Martingales on Trees and the Empire Chromatic Number of Random Trees

Colin Cooper[1], Andrew R. McGrae[2], and Michele Zito[2]

[1] Department of Computer Science, Kings' College, London WC2R 2LS, UK
e-mail: colin.cooper@kcl.ac.uk
[2] Department of Computer Science, University of Liverpool, Liverpool, L69 3BX, UK
e-mail: {A.R.A.McGrae, M.Zito}@liverpool.ac.uk

Abstract. We study the empire colouring problem (as defined by Percy Heawood in 1890) for maps whose dual planar graph is a tree, with empires formed by exactly r countries. We prove that, for each fixed $r > 1$, with probability approaching one as the size of the graph grows to infinity, the minimum number of colours for which a feasible solution exists takes one of seven possible values.

1 Introduction

Assume that the n vertices of a graph G are partitioned into blocks $B_1, B_2, \ldots$ of size $r > 1$, so that B_i contains vertices labelled $(i-1)r + 1, (i-1)r + 2, \ldots, ir$. The *$r$-empire chromatic number* of G is the minimum number of colours $\chi_r(G)$ needed to colour the vertices of G in such a way that all vertices in a same block receive the same colour, but pairs of blocks connected by at least one edge of G are coloured differently. Let $\mathcal{T}_n$ denote a random n-vertex labelled tree (as defined, say, in [3]). Building on previous work by two of us [11], in this paper we prove that with probability approaching one as n tends to infinity (or *asymptotically almost surely* (a.a.s.) as we will say from now on) $\chi_r(\mathcal{T}_n)$ takes one of seven possible values, for fixed $r > 1$. This significantly improves the results in [11], where it was proved that there exists a positive integer $s_r = O(r/\log r)$ such that

$$s_r < \chi_r(\mathcal{T}_n) \leq 2r \qquad a.a.s.$$

The proof of our results srelies on a martingale argument and on tight estimates on the first two moments of a random variable counting the number of r-empire colourings of $\mathcal{T}_n$ using at most s colours. The key ingredient in the argument is a new martingale construction on the set of n-vertex labelled trees which may be of independent interest.

The reader at this point may question the reasons for studying this type of colourings. Our interest in the problem comes from its relationship with other important colouring problems. First of all its decision version reduces easily to the classical vertex colouring problem, but it is not clear to what extent the two are equivalent. Secondly, if G is planar, the blocks represent groups of countries belonging to the same "empire" and a colouring in the sense described above corresponds to a map colouring that gives all countries in the same empire the same colour and different colours to adjacent empires. This *r-empire colouring problem* was defined by Heawood [4] in the same paper in which he refuted a previous "proof" of the famous Four Colour Theorem (note that the 1-empire colouring problem is just planar graph colouring). It has been proved that $6r$ colours are always sufficient, and in some cases necessary to solve this problem [6], however not much is known about the distribution of the values of $\chi_r(G)$ over the set of all planar graphs on n vertices. Our work is a first step in this direction, and it indicates that, at least for trees, the worst-case predictions are very pessimistic: $2r$ colours are necessary and sufficient to colour all trees [11, Theorem 1], but, in fact, $\chi_r(\mathcal{T}_n) = O(r/\log r)$ a.a.s. Finally, the empire colouring problem is also related to problem of colouring graphs of given *thickness* [5] (a graph has thickness t if t is the minimum integer such that its edges can be partitioned into at least t planar graphs). It is known that the chromatic number of any graph of thickness two must be between eight and twelve but the exact solution of this problem is not known.

The rest of the paper is organized as follows. In Section 2 we outline our results and proof strategy. In Section 3 we list a number of key properties of $\mathcal{T}_n$ and its r-empire colourings. Section 4 is devoted to the definition of a martingale on the set of n-vertex labelled trees. In Section 5 we prove that $\chi_r(\mathcal{T}_n)$ is concentrated in an interval of size six a.a.s. Finally, in Section 6 we give tight estimates on the actual location of such interval.

