The cover time of two classes of random graphs

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October 4, 2005

1 Introduction

Let $G = (V, E)$ be a connected graph, let $|V| = n$, and $|E| = m$. A *random walk* $\mathcal{W}_u$, $u \in V$ on the undirected graph $G = (V, E)$ is a Markov chain $X_0 = u, X_1, \dots, X_t, \dots \in V$ associated to a particle that moves from vertex to vertex according to the following rule: the probability of a transition from vertex i, of degree d_i, to vertex j is $1/d_i$ if $\{i, j\} \in E$, and 0 otherwise. For $u \in V$ let C_u be the expected time taken for $\mathcal{W}_u$ to visit every vertex of G. The *cover time* C_G of G is defined as $C_G = \max_{u \in V} C_u$. The cover time of connected graphs has been extensively studied. It is a classic result of Aleliunas, Karp, Lipton, Lovász and Rackoff [2] that $C_G \leq 2m(n-1)$. It was shown by Feige [11], [12], that for any connected graph G

$$(1 - o(1))n \ln n \leq C_G \leq (1 + o(1))\frac{4}{27}n^3.$$

The lower bound is achieved by (for example) the complete graph K_n, whose cover time is determined by the Coupon Collector problem.

In a previous paper [10] we studied the cover time of random graphs $G_{n,p}$ when $np = c \ln n$ where $c = O(1)$ and $(c - 1) \ln n \to \infty$. This extended a result of Jonasson, who proved in [16] that when the expected average degree $(n - 1)p$ grows faster than $\ln n$, **whp** a random graph has the same cover time (asymptotically) as the complete graph K_n, whereas, when $np = \Theta(\ln n)$ this is not the case.

Theorem 1 *[10] Suppose that $np = c \ln n = \ln n + \omega$ where $\omega = (c - 1) \ln n \to \infty$ and $c \geq 1$. If $G \in G_{n,p}$, then* **whp**[1]

$$C_G \sim c \ln\left(\frac{c}{c-1}\right) n \ln n.$$

The notation $A_n \sim B_n$ means that $\lim_{n\to\infty} A_n/B_n = 1$.

We first consider random regular graphs:

Theorem 2 *Let $r \geq 3$ be constant. Let $\mathcal{G}_r$ denote the set of r-regular graphs with vertex set $V = \{1, 2, \dots, n\}$. If G is chosen randomly from $\mathcal{G}_r$, then* **whp**

$$C_G \sim \frac{r-1}{r-2}\, n \ln n.$$

Aldous [1] found the cover time of certain Cayley graphs. Once we have proved Theorem 2 we will see that some of Aldous's resuts can be obtained fairly easily. This connection will be discussed in the full paper.

We turn our attention to the preferential attachment graph $G_m(n)$ introduced by Barabási and Albert [4] as a simplified model of the WWW. The preferential attachment graph $G_m(n)$ is a random graph formed by adding a new vertex at each time step, with m edges which point to vertices selected at random with probability proportional to their degree. Thus at

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[1] A sequence of events $\mathcal{E}_n$ occurs *with high probability* **whp** if $\lim_{n\to\infty} \mathbf{Pr}(\mathcal{E}_n) = 1$.

time n there are n vertices and mn edges. We use the generative model of [7] (see also [8]) and build a graph sequentially as follows:

- At each time step t, we add a vertex v_t, and we add an edge from v_t to some other vertex u, where u is chosen at random according to the distribution:
$$\mathbf{Pr}(u = v_i) = \begin{cases} \frac{d_t(v_i)}{2t-1}, & \text{if } v_i \neq v_t; \\ \frac{1}{2t-1}, & \text{if } v_i = v_t; \end{cases}$$
where $d_t(v)$ denotes the degree of vertex v at time t.
- For some constant m, every m steps we contract the most recently added m vertices to form a single vertex.

Let $G_m(n)$ denote the random graph at time step mn after n contractions of size m. Thus $G_m(n)$ has n vertices and mn edges and may be a multi-graph.

We prove

Theorem 3 **Whp** *the preferential attachment graph $G = G_m(n)$ satisfies*
$$C_G \sim \frac{2m}{m-1}\, n \ln n.$$

The next section contains the heart of the proof of our Theorems. In it we establish a good estimate of the probability that the first visit of $\mathcal{W}$ to a vertex v takes place at a time t. Once this is done, we can proceed to the proof of Theorem 2 in Section 3.

2 The first visit time lemma.

2.1 Convergence of the random walk

In this section G denotes a fixed connected graph with n vertices. u is some arbitrary vertex from which a walk $\mathcal{W}_u$ is started. Let $\mathcal{W}_u(t)$ be the vertex reached at step t, let P be the matrix of transition probabilities of the walk and let $P_u^{(t)}(v) = \mathbf{Pr}(\mathcal{W}_u(t) = v)$. Let π be the steady state distribution of the random walk $\mathcal{W}_u$.

2.2 Generating function formulation

Fix two vertices u, v. Let h_t be the probability $\mathbf{Pr}(\mathcal{W}_u(t) = v) = P_u^{(t)}(v)$, that the walk $\mathcal{W}_u$ visits v at step t. Let $H(s)$ generate h_t.

Similarly, considering the walk $\mathcal{W}_v$, starting at v, let r_t be the probability that this walk returns to v at step $t = 0, 1, ...$ Let $R(s)$ generate r_t. We note that $r_0 = 1$.

