The cover times of random walks on hypergraphs.

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Abstract

Random walks in graphs have been applied to various network exploration and network maintenance problems. In some applications, however, it may be more natural, and more accurate, to model the underlying network not as a graph but as a hypergraph, and solutions based on random walks require a notion of random walks in hypergraphs. While random walks in graphs have been extensively studied, there have been very few results showing properties of random walks in hypergraphs.

At each step, a random walk on a hypergraph moves from its current position v to a random vertex in a randomly selected hyperedge containing v. We consider two definitions of cover time for random walks on a hypergraph H. If the walk sees only the vertices it moves between, then the usual definition of cover time, $C(H)$, applies. If the walk sees the complete edge during the transition, then an alternative definition of cover time, the inform time $I(H)$ is used. The notion of inform time is a reasonable model of passive listening which fits the following types of situations. The particle is a rumor passing between friends, which is overheard by other friends present in the group at the same time. The particle is a message transmitted randomly from location to location by a directional transmission in an ad-hoc network, but all receivers within the transmission range can hear.

In this paper we give an expression for $C(H)$ which is tractable for many classes of hypergraphs, and calculate $C(H)$ and $I(H)$ exactly for random r-regular, s-uniform hypergraphs. We find that for such hypergraph **whp** $C(H)/I(H) = \Omega(s)$ for large s.

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1 Introduction

The idea of a random walk on a hypergraph is a natural one. The particle making the walk picks a random edge incident with the current vertex. The particle enters the edge, and exits via a random endpoint, other then the vertex of entry. Two alternative definitions of cover time are possible for this walk. Either the particle sees only the vertices it visits, or it inspects all vertices of the hyperedge during the transition across the edge.

A random walk on a hypergraph thus serves as a model of the following process. The vertices of a network are associated into groups, and these groups define the edges of the network. In the simplest case, the network is a graph so the groups are exactly the edges of the graph. In general, the groups may be larger, and represent friends, a family, a local computer network, all receivers within transmission range of a directed transmission in an ad-hoc network, etc. In this case the network is modeled as a hypergraph, the hyperedges being the group relationships. An individual vertex can be in many groups, and two vertices are neighbours if they share a common hyperedge. Within the network a particle (message, rumor, infection, etc.) is moving randomly from vertex to neighboring vertex. When this transition occurs all vertices in a given group are somehow affected (infected, informed) by the passage of the particle *within the group.* Examples of this type of process include the following. The particle is an infection passed from person to person and other family members also become infected with some probability. The particle is a virus traveling on a network connection in an intra-net. The particle is a message transmitted randomly from location to location by a directional transmission in an ad-hoc network, and all receivers within the transmission range can hear. The particle is a rumor passing between friends, which may be overheard by other friends present in the group at the same time.

Let $H = (V(H), E(H))$ be a hypergraph. For $v \in V = V(H)$ let $d(v)$ be the degree of v, i.e. the number of edges $e \in E$ incident with v, and let $d(H) = \sum_{v \in V} d(v)$ be the total degree of H. For $e \in E = E(H)$ let $|e|$ be the size of hyperedge e, i.e. the number of vertices $v \in e$, respecting multiplicity. Let $N(v)$ be the neighbour set of v, $N(v) = \{w \in V : \exists e \in E, e \supseteq \{v, w\}\}$. We regard $N(v)$ as a multi-set in which each $w \in N(v)$ has a multiplicity equal to the number of edges e containing both v and w. A hypergraph is r regular if each vertex is in r edges, and is s-uniform if every edge is of size s. A hypergraph is simple if no edge contains a repeated vertex, and no two edges are identical.

We assume a particle or message originated at some vertex u and, at step t, is moving randomly from a vertex v to a vertex w in $N(v)$. We model the problem conceptually as a random walk $\mathcal{W}_u = (W_u(0), W_u(1), \ldots, W_u(t), \ldots)$ on the vertex set of hypergraph H, where $W_u(0) = u$, $W_u(t) = v$ and $W_u(t+1) = w$.

Several models arise for reversible random walks on hypergraphs. Assume that the walk $\mathcal{W}$ is at vertex v, and consider the transition from that vertex. In the first model (Model 1), an edge e incident with v is chosen proportional to $|e| - 1$, where $|e|$ is the size of hyperedge e. The walk then moves to a random endpoint of that edge, other than v. This is equivalent to v choosing a neighbour w u.a.r. (uniformly at random) from the multi-set of neighbours $N(v)$, where vertex w is chosen according to its multiplicity. The stationary distribution of v in Model 1 is given by

$$\pi_v = \frac{\sum_{e : v \in e} (|e| - 1)}{\sum_{e \in E(H)} |e|(|e| - 1)}.$$

In the case of graphs this reduces to $\pi_v = d(v)/2m$, where $d(v)$ is the degree of v, and m is the number of edges in the graph. An alternative model (Model 2), is that when $\mathcal{W}$ is at v, edge e is chosen u.a.r. from the hyperedges incident with v, and *then* w is chosen u.a.r. from the vertices

$w \in e, w \neq v$. The stationary distribution of v in Model 2 is given by

$$\pi_v = \frac{d(v)}{\sum_{u \in V(H)} d(u)},$$

which corresponds to the familiar formula for graphs. If the hypergraph is uniform (all edges have the same size) then the models are equivalent.

Random walks on graphs are a well studied topic, for an overview see e.g. [1, 9]. Random walks on hypergraphs were used in [5] to cluster together electronic components which are near in graph distance for physical layout in circuit design. For that application, edges were chosen inversely proportional to their size, and then a random vertex within the edge was selected. A random walk model is also used for generalized clustering in [10]. As before, the aim is to partition the vertex set, and this is done via the Laplacian of the transition matrix. A paper which directly considers notions of cover time for random walks on hypergraphs is [3], where Model 2 is used. We mention the results shown in [3] later in this section after introducing some necessary terminology.

The (vertex) cover time $C(H)$ of a graph H, is given by $C(H) = \max_u C_u(H)$, where $C_u(H)$ is the expected time to visit all vertices of H for a walk starting at vertex u. The edge cover time $C_E(H)$ is similarly defined as $C_E(H) = \max_u C_{u,E}(H)$, where $C_{u,E}(H)$ is the expected time to visit all edges of H for a walk starting at vertex u.

For a hypergraph H, we define the vertex cover time $C(H)$, and the inform time $I(H)$ as follows.

The (vertex) cover time $C(H)$ of a hypergraph H, is given by $C(H) = \max_u C_u(H)$, where $C_u(H)$ is the expected time for the walk $\mathcal{W}_u$ to visit all vertices of H. Similarly, the edge cover time $C_E(H) = \max_u C_{u,E}(H)$, where $C_{u,E}(H)$ is the expected time to visit every edge of G for a walk starting at vertex u.

