Random regular graphs of non-constant degree

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July 5, 2003

Abstract

Let G_r denote graph chosen uniformly at random from the set of r-regular graphs with vertex set $\{1, 2, \ldots, n\}$ and $3 \leq r \leq c_0 n$ for some small constant c_0. We prove that with probability tending to 1 as $n \to \infty$, G_r is r-connected and G_r is Hamiltonian.

1 Introduction

The properties of random r-regular graphs on n vertices have received much attention. The recent survey of Wormald [21] and Chapter 9 of Janson, Łuczak and Ruciński [11] contain the most up to date results. By and large, results are most easily proved for the case where r is bounded as $n \to \infty$. The paper by Bollobás [3] considered the case where $r = O((\log n)^{1/2})$. McKay [15] and McKay and Wormald [16, 17] subsequently gave models which are useful for $r = o(n^{1/2})$ or $r = \Omega(n)$. We prove the following theorem: G_r denotes a graph chosen uniformly at random from the set $\mathcal{G}_r$ of r-regular graphs with vertex set $V = \{1, 2, \ldots, n\}$. Its edge set is denoted by E.

Theorem 1 *Assume $3 \leq r \leq c_0 n$ for some small positive absolute constant. Then with probability tending to 1 as $n \to \infty$,*

(a) *G_r is r-connected.*

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(b) *G_r is Hamiltonian.*

Using different methods, based on the concentration of the sizes of eigenvalues, Krivelevich, Sudakov, Vu and Wormald [12] have shown a similar result for $r \geq n^{1/2} \log n$. Thus G_r is r-connected and Hamiltonian **whp**[1] for all $3 \leq r \leq n-3$.

These results are well known for r constant. Result (a) is from Bollobás [4] and (b) is from Robinson and Wormald [19, 20], Bollobás [5], Fenner and Frieze [7]. For $r = o(n^{1/2})$ such results could probably be proved with the help of the models of [15] and [16]. In fact this was done, up to $r = o(n^{1/5})$, in unpublished work by Frieze [8] and for $r \leq n^{.002}$ by Łuczak [14]. Hence the results here are most interesting for the case $r = \Omega(n^{1/2})$.

2 Generating graphs with a fixed degree sequence.

In this section we let $\mathcal{G}_{\mathbf{d}} = \{$graphs with vertex set $[n]$ and degree sequence $\mathbf{d}\}$. $\mathbf{d} = (d_1, d_2, \ldots, d_n)$. Let $2D = (d_1 + d_2 + \cdots + d_n)$. The standard method for generating uniformly distributed simple graphs with a given degree sequence is the configuration model of Bollobás [3] which is a probabilistic interpretation of the counting formula of Bender and Canfield [2].

Definition: Let $W = [2D]$ and let $W_1, W_2, ..., W_n$ be a partition of W satisfying $|W_i| = d_i$, $i = 1, 2, \ldots, n$ and $\max W_i < \min W_j$ for $i < j$. Let Ω be the set of all $\frac{(2D)!}{D!2^D}$ partitions of Ω into D 2-element sets. An element F of Ω is a *configuration*. The constituent 2-element sets of F are referred to as the *edges* of F.

Define $\phi : W \to [n]$ by $\phi(w) = i$ for $w \in W_i$. Let $\gamma(F)$ denote the multigraph with vertex set $[n]$ and edge multiset $E_F = \{(\phi(x), \phi(y)) : (x, y) \in F\}$. Ω^* denotes those configurations F for which $\gamma(F)$ is simple.

In the configuration model, the probability that the underlying multigraph is simple is $\exp\{-\Theta(d^2)\}$, where $d = 2D/n$ is the average degree. This is small if d is large, in particular if d grows with n. On the other hand it is not terribly small and we can prove many things directly via configurations and then inflate the probability estimates accordingly. In one crucial case, Lemma 2, we work directly in the space of simple graphs, using *swaps*. Swaps were used successfully for the configuration model in [15], [16, 17] and [8] while Kučera e.g. [13] has used swaps to work directly in the space of simple graphs.

The following lemma will be needed later. For $r = o(n^{1/2})$ it follows from [16, 17].

[1] A sequence of events $\mathcal{E}_n$ is said to occur *with high probability* (**whp**) if $\lim_{n\to\infty} \mathbf{Pr}(\mathcal{E}_n) = 1$.