Let $f_t(u{\to}v)$ be the probability that the first visit of the walk $\mathcal{W}_u$ to v occurs at step t. If $u \neq v$ then $f_0(u{\to}v) = 0$. Let $F(s)$ generate $f_t(u{\to}v)$. Thus

$$H(s) = F(s)R(s). \tag{1}$$

Let T be the smallest positive integer such that for $t \geq T$,

$$\max_{x \in V} |P_u^{(t)}(x) - \pi_x| \leq n^{-3} \tag{2}$$

For $R(s)$ let

$$R_T(s) = \sum_{j=0}^{T-1} r_j s^j. \tag{3}$$

Thus $R_T(s)$ generates the probability of a return to v during steps $0, ..., T-1$ of a walk starting at v. Similarly for $H(s)$, let

$$H_T(s) = \sum_{j=0}^{T-1} h_j s^j. \tag{4}$$

2.3 First visit time: Single vertex v

The following lemma should be viewed in the context that G is an n vertex graph which is part of a sequence of graphs with n growing to infinity. We prove it in greater generality than is needed for the proof of Theorem 2.

Lemma 4 *Let T be as defined in (2) and*

(5) $$\lambda = \frac{1}{K_1 T}$$

for sufficiently large K_1.

Suppose that for some constant $0 < \theta < 1$,

(a) $H_T(1) < \theta R_T(1)$.

(b) $\min_{|s| \le 1+\lambda} |R_T(s)| \ge \theta$.

(c) $T\pi_v = o(1)$, $T\pi_v = \Omega(n^{-2})$.

Let

(6) $$p_v = \frac{\pi_v}{R_T(1)(1+O(T\pi_v))},$$

(7) $$c_{u,v} = 1 - \frac{H_T(1)}{R_T(1)(1+O(T\pi_v))},$$

where the values of the $1 + O(T\pi_v)$ terms are given implicitly in (14), (17) respectively. Then

(8)
$$f_t(u \to v) = c_{u,v} \frac{p_v}{(1+p_v)^{t+1}} + O(e^{-\lambda t/2}) \quad \text{for all } t \ge T.$$

Proof Write

(9) $$R(s) = R_T(s) + \widehat{R}_T(s) + \frac{\pi_v s^T}{1-s},$$

where $R_T(s)$ is given by (3) and

$$\widehat{R}_T(s) = \sum_{t \ge T} (r_t - \pi_v) s^t$$

generates the error in using the stationary distribution π_v for r_t when $t \ge T$. Similarly, let

(10) $$H(s) = H_T(s) + \widehat{H}_T(s) + \pi_v \frac{s^T}{1-s}.$$

Note that for $Z = H, R$ and $|s| \le 1 + o(1)$,

(11) $$|\widehat{Z}(s)| = o(n^{-2}).$$

This is because the variation distance between the stationary and the t-step distribution decreases exponentially with t.

Using (9), (10) we rewrite $F(s) = H(s)/R(s)$ from (1) as $F(s) = B(s)/A(s)$ where

(12) $$A(s) = \pi_v s^T + (1-s)(R_T(s) + \widehat{R}_T(s)),$$

(13) $$B(s) = \pi_v s^T + (1-s)(H_T(s) + \widehat{H}_T(s)).$$

For real $s \ge 1$ and $Z = H, R$, we have

$$Z_T(1) \le Z_T(s) \le Z_T(1) s^T.$$

Let $s = 1 + \beta\pi_v$, where $\beta > 0$ is constant. Since $T\pi_v = o(1)$ we have

$$Z_T(s) = Z_T(1)(1 + O(T\pi_v)).$$

$T\pi_v = o(1)$ and $T\pi_v = \Omega(n^{-2})$ and $R_T(1) \ge 1 + r_2 > 1 + \frac{1}{n}$ implies that

$$A(s) = \pi_v(1 - \beta R_T(1)(1 + O(T\pi_v))).$$

It follows that $A(s)$ has a real zero at s_0, where

(14) $$s_0 = 1 + \frac{\pi_v}{R_T(1)(1+O(T\pi_v))} = 1 + p_v,$$

say. We also see that

(15) $$A'(s_0) = -R_T(1)(1 + O(T\pi_v)) \ne 0$$

and thus s_0 is a simple zero (see e.g. [9] p193). The value of $B(s)$ at s_0 is

(16)
$$B(s_0) = \pi_v \left(1 - \frac{H_T(1)}{R_T(1)(1+O(T\pi_v))} + O(T\pi_v)\right) \ne 0.$$

Thus, from (6), (7)

(17) $$\frac{B(s_0)}{A'(s_0)} = -p_v c_{u,v}.$$

Thus the (see e.g. [9] p195) the principal part of the Laurent expansion of $F(s)$ at s_0 is

(18) $$f(s) = \frac{B(s_0)/A'(s_0)}{s - s_0}.$$

To approximate the coefficients of the generating function $F(s)$, we now use a standard technique for the asymptotic expansion of power series (see e.g.[19] Th 5.2.1).

We prove below that $F(s) = f(s)+g(s)$, where $g(s)$ is analytic in C_λ and that $M = \max_{s\in C_\lambda} |g(s)| = O(1)$.

Let $a_t = [s^t]g(s)$, then (see e.g.[9] p143), $a_t = g^{(t)}(0)/t!$. By the Cauchy Inequality (see e.g. [9] p130) we have that $|g^{(t)}(0)| \le Mt!/(1+\lambda)^t$ and thus

$$|a_t| \le \frac{M}{(1+\lambda)^t} = O(e^{-t\lambda/2}).$$

As $[s^t]F(s) = [s^t]f(s) + [s^t]g(s)$ and $[s^t]1/(s-s_0) = -1/(s_0)^{t+1}$ we have

$$[s^t]F(s) = \frac{-B(s_0)/A'(s_0)}{s_0^{t+1}} + O(e^{-t\lambda/2}). \tag{19}$$

Thus, we obtain

$$[s^t]F(s) = c_{u,v}\frac{p_v}{(1+p_v)^{t+1}} + O(e^{-t\lambda/2}),$$

which completes the proof of (8).

