= \sum_{j\geq\kappa} jL_j(t_k) \\
&= \Delta^2 k\xi + (1+k\lambda)\sum_{i\geq j} L_i \sum_{j\geq\kappa} j\Delta_{i,j}(0,t_k) \\
&= \Delta^2 k\xi + (1+k\lambda)nq(t_k).
\end{aligned}$$

Thus for $k \leq S$

$$|Q(t_k) - nq(t_k)| = \Delta^2 k\xi + k\lambda nq(t) = o(\tau).$$

Hence

$$|A(t_k) - na(t_k)| = o(\tau). \tag{20}$$

By condition S2 $na(t) \geq K\tau$ for $t \leq \sigma_0$. Thus $A(t_k) \geq r\tau$.

Upper Bound. The expected number of new points added to $A(t)$ during step t of Phase I is

$$(1+o(1))(r-1)(\kappa-1)\frac{\kappa L_\kappa(t)}{\theta n - rt}.$$

We prove below that **whp**

$$(r-1)\kappa(\kappa-1)\frac{L_\kappa(\sigma_0)}{\theta n - r\sigma_0} = 1 - c + o(1), \tag{21}$$

where $0 < c \leq 1$ constant is given by $c = -na'(\widehat{\sigma})$. We complete the proof of the upper bound as follows: Let $F(s)$ be the number of new points added to $A(s)$ at step $s = \sigma_0+1, ..., \sigma_0+T$, where $T = K\tau$. Thus $\sum F(s)$ is stochastically dominated by the sum of non-negative independent random variables bounded above by $(r-1)(\kappa-1)$ and with expectation

$$(1+o(1))(r-1)\kappa(\kappa-1)\frac{L_\kappa(\sigma_0) + (s-\sigma_0)(r-1)}{\theta n - rs}.$$

If $(1-c) = o(1)$ then **whp** $\sum_{s=\sigma_0+1}^{\sigma_0+T} F(s) = o(T)$ and Phase I ends in $A(\sigma_0) + O(T) = O(K\tau)$ steps. If $(1-c) > 0$ constant then

$$\mathbf{Pr}(\sum_s F(s) \geq (1+\epsilon)T(1-c)) \leq \exp\left(-\frac{\epsilon^2 T(1-c)^2}{3(r\kappa)^2}\right),$$

and **whp**

$$A(\sigma_0+T) \leq A(\sigma_0) + (1+\epsilon)T(1-c) - T.$$

As (**whp**) $A(\sigma_0) = O(\tau)$, Phase I halts after at most $O(K\tau)$ steps.

We next prove (21). Let $I = [\sigma_0 - \tau, ..., \sigma_0 - 1]$. Note that

$$\begin{aligned}\mathbf{E}(A(\sigma_0) \mid A(\sigma_0 - \tau)) =& A(\sigma_0 - \tau) - \tau - (1 + o(1))(r-1)\sum_{t\in I} \frac{A(t)}{\theta n - rt} \\ &+ (1+o(1))(\kappa - 1)(r-1)\sum_{t \in I} \frac{\kappa L_\kappa(t)}{\theta n - rt}.\end{aligned}$$

By Lemma 6 we have **whp**

$$|A(\sigma_0) - \mathbf{E}(A(\sigma_0) \mid A(\sigma_0 - \tau))| = o(\tau).$$

Also

$$\begin{aligned}|A(t+1) - A(t)| &\le \max(r, (\kappa-1)(r-1) - 1) \\ |L_\kappa(t+1) - L_\kappa(t)| &\le (r-1).\end{aligned}$$

Thus for $X = A, L_\kappa$ we have

$$\sum_{t \in I} \frac{X(t)}{\theta n - rt} = \frac{\tau(X(\sigma_0) + O(\tau))}{\theta n - r\sigma_0}.$$

From (20), **whp**

$$A(\sigma_0) - A(\sigma_0 - \tau) = na(\sigma_0) - na(\sigma_0 - \tau) + o(\tau).$$