Lemma 1 *Suppose $d_j = r$ for $r = 1, 2, \dots, n$. If F is chosen randomly from Ω then*

$$\mathbf{Pr}(F \in \Omega^*) \geq e^{-20r^2}.$$

Proof Consider the following algorithm from Frieze and Łuczak [10]:

Algorithm GENERATE

begin

$W := (x_1, x_2, \dots, x_{2D-1}, x_{2D})$.

$F_0 := \emptyset$

For $i = 1$ **to** D **do**

begin

$$F_i := \begin{cases} F_{i-1} \cup \{(x_{2i-1}, x_{2i})\} & \left(\text{With probability } \frac{1}{2i-1}\right) \ \mathbf{A} \\ \\ F_{i-1} \cup \{(x_{2i-1}, z_1), (x_{2i}, z_2)\} - (z_1, z_2) & \left(\text{With probability } \frac{2i-2}{2i-1}\right) \ \mathbf{B} \end{cases}$$

where (z_1, z_2) is chosen randomly from F_{i-1} where the order of z_1, z_2 is random.

end

Output $F := F_D$.

end

We will prove that GENERATE produces a random member of Ω and then describe a sequence $x_1, x_2, \dots, x_{2D}$ from which we can prove the lemma.

Let $W^{(i)} = \{x_1, x_2, \dots, x_{2i}\}$ and let Ω_i be the set of configurations of $W^{(i)}$. We show inductively that F_i is a random member of Ω_i. This clearly true for $i = 1$ and so assume that for some $i \geq 2$ we have that F_{i-1} is chosen randomly from Ω_{i-1}.

Now consider a bipartite graph H with vertex partition Ω_{i-1}, Ω_i and an edge (F, F') if either $F' = F \cup \{x_{2i-1}, x_{2i}\}$ or $F' = (F \setminus \{a, b\}) \cup \{\{a, x_{2i-1}\}, \{\{b, x_{2i}\}\}$ for some $\{a, b\} \in F$. Then each $F \in \Omega_{i-1}$ has degree $2i - 1$ in H and each $F \in \Omega_i$ has degree 1. Our algorithm chooses a random F from Ω_{i-1} (induction) and then choose a random H-edge leaving it. This implies uniformity.

Now for the ordering. Let us specify that x_i is always chosen as one of the remaining points for which $\phi(x_i)$ occurs as little as possible in the given sequence $(x_1, \dots, x_{i-1})$. Let $\Omega_i^* = \{F \in \Omega_i : \ \gamma(F) \text{ is simple}\}$. Let

$$\Delta_i \leq \frac{2i}{n} + 1 \tag{1}$$

denote the maximum degree in F_i. Then

$$\mathbf{Pr}(F_i \in \Omega_i^* \mid F_{i-1} \in \Omega_{i-1}^*) \geq \begin{cases} 1 & i \leq n/2 \\ \left(1 - \frac{\Delta_i^2}{(2i-1)(2i-3)}\right)\left(1 - \frac{4(\Delta_i+1)}{n}\right) & n/2 < i \leq D \end{cases} \tag{2}$$

If $i \leq n/2$ then F_i induces a matching. If $i > n/2$ and the ith step of GENERATE is type A then the probability that this creates a parallel edge is at most $\frac{\Delta_i^2}{2i-3}$, this being a bound on the probability that F_{i-1} contains an edge parallel to (x_{2i-1}, x_{2i}). If the ith step is type B then a loop is created with probability at most $\frac{4}{n}$ and a parallel edge is created with probability at most $\frac{4\Delta_i}{n}$.

The result follows from (2) and crude estimations. $\square$

At this point we describe an alternate simpler algorithm for constructing a random configuration. It is helpful to have this for the estimation of the probability of certain events.

Algorithm CONSTRUCT
begin
  $F_0 \leftarrow \emptyset$; $R_0 \leftarrow W$
  For $t = 1$ **to** D **do**
  begin
    Choose $u_t \in R_{t-1}$ *arbitrarily*
    Choose v_t *randomly* from $R_{t-1} \setminus \{u_t\}$
    $F_t \leftarrow F_{t-1} \cup \{\{u_t, v_t\}\}$; $R_t \leftarrow R_{t-1} \setminus \{u_t, v_t\}$
  end
  Output $F := F_D$.
end

3 r-Connectivity

We prove Theorem 1(a). Since the result is already known for r constant, we henceforth assume that $r \geq 10^6$ and at most $c_0 n$, where c_0 is sufficiently small.