We now prove that s_0 is the only zero of $A(s)$ inside the circle C_λ. We use Rouché's Theorem (see e.g. [9]), the statement of which is as follows: *Let two functions $\phi(z)$ and $\gamma(z)$ be analytic inside and on a simple closed contour C. Suppose that $|\phi(z)| > |\gamma(z)|$ at each point of C, then $\phi(z)$ and $\phi(z)+\gamma(z)$ have the same number of zeroes, counting multiplicities, inside C.*

Let the functions $\phi(s), \gamma(s)$ be given by $\phi(s) = (1-s)R_T(s)$ and $\gamma(s) = \pi_v s^T + (1-s)\widehat{R}_T(s)$.

$$|\gamma(s)|/|\phi(s)| \le \frac{\pi_v(1+\lambda)^T}{\theta} + \frac{|\widehat{R}_T(s)|}{\theta} = o(1).$$

As $f(s)+g(s) = A(s)$ we conclude that $A(s)$ has only one zero inside the circle C_λ. This is the simple zero at s_0. □

Corollary 5 *Let $\boldsymbol{A}_t(v)$ be the event that $\mathcal{W}_u$ has not visited v by step t. Then for $t \ge T$,*

$$\mathbf{Pr}(\boldsymbol{A}_t(v)) = \frac{c_{u,v}}{(1+p_v)^t} + O(e^{-\lambda t/2}).$$

Proof We use Lemma 4 and

$$\mathbf{Pr}(\boldsymbol{A}_t(v)) = \sum_{\tau>t} f_\tau(u\to v).$$

□

3 Random regular graphs

From here on, we replace $R_T(1), H_T(1)$ by the notation R_v, H_v.

We start with some typical properties of a random regular graph. Let

$$\omega = \lfloor \ln\ln\ln n \rfloor.$$

Say a cycle C is *small* if $|C| \le 2\omega+1$. An r-regular graph G is *nice* if

P1. G is connected.

P2. The second eigenvalue of the adjacency matrix of G is at most $2\sqrt{r-1}+\epsilon$, where $\epsilon > 0$ is arbitrarily small ($\epsilon = 1/10$ is small enough). This implies that $T = O(\ln n)$.

P3. There are at most $r^{2\omega}$ vertices on small cycles.

P4. No pair of small cycles are within distance 3ω of each other.

In the full paper we show

Theorem 6 *Let $r \ge 3$ be a constant and let G be chosen uniformly from the set $\mathcal{G}_r$ r-regular graphs with vertex set $[n]$. Then G is nice* **whp**.

Assume from now on that G is a nice regular graph.

For $v \in V$ let H_v be the sub-graph induced by the vertices at distance 2ω or less from v.

Definition 1 *We say v is locally tree-like if H_v is a tree.*

Lemma 7 *If v is locally tree-like then*

$$R_v = \frac{r-1}{r-2} + o(\omega^{-1}).$$

Proof Let T_r be the infinite r-regular tree, rooted at v. Let $\mathcal{X}$ be a random walk on T_r starting at v. Let ρ_i be the probability that $\mathcal{X}$ is at v at step i. Now we can project the walk $\mathcal{X}$ onto a walk on $\{0, 1, 2, \ldots, \}$ where the particle moves right with probability $q = \frac{r-1}{r}$ and left with probability $p = \frac{1}{r}$. Let E_i be the expected number of visits to 0 for such a walk starting at i. Then

$$E_0 = 1 + E_1 = 1 + E_0 p/q.$$

This is because E_1 is E_0 times the expected number of visits to 0 between right moves from 1. Solving gives

$$\sum_{i=0}^{\infty} \rho_i = E_0 = \frac{r-1}{r-2}. \tag{20}$$

Note next that $\rho_{2i+1} = 0, \rho_{2i} \leq \gamma_i = \binom{2i}{i}\left(\frac{r-1}{r^2}\right)^i$ bounds the number of walks that place the particle at 0 at time $2i$). Therefore,

$$\sum_{i=\omega+1}^{\infty} \rho_i \leq \sum_{j=\omega/2} \gamma_j = o(\omega^{-1}). \tag{21}$$

We compare this with R_v. First observe that $r_i = \rho_i$ for $i \leq \omega$. Then from e.g. [15], we see that

$$\sum_{i=\omega+1}^{T} r_i \leq \sum_{i=\omega+1}^{T} (\pi_v + \lambda_{\max}^i) = o(\omega^{-1}). \tag{22}$$

The lemma now follows from (20) and (21). □

Lemma 8 *If v is locally tree-like then $|R_T(s)| \geq 1/(2(1+e))$ for $|s| \leq 1 + \lambda$*

Proof $R_T(s)$ generates the expected number of returns to v in T. Assuming v is locally tree-like

$$R_T(s) = \frac{1}{1 - F(s)} + Q(s)$$

where $F(s)$ generates the first return probability in the tree T_r and $Q(s)$ is a correction. Thus

$$|R_T(s)| \geq \frac{1}{|1 - F(s)|} - |Q(s)|.$$

and

$$|1 - F(s)| \leq 1 + |F(s)| \leq 1 + (1+\lambda)^T,$$

as $|F(s)| \leq \sum_{j \geq 0} \gamma_j |s|^{2j} < 1$.

Write $Q(s) = Q_1(s) - Q_2(s)$ where

$$Q_1(s) = q_{\omega+1}s^{\omega+1} + \cdots + q_{T-1}s^{T-1}$$

corrects returns due to non tree-like structure of G at steps $\omega + 1, \ldots T - 1$. $Q_2(s) = r_T s^T + \ldots + r_t s^t + \ldots$ is the tail of $M(s) = 1/(1 - F(s))$ above $T - 1$. $\gamma_j \leq \left(\frac{4(r-1)}{r^2}\right)^j$ implies that the radius of convergence of $M(s)$ is $d > 1$, $r_t = O(d^{-t})$. It was proved in (22) that $Q_1(1) = O(1/\omega)$ so that $|Q_1(s)| \leq (1 + \lambda)^T O(1/\omega)$ and for $1 + \lambda < d$

$$|Q_2(s)| = O\left(\frac{((1+\lambda)/d)^T}{1 - (1+\lambda)/d}\right) = O\left(\frac{1+\lambda}{d}\right)^T.$$