Suppose that the walk $\mathcal{W}_u$ is at vertex v. Using e.g. Model 2, the walk first selects an edge e incident with v and then makes a transition to $w \in e$. The vertices of e are said to be *informed* by this move. The *inform time* $I(H)$, introduced in [3] as the *radio cover time*, is the maximum over start vertices u, of the expected time at which all vertices of the graph are informed. More formally, let $\mathcal{W}_u(t) = (W_u(0), W_u(1), ..., W_u(t))$ be the trajectory of the walk. Let $e(j)$ be (the vertex set of) the edge $e(j)$ used for the transition $W(j), W(j+1)$ at step j. Let $\mathcal{S}_u(t) = \cup_{j=0}^{t-1} e(j)$ be the set of vertices spanned by the edges of $\mathcal{W}_u(t)$. Let $\boldsymbol{I}_u$ be the step t at which $\mathcal{S}_u(t) = V$ for the first time, and let $I(H) = \max_u \mathbf{E}(\boldsymbol{I}_u)$. We use the name "inform time" rather than the name "radio cover time" used in [3] to indicate the relevance of this term beyond the radio networks.

Several upper bounds on the cover time $C(H)$ are readily obtainable. The first mimics the upper bound of $O(nm)$ steps for graphs, [2], based on a twice round the spanning tree argument. For Model 1, if we replace each edge e by a graph consisting of a clique of size $\binom{|e|}{2}$ this gives an upper bound of $O(nm\overline{s^2})$ for connected hypergraphs. Here $\overline{s^2}$ is the expected squared edge size $(\sum_{e \in E(H)} |e|^2)/m$. Thus $C(H) = O(n^3 m)$. A better bound of $O(nm\overline{s}) = O(n^2 m)$ was shown in [3] for Model 2.

Similarly, a Matthews type bound of $O(\log n \cdot \max_{u,v} \mathbf{E}(\boldsymbol{H}_{u,v}))$ on the cover time exists, where $\mathbf{E}(\boldsymbol{H}_{u,v})$ is the expected hitting time of v starting from u. We contribute a bound on the cover time of a hypergraph given in Theorem 1, which allows us to calculate $C(H)$ for many classes of hypergraphs H. To prove this bound, we first observe that we can always write $\mathbf{E}(\boldsymbol{H}_{u,v}) = O(T + \mathbf{E}_\pi(\boldsymbol{H}_v))$, where T is a suitable mixing time and $\mathbf{E}_\pi(\boldsymbol{H}_v)$ is the expected hitting time of vertex v from stationarity. Then we bound $\mathbf{E}_\pi(\boldsymbol{H}_v)$ and apply Matthews' bound.

Theorem 1. *Let H be connected and aperiodic with stationary distribution π. Let P denote the* *transition matrix for a random walk on H. Let T be a mixing time such that $|P_u^{(T)}(v) - \pi_v| \leq \pi_v$ for all $u, v \in V$, and suppose that $\max \pi_v = o(1)$. For a walk starting from v, let $R_v(T)$ be the expected number of returns to v during T. Then*

$$C(H) = \log n \cdot O\left(T + \max_v \frac{R_v(T)}{\pi_v}\right). \tag{1}$$

This bound for $C(H)$ can be evaluated for many classes of random hypergraphs. For example, for random r-regular, s-uniform hypergraphs $\mathcal{G}(n, r, s)$, and random s-uniform hypergraphs $G_{n,p,s}$ where each edge occurs independently with probability p. Let $r \geq 2$ and $s \geq 3$ in $\mathcal{G}(n, r, s)$, and let $p \geq C \log n / \binom{n-1}{s-1}$ in $G_{n,p,s}$, where $C > 1$. Then **whp** the required mixing time is $T = o(n)$, $\pi_v = \Theta(1/n)$, and $R_v(T) = 1 + O(1)$, so Theorem 1 implies that **whp** $C(H) = O(n \log n)$ for these classes of graphs. The proof details are not given here.

The calculation of inform time $I(H)$ seems more challenging. Avin *et al.* [3] consider a special type of *directed hypergraphs*, called *radio hypergraphs*, and analyse $I(H)$ on *one-* and *two-dimensional mesh radio hypergraphs*, which are induced by a cycle and a square grid on a torus, respectively. Their result for the two-dimensional mesh can be stated in the following way. For a random walk on a $\sqrt{n} \times \sqrt{n}$ grid such that in each step all vertices within distance k from the current vertex are informed and the walk moves to a random vertex in this k-neighbourhood, the inform time is $I(H) = O((n/k^2) \log(n/k^2) \log n)$.

In this paper we calculate precisely $C(H)$, $I(H)$ and $C_E(H)$ for the case of simple random r-regular, s-uniform hypergraphs H (a simple hypergraph does not have multiple edges). As far as we know, this is the first analysis of cover time and inform time for random walks on classes of general (undirected) hypergraphs. The proof of the following theorem is the main technical contribution of this paper. Throughout this paper "log" stands for the natural logarithm.

Theorem 2. *Suppose that $r \geq 2$ and $s \geq 3$ are constants and H is chosen u.a.r. from the set of all simple r-regular, s-uniform hypergraphs with vertex set $V = [n]$. Then* **whp** *as $n \to \infty$,*

$$C(H) \sim \left(1 + \frac{1}{(r-1)(s-1)-1}\right) n \log n,$$

$$I(H) \sim \left(1 + \frac{s-1}{(r-1)(s-1)-1}\right) \frac{n}{s-1} \log n,$$

and

$$C_E(H) \sim \left(1 + \frac{s-1}{(r-1)(s-1)-1}\right) \frac{rn}{s} \log n.$$

Our proof of the above theorem applies also if s and/or r grow (slowly) with n. In particular, we have the following corollary.

Corollary 3. *If $r \geq 2$, $s \to \infty$, and $(rs)^{4 \log\log\log n} = o(\log n)$, then*

$$C(H) \sim n \log n \quad \text{and} \quad I(H) \sim \frac{r}{r-1} \frac{n}{s} \log n.$$

Thus in this case, seeing s vertices at each step of the walk leads to an $\Omega(s)$ speed up in cover time. In the case of graphs, $I(H) = C(H)$, and $C_E(H) \geq C(H)$. For hypergraphs, clearly $I(H) \leq C(H)$. However there is the possibility that $C_E(H) \leq C(H)$, as every edge can be visited without visiting every vertex. We must have $I(H) \leq C_E(H)$ as a vertex is informed whenever the walk covers an edge containing that vertex. Indeed, intuitively we should have

$C_E(H)$ about r times $I(H)$, if every vertex has degree r. We note that our theorem gives $C_E(H) \sim r\,((s-1)/s)\,I(H)$.