Lemma 2 *For $G_r \in \mathcal{G}_r$ let $\mathcal{Q}$ be the events that for all vertices $v, w \in V$*

(a) *If $r = o(n)$ then $|N(v) \cap N(w)| \leq 10 + o(r)$.*

(b) *If $\log^2 n \leq r \leq n$ then $|N(v) \cap N(w)| \leq r^2/n + 4\sqrt{r \log n}$.*

Then $\mathbf{Pr}(\overline{\mathcal{Q}}) = O(1/n^2)$.

Proof It will be convenient for this lemma to let $\mathcal{G}_r$ be the set of simple r-regular graphs on $n+1$ vertices. Let $v, w \in [n+1]$.

Let $\mathcal{F} = \{G : G = G_r - v,\ G_r \in \mathcal{G}_r,\ N(v) = S\}$ be the graphs G on n vertices formed by deleting v, and with a fixed set of vertices $N(v) = S$ of degree $r - 1$.

Let $\mathcal{N}(R,T)$ be the set of graphs in $\mathcal{F}$ with $N(w) \cap S = R$ and $N(w) - S = T$. Let $|R| = k,\ k < r$ and let $x \in S - R,\ a \in T$. We consider a bipartite graph $\mathcal{B}$ with left vertex set $\mathcal{N}(R,T)$ and right vertex set $\mathcal{N}(R+x, T-a)$.

If $G \in \mathcal{N}(R,T)$ and $(w,a),(x,b)$ are edges of G we make a *swap* $(wa, xb) \to (wx, ab)$ in which edges $(w,a),(x,b)$ are replaced by $(w,x),(a,b)$ provided the resulting graph G' is simple. These swaps define the edges of $\mathcal{B}$.

Let $\nu(a,x;G)$ be the number of edges in G from a to $N(x)$ and let $\delta(a,x;G) = 1$ if $a \in N(x)$.

Considering the swap $G : (wa, xb) \to (wx, ab)$ giving G' we have

$$d_L(G) = |N(x)| - \nu(a,x;G) - \delta(a,x;G)$$

for G' is simple iff $b \neq a$ and $b \notin N(a)$. Here $|N(x)| = r-1$ as $x \in S$. The swap leaves $\delta(a,x;G') = \delta(a,x;G)$ and $\nu(a,x;G') = \nu(a,x;G)$.

Considering the swap $G' : (wx, ab) \to (wa, xb)$ giving G we have

$$d_R(G') = |N(a)| - \nu(a,x;G') - \delta(a,x;G').$$

The graph $\mathcal{B}$ consists of components within which δ, ν are invariant. Let the components have bipartition size (N_L, N_R). We now prove that $N_L \geq N_R$. In any component with edges we have $d_R = d_L + 1$ so that $N_R = N_L d_L/(d_L+1)$. This also covers the case $(N_L, N_R) = (1,0)$ with $d_L = 0$ of isolated vertices in the left bipartition. The case $(N+L, N_R) = (0,1)$ of isolated vertices in the right bipartition, cannot occur, for

$$\nu(a,x;G') + \delta(a,x;G') = |N(x) - w| \leq r-2$$

and so

$$d_R(G') = |N(a)| - \nu - \delta \geq 2.$$

Thus

$$|\mathcal{N}(R,T)| \geq |\mathcal{N}(R+x, T-a)|.$$

The size of $\mathcal{N}(R,T)$ is invariant for all $R, T, |R| = k$ by a simple symmetry argument. Let $|\mathcal{N}(R,T)| = \eta(k)$. Let $f(k)$ be the number of graphs in $\mathcal{F}$ with $|N(w) \cap S| = k$. Then $f(k) = \binom{r}{k}\binom{n-r}{r-k}\eta(k)$. Let $T = \sum_{j=0\ldots r} f(j)$ then T is the number of simple graphs $|\mathcal{F}|$ and

$$\mathbf{Pr}(w \text{ has at least } k \text{ neighbours in } S) = \sum_{j \geq k} \frac{f(j)}{T}.$$

A hypergeometric random variable X, with $\mathbf{Pr}(X = j) = \binom{r}{j}\binom{n-r}{r-j}/\binom{n}{r}$ has mean $\mu = r^2/n$ and variance $r^2/n(1+O(r/n))$. Let $T^+ := \sum_{j=0\ldots k} f(j)$. As $\eta(k) \geq \eta(k+1)$ we have

$$T^+ \geq \eta(k) A \binom{n}{r}$$

where $A = 1 - o(1)$ as we will choose k in the upper tail of the distribution so that $\mathbf{Pr}(X \geq k) = o(1)$. As $T \geq T^+$,

$$\mathbf{Pr}(w \text{ has at least } k \text{ neighbours in } S) \leq \sum_{j \geq k} \frac{f(j)}{T^+} \leq \frac{1}{A} \sum_{j \geq k} \frac{\binom{r}{j}\binom{n-r}{r-j}}{\binom{n}{r}}.$$