For $h = O(\tau)$ by condition S3

$$a(\widehat{\sigma} + h) = ha'(\widehat{\sigma})(1 + o(1)).$$

Putting this all together gives (21). $\square$

Let $M(t)$ be the number of vertices of residual degree at least κ and let

$$m(t) = \sum_{i \ge \kappa} l_i \sum_{j=\kappa}^{i} \binom{i}{j} x^j \left(1 - x\right)^{i-j}. \tag{22}$$

Let $L_{i,j}(t)$ be the number of vertices v such that $d(v,0) = i,\ d(v,t) = j \ge \kappa$ and let

$$l_{i,j}(t) = l_i \binom{i}{j} x^j (1-x)^{i-j}. \tag{23}$$

The following corollary summarizes results deriving from the previous lemmas.

Lemma 8. *With probability* $1 - o(1)$ *for all* $\kappa \le j \le i \le \Delta$ *and for all* $t \le \sigma_1$

$$\begin{aligned}|L_{i,j}(t) - nl_{i,j}(t)| &= O(\tau) \\ |L_i(t) - nl_i(t)| &= O(\tau) \\ |M(t) - nm(t)| &= O(\tau) \\ |A(t) - na(t)| &= O(\tau) \\ |Q(t) - nq(t)| &= O(\tau).\end{aligned}$$

We next consider the case where $\widehat{\sigma} > \theta n/r - \tau \log n$.

Lemma 9. *Let $\Delta = o(n^{1/24})$ and let $r = 2,\ \kappa \geq 3$ or $r \geq 3,\ \kappa \geq 2$, then* **whp** *a κ-core of size $O(\tau \log n)$ is empty.*

Proof We use a crude counting argument in the configuration model. Let $C(s,t)$ be the probability there exists a κ-core of s vertices and t edges. Let $d(v,S)$ denote the degree of vertex v in $S \subseteq V$ and let

$$\Omega(s,t) = \{S : |S| = s,\ S = \{v_1, ..., v_s\},\ d(v_i,S) \geq \kappa,\ \sum_{v_i \in S} d(v_i,S) = rt\}.$$

Thus $C(s,t) = O(D(s,t))$ where

$$D(s,t) = \sum_{\Omega(s,t)} \binom{d(v_1) + \cdots + d(v_s)}{rt} \frac{\binom{\theta n/r}{t}}{\binom{\theta n}{rt}}.$$

Let $1 \leq s \leq \tau \log n$. Replacing $d(v_i)$ by Δ we obtain

$$\begin{aligned} D(s,t) &\leq \binom{n}{s} \frac{(\Delta s)^{rt}}{rt!} \frac{\binom{\theta n/r}{t}}{\binom{\theta n}{rt}} \\ &\leq \left(\frac{ne}{s}\right)^s \left(\frac{\Delta se}{\theta n}\right)^{rt} \left(\frac{\theta ne}{rt}\right)^t. \end{aligned}$$

Note that $rt = \kappa s + j,\ j \geq 0$, and thus for $r \geq 2$

$$\left(\frac{\Delta se}{\theta n}\right)^r \frac{\theta ne}{rt} = O\left(\Delta \left(\frac{\Delta s}{n}\right)^{r-1}\right) = O\left(n^{1/24} \left(\frac{\log^2 n}{n^{17/72}}\right)^{r-1}\right) = o(1).$$

Thus

$$\begin{aligned} \sum_s \sum_{rt \geq \kappa s} D(s,t) &\leq \sum_s \left(\left(\frac{ne}{s}\right) \left(\frac{\Delta se}{\theta n}\right)^\kappa \left(\frac{\theta ne}{rs}\right)^{\kappa/r}\right)^s \\ &\leq \sum_s O\left(\Delta^\kappa \left(\frac{s}{n}\right)^{\kappa\left(1-\frac{1}{r}\right)-1}\right)^s, \end{aligned}$$

which is $o(1)$ provided $\kappa \geq 3$ when $r = 2$, or $\kappa \geq 2$ when $r \geq 3$. □

The case when $\widehat{\sigma} < \tau \log n$ follows from the upper bound proof in Lemma 7.

This completes the proof of Theorem 1 and the main part of Theorem 2.