Thus, with $\alpha = (1 + \lambda)^T$

$$\begin{aligned} |R_T(s)| &\geq \frac{1}{|1 - F(s)|} - |Q(s)| \\ &\geq \frac{1}{1+\alpha} - \alpha O\left(\frac{1}{\omega}\right), \end{aligned}$$

Thus for $\lambda = 1/T$ we have $|R_T(s)| \geq 1/(2(1 + e))$ for $|s| \leq 1 + \lambda$. □

Finally we note:

Lemma 9 *For nice graphs, $\frac{H_v}{R_v} \leq \frac{9}{10}$.*

Proof Let f'_t be the probability that $\mathcal{W}_u$ has a first visit to v at time t. As $H(s) = F(s)R(s)$ we have

$$\begin{aligned} H_T(1) &\leq \mathbf{Pr}(\mathcal{W}_u \text{ visits } v \text{ by time } T-1)R_T(1) \\ &= R_T(1) \sum_{t=1}^{T-1} f'_t. \end{aligned}$$

Now if $\tau_0 = \lfloor 2 \ln \lambda_{\max}^{-1} \ln \ln n \rfloor$ then

$$\sum_{t=\tau_0}^{T-1} f'_t \leq \sum_{t=\tau_0}^{T-1} (\pi_v + \lambda_{\max}^t) = o(1).$$

We now estimate $\sum_{t=0}^{\tau_0} f'_t$, the probability that $\mathcal{W}_u$ visits v by time τ_0. Let $v_1, v_2, \ldots, v_r$ be the neighbours of v and let w be the first neighbour of v visited by $\mathcal{W}_u$. Then

$$\mathbf{Pr}(\mathcal{W}_u \text{ visits } v \text{ by time } \tau_0) = \sum_{i=1}^{r} \mathbf{Pr}(\mathcal{W}_u \text{ visits } v \text{ by time } \tau_0 \mid w = v_i)\mathbf{Pr}(w = v_i) \leq \sum_{i=1}^{r} \mathbf{Pr}(\mathcal{W}_{v_i} \text{ visits } v \text{ by the time } \tau_0)\mathbf{Pr}(w = v_i).$$

So it suffices to prove the lemma when u is a neighbour of v. If $G_l(u)$ is a tree then we can argue as in Lemma 7. Let ψ be the probability that a particle at the root of T_r ever returns to the root. The expected number of visits is

$$\frac{r-1}{r-2} = \sum_{k=1}^{\infty} k\psi^{k-1}(1-\psi) = \frac{1}{1-\psi}.$$

So $\psi = \frac{1}{r-1}$ and

$$\mathbf{Pr}(\mathcal{W}_u \text{ does not visit } v \text{ by time } \tau_0) \geq \frac{r-1}{r}(1-\psi-o(1)) = \frac{r-2}{r} - o(1).$$

If $G_l(u)$ contains a cycle C then let $e = (\xi, \eta)$ be an edge of C not incident with u and let T_u be the tree $G_l(u) - e$. Let $N'(u) = \{u_1, u_2, \ldots, u_s\}$, $s \in \{r-2, r-1\}$ be the neighbours of u which are not on a shortest path from ξ or η to u in T_u. $|N'(u) \setminus \{v\}| \geq r-3$ and so

$$\mathbf{Pr}(\mathcal{W}_u \text{ does not visit } v \text{ by time } \tau_0) \geq \frac{r-3}{r}(1-\psi-o(1)) = \frac{(r-2)(r-3)}{r(r-1)} - o(1).$$

This leaves the case $r = 3$ and $N'(u) = \{v\}$. With probability $\frac{2}{3}$ we have $\mathcal{W}_u(1) \neq v$. If ξ or η is reached (possibly $N(u) = \{v, \xi, \eta\}$), then with probability $\frac{1}{3}$ the next move is away from u and $1 - \psi - o(1)$ bounds the probability that there is no return to ξ or η. Hence

$$\mathbf{Pr}(\mathcal{W}_u \text{ does not visit } v \text{ by time } \tau_0)) \geq \frac{2}{9}(1-\psi-o(1))$$

completing the proof of the lemma. □

4 Cover time of nice graphs

We now prove that

$$C_G \sim \frac{r-1}{r-2} n \ln n.$$

Assume that $u, v \in V$ and that v is tree-like. Section 3 establishes that the conditions of Lemma 4 hold, and gives values for the parameters c_{uv}, p_v given by (6), (7).

Hence, the probability that $\mathcal{W}_u$ has not visited v by some step $t \geq T$ (see Corollary 5) is given by

$$\mathbf{Pr}(\boldsymbol{A}_t(v)) = (1+o(1))c_{uv}e^{-tp_v} + O(\lambda^{-1}e^{-\lambda t/2}).$$

Here $c_{uv} = \Theta(1)$, $\lambda = \Theta(1/\ln n)$ and

$$p_v = \frac{r-2}{(r-1)n}(1+o(\omega^{-1})).$$

4.1 Upper bound on cover time

Let $t_0 = \lceil (1+\sigma^{-1})\frac{r-1}{r-2} n \ln n \rceil$. We prove that for nice graphs, for any vertex $u \in V$,

$$C_u \leq t_0 + o(t_0). \tag{23}$$

Let $T_G(u)$ be the time taken to visit every vertex of G by the random walk $\mathcal{W}_u$. Let U_t be the number of vertices of G which have not been visited by $\mathcal{W}_u$ at step t. We note the following:

$$\begin{aligned} (24)\; C_u = \mathbf{E}\, T_G(u) &= \sum_{t>0} \mathbf{Pr}(T_G(u) \geq t), \\ (25)\; \mathbf{Pr}(T_G(u) > t) &= \mathbf{Pr}(U_t > 0) \leq \min\{1, \mathbf{E}\, U_t\}. \end{aligned}$$

It follows from (24), (25) that for all t

$$C_u \leq t + \sum_{s \geq t} \mathbf{E}\, U_s = t + \sum_{v \in V} \sum_{s \geq t} \mathbf{Pr}(\boldsymbol{A}_s(v)). \tag{26}$$

Let V_1 be the set of locally tree-like vertices and let $V_2 = V - V_1$. If G is nice then $|V_2| \leq r^{3\omega}$ for there are at most r^{ω} vertices within distance ω of a particular vertex in a small cycle, and at most $r^{2\omega}$ vertices on small cycles.