If $r \leq \log^2 n$ let $k = \alpha r + 10$, $\alpha = 1/\log\log n$. The hypergeometric distribution is stochastically dominated by binomial $B(r, r/n)$ in the upper tail, provided $r = o(\sqrt{n})$. Thus

$$\mathbf{Pr}(X \geq \alpha r + 10) \leq \left(\frac{er^2}{(\alpha r + 10)n}\right)^{\alpha r + 10} = o(n^{-4})$$

follows from a standard bound on the tail of the binomial.

For $\log^2 n \leq r \leq n$ let $k = r^2/n + 4\sqrt{r \log n}$. We can apply Azuma's inequality to the $0, 1$ sequence of observations of the sampling process of X, with $c_i = 1$ to infer that

$$\mathbf{Pr}(X \geq r^2/n + 4\sqrt{r \log n}) = o(n^{-4}).$$

$\square$

Lemma 3 *Let $G_r \in \mathcal{G}_r$. Let $\mathcal{Q}$ be the event that:*

No set of vertices $U \subset V$ of G_r, $1 \leq |U| \leq n/70$, induces more than $r|U|/12$ edges.

Then $\mathbf{Pr}(\mathcal{Q}) = 1 - O(1/n^2)$.

Proof Let $\beta = 1/12$ and $\theta = 1/70$.

Case of $G \in \mathcal{G}_r$ and $|U| < 2\beta r$.

Let $|U| = u$. In a simple r-regular graph a set of size u induces at most $\binom{u}{2}$ edges and

$$\binom{u}{2} > \beta r u \implies u > \lceil 2\beta r + 1 \rceil.$$

Case of $2\beta r \leq |U| \leq \theta n$.

Let $\mathcal{E} = \{F \in \Omega :$ No vertex set $U,\ 2\beta r \leq |U| \leq \theta n$ induces more than $\beta r|U|$ edges $\}$.

In Ω the number of edges X falling inside a set U is dominated by a binomial random variable $Y \sim B(ur, u/(n-u))$ in which each configuration point of U independently selects a pairing on the assumption that all configuration points of U are available, and that ru configuration points of $V \setminus U$ are unavailable. Now, $\mu_Y = ru^2/(n-u)$ and

$$\begin{aligned} \mathbf{Pr}(Y \geq \beta r u) &= \mathbf{Pr}(Y \geq (\beta(n-u)/u)\mu_Y) \\ &\leq \left(\frac{ue}{\beta(n-u)}\right)^{\beta r u}, \end{aligned}$$

where $\beta(n-u)/eu > 2$ provided $u < n/(1+2e/\beta)$. This is satisfied by $\beta = 1/12$ and $\theta = 1/70$. Thus

$$\mathbf{Pr}\left(\overline{\mathcal{E}}\right) \leq \sum_{u=2\beta r}^{\theta n} \binom{n}{u} \left(\frac{13ue}{n}\right)^{\beta r u} \leq \sum_{u=2\beta r}^{\theta n} \left(\frac{ne}{u}\right)^{u} \left(\frac{13ue}{n}\right)^{\beta r u} \leq$$
$$\sum_{u=2\beta r}^{\theta n} \left(\frac{13ue}{n}\right)^{\beta r u/2} \leq 2\left(\frac{26e\beta r}{n}\right)^{\beta^2 r^2} \leq \exp\left\{-Kr^2 \log\frac{n}{r}\right\}$$

for some $K > 0$.

Using Lemma 1 we see that the result holds **whp** in $\mathcal{G}_r$. $\square$

We can now complete the proof of Theorem 1(a).

Suppose G_r is not r-connected. Suppose there is a separator $X, |X| = x \leq r-1$ where $G_r - X = A + B$ and $|A| = a \leq |B| = b$.

Case 1: $2 \leq a \leq r/2$.
Let $u, v \in A$ be arbitrary. If $r = o(n)$ then by Lemma 2,

$$|N(u) \cup N(v)| \geq 2r - |N(u) \cap N(v)| \geq 2r - o(r) - 10$$

However

$$|A \cup N(A)| = |A| + |X| \leq a + r - 1 < 3r/2$$

which is a contradiction.