4 Results on 2-cores of graphs

We estimate the number of small tree components, and in the case of smooth sequences the size of the mantle and number of edges in small components. Also, for 2-cores of graphs we establish simpler conditions which imply Theorems 1, 2.

Let $\boldsymbol{d}$ be a degree sequence satisfying Theorem 2. The number of vertices $V_2(G)$ in the 2-core of an r-uniform hypergraph $G \in \boldsymbol{G}(\boldsymbol{d})$ is **whp** $V_2(G) = \nu n + O(\tau)$ where

$$\nu = 1 - \theta \widehat{x} \left(1 - \widehat{x}^{1/(r-1)}\right) - \sum_{i \geq 0} l_i (1-\widehat{x})^i. \tag{24}$$

We now consider isolated tree components in graphs ($r = 2$). Let $R(0)$ count isolated vertices of G, (ie. $d_i = 0$), and for $T > 0$ let $R(T)$ also count vertices whose last remaining configuration point was selected as the terminal point y_t of the edge $e_t = (x_t, y_t)$ generated during steps $t \leq T$ of Phase I. At the stopping time σ of Phase I, $R(G) = R(\sigma)$ is the number of isolated tree components.

Lemma 10. **whp** $R(G) = n\psi + O(\tau)$, *where*

$$\psi = \sum_{i \geq 0} l_i (1-\widehat{x})^i - \frac{\theta}{2}(1-\widehat{x})^2. \tag{25}$$

Proof In the case of 2-cores of graphs, $A(t)$ is also the number of vertices of degree 1. The balance equation for vertices is

$$n = M(t) + A(t) + t + R(t),$$

where $M(t)$ is the number of vertices of residual degree at least 2 at the end of step t. At σ, $A(\sigma) = 0$ so that $R(\sigma) = n - \sigma - M(\sigma)$. The result follows from (24) which gives the value of $\nu = m(\widehat{\sigma})$. □

The structure of graphs with a smooth degree sequence can now be deduced from the above results and the work of Molloy & Reed [18, 19].

A sequence $(L_i,\ i = 1, 2, ...)$, giving the number of vertices of degree i, is *smooth* in the sense used by [18, 19] if $\lim_{n\to\infty} L_i/n = \lambda_i$ where λ_i is constant and $l_i = \lambda_i + o(1)$. Thus $\lambda_i = \lim_{n\to\infty} l_i$ should such a limit exist. [18, 19] gives the size of the giant component for such sequences.

For 2-cores, rearranging the definition of $q(x)$ in (4) gives

$$q(x) = \theta x - x \sum_{i \geq 1} i l_i (1-x)^{i-1}.$$

Let $\gamma = 1 - \widehat{x}$ and $q_{i-1} = p_i = i l_i / \theta$. The condition $a(\widehat{\sigma}) = 0$ can be written $q(\widehat{x}) = \theta \widehat{x}^2$. This is satisfied if $\gamma = 1$ or if

$$\gamma = \sum_{i \geq 0} q_i \gamma^i. \tag{26}$$

Equation (26) is the equation for the extinction probability of an i.i.d. branching process X with distribution of progeny $\{q_i : i \geq 0\}$ and expected value $\mathbf{E}X = d$. Provided $q_0 > 0$, such a branching process only has a solution $\gamma \neq 1$ if $d > 1$. The condition $d > 1$ corresponds to the **whp** existence of a non-trivial 2-core (ie. $\widehat{x} > 0$). This condition can be written as $\sum_{i \geq 0} i(i-1)l_i > \theta$ or equivalently, using $\theta = \sum il_i$ as

$$\sum_{i \geq 0} i(i-2)l_i > 0.$$

This is the Molloy-Reed condition for the **whp** existence of a giant component.