2 Results and Proof Methods

Let Z be a random variable defined on some combinatorial structure $\mathcal{G}$. If $\mathcal{G}$ can be defined in some "orderly" way (formalized by the notion of filter discussed in Section 4) the difference $Z - \mathrm{E}Z$ can be estimated by associating a sequence of random variables $X_0, X_1, \ldots X_t$ called Doob martingale (see Section 4) to Z in such a way that $X_0 = \mathrm{E}Z$ and $X_t = Z$ and no X_i is "too far" from X_0. More specifically, if the martingale differences $X_i - X_{i-1}$ are small one can get upper bounds on the probability that Z is far from $\mathrm{E}Z$ which become smaller and smaller and the size of the structures considered becomes large. In this way (deterministic) information about $\mathrm{E}Z$ can be used to make a.a.s. statements about Z.

This approach has been applied successfully to various problems including the occupancy [7], the analysis of quicksort [10], and, in a less direct way, the vertex colouring of the random graph $\mathcal{G}_{n,p}$ [15]. We briefly outline the proof strategy used in the last paper as it is particularly relevant to our problem. To obtain information about the range of variability of $\chi(\mathcal{G}_{n,p})$ the authors proved that (a) any subgraph of $\chi(\mathcal{G}_{n,p})$ with less than some $z = z(n,p)$ vertices is relatively sparse and, therefore, easy to colour with just three colours. and that (b) if k is such that a large enough proportion of the graphs considered is k-colourable then the set of vertices that fail to be coloured is small (in particular it's $o(z)$). We will adopt a similar approach here. We will start by proving that any subgraph induced by a small collection of blocks in $\mathcal{T}_n$ (or any subgraphs obtained from it by removing any number of blocks) has small minimum degree (this is Lemma 2 in Section 3). Then we will provide an orderly definition of the class of labelled n-vertex trees and, using a suitably defined Doob martingale on such structures, prove that at least $n - 3r\sqrt{(n-1)\log\omega(n)}$ of the vertices of $\mathcal{T}_n$ can be coloured with $s_r + 1$ colours with probability at least $1 - 1/\omega(n)$ (see Lemma 4 in Section 5). These two results are sufficient to prove the following theorem.

Theorem 1. *For fixed $r > 1$ the r-empire chromatic number of $\mathcal{T}_n$ is a.a.s. concentrated over six consecutive integers.*

One problem with the martingale approach outlined above is that it does not give precise information on the location of these concentration intervals. To get that, one often needs to rely on additional results. Using asymptotic information about the first two moments of the number of r-empire colourings with s colour classes obtained in [11] (see Section 3) we will be able to prove the following result.

Theorem 2. *For every fixed integer $r > 1$, $\chi_r(\mathcal{T}_n)$ satisfies*

$$s_r \le \chi_r(\mathcal{T}_n) \le s_r + 6.$$

Furthermore if $r > \frac{2(s_r-1)^2}{2s_r-3}\log s_r$ then $s_r + 1 \le \chi_r(\mathcal{T}_n) \le s_r + 6$.

Table 1 gives the range for $\chi_r(\mathcal{T}_n)$ for the first few values of r. Note that by [11, Theorem 1] $\chi_r(\mathcal{T}_n) \le 2r$ for every fixed positive r. Thus

Theorem 2 improves the results in [11] for $r \geq 6$. Also, we found that $r = 43$ is the smallest value for which the concentration range has length seven: $\chi_{43}(\mathcal{T}_n) \in \{16, 17, 18, 19, 20, 21, 22\}$.