For $v \in V_1$ we have

$$\begin{aligned}
&\sum_{s \geq t_0} \mathbf{Pr}(\boldsymbol{A}_s(v)) \\
&\leq (1+o(1))e^{-t_0 p_v} \sum_{s \geq t_0} e^{-(s-t_0)p_v} + O(\lambda^{-2} e^{-\lambda t_0/2}) \\
&\leq \pi_v^{-1} e^{-t_0 p_v} \\
&\leq 3\frac{r-1}{r-2}.
\end{aligned}$$

Furthermore, we see that in particular,

$$\mathbf{Pr}(\boldsymbol{A}_{5n}(v)) \leq 2e^{-1}. \tag{27}$$

Suppose next that $v \in V_2$. We can find $w \in V_1$ such that $dist(v,w) \leq \omega$. So from (27), with $\nu = 5n + \omega$, we have

$$\mathbf{Pr}(\boldsymbol{A}_\nu(v)) \leq 1 - (1-2e^{-1})r^{-\omega}$$

since if our walk visits w, it will with probability at least $r^{-\omega}$ visit v within the next ω steps. Thus if $\gamma = (1-2e^{-1})r^{-\omega}$,

$$\begin{aligned}
\sum_{s \geq t_0} \mathbf{Pr}(\boldsymbol{A}_s(v)) &\leq \sum_{s \geq t_0} (1-\gamma)^{\lfloor s/\nu \rfloor} \leq \sum_{s \geq t_0} (1-\gamma)^{s/(2\nu)} \\
&= \frac{(1-\gamma)^{t_0/(2\nu)}}{1-(1-\gamma)^{1/(2\nu)}} \leq 3\nu\gamma^{-1}.
\end{aligned}$$

Thus, for all $u \in V$,

$$\begin{aligned}
C_u &\leq t_0 + 3\frac{r-1}{r-2}|V_1| + 3|V_2|\nu\gamma^{-1} \\
&= t_0 + O(r^{4\omega} n) = t_0 + o(t_0).
\end{aligned}$$

4.2 Lower bound on cover time

For any vertex u, we can find a set of vertices S such that at time $t_1 = t_0(1-\epsilon)$, $\epsilon \to 0$, the probability the set S is covered by the walk $\mathcal{W}_u$ tends to zero. Hence $T_G(u) > t_1$ **whp** which implies that $C_G \geq t_0 - o(t_0)$.

We construct S as follows. Let $S \subseteq V_1$ be some maximal set of locally tree-like vertices all of which are at least distance $2\omega + 1$ apart. Thus $|S| \geq (n - r^{3\omega})r^{-(2\omega+1)}$.

Let $S(t)$ denote the subset of S which has not been visited by $\mathcal{W}_u$ after step t. Now, provided $t \geq T$

$$\mathbf{E}\ |S(t)| \geq (1-o(1)) \sum_{v \in S} \left(\frac{c_{u,v}}{(1+p_v)^t} + o(n^{-2}) \right).$$

Let u be a fixed vertex of S. Let $v \in S$ and let H_v be given by (4), then (32) implies that

$$H_v \leq \sum_{t=\omega}^{T-1} (\pi_v + \lambda_{\max}^t) = o(1). \tag{28}$$

Thus $c_{uv} = 1 - o(1)$. Setting $t = t_1 = (1-\epsilon)t_0$ where $\epsilon = 2\omega^{-1}$, we have

$$\begin{aligned}
\mathbf{E}\ |S(t_1)| &= (1+o(1))|S|e^{-(1-\epsilon)t_0 p_v} \\
&\geq n^{1/\omega}.
\end{aligned} \tag{29}$$

Let $Y_{v,t}$ be the indicator for the event that $\mathcal{W}_u$ has not visited vertex v at time t. Let $Z = \{v, w\} \subset S$. We can show (proof omitted) that that for $v, w \in S$

$$\mathbf{E}\ (Y_{v,t_1} Y_{w,t_1}) = \frac{c_{u,Z}}{(1+p_Z)^{t+2}} + o(n^{-2}), \tag{30}$$

where $c_{u,Z} \sim 1$ and $p_Z \sim 2(r-2)/(n(r-1))$. Thus

$$\mathbf{E}\ (Y_{v,t_1} Y_{w,t_1}) = (1+o(1))\mathbf{E}\ (Y_{v,t_1})\mathbf{E}\ (Y_{w,t_1}). \tag{31}$$

It follows from (29) and (31), that

$$\begin{aligned}
\mathbf{Pr}(S(t_1) \neq 0) &\geq \frac{(\mathbf{E}\ |S(t_1)|)^2}{\mathbf{E}\ |S(t_1)|^2} \\
&= \frac{1}{\frac{\mathbf{E}|S_{t_1}|(|S_{t_1}|-1)}{(\mathbf{E}|S(t_1)|)^2} + (\mathbf{E}\ |S_{t_1}|)^{-1}} = 1 - o(1).
\end{aligned}$$

□

5 Preferential Attachment Graph

5.1 The random graph $G_m(n)$

Lemma 10 (a) *Suppose that* $0 < \alpha < \beta < 2/3$.

Then

$$\mathbf{Pr}(\exists i \le n^{\alpha} :\ d(i) \le n^{1/2-\alpha/2-\beta}) = o(1).$$

(b)

$$\mathbf{Pr}(\exists s, t :\ d_t(s) \ge (t/s)^{1/2}(\ln n)^3) = O(n^{-3}).$$

□

Suppose that v is locally tree-like. We say that v is *locally regular* if H_v is a tree of depth 2ω, rooted at v, in which every non-leaf has branching factor m.