We sketch in some structural details on the size of the mantle and number of edges in small components. Theorem 1 of [18] gives the (**whp**) proportion[3] ϵ of vertices in the (unique) giant component as

$$\epsilon = 1 - \sum_{i \geq 0} \lambda_i \gamma^i,$$

where $\gamma = 1 - \widehat{x}$ is the solution of (26). But $\epsilon = \mu + \nu$, where μ is the proportion of vertices in the mantle of the giant (arborescences rooted on the 2-core), and ν the proportion of vertices in the 2-core, given by (24) is $\nu = 1 - \theta\gamma(1-\gamma) - \sum_{i \geq 0} \lambda_i \gamma^i$. Thus the proportional size of the mantle is $\mu = \theta\gamma(1-\gamma)$.

The proportion of edges δ in the giant is given by $\delta = \mu + \eta$ where η is given by $\theta/r - \widehat{\sigma}/n$. Thus the proportion of edges on small components, ϕ, is $\phi = \frac{\theta}{2} - \delta = \frac{\theta}{2}\gamma^2$. The papers [18, 19] show that **whp** the number of vertices on small non-tree components is $o(n^{1/4} \log^3 n)$.

4.1 Conditions implying the Separation Property

Lemma 11. *Let $d = \sum_{i \geq 0}(i-1)il_i/\theta$. For $\kappa = 2, r = 2$, provided $\Delta = o(n^{1/24})$, $l_1 > 0$ and $d > 1$ then the conditions of Theorems 1, 2 hold.*

Proof We first prove that $\tau \log n \leq \widehat{\sigma} \leq \theta n/r - \tau \log n$. Let $z = 1 - (1 - 2t/\theta n)^{\frac{1}{2}}$. Let $g(z) = \sum_{i \geq 0}((i+1)l_{i+1}/\theta)z^i$. As $d > 1$ and $l_1 > 0$ the smallest solution of the equation $z = g(z)$ is the unique solution γ in $(0, 1)$.

We next prove the Separation Property. We have that

$$a(z) = \theta(1-z)(g(z) - z).$$

[3] When we refer to the *proportion* ξ of vertices (resp. edges, components etc) with the stated property, it is to be understood that the associated random variable $X(G) = n\xi + O(\tau)$ **whp**.

Certainly $g(z) > z$ for $z < \gamma$. The function $g(z) - z$ is monotone decreasing in $[0, \gamma + \epsilon]$ for some $\epsilon > 0$. This follows because $g'(z)$ is an increasing function of z, and at $z = \gamma$, the slope of $y = g(z)$ is less than the slope of $y = z$. Thus $0 < g'(\gamma) < 1$. It follows from $g'(\gamma) < 1$ and $dz/dt = 1/((1-z)\theta n)$ that

$$a'(\widehat{\sigma}) = \theta(1-\gamma)(g'(\gamma) - 1)z'(\widehat{\sigma}) = (g'(\gamma) - 1)/n.$$

That $a(t)$ has a uniform Taylor expansion about $\widehat{\sigma}$ is similarly proved. □

5 Poisson degree sequences

As a sanity check, we reassure ourselves that the results of Theorems 1, 2 and Lemma 8 for a Poisson $Po(c)$ degree sequence give the standard results for the κ-core of the hypergraph models $G_{n,p}, G_{n,m}$ (see eg: [11, 12, 14, 22, 23]). The degree sequence $(L_i,\ i \geq 0)$ of random hypergraphs $G_{n,p},\ p = c/n$ and $G_{n,m},\ m = \lceil cn/r \rceil$ is (**whp**) approximately the Poisson sequence $(n\lambda_i,\ i \geq 0)$ where $\lambda_i = (c^i/i!)e^{-c}$.

What kind of error do we introduce by replacing (L_i) by $(n\lambda_i)$? We prove below that the error in $\widehat{\sigma}$ and the properties $X(G)$ of the cores of G (number of vertices, edges, degree sequence) is $O(\sqrt{n} \log^5 n)$, which is within the $O(\tau)$ bounds of this paper.

We consider the case of $G_{n,p}, p = c/n$. Let λ refer to $(n\lambda_i)$ and l to (L_i). The solution of $a(t, \lambda) = 0$ is at $x^{1/(r-1)} = f(x)$ where $x = (1 - rt/\theta n)^{(r-1)/r}$ and

$$f(x) = 1 - e^{-cx}\left(1 + \cdots + \frac{(cx)^{\kappa-2}}{(\kappa-2)!}\right).$$

We next establish that, for suitable λ, $a(t, \lambda)$ satisfies the separation properties.