It was shown in [3] that for any n-vertex hypergraph H, $I(H) = O(\log n \cdot \max_{u,v} \mathbf{E}(\tilde{\boldsymbol{H}}_{u,v}))$, where $\mathbf{E}(\tilde{\boldsymbol{H}}_{u,v})$ is the expected time when vertex v is informed starting from u, called the radio hitting time in [3]. For a random walk on an s-uniform hypergraph, in each period of $2 \cdot \max_x \mathbf{E}(\tilde{\boldsymbol{H}}_{x,v}))$ steps, the probability that vertex v is visited is at least $1/(2s)$ (there is a transition within an edge containing v with probability at least $1/2$ and each such transition either leaves from v or goes to v with probability $1/(s-1)$). Hence $\mathbf{E}(\boldsymbol{H}_{u,v}) = O(s \cdot \max_x \mathbf{E}(\tilde{\boldsymbol{H}}_{x,v}))$, implying that $C(H) = O(s \log n \cdot \max_{u,v} \mathbf{E}(\tilde{\boldsymbol{H}}_{u,v}))$. Thus the speed-up of the inform time over the cover time is always $O(s \log n)$, and is $O(s)$, if $I(H) = \Theta(\log n \cdot \max_{u,v} \mathbf{E}(\tilde{\boldsymbol{H}}_{u,v}))$.

2 Proof of Theorem 1

To prove Theorem 1, we use the bound $C(H) = O(\log n \cdot \max_{u,v} \mathbf{E}(\boldsymbol{H}_{u,v}))$, the observation that $\mathbf{E}(\boldsymbol{H}_{u,v}) = O(T + \mathbf{E}_\pi(\boldsymbol{H}_v))$, and the bound on $\mathbf{E}_\pi(\boldsymbol{H}_v)$ given in Lemma 4 below. The quantity $\mathbf{E}_\pi(\boldsymbol{H}_v)$, expected hitting time of a vertex v from the stationary distribution π, can be expressed as $\mathbf{E}_\pi(\boldsymbol{H}_v) = Z_{vv}/\pi_v$, where

$$Z_{vv} = \sum_{t=0}^{\infty} (P_v^{(t)}(v) - \pi_v), \tag{2}$$

see e.g. [1]. Let P denote the transition matrix for a random walk on H, and, for a walk $\mathcal{W}_v$ starting from v define

$$R_v(T) \;=\; \sum_{t=0}^{T-1} P_v^{(t)}(v). \tag{3}$$

Thus $R_v(T)$ is the expected number of returns made by $\mathcal{W}_v$ to v during T steps, in the hypergraph H. We note that $R_v \geq 1$, as $P_v^{(0)}(v) = 1$.

Lemma 4. *Let T be a mixing time of a random walk $\mathcal{W}_u$ on H satisfying $|P_u^{(T)}(x) - \pi_x| \leq \pi_x$ for all $u, x \in V$. Then, assuming $\pi_v = o(1)$,*

$$\mathbf{E}_\pi(\boldsymbol{H}_v) \leq 2T + \frac{R_v(T)}{\pi_v}. \tag{4}$$

Proof Let $D(t) = \max_{u,x} |P_u^{(t)}(x) - \pi_x|$. It follows from e.g. [1] that $D(s+t) \leq 2D(s)D(t)$. Hence, since $\max_{u,x} |P_u^{(T)}(x) - \pi_x| \leq \pi_v$, then for each $k \geq 1$, $\max_{u,x} |P_u^{(kT)}(x) - \pi_x| \leq (2\pi_v)^k$. Thus

$$\begin{aligned} Z_{vv} = \sum_{t=0}^{\infty} (P_v^{(t)}(v) - \pi_v) \;&=\; \sum_{t<T} (P_v^{(t)}(v) - \pi_v) \;+\; T \sum_{k\geq 1} (2\pi_v)^k \\ \leq\; R_v(T) + 2T\pi_v. \end{aligned}$$

□

3 Proof of Theorem 2: preliminaries

We explain the proof of the value of $C(H)$ of Theorem 2; the proofs of $I(H)$ and $C_E(H)$ are similar. We reduce the walk $\mathcal{W}_u(H)$ on the hypergraph H to an equivalent walk $\mathcal{W}_u(G)$ on a graph $G(H)$. This is done in Section 3.4.

In Section 3.1 we state a lemma (Lemma 5) on which the proof of Theorem 2 is based. Lemma 5 gives the probability that a random walk $\mathcal{W}_u(t)$ on a graph G, does not visit a given vertex v within t steps after a suitably defined mixing time T. This lemma is proved in [6]. To use Lemma 5, we have to calculate the parameter p_v of the walk in the associated graph $G(H)$ defined in (11). We can calculate p_v, if v is a *tree-like vertex*. In Section 3.2 we define this property of a tree-like vertex of a hypergraph, which holds for most vertices of H, **whp**. In Section 3.3 we outline the *configuration model* for generating a random r-regular s-uniform hypergraph H, which we use to prove that most vertices of H are indeed tree-like, **whp**.

In Section 3.5, we establish the conductance of the graph $G(H)$, and hence the value of the mixing time T on graph $G(H)$ (using the relation between the conductance and the mixing time given in Section 3.1). We also prove that the conditions of Lemma 5 hold for the associated graph $G(H)$, provided v is tree-like, and for such vertices, derive the parameter p_v. In Section 4 we prove the formula for $C(H)$ stated in Theorem 2, by establishing an upper bound on $C(H)$ in Section 4.1, and a lower bound in Section 4.2. In Section 5 we sketch how the calculations of $C(H)$ can be adapted to derive the formulas for $I(H)$ and $C_E(H)$ given in Theorem 2.

3.1 Random walk background

Let $G = (V, E)$ denote a fixed connected graph with n vertices and m edges. Let P be the matrix of transition probabilities of the walk and let $P_u^{(t)}(v) = \mathbf{Pr}(W_u(t) = v)$. We assume the random walk $\mathcal{W}_u$ on G is ergodic, and thus the random walk has stationary distribution π, where $\pi_v = d(v)/(2m)$, and $d(v)$ is the degree of vertex v.

Let Φ_G be the conductance of G i.e. $\Phi_G = \min_{S\subseteq V, \pi_S \le 1/2} \Phi_G(S)$ where

$$\Phi_G(S) = \frac{\sum_{x\in S} \pi_x P(x, \bar{S})}{\pi_S}. \tag{5}$$

Then, with $\Phi = \Phi_G$,

$$|P_u^{(t)}(x) - \pi_x| \le (\pi_x/\pi_u)^{1/2} e^{-\frac{\Phi^2}{2} t}. \tag{6}$$

Let T be such that, for $t \ge T$

$$\max_{u,x\in V} |P_u^{(t)}(x) - \pi_x| \le n^{-3}. \tag{7}$$

If this inequality holds, we say the distribution of the walk is in *near stationarity*.

We consider the returns to vertex v made by a walk $\mathcal{W}_v$, starting at v. Let $r_t = \mathbf{Pr}(W_v(t) = v)$ be the probability that the walk returns to v at step $t = 0, 1, ...$ In particular note that $r_0 = 1$, as the walk starts on v. For a walk $\mathcal{W}_v$ starting from v define

$$R_v(T, z) \ = \ \sum_{t=0}^{T-1} P_v^{(t)}(v) z^t. \tag{8}$$

Thus $R_v(T, 1) = R_v(T)$ in (3).

Let $v \in V$. We list the conditions required by Lemma 5.
(o) T is such that $\max_{u,x} |P_u^{(T)}(x) - \pi_x| \le 1/n^3$.
(i) For some constant $\theta > 0$ we have:

$$\min_{|z|\le 1+1/KT} |R_v(T, z)| \ge \theta, \tag{9}$$

where K is a large constant.
(ii) $T\pi_v = o(1)$ and $T\pi_v = \Omega(n^{-2})$.