If $r = cn, c \leq 1/4$ we can infer that $|N(u) \cup N(v)| \geq (1-o(1))7r/4$ which gives a similar contradiction via Lemma 2.

Case 2: $r/2 \leq a \leq n/80$.
$|A \cup X| \leq a + r - 1$ and $A \cup X$ contains at least $ar/2$ edges. Putting $\beta = 1/12$ and using Lemma 3

$$\frac{ar}{2} \leq \beta r(a + r - 1) \implies a < r/5$$

which is a contradiction.

Case 3: $n/80 \le a \le \lceil n/2 \rceil$.
If configuration F is chosen randomly from Ω then the probability of the existence of such a separator is at most

$$\binom{n}{a}\binom{n-a}{r-1}\left(1-\frac{b}{n}\right)^{ar/2}.$$

Thus the probability in $\mathcal{G}_r$ is at most

$$e^{20r^2}4^n e^{-a(n-(a+r))r/2n} = e^{-rn/500}$$

for c_0 sufficiently small. □

4 Hamilton cycles in r-regular graphs

We prove Theorem 1(b) with the same assumptions on the size of r.

Definition: Let $\mathcal{G}_r^*$ denote the subset of $\mathcal{G}_r$ consisting of those graphs G with the following properties.

C1: All sets of vertices U of size at most $n/70$ induce at most $r|U|/12$ edges.

C2: The graph G is connected.

Lemma 3 and Theorem 1(a) imply that

Lemma 4 $|\mathcal{G}_r^*| = (1-o(1))|\mathcal{G}_r|$

Definition: Suppose $R \subseteq E$. For $v \in V$ let $d_R(v)$ denote the number of edges of R which are incident with v. A set of edges R is *deletable* from G, $(R \in \mathrm{Del}(G))$, if

D1: R avoids at least one longest path of G.

D2: $d_R(v) \in [\frac{r}{4}, \frac{r}{2}]$ for every $v \in V$.

Lemma 5 *Let $G \in \mathcal{G}_r$ and let R be a random subset of the edges of G where each edge of G is placed into R independently with probability 1/6. then*

$$\mathbf{Pr}(R \textit{ is deletable} \mid G) \ge e^{-n}$$

Proof Let P be some fixed longest path of G.

$$\mathbf{Pr}(R \text{ avoids } P \mid G) = \left(\frac{5}{6}\right)^{|P|} \geq \left(\frac{5}{6}\right)^{n}.$$

For the proof of (D2) we condition on (D1). We use the Lovász Local Lemma (symmetric version) (see for example Alon and Spencer [1]) to show that

$$\mathbf{Pr}(D2 \mid D1) \geq e^{-n/2}.$$

Let A_v be the event $\{d_R(v) \notin [\frac{r}{4}, \frac{r}{2}]$. Then $\mathbf{Pr}(A_v) \leq e^{-r/100}$ and the dependency graph has degree at most r. So for large r we can apply the lemma to show that conditional on D_1,

$$\mathbf{Pr}\left(\bigcap_{v \in V} \bar{A}_v\right) \geq (1 - 2e^{-r/100})^n.$$

$\square$

Definition: A set of edges S is *addable* to a graph H, ($S \in \mathrm{Add}(H)$), if $H + S$ has the same length longest path as H, and no longest path of H is closed to a cycle by S.

Let

$$\begin{aligned}
\mathcal{N} &= \{G \in \mathcal{G}^* : G \text{ is not Hamiltonian }\}\\
\mathcal{E} &= \{(G, R) : G \in \mathcal{N}, R \in \mathrm{Del}(G)\}\\
\Psi &= \{H : H = G - R,\ (G, R) \in \mathcal{E}\}\\
\mathcal{F} &= \{(G, S) : G \in \mathcal{G}_r, G - S \in \Psi, S \in \mathrm{Add}(G - S)\}.
\end{aligned}$$

We note that $\mathcal{E} \subseteq \mathcal{F}$. For, let $(G, R) \in \mathcal{E}$ so that $G - R \in \Psi$, and let P be any longest path of G avoided by R. By (C2), G is connected, so P cannot be contained in any cycle, as this would imply either that G was Hamiltonian, or that P was not a longest path. Thus R is addable for $G - R$ and $((G - R) + R, R) \in \mathcal{F}$.