If a solution $\widehat{x} > 0$ of $x^{1/(r-1)} = f(x)$ exists then $\widehat{x} < 1$. This means that $\widehat{\sigma}(\lambda) = \theta n/r(1 - x^{r/(r-1)}) = d_1 n$ and $\theta n/r - \widehat{\sigma}(\lambda) = d_2 n$ where $d_1, d_2 > 0$ constant. Certainly S2 is satisfied.

We now prove S3 is satisfied above the threshold value. Referring to the discussion below (9), as $\theta = c$ increases above c^*, the curve $(y/c)^{1/(r-1)}$ cuts $f(y)$ at two points. The larger value $y = c\widehat{x}$ is above the threshold value $\widehat{y}(c^*)$ derived from (10). Thus $f(y) > (r-1)(y^{\kappa-1}/(\kappa-2)!)e^{-y} = (r-1)x\partial f(x)/\partial x$. This follows because for $r \geq 3$ or for $r = 2$ and $\kappa \geq 3$ we have $\widehat{y}(c^*) > 1$ and hence the curve $(r-1)(y^{\kappa-1}/(\kappa-2)!)e^{-y}$ is decreasing at $\widehat{y}$ whereas the other curves are monotone increasing. As

$$\frac{\partial a(x)}{\partial x} = \frac{c}{\rho}\left(x^{1/(r-1)} - f + \frac{1}{r}\left(f - (r-1)x\frac{\partial f(x)}{\partial x}\right)\right)$$

we see that $\partial a(x)/\partial x > 0$ at $\widehat{x}$ and

$$\frac{\partial a(t)}{\partial t} = \frac{\partial a(x)}{\partial x}\frac{\partial x}{\partial t} = -\frac{\rho r}{cnx^{1/(r-1)}}\frac{\partial a(x)}{\partial x} < 0$$

at $t = \widehat{\sigma}$ as required in S3.

Say (L_i) is *nice* if for $i = \Theta(\log n)$, $L_i = 0$, and for $i = o(\log n)$, the number of vertices of degree i is

$$L_i = n\frac{c^i}{i!}e^{-c} + O(i\sqrt{n}\log n).$$

whp(L_i) is nice. We prove that nice sequences satisfy the Separation Properties and that

$$\widehat{\sigma}(l) = \widehat{\sigma}(\lambda) + O\left(\sqrt{n}\log^5 n\right). \tag{27}$$

Note that $|\theta(l) - \theta(\lambda)| = O(\log^3 n/\sqrt{n})$ and $|a(t,l) - a(t,\lambda)| = O(\log^4 n/\sqrt{n})$. Assume $\widehat{\sigma}(\lambda) < \widehat{\sigma}(l)$. Let $h = \log^5 n\sqrt{n}$. By S3 for $a(t,\lambda)$ we have $a(\widehat{\sigma}(\lambda) + h, \lambda) = -K(1+o(1))\log^5 n/\sqrt{n}$. But $|a(t,l) - a(t,\lambda)| = O(\log^4 n/\sqrt{n})$ and thus $a(\widehat{\sigma}(\lambda) + h, l) < 0$ which proves (27). It also proves that $\widehat{\sigma}(l)$ is suitably bounded away from $0, \theta n/r$ and Theorem 2 holds. Furthermore it follows that $|a'(t,l) - a'(t,\lambda)| = O(\log^5 n/\sqrt{n})$ for $t \leq (1+\epsilon)\widehat{\sigma}(\lambda)$ where $\epsilon > 0$ constant, giving S3. Finally for any property X of Theorem 2

$$|X(l) - X(\lambda)| = O(\sqrt{n}\log^5 n) = o(\tau).$$

5.1 2-cores of Poisson hypergraphs

In the case of 2-cores of r-uniform Poisson hypergraphs the size of the 2-core is determined by the largest solution $\widehat{x}$ in $[0,1]$ to

$$\widehat{x} = (1 - e^{-c\widehat{x}})^{r-1}. \tag{28}$$

The (limiting) proportion of edges, η and vertices, ν, in the 2-core is given by

$$\eta = \frac{c}{r}\widehat{x}^{\frac{r}{r-1}} \tag{29}$$

$$\nu = \widehat{x}^{\frac{1}{r-1}} - c\widehat{x} + c\widehat{x}^{\frac{r}{r-1}}. \tag{30}$$

We note that ν can also be written as $\nu = 1 - e^{-c\widehat{x}}(1 + c\widehat{x})$.