Lemma 5. [6] *Assume conditions (o), (i), (ii) above hold for a graph G. Let $\mathcal{A}_v(t)$ be the event that a walk $\mathcal{W}_u$ on graph G, does not visit vertex v at steps $T, T+1, \ldots, t$. Then,*

$$\mathbf{Pr}(\mathcal{A}_v(t)) = \frac{(1+O(T\pi_v))}{(1+p_v)^t} + O(T^2\pi_v e^{-t/KT}), \tag{10}$$

where p_v is given by the following formula, with $R_v = R_v(T)$:

$$p_v = \frac{\pi_v}{R_v(1+O(T\pi_v))}. \tag{11}$$

3.2 Tree-like vertices

To use Lemma 5, we need the parameter R_v for (11). To calculate R_v, the expected number of returns made by $\mathcal{W}_v$ to vertex v during T steps, we need to identify the local structure of a typical vertex of a random hypergraph H. Let

$$\omega = \log\log\log n.$$

A sequence $v_1, v_2, \ldots, v_k \in V$ is said to define a *path* of *length* $k-1$ if there are *distinct* edges $e_1, e_2, \ldots, e_{k-1} \in E$ such that $\{v_i, v_{i+1}\} \subseteq e_i$ for $1 \leq i \leq k-1$. A sequence $v_1, v_2, \ldots, v_k \in V, k \geq 3$ is said to define a *cycle* of *length* k if there are *distinct* edges $e_1, e_2, \ldots, e_k \in E$ such that $\{v_i, v_{i+1}\} \subseteq e_i$ for $1 \leq i \leq k$, with $v_{k+1} = v_1$. A path/cycle is short if it has length at most ω.

A vertex $v \in V(H)$ is said to be *locally-tree-like to depth* k if there does not exist a path from v of length at most k to a cycle of length at most k. An edge $e \in E(H)$ is locally-tree-like to depth k, if it contains only vertices which are locally-tree-like to depth k. A vertex, or en edge, is *tree-like* if it is locally-tree-like to depth ω.

We argue that almost all vertices of H are tree-like. The proofs are in Appendix A3.1.

Lemma 6. Whp *there are at most $(rs)^{3\omega}$ vertices in H that are not tree-like.*

Lemma 7. Whp *there are no short paths joining distinct short cycles.*

3.3 Configuration model

We will need a workable model of an r-regular s-uniform hypergraph. We use a hypergraph version of the configuration model of Bollobás [4]. A configuration $C(r,s)$ consists of a partition of rn labeled points $\{a_{1,1}, ..., a_{1,r}, \cdots, a_{n,1}, ..., a_{n,r}\}$ into unordered sets $E_i,\ i = 1, ..., rn/s$ of size s. We assume naturally that s divides rn. We refer to these sets as the hyperedges of the configuration, and to the sets $v_i = \{a_{i,1}, ..., a_{i,r}\}$ as the vertices. By identifying the points of v_i, we obtain an r-regular, s-uniform (multi-)hypergraph $H(C)$. In general, many configurations map to one underlying hypergraph $H(C)$. Considering the set $\mathcal{C}(r,s)$ of all configurations $C(r,s)$ with the uniform measure, the measure $\mu(H(C))$ depends only on the number of parallel edges (if any) at each vertex, and as an example all *simple* hypergraphs i.e. those without multiple edges have equal measure in $H(C)$. The probability a u.a.r. sampled configuration is simple is bounded below by a constant dependent only on r and s.

For the values of r, s considered in this paper, the probability $H(C)$ is simple is $\Omega(e^{-(r-1)(s-1)})$. It follows that any almost sure property of $H(C)$ is also an almost sure property of simple hypergraphs G.

3.4 Construction of an equivalent (contracted) graph

To calculate the quantities $C(H), I(H)$ and $C_E(H)$ we replace the hypergraph H with graphs $G(H)$, $\Gamma(v)$ and $\Gamma(e)$, where v and e are a tree-like vertex and edge of G, respectively. The precise construction of these graphs is as follows:

Clique graph $G(H)$. To estimate the cover time of hypergraph H we define a (multi-)graph $G(H)$ with the same cover time. To obtain $G(H)$ from H we replace each hyperedge $e \in E(H)$ by a clique of size $|e|$ on the vertex set of e. This transforms the hypergraph H into a multi-graph $G(H)$, which we call the clique graph of H. Formally, $G(H) = (V, F)$ where $F = \bigcup_{e \in E} \binom{e}{2}$.

We can think of $\mathcal{W}_u$ as a walk on $G(H)$. Thus, the cover time of $G(H)$ is the cover time of H.

Inform-Contraction graph $\Gamma(v)$. This graph will be used in the analysis of the inform time $I(H)$. Let S_v be the multi-set of edges $\{w, x\}$ in $G(H)$, not containing v, but which are contained in hyperedges incident with vertex v in H i.e.

$$S_v = \{\{w, x\} : \exists e \in E, v \in e, \text{ and } w, x \in e \setminus \{v\}\}.$$

Since H is r-regular and s-uniform, each S_v has size $r\binom{s-1}{2}$.

A vertex v is informed if either (i) v is visited or (ii) S_v is visited by $\mathcal{W}_u$. To compute the probability that v or S_v is visited we subdivide each edge $f = \{w, x\}$ of S_v by introducing an artificial vertex a_f. Thus f is replaced by $\{w, a_f\}, \{a_f, x\}$. Call the resulting graph $G_v(H)$. Let $D_v = \{v\} \cup \{a_f : f \in S_v\}$ and note that D_v is an independent set in $G_v(H)$. Now contract D_v to a single vertex $\gamma = \gamma(D_v)$. Let $\Gamma(v)$ be the resulting multi-graph. The degree of γ is then

$$d(\gamma) = r\left(2\binom{s-1}{2} + (s-1)\right) = r(s-1)^2.$$

Furthermore,

$$d(\Gamma(v)) = d(G_v(H)) = r(s-1)n + r(s-1)(s-2).$$

For a random walk in $\Gamma = \Gamma(v)$ the stationary distribution of γ is thus

$$\pi(\gamma) = \frac{s-1}{n+s-2}. \tag{12}$$

Note that $m_v = |E(\Gamma(v)| = |E(G(H))| + r\binom{s-1}{2} = r(s-1)n/2 + r\binom{s-1}{2}$.

Suppose now that $\mathcal{X}_u$ is a random walk in $G_v(H)$ starting at $u \notin D_v$. For $t \geq T$, let $\mathcal{B}_v(t)$ be the event that the walk $\mathcal{W}_u$ in $G(H)$ does not visit $S_v \cup \{v\}$ at steps $T, T+1, \ldots, t$. Then $\mathcal{B}_v(t)$ is equivalent to $\wedge_{x \in D_v} \mathcal{A}_x(t)$ defined with respect to $\mathcal{X}_u$.