Table 1. The concentration intervals of $\chi_r(\mathcal{T}_n)$ for $r \leq 12$.

r	2	3,4	5,6,7	8,9	10,11,12
$\chi_r(\mathcal{T}_n)$	3,4,5,6,7,8	4,5,6,7,8,9	5,6,7,8,9,10	6,7,8,9,10,11	7,8,9,10,11,12

The observant reader will have noticed that, although Theorem 1 and Theorem 2 prove that $\chi_r(\mathcal{T}_n)$ a.a.s. belongs to a set of values of small fixed size, they are weaker than analogue statements [1, 2, 9] proved in the past for other models of random graphs. We speculate this may be due to the particular features of the problem at hand. The precise information on the moments of the relevant random variables helps arguing that six consecutive values suffice most of the times. However the main weakness seems to be in the fact that the martingale defined in Section 4, in a sense, builds the tree one edge at a time, whereas the r-empire colourings puts constraints to blocks of vertices.

3 Relevant Properties of Random Trees

To make our concentration argument go through we need asymptotic information about the number of r-empire colourings of $\mathcal{T}_n$ using a specific number of colours and we need to prove that certain "small" induced subgraphs of $\mathcal{T}_n$ are sparse. Results in [11] address the first issue. A colouring is *balanced* if its colour classes have the same size. Let $W_{r,s}(\mathcal{T}_n)$ be the number of balanced r-empire colourings of $\mathcal{T}_n$ using s colours. For each integer $r \geq 2$, and $s \geq 2$ let $c_{s,r} = s^{\frac{1}{r}-1}(s-1)$ and $a_n = n^{-\frac{s-1}{2}}(c_{s,r})^n$. The following result was proved in [11].

Theorem 3. *For each integer $r \geq 2$, $s \geq 2$, and $k \geq 1$, there exists a positive real number $C_{s,r,k}$, independent of n, such that*

$$\mathrm{E}W_{r,s}(\mathcal{T}_n)^k \sim C_{s,r,k} \times (a_n)^k.$$

The next two results, also from [11], follow from Theorem 3 and, respectively, Markov's and Chebyshev's inequalities.

Theorem 4. *For each fixed $r > 1$ there exists a positive integer s_r such that $W_{r,s}(\mathcal{T}_n) = 0$ a.a.s. for $s \leq s_r$.*

Theorem 5. *For each fixed $r > 1$, $W_{r,s}(\mathcal{T}_n) > 0$ with probability at least $b_{r,s}(n) > 0$ for $s > s_r$. Furthermore there exist positive constants $b_{r,s}$ such that $b_{r,s}(n) \to b_{r,s}$ as n tends to infinity.*

We move now to the density properties of small subgraphs of $\mathcal{T}_n$. We start by arguing that small collections of blocks in $\mathcal{T}_n$ don't span too many edges.

Lemma 1. *For any $\delta > 0$, let integers h and m be such that $h < n^{3/5-\delta}$ and $m < rh$. The probability that a given collection of h blocks in $\mathcal{T}_n$ spans m specific edges is at most*

$$\left(1+\frac{2}{n}\right)^{rh-1-m}\left(\frac{\mathrm{e}}{n}\right)^{m}.$$

Remark. Note that the result does not exclude the possibility that other edges may belong to the graph induced by the vertices in the given blocks. In the proof we will use $f \in E(H)$ to signify that the collection H spans edge f (and possibly others). On the other hand $E(H) = \{f\}$ will denote the fact that f is *the only* edge spanned by the vertices in H.

Proof. The probability of interest can be estimated by counting the number of labelled trees on n vertices for which a particular collection H of h blocks spans m particular edges. Using inclusion-exclusion we have

$$\Pr[f_1,\ldots,f_m \in E(H)] = \sum_{k=0}^{rh-1-m}\binom{rh-1-m}{k} \times(-1)^k \Pr[|H| = h \wedge E(H) = \{f_1,\ldots,f_m,g_1,\ldots,g_k\}] \quad (1)$$

We next argue that, in fact, the right-hand side of (1) is asymptotic to $\Pr[E(H) = \{f_1,\ldots,f_m\}]$ as the addition of more edges makes the event less and less likely. First note that we have

$$\Pr[f_1,\ldots,f_m \in E(H)] \le \sum_{k=0}^{rh-1-m}\binom{rh-1-m}{k}\Pr[E(H) = \{f_1,\ldots,f_m,g_1,\ldots,g_k\}].$$