Lemma 11 Whp, *$G_m(n)$ contains at least $n^{1-o(1)}$ locally regular vertices.*

A small cycle is *light* if it contains no vertex $v \le n^{1/10}$ (it has no "heavy" vertices), otherwise it is *heavy*.

Lemma 12 Whp *$G_m(n)$ does not contain a small cycle within 10ω of a light cycle.*

Lemma 13 Whp *$G_m(n)$ does not contain a vertex $v \ge n^{3/5}$ which is within distance 10ω of 2 distinct small cycles.*

We also need to deal with the possibility that $G_m(n)$ contains many cycles.

Lemma 14 Whp *$G_m(n)$ contains at most $(\ln n)^{5\omega}$ small cycles.*

Lemma 15 Whp *there are at most $O(n^{1/2+o(1)})$ non tree-like vertices.*

The *conductance* Φ of the walk $\mathcal{W}_u$ is defined by

$$\Phi = \min_{\pi(S)\le 1/2} \frac{e(S:\overline{S})}{d(S)}.$$

Mihail, Papadimitriou and Saberi [18] proved that the *conductance* Φ of the walks $\mathcal{W}$ are bounded below by some absolute constant. Now it follows from Jerrum and Sinclair [15] that

$$(32) \qquad |\tilde{P}_u^{(t)}(x) - \pi_x| \le (\pi_x/\pi_u)^{1/2}(1-\Phi^2/2)^t.$$

For sufficiently large K, the RHS above will be $O(n^{-10})$ at τ_0. We remark that there is a technical point here. The result of [15] assumes that the walk is *lazy*, and only makes a move to a neighbour with probability 1/2 at any step. This halves the conductance but we still have $T = O(\ln n)$ in (2). It doubles the covertime. It also asymptotically doubles the values R_v. Otherwise, it has a negligible effect on the analysis and we will ignore this for the rest of the paper and continue as though there are no lazy steps.

6 Cover time of $G_m(n)$

6.1 Parameters

Asume now that $G_m(n)$ (i) has $n^{1-o(1)}$ locally regular vertices, (ii) $d(s) \ge n^{1/4}$ for $s \le n^{1/10}$, (iii) $d(s) \ge n^{1/25}$ for $s \le n^{3/5}$, (iv) no small cycle close to a light cycle, (v) no $v \ge n^{3/5}$ within distance 10ω of 2 distinct small cycles, (vi) $O((\ln n)^{5\omega})$ small cycles and (vii) $O(n^{1/2+o(1)})$ non tree-like vertices.

Lemma 16 *Suppose that v is locally-tree-like. Then*

(a) $R_v \le \frac{d(v)}{m-1}$.

(b) $d(v) \ge m+1$ *implies* $R_v \le \frac{d(v)(m+m^{-1}-1)}{d(v)(m+m^{-1}-2)+m^{-1}-1}$

Proof We first define an infinite tree T_v^* by taking the tree T_v' defined by the first $\omega+1$ levels of H_v and then rooting a copy of the infinite tree T_m^{∞} which has branching factor m from each leaf of T_v'. Thus if v is locally regular, T_v itself is an infinite tree with branching factor m, rooted at v.

Let R_v^* be the expected number of visits to v for infinite random walks $\mathcal{W}_v^*$ on T_v^*, started at v, making null moves with probability γ and making no null moves respectively. We argue first that

$$(33) \qquad |R_v - R_v^*| = o(\omega^{-1}).$$

Let $r_t^* = \mathbf{Pr}(\mathcal{W}_v^*(t) = v)$. Then

$$|R_v - R_v^*| \leq \sum_{t=\omega+1}^{T} r_t + \sum_{t=\omega+1}^{\infty} r_t^*$$

(34)

$$\leq o(\omega^{-1}) + \sum_{t=\omega/10}^{\infty} 1_{t\ even} \binom{t}{t/2} \frac{(m^2-m)^{t/2}}{(m^2-m+1)^t}$$

$$= o(\omega^{-1}).$$

Explanation of (34): The first term follows directly from (32): $\sum_{t=\omega+1}^{T} r_t \leq T\pi_v + \sum_{t=\omega+1}^{T}(1-\Phi^2/2)^t$. We bound the second sum by considering a walk $\mathcal{Y}$ on $\{0,1,2,\ldots,\}$ which at each time step moves right with probabilty $q_{odd} = \frac{m-1}{m}$ when at odd values, $q_{even} = \frac{m}{m+1}$ when at even values and moves left with probablity $p_{parity} = 1 - q_{parity}$, *parity=odd,even*, except at the origin, when it always moves right. We couple $\mathcal{Y}(t)$ with the distance of $\mathcal{W}_v^*(t)$ from v. Our choice for p_{odd}, p_{even} is determined by the fact that if a vertex w in the first $\omega+1$ levels of H_v has branching factor $m-1$ then its ancestors have branching factor $\geq m$. We maximise R_v by keeping branching factors small and so the largest R_v is achieved by having branching factors $m-1, m$ alternating on any path from v. This leads to $\mathcal{Y}$.

Thus r_t^* is at most the probability that $\mathcal{Y}(t) = 0$. We now bound this latter probability. We observe that it is bounded by the probability that another walk $\mathcal{Y}_1$ is at the origin after t steps. Here $\mathcal{Y}_1$ is the walk on $\{0, \pm1, \pm2, \ldots,\}$ where the particle moves right with probabilities q_{odd}, q_{even} and left with probabilities p_{odd}, p_{even} i.e. there is no barrier at the origin. We can couple $\mathcal{Y}, \mathcal{Y}_1$ so that $\mathcal{Y}(t) \geq |\mathcal{Y}_1(t)|$. When $\mathcal{Y}_1(t) > 0$ we can move them in the same direction and when $\mathcal{Y}_1 < 0$ then we can move $\mathcal{Y}$ further from the origin whenever $\mathcal{Y}_1$ moves further from the origin.