Edge-Contraction graph $\Gamma(e)$. This graph will be used in the analysis of the edge cover time $C_E(H)$. Starting from $G(H)$, and given $e \in E(H)$ form $G_e(H)$ as follows. For each of the edges $f = \{u, v\} \in \binom{e}{2}$, subdivide f using a new vertex a_f. Thus f is replaced by $\{u, a_f\}, \{a_f, v\}$. The set $D_e = \{a_f : f \subseteq e \in E(H)\}$ gives rise to $G_e(H)$, similarly as for $G_v(H)$ above. Contract D_e to a vertex γ to form a multi-graph $\Gamma(e)$, similalry to $\Gamma(v)$. The degree of γ is then

$$d(\gamma) = s(s-1).$$

Furthermore,

$$d(\Gamma(e)) = d(G_e(H)) = rn(s-1) + s(s-1).$$

For a random walk in $\Gamma = \Gamma(e)$ the stationary distribution of γ is thus

$$\pi(\gamma) = \frac{s}{rn+s}. \tag{13}$$

Suppose now that $\widehat{\mathcal{X}}_u$ is a random walk in $G_e(H)$ starting at $u \notin D_e$. For $t \geq T$, let $\mathcal{B}_e(t)$ be the event that the walk $\mathcal{W}_u$ in $G_e(H)$ does not visit D_e at steps $T, T+1, \ldots, t$.

The following lemma is established in the Appendix A1. It is used in conjunction with Lemma 5, which gives $\mathbf{Pr}(\mathcal{A}_\gamma(t); \Gamma)$.

Lemma 8. *Let $x = v$ or e, and let $\Gamma = \Gamma(v)$ or $\Gamma(e)$, respectively. Let $\mathcal{Y}_u$ be a random walk in Γ starting at $u \neq \gamma$. Let T be a mixing time satisfying (7) in both $G_x(H)$ and Γ. Then*

$$\mathbf{Pr}(\mathcal{A}_\gamma(t); \Gamma) = \mathbf{Pr}(\mathcal{B}_x(t); G(H)) \left(1 + O\left(\frac{s}{n}\right)\right),$$

where the probabilities are those derived from the walk in the given graph.

3.5 Conditions and parameters for Lemma 5

Let $R_v = R_v(T)$ be as defined in (3). To establish Theorem 2 we need precise estimates of R_v in $G(H)$ and R_γ in $\Gamma(v)$ and $\Gamma(e)$. Once this is done the theorem will follow from Lemma 5. But first we have to estimate the value of T in (7). This is done via lower bounds on conductance of graphs $G(H)$, $\Gamma(v)$ and $\Gamma(e)$, derived in Appendix A2, and using (6).

Lemma 9. *The conductance Φ_G of the graph $G(H)$ is $\Omega(1/s)$* **whp**.
The conductance Φ_Γ of the graph $\Gamma = \Gamma(v), \Gamma(e)$ is $\Omega(1/s)$ **whp**.

We then apply (6) to check (7), and obtain the following lemma.

Lemma 10. *Let $s = o(\log n)$, and let $T = A \log^2 n$, where A is a large constant. Then* **whp** *T satisfies the mixing time condition (7) in each of the graphs $G(H), \Gamma(v), \Gamma(e)$.*

To apply Lemma 5 it remains to check that the technical condition (i) holds (the condition (ii) is clear since $T = O(\log^2 n)$) and to obtain the value of p_v in (11). The results are given in the next lemma, and derived in Appendix A3.2-A3.4 (the formulas for p_v, $p_{\gamma(v)}$ and $p_{\gamma(e)}$) and Appendix A4 (the proof of the condition (i) of Lemma 5).

Lemma 11.

(i) *Let v be tree-like in H, then in $G(H)$ the value of p_v is given by*

$$p_v = (1 + o(1)) \frac{1}{n} \frac{(r-1)(s-1) - 1}{(r-1)(s-1)}. \tag{14}$$

(ii) *Let v be tree-like in H, then in $\Gamma(v)$ the value of $p_{\gamma(v)}$ is given by*

$$p_\gamma = (1 + o(1)) \frac{s-1}{n} \frac{(r-1)(s-1) - 1}{r(s-1) - 1}. \tag{15}$$

(iii) *Let e be tree-like in H, then in $\Gamma(e)$ the value of $p_{\gamma(e)}$ is given by*

$$p_\gamma = (1 + o(1)) \frac{s}{rn} \frac{(r-1)(s-1) - 1}{r(s-1) - 1}. \tag{16}$$

(iv) *Let v (resp. e) be a tree like vertex (resp. edge) in H. Then v (resp $\gamma(v)$, $\gamma(e)$) satisfies the conditions of Lemma 5 in $G(H)$, (resp. $\Gamma(v)$, $\Gamma(e)$).*

4 Proof of Theorem 2: computation of the cover time $C(H)$

4.1 Upper bound on cover time $C(H)$

We are assuming from now on that the hypergraph H satisfies the conditions stated in Lemmas 6 and 7, and that the mixing time T satisfying (7) is $O(\log^2 n)$ (see Lemma 10).

Let $t_0 = (1+o(1))\frac{(r-1)(s-1)}{(r-1)(s-1)-1} n \log n$ where the $o(1)$ term is large enough so that all inequalities below are satisfied. Let $T_G(u)$ be the time taken to visit every vertex of G by the random walk $\mathcal{W}_u$. Let U_t be the number of vertices of $G = G(H)$ which are not visited by $\mathcal{W}_u$ in the interval $[T, t]$. We note the following:

$$\begin{aligned} C_u = C_u(H) = \mathbf{E} T_G(u) &= \sum_{t>0} \mathbf{Pr}(T_G(u) > t), & (17) \\ \mathbf{Pr}(T_G(u) > t) &= \mathbf{Pr}(U_t > 0) \le \min\{1, \mathbf{E} U_t\}. & (18) \end{aligned}$$

It follows from (17) and (18) that for all $t \ge T$

$$C_u \le t + \sum_{\sigma \ge t} \mathbf{E} U_\sigma = t + \sum_{v \in V} \sum_{\sigma \ge t} \mathbf{Pr}(\mathcal{A}_v(\sigma)). \tag{19}$$

Let V_1 be the set of tree-like vertices and let $V_2 = V - V_1$. We apply Lemma 5. For $v \in V_1$, from (14) we have $np_v \sim \frac{(r-1)(s-1)-1}{(r-1)(s-1)}$. Hence, with $\lambda = 1/(KT)$,

$$\begin{aligned} \sum_{\sigma \ge t_0} \mathbf{Pr}(\mathcal{A}_v(\sigma)) &\le (1+o(1)) e^{-t_0 p_v} \sum_{\sigma \ge t_0} e^{-(\sigma - t_0) p_v} + O(e^{-\lambda t_0 / 2}) \\ &\le 2 p_v^{-1} e^{-t_0 p_v} \le 5. \end{aligned}$$