From now on call $T(k)$ the kth term in the sum above. By an old result of Moon [12, Theorem 2] we have that

$$\Pr[E(H) = \{f_1,\ldots,f_m,g_1,\ldots,g_k\}] = n^{-m-k}\prod_{i=1}^{rh-m-k}|C_i|$$

(where $C_1, \ldots, C_{rh-m-k}$ are the components formed on the vertices of H by the edges $f_1, \ldots, f_m, g_1, \ldots, g_k$). We claim that, for each integer $k \in \{0, \ldots, rh-2-m\}$, assuming that edge g_{k+1} connects components C_{j_1} and C_{j_2}, we have

$$T(k+1) = \frac{rh-1-m-k}{(k+1)n} \left(\frac{|C_{j_1}| + |C_{j_2}|}{|C_{j_1}| \cdot |C_{j_2}|} \right) T(k). \tag{2}$$

To believe this claim note that the addition of g_{k+1} reduces the number of components by one because two components (possibly two isolated vertices) of the forest on $m+k$ edges spanning H, say C_{j_1} and C_{j_2}, get connected.

Since $1 \le |C_i| \le rh$ and $|C_{j_1}| + |C_{j_2}| \le rh$, implies

$$\frac{|C_{j_1}| + |C_{j_2}|}{|C_{j_1}| \cdot |C_{j_2}|} \le 2$$

equation (2) leads to a recurrence providing an upper bound on $T(k+1)$ in terms of $T(k)$. Solving it with initial condition $T(0) = n^{-m} \prod_{i=1}^{rh-m} |C_i|$ gives

$$T(k) \le \left(\frac{2}{n}\right)^k \binom{rh-1-m}{k} n^{-m} \prod_{i=1}^{rh-m} |C_i|.$$

Thus, by the binomial theorem,

$$\Pr[f_1, \ldots, f_m \in E(H)] \le \left(1 + \frac{2}{n}\right)^{rh-1-m} n^{-m} \prod_{i=1}^{rh-m} |C_i|.$$

The result follows from the inequality

$$\prod_{i=1}^{rh-m} x_i \le \left(\frac{rh}{rh-m}\right)^{rh-m},$$

which is true of any set of positive integers $x_1, \ldots, x_{rh-m}$ adding to rh.

□

The following is the relatively simple consequence of Lemma 1 that will be used in the proof of Theorem 1.

Lemma 2. *For any $\delta > 0$, any collection of less than $n^{3/5-\delta}$ blocks in $\mathcal{T}_n$ has average degree less than five a.a.s.*

Proof. In this proof H denotes a collection of blocks in $V(\mathcal{T}_n)$ and $|H|$ the number of blocks in H. We will prove that the probability of

$$\mathrm{BAD} \equiv \exists H \; |H| < n^{\frac{3}{5}-\delta} \wedge |E(H)| \geq \frac{5}{2}|H|$$

tends to zero as n tends to infinity. Let $M = \binom{rh}{2}$. By a simple union bound,

$$\Pr[\mathrm{BAD}] \leq \sum_{h=2}^{\lfloor n^{\frac{3}{5}-\delta} \rfloor} \binom{n/r}{h} \sum_{m=\lceil 5h/2 \rceil}^{rh-1} \binom{M}{m} \Pr[f_1, \ldots, f_m \in E(H)]$$

(here f_j are edges connecting two vertices belonging to some blocks of H). By Lemma 1

$$\Pr[f_1, \ldots, f_m \in E(H)] \leq (1 + o(1)) \left(\frac{\mathrm{e}}{n}\right)^m .$$