Some threshold values $(c^*, \widehat{x})$ for emerging 2-cores of r-uniform hypergraphs are tabulated below.

r	3	4	5	6	7
c^*	2.45	3.1	3.5	3.8	4.0
$\widehat{x}$	0.52	0.62	0.67	0.70	0.72
ν	0.36	0.57	0.68	0.74	0.78
η	0.31	0.41	0.42	0.41	0.39

The relationship (28) and the results in the table are given in the paper [20].

From (23), the proportion of vertices of the 2-core with degree $j \geq \kappa$ (in the 2-core) is $Po(c\widehat{x})$, and the proportion of vertices of the 2-core with degree $J \geq 0$ in the mantle is

$$\nu \, \frac{(c(1-\widehat{x}))^J}{J!} \, e^{-c(1-\widehat{x})}.$$

For graphs, the detailed structure of the mantle is given in [21].

5.2 Upper bounds for the rank of random Boolean matrices

Consider a $n \times m$ matrix whose columns consist of the vertex-edge incidence vectors of a random r-uniform hypergraph on n vertices. The m columns are independent and sampled with replacement. All columns have exactly r entries which are 1, and $n - r$ entries which are 0. The probability of any column is $1/\binom{n}{r}$.

The rank of such random matrices over *GF(2)* is sharply concentrated, but it is difficult to obtain the expected value of the rank as a function of m. The following question has been studied extensively. At what value of m does the first linear dependence occur among the columns (**whp**)?

A lower bound for the first linear dependence can be obtained from the smallest m such that the expected number of dependencies among the columns does not tend to zero (see e.g.[5] or [15]). The number of entries per row is well approximated by a $Po(c)$, $c = rm/n$ degree sequence. Let $n_0(m)$ is the number of rows whose entries are all zero. A trivial upper bound can be found when the number of columns m exceeds the number of non-zero rows $n - n_0(m) \sim n(1 - e^{-c})$.

An improved upper bound is obtained by finding the smallest value of m for which the number of edges in the 2-core exceeds the number of vertices **whp**. The solution to the equation $\eta/\nu = 1$ is obtained from (28), (29), (30). The table below gives the details.

r (non-zero entries per column)	3	4	5	6
m/n lower bound ([5], [15])	0.8895	0.9672	0.9892	0.9962
m/n upper bound (from $m = n - n_0$)	0.9404	0.9801	0.9930	0.9974
m/n upper bound (from $\eta/\nu = 1$ in 2-core)	0.9179	0.9768	0.9924	0.9974

Acknowledgement

Many thanks to an anonymous referee who greatly improved the content and exposition of this paper with many helpful remarks and suggestions.

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6 Appendix. Proof of Lemma 4: Distribution of residual degree

Let $0 \le s \le t \le (\theta n - d(v,0))/r$. Let $\rho = (r-1)/r$, and let

$$\Delta_{k,k-j}(s,t) = \binom{k}{j}\left(\frac{\theta n - rt}{\theta n - rs}\right)^{\rho(k-j)}\left(1 - \left(\frac{\theta n - rt}{\theta n - rs}\right)^{\rho}\right)^{j}.$$

Lemma 12.

$$\pi_{k,k}(s,t;v) = \Delta_{k,k}(s,t)\ \left(1 + O\left(\tfrac{rk^3}{\theta n - rt} + \tfrac{1}{\theta n - rs}\right)\right). \tag{31}$$