Furthermore, we see also that,

$$\mathbf{Pr}(\mathcal{A}_v(3n)) \le (1+o(1)) e^{-3np_v} \le e^{-1}. \tag{20}$$

Suppose next that $v \in V_2$. It follows from Lemmas 6 and 7 that we can find $w \in V_1$ such that $dist(v, w) \le \omega$. So from (20), with $\nu = 3n + \omega$, we have

$$\mathbf{Pr}(\mathcal{A}_v(\nu)) \le 1 - (1 - e^{-1})(rs)^{-\omega},$$

since if our walk visits w, it will with probability at least $(rs)^{-\omega}$ visit v within the next ω steps. Thus if $\zeta = (1 - e^{-1})(rs)^{-\omega}$,

$$\begin{aligned} \sum_{\sigma \ge t_0} \mathbf{Pr}(\mathcal{A}_v(\sigma)) &\le \sum_{\sigma \ge t_0} (1-\zeta)^{\lfloor \sigma/\nu \rfloor} \le \sum_{\sigma \ge t_0} (1-\zeta)^{\sigma/(2\nu)} \\ &= \frac{(1-\zeta)^{t_0/(2\nu)}}{1 - (1-\zeta)^{1/(2\nu)}} \le 3\nu\zeta^{-1}. & (21) \end{aligned}$$

Thus, for all $u \in V$,

$$\begin{aligned} C_u &\le t_0 + 5|V_1| + 3|V_2|\nu\zeta^{-1} & (22) \\ &= t_0 + O((rs)^{4\omega} n) = t_0 + o(t_0), \end{aligned}$$

assuming for the last bound that $(rs)^{4\omega} = o(\log n)$.

4.2 Lower bound on cover time $C(H)$

For any vertex u, we can find a set of vertices S, such that at time $t_1 = t_0(1-o(1))$, the probability the set S is covered by the walk $\mathcal{W}_u$ tends to zero. Hence $T_G(u) > t_1$ **whp** which implies that $C_G \geq t_0 - o(t_0)$. We construct S as follows. Let $S \subseteq V_1$ be some maximal set of locally tree-like vertices all of which are at least distance $2\omega + 1$ apart. Thus $|S| \geq (n - (rs)^{3\omega})(rs)^{-(2\omega+1)}$.

Let $S(t)$ denote the subset of S which has not been visited by $\mathcal{W}_u$ in the interval $[T, t]$. Now,

$$\mathbf{E}|S(t)| \geq (1 - o(1)) \sum_{v \in S} \left(\frac{1 + o(1)}{(1 + p_v)^t} + o(n^{-2}) \right).$$

Setting $t_1 = (1 - \epsilon)t_0$ where $\epsilon = 2\omega^{-1}$, we have

$$\mathbf{E}|S(t_1)| \quad = \quad (1 + o(1))|S|e^{-(1-\epsilon)t_0 p_v} \quad \geq \quad (1 + o(1)) \frac{n^{2/\omega}}{(rs)^{2\omega+1}} \quad \geq \quad n^{1/\omega}. \tag{23}$$

Let $Y_{v,t}$ be the indicator for the event that $\mathcal{W}_u$ has not visited vertex v at time t. Thus $\sum_{v \in S} Y_{v,t} = |S(t)|$. Let $Z = \{v, w\} \subset S$. We will show (below) that

$$\mathbf{E}(Y_{v,t_1} Y_{w,t_1}) = \frac{1 + o(1)}{(1 + p_Z)^{t_1+2}} + o(n^{-2}), \tag{24}$$

where $p_Z \sim p_v + p_w$. Thus

$$\mathbf{E}(Y_{v,t_1} Y_{w,t_1}) = (1 + o(1))\mathbf{E}(Y_{v,t_1})\mathbf{E}(Y_{w,t_1}). \tag{25}$$

We have

$$\mathbf{Pr}(|S(t_1)| > T) \geq \frac{(\mathbf{E}|S(t_1)| - T)^2}{\mathbf{E}((|S(t_1)| - T)^2)} = \left(\frac{\mathbf{E}((|S_{t_1}| - T)(|S_{t_1}| - T - 1))}{(\mathbf{E}|S(t_1)| - T)^2} + (\mathbf{E}|S_{t_1}| - T)^{-1} \right)^{-1}$$

and it follows from (23) and (25) that the right-hand side above is equal to $1 - o(1)$. Since at most T/ω of $S(t_1)$ can be visited in the first T steps, the probability that not all vertices are covered at time t_1 is equal to $1 - o(1)$, so $C(H) \geq t_1$.

Proof of (24). Let G^* be obtained from G by merging v, w into a single node Z. This node has degree $2r(s - 1)$ and every other node has degree $r(s - 1)$.

There is a natural measure preserving mapping from the set of walks in G which start at u and do not visit v or w, to the corresponding set of walks in G^* which do not visit Z. Thus the probability that $\mathcal{W}_u$ does not visit v or w during $[T, t]$ is equal to the probability that a random walk $\widehat{\mathcal{W}}_u$ in G^* which also starts at u does not visit Z in the first t steps.

We apply Lemma 5 to G^*. That $\pi_Z = \frac{2}{n}$ is clear. The derivation of $R_T(1)$ in Appendix A3.2 is also valid. The vertex Z is tree-like up to distance ω in G^*. The fact that the root vertex of the corresponding infinite structure is in $2r$ cliques does not affect the calculation of $R_T(1)$.

5 Proof of Theorem 2: computation of $I(H)$ and $C_E(H)$

This is very similar to the previous sections 4.1, 4.2 and so we will be light on details.

We briefly outline the upper bound proof for $I(H)$. Let $I_u(H)$ be the expected time for $\mathcal{W}_u$ to inform all vertices. Then for $t \geq T$, similarly to (19),

$$I_u(H) \leq t + \sum_{v \in V} \sum_{\sigma \geq t} \mathbf{Pr}(\mathcal{B}_v(\sigma))$$

where $\mathcal{B}_v(\sigma)$ is the event that vertex v is not informed in the interval $[T, \sigma]$.

Let $t_0 = (1 + o(1))\left(1 + \frac{s-1}{(r-1)(s-1)-1}\right)\frac{n}{s-1}\log n$. For tree-like vertices we use p_γ from (15), and apply Lemma 5. For non-tree-like vertices we use the argument for (21) and obtain, as in (22),

$$I_u(H) \le t_0 + 5|V_1| + 3|V_2|\nu\zeta^{-1} = t_0 + o(t_0).$$

We briefly outline the upper bound proof for $C_E(H)$. Let $C_{E,u}(H)$ be the expected time for $\mathcal{W}_u$ to cover all edges. Then for $t \ge T$,

$$C_{E,u}(H) \le t + \sum_{e\in E}\sum_{\sigma\ge t}\mathbf{Pr}(\mathcal{B}_e(\sigma))$$

where $\mathcal{B}_e(\sigma)$ is the event that edge e is not covered in the interval $[T, \sigma]$.