Hence

$$\begin{aligned}
\Pr[\mathrm{BAD}] &\leq (1+o(1)) \sum_{h=2}^{\lfloor n^{3/5-\delta} \rfloor} \binom{n/r}{h} \sum_{m=\lceil 5h/2 \rceil}^{rh-1} \binom{M}{m} \left(\frac{\mathrm{e}}{n}\right)^m \\
&\leq (1+o(1)) \sum_{h=2}^{\lfloor n^{3/5-\delta} \rfloor} \left(\frac{n\mathrm{e}}{rh}\right)^h \sum_{m=\lceil 5h/2 \rceil}^{rh-1} \left(\frac{M\mathrm{e}^2}{mn}\right)^m \\
&\leq (1+o(1)) \sum_{h=2}^{\lfloor n^{3/5-\delta} \rfloor} \left(\frac{n\mathrm{e}}{rh}\right)^h \sum_{m=\lceil 5h/2 \rceil}^{rh-1} \left(\frac{r^2 h\mathrm{e}^2}{5n}\right)^m
\end{aligned}$$

where the last inequality holds because $M \leq (rh)^2/2$ and $m > 5h/2$. Now, the argument of the inner sum is less than one for sufficiently large n. Therefore

$$\begin{aligned}
\Pr[\mathrm{BAD}] &\leq (1+o(1)) \sum_{h=2}^{\lfloor n^{3/5-\delta} \rfloor} \left(\frac{n\mathrm{e}}{rh}\right)^h rh \left(\frac{r^2 h\mathrm{e}^2}{5n}\right)^{\frac{5h}{2}} \\
&= (1+o(1)) \sum_{h=2}^{\lfloor n^{3/5-\delta} \rfloor} rh \left[\left(\frac{r^4\mathrm{e}^6}{5^{5/2}}\right) \left(\frac{h}{n}\right)^{3/2}\right]^h \\
&\leq (1+o(1)) \sum_{h=2}^{\lfloor n^{3/5-\delta} \rfloor} rh \left[\left(\frac{r^4\mathrm{e}^6}{5^{5/2}}\right) \left(\frac{1}{n}\right)^{\frac{3}{5}+\frac{3\delta}{2}}\right]^h .
\end{aligned}$$

Thus

$$\Pr[\mathrm{BAD}] \le (1+o(1)) r n^{\frac{6}{5}-2\delta} \left[\left(\frac{r^4 \mathrm{e}^6}{5^{5/2}} \right) \left(\frac{1}{n} \right)^{\frac{3}{5}+\frac{3\delta}{2}} \right]^2 \le C n^{-\delta}$$

□

4 Martingales

In this section we describe our martingale construction over the set of all labelled trees on n vertices. We start by recalling few relevant definitions and results (the interested reader may consult [8, Section 6.7] or [14, Chapter 4] for a more extensive treatment of the topic). A σ-field $(\Omega, \mathcal{F})$ is a set Ω, along with a family $\mathcal{F}$ of subsets that contains the empty set and is closed under complementation and countable unions.

Definition 1. *Given the σ-field $(\Omega, 2^\Omega)$, a* filter *is a sequence $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \ldots \subseteq \mathcal{F}_{n-1} \subseteq \mathcal{F}_n = 2^\Omega$ such that*

1. *$\mathcal{F}_0 = \{\emptyset, \Omega\}$,*
2. *$(\Omega, \mathcal{F}_i)$ is a σ-field.*

One way to define a filter is to form each $\mathcal{F}_i$ by means of a partition $(P^i_j)_{j \in J_i}$ of Ω: $\mathcal{F}_i$ is then the collection of all sets that may be defined as unions of blocks in the partition. Note that the partition defining $\mathcal{F}_{i+1}$ is a refinement of the one defining $\mathcal{F}_i$. Filters are used in the definition of martingale which we state for completeness.