Finally, consider the walk $\mathcal{Y}_2$ that $\mathcal{Y}_1$ induces on the even integers. The non-trivial moves are right with probabilty $\frac{m^2-m}{m^2-m+1}$ and left with probability $\frac{1}{m^2-m+1}$. The probability that t non-trivial moves yields a return is precisely $1_{t\ even}\binom{t}{t/2}\frac{(m^2-m)^{t/2}}{(m^2-m+1)^t}$. Now the probability that there are at most $\omega/10$ non-trivial moves is exponentially small, in ω and this can be absorbed into the $o(\omega^{-1})$ term.

Let b_w, $w \in T_v^*$ be the branching factor at w i.e. $b_v = d_v$ and $b_w = d_w - 1$ if w is not the root. Further, if w is in the first ω levels let $b_w = b_w^+ + b_w^-$ where b_w^+ is the number of descendants w' of w with $w > w'$ i.e. w chose w' in the construction of $G_m(n)$. If w is at a higher level, we take $b_w = b_w^+ = m$ and $b_w^- = 0$.

Let $\widehat{T}_w$ be the sub-tree of T_v^* rooted at vertex w. (Thus $\widehat{T}_v = T_v^*$). Let ρ_w denote the probability that a random walk on $\widehat{T}_w$ which starts at w ever returns to w. Our aim is to estimate ρ_v and use

$$R_v^* = \frac{1}{1-\rho_v}. \tag{35}$$

Let $C(w)$ denote the children of w in T_v^*. We use the following recurrence: The parameter k counts the number of returns to x.

$$\rho_w =$$

$$1 - \frac{1}{b_w} \sum_{x \in C(w)} \sum_{k \geq 0} \left(1 - \frac{1}{d_x}\right) \left(\rho_x \left(1 - \frac{1}{d_x}\right)\right)^k (1-\rho_x)$$

$$= 1 - \frac{1}{b_w} \sum_{x \in C(w)} \frac{\left(1 - \frac{1}{d_x}\right)(1-\rho_x)}{1 - \rho_x\left(1 - \frac{1}{d_x}\right)}$$

$$= \frac{1}{b_w} \sum_{x \in C(w)} \frac{1}{b_x + 1 - \rho_x b_x}.$$

We see immediately that if T_v^* is a regular tree with branching factor $m \geq 2$ then, with $\rho_w = \rho$ for all w,

$$\rho = \frac{1}{m+1-\rho m} \text{ and hence } \rho = \frac{1}{m}$$

and this deals with the locally regular case.

Now define b_w^+ to be the number of children x of w with $x < w$. These are the children chosen by w. Let $b_w^- = b_w - b_w^+$.

We will now prove the following by induction on $\omega + 1 - \ell_w$, where $\ell_w \leq \omega + 1$ is the level of w in the tree.:

(a) $b_w = m-1$ implies $\rho_w \leq \frac{1}{m}$.

(b) $b_w \geq m+1$ implies $\rho_w \leq \frac{1}{b_w}\left(1+\frac{b_w-m}{m+m^{-1}-1}\right)$.

(c) $b_w = b_w^+ = m$ implies $\rho_w \leq \frac{1}{m}$.

(d) $b_w = b_w^+ + 1 = m$ implies

$$\rho_w \leq \frac{1}{m}\left(\frac{m-1}{m}+\frac{m}{m^2-m+1}\right)$$

The base case will be $\ell_w = \omega+1$. For which, Case (c) applies and the induction hypothesis holds from the locally regular case.

The lemma follows from this since only cases (b),(c),(d) can apply to the root v, in which case $b_v = d(v)$.

Let us now go through the inductive step. Let us assume these conditions apply to $x \in C(w)$ and then we find that in these cases:

(a) $b_x + 1 - b_x\rho_x \geq m + \frac{1}{m} - 1$.

(b) $b_x + 1 - b_x\rho_x \geq m + (b_x - m)\left(1 - \frac{1}{m+m^{-1}-1}\right) \geq m$.

(c) $b_x + 1 - b_x\rho_x \geq m$.

(d) $b_x + 1 - b_x\rho_x \geq m + \frac{1}{m} - \frac{m}{m^2-m+1}$.

Case (a): In this case $b_w = b_w^+$ and only cases (b),(c) are possible for $x \in C(w)$. In which case $b_x + 1 - b_x\rho_x \geq m$ for $x \in C(w)$.

Case (b): In $C(w)$ we have b_w^+ cases of (b) or (c) and b_w^- cases of (a),(b) or (d). In the first case we have $b_x + 1 - b_x\rho_x \geq m$. In the second case we have $b_x + 1 - b_x\rho_x \geq m + m^{-1} - 1$. Thus

$$\rho_w \leq \frac{1}{b_w}\left(\frac{b_w^+}{m} + \frac{b_w^-}{m+m^{-1}-1}\right)$$

Sub-case (i): $b_w^+ = m$.

$$\rho_w \leq \frac{1}{b_w}\left(1 + \frac{b_w - m}{m+m^{-1}-1}\right).$$

Sub-case (ii): $b_w^+ = m-1$.

$$\rho_w \leq \frac{1}{b_w}\left(1 - \frac{1}{m} + \frac{b_w - m + 1}{m+m^{-1}-1}\right).$$

Case (c): This follows as in Case (a).

Case (d): In $C(w)$ we have $m-1$ cases of (b) or (c) and one case of (a),(b) or (d). Thus

$$\rho_w \leq \frac{1}{m}\left(\frac{m-1}{m} + \frac{1}{m+m^{-1}-1}\right)$$

as is to be shown. □

Lemma 17 *Suppose that either*
(i) G_v contains a unique light cycle C_v, that $v \notin C_v$ and that the shortest path $P = (w_0 = v, w_1, \ldots, w_k)$ from v to C_v is such that $\max\{d(w_1), \ldots, d(w_k)\} \geq \omega^3$, *or*
(ii) that H_v contains only heavy cycles. Then

(a) $R_v \leq \frac{d(v)}{m-1}$.