Let $t_0 = (1 + o(1))\left(1 + \frac{s-1}{(r-1)(s-1)-1}\right)\frac{2rn+s}{2s}\log n$. For tree-like edges we use p_γ from (16), and apply Lemma 5. For non-tree-like edges we use the argument for (21) and obtain as we did for (22), where $E = E_1 \cup E_2$ is a partition of E into tree-like edges and the rest,

$$C_{E,u}(H) \le t_0 + 5|E_1| + 3|E_2|\nu\zeta^{-1} = t_0 + o(t_0).$$

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Appendix

A1 Proof of Lemma 8

Lemma 12. *Let $x = v, e$. Let $\Gamma = \Gamma(v), \Gamma(e)$. Let $\mathcal{Y}_u$ be a random walk in Γ starting at $u \neq \gamma$. Let T be a mixing time satisfying (7) in both $G_x(H)$ and Γ. Then*

$$\mathbf{Pr}(\mathcal{A}_\gamma(t); \Gamma) = \mathbf{Pr}(\mathcal{B}_x(t); G(H)) \left(1 + O\left(\frac{s}{n}\right)\right),$$

where the probabilities are those derived from the walk in the given graph.

Proof

We give the proof for $\Gamma(v)$. The proof for $\Gamma(e)$ is similar. Let $Y_y(j)$ (resp. $X_y(j)$) be the position of walk $\mathcal{Y}_y$ (resp. $\mathcal{X}_y(j)$) at step j. Let $\Lambda = G(H), \Gamma(v)$ and let $P_u^s(z; \Lambda)$ be the transition probability in Λ, for the walk to go from u to z in s steps.

$$\begin{aligned}
\mathbf{Pr}(\mathcal{A}_\gamma(t); \Gamma(v)) &= \sum_{y \neq \gamma} P_u^T(y; \Gamma(v))\, \mathbf{Pr}(Y_y(\sigma - T) \neq \gamma,\ T \leq \sigma \leq t; \Gamma(v)) \\
&= \sum_{y \neq \gamma} \left(\frac{d(y)}{2m}(1 + O(n^{-3}))\right) \mathbf{Pr}(Y_y(\sigma - T) \neq \gamma,\ T \leq \sigma \leq t; \Gamma(v)) \qquad (26) \\
&= \sum_{z \notin D_v} \left(P_u^T(z; G_v(H))(1 + O(s/n))\right) \mathbf{Pr}(X_x(\sigma - T) \notin D_v,\ T \leq \sigma \leq t; G_v(H)) \qquad (27) \\
&= \mathbf{Pr}(\wedge_{x \in D_v} \mathcal{A}_x(t); G_v(H))(1 + O(s/n)).
\end{aligned}$$

Equation (26) follows from (7). Equation (27) follows because there is a natural measure preserving map ϕ between walks in $G_v(H)$ that start at $y \notin D_v$ and avoid D_v and walks in $\Gamma(v)$ that start at $y \neq \gamma$ and avoid γ. □

A2 Mixing time of the random walk

We estimate the conductance of graphs $G(H)$, $\Gamma(v)$ and $\Gamma(e)$. We need the following lemma.

Lemma 13. *Suppose that $r = 2, s \geq 3$ or that $r \geq 3$. Let $C(r, s)$ be sampled u.a.r. from $\mathcal{C}(r, s)$. Then there exists $\epsilon > 0$ such that* **whp** *there is no set of $t \leq n/2$ vertices that contain $rt(1-\epsilon)/s$ edges or more.*

Proof Let $N(t, k, r, s)$ be the expected number of sets of configuration vertices of size t which induce at least k hyperedges, where $ks = rt(1 - \epsilon)$. We will prove the lemma by showing that in the configuration model

$$\sum_{t=1}^{n/2} N(t, k, r, s) = o(n^{-1/10}).$$

Now

$$N(t, k, r, s) \leq \binom{n}{t} \binom{rt}{ks} \frac{F(ks) F(rn - ks)}{F(rn)},$$

where $F(a) = a!/((a/s)!(s!)^{(a/s)})$ for $s \mid a$. Note that if $s \mid a, b$ and $a > b$ then

$$\frac{F(b)F(a-b)}{F(a)} = \frac{\binom{a/s}{b/s}}{\binom{a}{b}} = O(\sqrt{s}) \left(\frac{b}{a}\right)^{b(s-1)/s} \left(1 - \frac{b}{a}\right)^{(a-b)(s-1)/s}, \tag{28}$$

and that

$$\begin{aligned}
\binom{n}{t} &= O(1)\sqrt{\frac{n}{t(n-t)}} \left(\frac{n}{t}\right)^t \left(1 - \frac{t}{n}\right)^{t-n} \\
\binom{rt}{ks} &= O(1)\sqrt{\frac{rt}{ks(rt-ks)}} \left(\frac{rt}{ks}\right)^{ks} \left(1 - \frac{ks}{rt}\right)^{ks-rt}.
\end{aligned}$$

Thus, assuming $t \leq n/2$ and that $ks = rt(1-\epsilon)$ where $\epsilon > 0$ constant,

$$\begin{aligned}
N(t,k,r,s) =& O\left(\frac{1}{\sqrt{ks}}\right) \left(\frac{rn}{ks}\right)^k \left(\frac{n}{t}\right)^{t-ks} \left(1 - \frac{ks}{rn}\right)^{(rn-ks)(s-1)/s} \left(1 - \frac{ks}{rt}\right)^{ks-rt} \left(1 - \frac{t}{n}\right)^{t-n} \\
=& O(1) \left(\frac{n}{(1-\epsilon)t}\right)^{rt(1-\epsilon)/s} \left(\frac{t}{n}\right)^{rt(1-\epsilon)-t} \left(\frac{1}{\epsilon}\right)^{\epsilon rt} \left(1 - \frac{t(1-\epsilon)}{n}\right)^{r(n-t(1-\epsilon))\frac{s-1}{s}} \left(1 - \frac{t}{n}\right)^{t-n} \\
=& O(1) \left(\left(\frac{1}{1-\epsilon}\right)^{(1-\epsilon)/s} \left(\frac{1}{\epsilon}\right)^{\epsilon} \left(1 - \frac{t(1-\epsilon)}{n}\right)^{\epsilon(1-1/s)} \left(\frac{t}{n}\right)^{(1-\epsilon)(1-1/s)-1/r} \right)^{tr}
\end{aligned} \tag{29}$$

$$\times \left(\frac{(1-t(1-\epsilon)/n)^{r(1-1/s)}}{1-t/n}\right)^{n-t}. \tag{30}$$

To establish an upper bound, we first consider the term (30). We write

$$\begin{aligned}
\frac{(1-t(1-\epsilon)/n)^{r(1-1/s)}}{1-t/n} &= \left(1 + \frac{\epsilon t}{n-t}\right)^{r(1-1/s)} \left(1 - \frac{t}{n}\right)^{r(1-1/s)-1} \\
&\leq \exp\left\{\frac{t}{n}\left(1 - r(1-1/s)(1-2\epsilon)\right)\right\}.
\end{aligned}$$