Definition 2. *Let $(\Omega, \mathcal{F}, \Pr)$ be a probability space with a filter $(\mathcal{F}_i)_{i \ge 0}$. Suppose that $X_0, X_1, \ldots$ are random variables such that, for each $i \ge 0$, X_i is $\mathcal{F}_i$-measurable. The sequence $X_0, \ldots, X_n$ is a martingale provided that, for all $i \ge 0$,*

$$\mathrm{E}(X_{i+1} \mid \mathcal{F}_i) = X_i.$$

Doob martingales (see for instance [8]) give a simple way to construct a martingale from any given random variable Z. The following result, known as Azuma's inequality (see, for instance, [14, Theorem 4.16]), gives bounds on the probability that $|Z - \mathrm{E}Z|$ is large using a Doob martingale.

Theorem 6. *Let $(\Omega, \mathcal{F})$ be a σ-field and $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \ldots \subseteq \mathcal{F}_n$ a filter. Let the random variable Z be $\mathcal{F}_n$-measurable, and let $X_0, X_1, \ldots, X_n$ be* *the martingale obtained by setting* $X_i = \mathrm{E}(Z \mid \mathcal{F}_i)$. *Suppose that for each* $i = 1, \ldots, n$ *there is a constant* c_i *such that*

$$|X_i - X_{i-1}| \le c_i. \tag{3}$$

Then, for any $t > 0$,

$$\Pr[Z - \mathrm{E}Z < -t] \le \exp\left\{-2t^2 / \sum_{i=1}^{n} (c_i)^2\right\},$$
$$\Pr[Z - \mathrm{E}Z > t] \le \exp\left\{-2t^2 / \sum_{i=1}^{n} (c_i)^2\right\}.$$

Martingales on random trees. Let $\mathcal{T}_n$ be a random tree on n vertices. It is well-known (see for instance [13]) that a bijection maps each n-vertex tree T labelled over the set $\{1, \ldots, n\}$ to a unique sequence of $n-2$ numbers in the same set, referred to as its *Prüfer code*. We denote by $C(T)$ the Prüfer code of T (in fact it is handy to work with sequences of the form $(c_1, \ldots, c_{n-2}, n)$). Prüfer codes are very handy because they define a very simple encoding of random trees into sequences of positive integers. For our purposes, however, we also need to define another, related sequence. Let $L(T) = (u_1, \ldots, u_{n-1})$ be the permutation of the set $\{1, \ldots, n-1\}$ corresponding to the order in which leaves are peeled off T to get $C(T)$. Note that the pairs (u_i, c_i), for $i \in \{1, \ldots, n-1\}$ define the edges of T. Thus if $n = 8$ and $C(T) = (3, 1, 2, 6, 6, 1, 8)$ then $L(T) = (4, 3, 5, 2, 7, 6, 1)$.

Given $L(T)$, denote by S_L the set of Prüfer codes whose corresponding trees share the same sequence $L = L(T) = (u_1, \ldots, u_{n-1})$, and define a filter on $(S_L, 2^{S_L})$ by generating $\mathcal{S}_i$, for $i \in \{0, \ldots, n-1\}$, from a partition $(P^i_j)_{j \in J_i}$ of S_L such that P^i_j contains all Prüfer codes in S_L sharing the same prefix[3] $(c^j_1, c^j_2, \ldots, c^j_i)$. If $Z = Z(\mathcal{T}_n)$ is a random variable defined on the random tree $\mathcal{T}_n$, we may bound the difference $Z - \mathrm{E}Z$ using the Doob martingale associated with the filter $(\mathcal{S}_i)_{i \ge 0}$, provided we have some means of estimating the martingale differences in (3). This is possible because of the following result which translates small differences in Z on rather similar trees to small differences between consecutive elements of the martingale sequence.

Lemma 3. *Let* Z *be a random variable defined on* $\mathcal{T}_n$ *and let* $X_0, X_1, \ldots$ *be the Doob martingale defined on* Z. *Then, for each* $i > 0$

$$|X_i - X_{i-1}| \le |Z(T) - Z(T')|$$

[3] Here we assume that codes are ordered lexicographically.

where T and T' are two labelled trees on n vertices such that $C(T) \in S_{L(T')}$ and the Prüfer codes $C(T)$ a $C(T')$ only differ at position i.