(b) $d(v) \geq m+1$ *implies* $R_v \leq \frac{d(v)(m+m^{-1}-1)}{d(v)(m+m^{-1}-2)+m^{-1}-1}$

Proof
(a) Let w be the first vertex on the path from v to C_v which has degree at least ω^3. Let H_v' be obtained from H_v by deleting those vertices, other than w, whose only path to v in H_v goes through w. Let R_v' be the expected number of returns to v in a random walk of length ω on H_v' where w is an absorbing state. We claim that

$$R_v \leq R_v' + o(\omega^{-1}). \tag{36}$$

Once we verify this, the proof of (a) follows from the proof of Lemma 16 i.e. embed the tree $H'v$ in an infinite tree by rooting a copy of T_m^∞ at each leaf. To verify (36) we couple random walks on H_v, H_v' until w is visited. In the latter the process stops. In the former, we find that when at w, the probability we get closer to v in the next step is at most ω^{-3} and so the expected number of returns from now on is at most $\omega \times \omega^{-3}$ and (36) follows.

(b) Now consider the case where H_v contains only heavy cycles. We argue first that a random walk of length ω that starts at v might as well terminate if it reaches a vertex $w \leq n^{1/10}$, $w \neq v$. We can assume

$d(w) \geq n^{1/4}$. Now we can assume from Lemma 14 at least $n_0 = n^{1/4} - (\ln n)^{5\omega}$ of the edges incident with w are not in cycles contained in H_v. But then a walk that arrives at w has a more than $\frac{n_0}{n^{1/4}}$ chance of entering a sub-tree T_w of H_v rooted at w for which every vertex is separated from v by w. But then the probability of leaving T_w in ω steps is $O(\omega(\ln n)^{5\omega}/n^{1/4})$ and so once a walk has reached w, the expected number of further returns to v is $o(\omega^{-1})$. We can therefore remove T_w from H_v and then replace an edge (x, w) by an edge (x, w_x) and make all the vertices w_x absorbing. Repeating this argument, we are left with a tree to which we can apply the argument of Lemma 16. □

Note that if $v \in V_B$ then no bound on R_v has been established:

$$V_B = \{v : \; G_v \text{ contains a unique light cycle } C_v \text{ and the path from } v \text{ to } C_v \text{ contains no vertex of degree at least } \omega^3\}$$

However, for these it suffices to prove

Lemma 18 *If $v \in V_B$ then $R_v \leq (\ln n)^{2/3}$.*

We will also need to show the following:

Lemma 19 *If $v \geq n^{3/5}$ then*

(a) $|R_T(s)| \geq 1/(2(1+e))$ *for* $|s| \leq 1 + \lambda$.

(b) $H_v < C_m R_v + o(1)$ *where* $C_m < 1$.

One of the problems in proving this lemma arises from the existence of non-locally-tree-like vertices. This problem is ameliorated by restricting attention to $v \geq n^{3/5}$ and using Lemma 13. For $v < n^{3/5}$, we know $d(v) \geq n^{1/25}$ and so after a walk of length $(\ln n)^3$ there is an $\Omega(n^{-1/25})$ chance of being at v. Thus v will be visited in $O(n^{24/25}(\ln n)^5)$ time **qs**.

6.2 Upper bound on cover time

Let $t_0 = \lceil \frac{2m}{m-1} n \ln n \rceil$. We prove that **whp**, for $G_m(n)$, for any vertex $u \in V$, $C_u \leq t_0 + o(t_0)$.

Arguing as in (26)

$$C_u \leq t + o(1) + \sum_{v \in V_B} \sum_{s \geq t} \mathbf{Pr}(\boldsymbol{A}_s(v)) + (1 + O(T\pi_v)) \times \sum_{\substack{v \in V \setminus V_B \\ v \geq n^{3/5}}} \left(\frac{R_v}{\pi_v} e^{-(1+O(T\pi_v))t\pi_v/R_v} + O(\lambda e^{-\lambda t/2}) \right). \tag{37}$$

Let $t_1 = (1+\epsilon)t_0$ where $\epsilon = n^{-1/3}$ can be assumed by Lemma 10 to satisfy $T\pi_v = o(\epsilon)$ for all $v \in V$.

If $v \notin V_B$, $v \geq n^{3/5}$ then by Lemmas 16(a) and 17(a),

$$t_1(1 + O(T\pi_v))\pi_v/R_v \geq \frac{2m}{m-1} n \ln n \cdot \frac{d(v)}{2mn} \cdot \frac{m-1}{d(v)} = \ln n. \tag{38}$$

Plugging (38) into (26) and using $R_v \leq 5$ (Lemmas 16(b) and 17(b)) and $\pi_v \geq \frac{1}{2n}$ for all $v \in V \setminus V_B$ we get

$$C_u \leq t_1 + 10n + o(n) + \sum_{v \in V_B} \sum_{s \geq t} \mathbf{Pr}(\boldsymbol{A}_s(v)) \tag{39}$$

It remains to deal with $v \in V_B, v \geq n^{3/5}$ We first observe that

$$|V_B| \leq (\ln n)^{5\omega} \omega^{3\omega} \leq (\omega \ln n)^{5\omega} \tag{40}$$

Using Lemma 18 we have

$$\sum_{v \in V_B} \sum_{s \geq t_1} \mathbf{Pr}(\boldsymbol{A}_s(v)) \leq (\omega \ln n)^{5\omega} \left(2n\omega e^{-(1+o(1))t_1/(2n(\ln n)^{2/3})} + O(\lambda e^{-\lambda t_1/2}) \right) = o(n).$$

This completes our proof of the upper bound on cover time.

6.3 Lower bound on cover time

Done in a similar way to that for regular graphs.

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