Proof When $t = s$ we require that $\pi_{k,k}(s,s) = 1$ and (31) satisfies this. Assume $t > s$. During step t of algorithm CONSTRUCT-$[v]$, configuration points of vertex v can only be selected as terminal points $y_{t,i},\ i=2,...,r$. The choice of the point $y_{t,i}$ is made uniformly from a set of size $\theta n-(r(t-1)+(i-1))$, thus

$$\mathbf{Pr}(\phi(y_{t,i})\neq v)=1-\frac{k}{\theta n-(r(t-1)+(i-1))}.$$

Thus

$$\begin{aligned}
\pi_{k,k}(s,t) &= \prod_{\tau=s}^{t-1}\prod_{i=1}^{r-1}\left(1-\frac{k}{\theta n-(r\tau+i)}\right)\\
&= \exp\left(-k(r-1)\sum_{\tau=s}^{t-1}\frac{1}{\theta n-r\tau}+O\left(\frac{k^2r(t-s)}{(\theta n-rs)(\theta n-rt)}\right)\right)\\
&= \exp\left(k\rho\log\frac{\theta n-rt}{\theta n-rs}+O\left(\frac{1}{\theta n-rs}\right)+O\left(\frac{k^2r(t-s)}{(\theta n-rs)(\theta n-rt)}\right)\right)\\
&= \left(\frac{\theta n-rt}{\theta n-rs}\right)^{\rho k}\left(1+O\left(\frac{rk^2}{\theta n-rt}\right)+O\left(\frac{1}{\theta n-rs}\right)\right). \qquad (32)
\end{aligned}$$

The last line follows because $(t-s)/(\theta n-rs)\leq t/\theta n<1$ as $rt\leq\theta n$. □

Lemma 13. *Let*

$$\begin{aligned}
\delta(k,s,t) &= \frac{rk^3}{\theta n-rt}, && (33)\\
\epsilon(j,s,t) &= \frac{j^2}{(t-s)}, && (34)\\
\gamma(j,s,t) &= \frac{j^3}{(t-s)}. && (35)
\end{aligned}$$

Let $j\geq 0$, then

$$\pi_{k,k-j}(s,t;v)=\Delta_{k,k-j}(s,t)\left(1+O(\delta(k,t)+\epsilon(j,s,t)+\gamma(j,s,t))\right). \qquad (36)$$

Proof

Let $(\tau_l,i_l),\ l=1,...,j,\ i=2,...,r$ denote the (step, iteration) at which the transition of vertex degree $k-l+1\to k-l$ occurs. There are two possibilities:
That all transitions occur at distinct steps τ (probability $p_{k,k-j}(s,t:1)$).
That at least 2 transitions occur during iterations $i=2,...,r$ of some step τ (probability $p_{k,k-j}(s,t:2)$).

We prove below that

$$\begin{aligned}
p(s,t:1) &= \Delta_{k,k-j}(s,t)\left(1+O(\epsilon(j,s,t))\right)\left(1+O(\delta(k,s,t))\right) && (37)\\
p(s,t:2) &\leq \Delta_{k,k-j}(s,t)O(\gamma(j,s,t)), && (38)
\end{aligned}$$

which will complete the proof of the lemma.

The probability that a configuration point of vertex v is chosen by CONSTRUCT-$[v]$ at iteration i of step τ is given by

$$\frac{l+1}{\theta n-(r(\tau-1)+(i-1))}.$$

Thus the probability of $a=1,...,r-1$ transitions from $l+a$ to l occurring at τ is

$$\binom{r-1}{a}\frac{(l+a)_{(a)}}{(\theta n-r\tau)^a}\left(1+O\left(\frac{ra}{\theta n-r\tau}\right)\right).$$