Proof. (Sketch) The proof is similar to that of [15, Theorem 5]. The relevant measure-preserving mapping in this case relates a (tree with) Prüfer code prefix $(c_1, c_2, \dots, c_{i-1}, c_i)$ to one of the form $(c_1, c_2, \dots, c_{i-1}, c'_i)$ provided the completions of such codes coincide. Also notice that, given a sequence $(c_1, c_2, \dots, c_i)$ of numbers in $\{1, \dots, n\}$, all trees T with $C(T) \in S_{L(T')}$ and such that all Prüfer codes share the same prefix $(c_1, c_2, \dots, c_i)$ are assigne the same probability n^{n-3-j}. □

5 Concentration

To be able to prove Theorem 1 we still need to argue that most vertices of $\mathcal{T}_n$ can be coloured with $s_r + 1$ colours.

In the following result we use the fact, which follows directly from Theorem 5, that s_r+1 is in fact the smallest integer s^* for which $\Pr[\chi_r(\mathcal{T}_n) \leq s^*] \geq 1/\omega(n)$ for any function tending to infinity arbitrarily slowly. Note that s^* cannot be less than $s_r + 1$ as $\Pr[\chi_r(\mathcal{T}_n) \leq s_r]$ is exponentially small by Theorem 4.

Lemma 4. *Let n be a positive integer and $\omega(n)$ a function of n which tends to infinity as n grows. With probability greater than $1 - 1/\omega(n)$ all but $3r\sqrt{(n-1)\log\omega(n)}$ of the vertices of $\mathcal{T}_n$ can be coloured using $s_r + 1$ colours.*

Proof. Let $Y_{s_r+1}(\mathcal{T}_n)$ be the minimal size of a set of vertices U in $\mathcal{T}_n$ such that $\mathcal{T}_n \setminus U$ is $s_r + 1$ colourable. Let T and T' be two instances of $\mathcal{T}_n$ belonging to the same S_L whose Prüfer codes differs in one position. Then

$$|Y_{s_r+1}(T) - Y_{s_r+1}(T')| \leq 3r$$

(as the symmetric difference of $E(T)$ and $E(T')$ spans $3r$ vertices). Thus by Theorem 6,

$$\Pr[Y_{s_r+1} - \mathrm{E}Y_{s_r+1} \geq \alpha\sqrt{n-1}] \leq \exp\left\{-2\alpha^2/9r^2\right\}, \tag{4}$$

$$\Pr[Y_{s_r+1} - \mathrm{E}Y_{s_r+1} \leq -\alpha\sqrt{n-1}] \leq \exp\left\{-2\alpha^2/9r^2\right\}. \tag{5}$$

Now, choose $\alpha = 3r\sqrt{\log\omega(n)}$ so that $\exp\{-2\alpha^2/9r^2\} = 1/\omega(n)$. It follows from (4) that $\mathrm{E}Y_{s_r+1} < \alpha\sqrt{n-1}$. Thus, using (5), we have

$$\Pr[Y_{s_r+1} \geq 2\alpha\sqrt{n-1}] < 1/\omega(n).$$

□

Proof of Theorem 1. By Lemma 4 a.a.s. all vertices of $\mathcal{T}_n$ outside some set U, $|U| < 3r\sqrt{(n-1)\log\omega(n)}$, can be coloured with $s_r + 1$ colours. Note that any empire of $\mathcal{T}_n$ is either in U or it has empty intersection with such set. We will prove that $\chi_r(\mathcal{T}_n) \leq s_r + 6$ a.a.s. To see this note that the set of blocks spanned by U has size less than $n^{3/5-\delta}$, for any $\delta < 1/10$. Hence, by Lemma 2, it is 5-colourable. The colouring of $\mathcal{T}_n$ can be completed using at most $s_r + 1$ new colours. □

6 Location, location, location

The results so far are interesting in that they improve (at least for $r > 6$) the results in [11]. However they have one significant weakness. They do not provide any information on the actual location of the concentration interval. Theorem 2 addresses this issue. It can be derived using the concentration result proved in the last section, and using information about the first two moments of $W_{r,s}(\mathcal{T}_n)$.