Lemma 6 *Let $H \in \Psi$. Let S be chosen randomly from the collection of sets of edges such that $H + S \in \mathcal{G}_r$. There exists a constant $\delta > 0$ such that*

$$\mathbf{Pr}(S \in \mathit{Add}(H)) \leq e^{-\delta r n}.$$

Proof Let $H = G - R$, where R is deletable for G.

Given y_0 let $P_{y_h} = y_0 y_1 ... y_h$ be a longest path starting at y_0 in H. A *Pósa rotation* $P_{y_h} \to P_{y_{i+1}}$, [18, 6] gives the path $P_{y_{i+1}} = y_0 y_1 ... y_i y_h y_{h-1} ... y_{i+1}$ formed from P_{y_h} by adding the edge $y_h y_i$ and erasing the edge $y_i y_{i+1}$.

Let the initial path P be x_0Py_0. Let $END(a)$ be any set of endpoint vertices formed by Pósa rotations, with a fixed, of a longest path aPb in H. We prove that $|END(a)| \geq n/70$.

The Pósa condition for the rotation endpoint set U of a longest path P requires that $|N(U)| < 2|U|$. Let $\nu = |U \cup N(U)|$ and let $u = |U|$. Thus $u > \nu/3$. The condition (D2) guarantees that $U \cup N(U)$ induces at least $ru/4 > r\nu/12$ edges and thus (C1) implies $u > n/70$.

We condition on the degree sequence $(r - d_1, ..., r - d_n)$ of H and choose a replacement set of edges S of size $D = (d_1 + d_2 + \cdots + d_n)/2$.

To generate S we generate F randomly from Ω and condition on $H + \gamma(F)$ being simple. The edges of $\gamma(F)$ will then be our random S. We argue that

$$Pr(H + \gamma(F) \text{ is simple}) \geq \frac{1}{n^2}e^{-4r^2}. \tag{3}$$

We consider running GENERATE and observe that

$$\mathbf{Pr}(H + \gamma(F_i) \text{ is simple} \mid H + \gamma(F_{i-1}) \text{ is simple}) \geq \left(1 - \frac{1}{2i-1}\right)\left(1 - \frac{4r+2}{n}\right).$$

Here $1 - \frac{1}{2i-1}$ is the probability of a type B step and $\frac{4r+2}{n}$ bounds the probability we create a loop or multiple edge. Now $\prod_{i=1}^{D}\left(1 - \frac{1}{2i-1}\right) \geq n^{-2}$ and $\left(1 - \frac{4r+2}{n}\right)^D \geq e^{-4r^2}$ and (3) follows.

Now let $Y = \{(x, y) : \exists a \in END(x_0), b \in END(a) \text{ such that } \phi(x) = a, \phi(y) = b\}$. Then $S \in Add(H)$ implies $F \cap Y = \emptyset$. Now by using CONSTRUCT to generate F we see that

$$\mathbf{Pr}(F \cap Y = \emptyset) \leq \left(1 - \frac{1}{280}\right)^{rn/280} \leq e^{-\delta_1 rn}$$

where $\delta_1 \geq 10^{-6}$.
(We can assume that we first try to pair up x such that $\phi(x) \in END(x_0)$.)

Thus

$$Pr(S \in Add(H)) \leq e^{-\delta_1 rn} \cdot n^2 \cdot e^{4r^2}$$

and the lemma follows.

□

We can now complete the proof of Theorem 1(b). Suppose G_r is chosen randomly from $\mathcal{G}_r$ and then R is chosen randomly.

$$\begin{aligned}
\mathbf{Pr}(\mathcal{E}) &= \sum_{G\in\mathcal{N}} \mathbf{Pr}((G,R):\ R \in Del(G)) \\
&\geq e^{-n}\mathbf{Pr}(\mathcal{N}). \\
\mathbf{Pr}(\mathcal{F}) &= \sum_{H\in\Psi} \mathbf{Pr}((H+S,S):\ S \in Add(H) \mid H)\ \mathbf{Pr}(H) \\
&\leq e^{-\delta rn}.
\end{aligned}$$

Now, $\mathcal{E} \subseteq \mathcal{F}$ implies that $\mathbf{Pr}(\mathcal{E}) \leq \mathbf{Pr}(\mathcal{F})$ and so

$$\mathbf{Pr}(\mathcal{N}) \leq e^{n-\delta rn} = o(1)$$

and the theorem follows from Lemma 4. □

As a final remark we note that by following Frieze [9] it would seem that we can, at the expense of complicating the proof, prove the existence of a polynomial time algorithm for finding a Hamilton cycle.

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