Case $p_{k,k-j}(s,t:1)$**.** Let $\tau_0=s$ and

$$T_1=\{\boldsymbol{\tau}=(\tau_1,...,\tau_j):s=\tau_0<\tau_1<\tau_2<\cdots<\tau_j\le t\}$$

be the set of possible sequences of transition points. From (32) we have,

$$\begin{aligned} p(s,t:1) &= \sum_{\boldsymbol{\tau}\in T_1}\left[\prod_{i=0}^{j-1}\left(\frac{\theta n-r(\tau_{i+1}-1)}{\theta n-r\tau_i}\right)^{(k-i)\rho}\frac{(r-1)(k-i)}{\theta n-r\tau_{i+1}}\left(1+O\left(\frac{r(k-i)^2}{\theta n-rt}\right)\right)\right]\\ &\quad\times\left(\frac{\theta n-rt}{\theta n-r\tau_j}\right)^{(k-j)\rho}\left(1+O\left(\frac{r(k-j)^2}{\theta n-rt}\right)\right).\end{aligned}$$

Thus

$$\begin{aligned} p(s,t:1) &= (k)_{(j)}\left(\frac{\theta n-rt}{\theta n-rs}\right)^{(k-j)\rho}\left(1+O\left(\frac{rk^2j}{\theta n-rt}\right)\right)\\ &\quad\times\frac{(r-1)^j}{(\theta n-rs)^{j\rho}}\sum_{\boldsymbol{\tau}\in T_1}\prod_{i=1}^{j}\frac{1}{(\theta n-r\tau_i)^{1-\rho}}.\end{aligned}$$

For $A>0$, $1/(1-x)^A$ is monotone increasing in x, so

$$\sum_{s+1}^{t}\frac{1}{(\theta n-r\tau)^A}=\int_s^t\frac{1}{(\theta n-r\tau)^A}d\tau+O\left(\frac{1}{(\theta n-rt)^A}\right).$$

For $i\ge 1$ let

$$f_i(t)=\sum_{\tau=s+1}^{t}\frac{1}{(\theta n-r\tau)^{i(1-\rho)}},$$

then

$$f_1(t)=\frac{1}{r-1}\left((\theta n-rs)^\rho-(\theta n-rt)^\rho\right)+O\left(\frac{1}{(\theta n-rt)^{1-\rho}}\right),$$

and, noting that $\rho = (r-1)/r \geq 1/2$,

$$f_2(t) = O\left(\frac{1}{(\theta n - rt)^{2-2\rho}}\right) + \begin{cases} \frac{1}{2}\left(\log(\theta n - 2s) - \log(\theta n - 2t)\right) & r = 2 \\ \\ \frac{1}{r-2}\left((\theta n - rs)^{2\rho-1} - (\theta n - rt)^{2\rho-1}\right) & r \geq 3 \end{cases}$$

For $j \geq 1$ let

$$F_j(t) = \sum_{\boldsymbol{\tau} \in T_1} \prod_{i=1}^{j} \frac{1}{(\theta n - r\tau_i)^{1-\rho}},$$

then $F_1(t) = f_1(t)$ and for $j \geq 2$

$$\frac{1}{j!}(f_1(t))^j \geq F_j(t) \geq \frac{1}{j!}(f_1(t))^j \left(1 - \binom{j}{2} \frac{f_2(t)}{(f_1(t))^2}\right).$$

For $B > 0$ constant

$$\left(\frac{\theta n - rt}{\theta n - rs}\right)^B = 1 - C\left(\frac{Br(t-s)}{\theta n - rs}\right), \tag{39}$$

for some constant $C > 0$. Thus

$$\frac{f_2(t)}{f_1(t)^2} = \frac{O(1)}{t-s}.$$

This completes the proof of (37).

Case $p_{k,k-j}(s,t:2)$**.** For $r = 2$ (graphs), $p(s,t:2) = 0$ always. Let $r \geq 3,\ j \geq 2$, then

$$\begin{aligned} p_{k,k-j}(s,t:2) &< (k)_{(j)} \left(\frac{\theta n - rt}{\theta n - rs}\right)^{\rho(k-j)} \frac{(r-1)^j}{(\theta n - rs)^{j\rho}} \\ &\quad \times j \frac{1}{(j-2)!}(f_1(t))^j \frac{f_2(t)}{(f_1(t))^2} \\ &\leq \Delta_{k,k-j}(s,t) \frac{j^3}{(t-s)} \end{aligned}$$

The last line follows from (39). This completes the proof of (38). Putting it all together proves the lemma. □