Proof of Theorem 2. Let

$$u_s = \frac{2(s-1)^2}{2s-3}\log s \qquad c_s = \frac{6s^3}{6s^2-3s+2}\log(s+1).$$

Theorem 3 implies that, for each integer $s > 1$, $\chi_r(\mathcal{T}_n) > s$ a.a.s. if $r > u_s$, and $\chi_r(\mathcal{T}_n) \leq s$ with some positive probability for $r < c_{s-1}$. Note that for $c_{s-1} < u_s < c_s$, for all positive integers s. Thus if s is the smallest integer such that $r < c_s$ two possibilities arise:

1. $u_s < r < c_s$, or
2. $c_{s-1} \leq r \leq u_s$.

In the first case we know by Theorem 4 that $\chi_r(\mathcal{T}_n) > s_r$ a.a.s. and, by Theorem 1, $\chi_r(\mathcal{T}_n) \leq s_r + 6$ a.a.s. In the second one we cannot exclude the possibility that $\chi_r(\mathcal{T}_n) = s_r$. □

References

1. D. Achlioptas and C. Moore. The chromatic number of random regular graphs. In K. Jansen, S. Khanna, J. D. P. Rolim, and D. Ron, editors, *Approximation, Randomization, and Combinatorial Optimization*, volume 3122 of *Lecture Notes in Computer Science*, pages 219–228. Springer-Verlag, 2004.
2. D. Achlioptas and A. Naor. The two possible values of the chromatic number of a random graph. *Annals of Mathematics*, 162, 2005.

3. M. Drmota. *Random Trees.* Springer-Verlag, 2009.
4. P. J. Heawood. Map colour theorem. *Quarterly Journal of Pure and Applied Mathematics*, 24:332–338, 1890.
5. J. P. Hutchinson. Coloring ordinary maps, maps of empires, and maps of the moon. *Mathematics Magazine*, 66(4):211–226, 1993.
6. B. Jackson and G. Ringel. Solution of Heawood's empire problem in the plane. *Journal für die Reine und Angewandte Mathematik*, 347:146–153, 1984.
7. A. Kamath, R. Motwani, K. Palem, and P. Spirakis. Tail bounds for occupancy and the satisfiability threshold conjecture. *Random Structures and Algorithms*, 7:59–80, 1995.
8. S. Karlin and H. M. Taylor. *A First Course in Stochastic Processes.* Academic Press, 1975.
9. T. Łuczak. A note on the sharp concentration of the chromatic number of random graphs. *Combinatorica*, 11(3):295–297, 1991.
10. C. McDiarmid and R. Hayward. Strong concentration for quicksort. In *SODA '92: Proceedings of the third annual ACM-SIAM symposium on Discrete algorithms*, pages 414–421, Philadelphia, PA, USA, 1992. Society for Industrial and Applied Mathematics.
11. A. R. McGrae and M. Zito. Colouring random empire trees. In E. Ochmański and J. Tyszkiewicz, editors, *Mathematical Foundations of Computer Science 2008*, volume 5162 of *Lecture Notes in Computer Science*, pages 515–526. Springer-Verlag, 2008.
12. J. W. Moon. Enumerating labelled trees. In F. Harary, editor, *Graph Theory and Theoretical Physics*, pages 261–272. Academic Press, 1967.
13. J. W. Moon. *Counting Labelled Trees*, volume 1 of *Canadian Mathematical Monographs.* Canadian Mathematical Congress, 1970.
14. R. Motwani and P. Raghavan. *Randomized Algorithms.* Cambridge University Press, 1995.
15. E. Shamir and J. Spencer. Sharp concentration of the chromatic number on random graphs $G_{n,p}$. *Combinatorica*, 7:121–129, 1987.