Now $r(1-1/s) \geq 4/3$ and so if $\epsilon < 1/8$ we find that the contribution of (30) is less than one. Considering (29), for $t \leq n/2$ it holds that $t/n \leq 1 - t(1-\epsilon)/n$, and thus (29) is $O(\Psi_t^{tr})$ where Ψ_t is given by

$$\Psi_t = \left(\frac{1}{1-\epsilon}\right)^{(1-\epsilon)/s} \left(\frac{1}{\epsilon}\right)^{\epsilon} \left(\frac{t}{n}\right)^{(1-2\epsilon)(1-1/s)-1/r}.$$

Provided $(1-2\epsilon)(1-1/s) - 1/r > 0$, Ψ is monotone increasing in t, and putting $t = n/2, r = 2, s = 3$,

$$\Psi_t \leq \left(\frac{1}{1-\epsilon}\right)^{(1-\epsilon)/3} \left(\frac{1}{\epsilon}\right)^{\epsilon} \left(\frac{1}{2}\right)^{(1-2\epsilon)(2/3)-1/2},$$

and choosing $\epsilon = 1/100$ we find $\Psi_t < 0.95$. Thus

$$\sum_{t=1}^{n/2} N(t,k,r,s) \leq \sum_{t=1}^{n/2} \Psi_t^{tr} = o(n^{-1/10}).$$

□

Going back to (5) we see that if G is a d-regular graph then $\Phi_G(S) = \frac{e(S:\overline{S})}{d|S|}$, were $e(S:\overline{S})$ denotes the number of edges with one endpoint in S and the other in $\overline{S} = V \setminus S$. Note that in this case $\pi(S) \leq 1/2$ if $|S| \leq n/2$.

The corollary below follows from Lemma 13, and the definition of the underlying clique graph $G(H)$ of a hypergraph H.

Corollary 14. *The conductance Φ_G of the graph $G(H)$ is $\Omega(1/s)$* **whp**.

Proof For $S \subseteq V$, $|S| = t \leq n/2$,

$$\Phi_G(S) = \frac{e(S : \overline{S})}{d|S|} \geq \frac{(s-1)\epsilon rt}{\binom{s}{2}rt} = \Omega(1/s). \tag{31}$$

□

Corollary 15. *The conductance Φ_Γ of the graph $\Gamma = \Gamma(v), \Gamma(e)$ is $\Omega(1/s)$* **whp**.

Proof Note first that contracting vertices cannot reduce conductance. This is because we minimise the same $\Phi(S)$ value over a smaller collection of sets S. It is a simple matter to see that subdividing at most $r\binom{s}{2}$ edges within S increases the degree of S by at most $rs(s-1)$ and thus $\Phi_\Gamma = \Omega(1/s)$. □

A3 Returns to a tree-like vertex

A3.1 Tree-like vertices

We argue next that almost all vertices of H are tree-like.

Lemma 16. **Whp** *there are at most $(rs)^{3\omega}$ vertices that are not tree-like.*

Proof We work with the model $H(C)$. The expected number of vertices on cycles of length k can be bounded above by

$$\sum_{k=3}^{\omega} skn^k \left(\frac{rs}{n}\right)^k \leq O\left(s\omega(rs)^\omega\right).$$

The Markov inequality implies that **whp** there are at most $(rs)^{2\omega}$ vertices on short cycles. For each such vertex there are at most $(rs)^\omega$ vertices reachable by a walk of length ω. □

Lemma 17. **Whp** *there are no short paths joining distinct short cycles.*

Proof If such a structure exists then there exists a walk $v_1, v_2, \ldots, v_k$ of length at most 3ω and a pair $i, j \in [k]$ and edges $f_1, f_2 \in E$ such that $v_1, v_i \in f_1$ and $v_k, v_j \in f_2$. The probability of this is at most

$$\sum_{k=5}^{3\omega} s^2k^2n^k \left(\frac{rs}{n}\right)^{k+1} = O\left(\frac{s^3\omega^3(rs)^{3\omega}}{n}\right).$$

□

A3.2 Returns in $G(H)$

For a vertex v and integer $k \geq 1$ let $N_k(v)$ denote the set of vertices w for which there is a path of length at most k from v to w. The following construction models an infinite extension of the neighbourhood of v in $G = G(H)$, for a tree-like vertex v. Let $\mathcal{T}_G^*$ be an infinite graph (with a tree-like structure) defined recursively as a root h joined to each vertex of $r-1$ disjoint cliques $C_1, C_2, \ldots, C_{r-1}$ of size $s-1$. Each vertex in $C_1 \cup \cdots \cup C_{r-1}$ is the root of a further disjoint copy of $\mathcal{T}_G^*$. For $\mathcal{T}_G$ we take a root vertex h and join it to each vertex of r disjoint cliques $C_1, C_2, \ldots, C_r$ of size $s-1$. Each vertex in $C_1 \cup \cdots \cup C_r$ is the root of a disjoint copy of $\mathcal{T}_G^*$. If v is tree-like, then provided $k \leq \omega$, the subgraph of $G(H)$ induced by $N_k(v)$ is isomorphic to the first k levels of $\mathcal{T}_G$.

We first compute the expected number of returns R_G to the root for a random walk on $\mathcal{T}_G$. We can then argue as in the proof of Lemma 7 of [6] that $R_v = R_G + o(1)$ for a tree-like vertex v of $G(H)$. We can project a walk on $\mathcal{T}_G$ onto the non-negative integers by mapping a vertex v of $\mathcal{T}_G$ to its distance Δ_v from the root. If $v \neq h$ then v has degree $(s-1)r$, and if the walk is at v, then it moves to a neighbour w where

$$\Delta_w = \begin{cases} \Delta_v + 1 & \text{probability } \frac{r-1}{r} \\ \Delta_v & \text{probability } \frac{s-2}{r(s-1)} \\ \Delta_v - 1 & \text{probability } \frac{1}{r(s-1)} \end{cases} \tag{32}$$

Now R_G is the expected number of returns to the origin of a random walk on the non-negative integers, with probabilities defined as in (32).

We note the following result (see e.g. [8]), for a random walk on the non-negative integers $\{0, 1, \ldots\}$ with transition probabilities at $k > 0$ of $q < p$ for moves left and right respectively. Starting at vertex 1, the probability of ultimate return to the origin 0 is

$$\rho = \frac{q}{p}. \tag{33}$$

It follows that if the walk always moves to 1 from the origin than the expected number of returns R to the origin is given by

$$R = \frac{1}{1-\rho} = \frac{p}{p-q}. \tag{34}$$

In which case we see that

$$R_v = R_G + o(1) = \frac{(r-1)(s-1)}{(r-1)(s-1)-1} + o(1).$$

Finally, for tree-like vertices v we have that the value of p_v in (11) is given by

$$p_v = (1 + o(1))\frac{1}{n}\frac{(r-1)(s-1)-1}{(r-1)(s-1)}. \tag{35}$$

A3.3 Returns in $\Gamma(v)$

The following construction models an infinite extension of the neighbourhood of γ in $\Gamma = \Gamma(v)$, for a tree-like vertex v. Let $\mathcal{T}_\Gamma$ be an infinite multi-graph consisting of a root h (corresponding to γ) joined to $r(s-1)$ distinct vertices $w_{i,j}, i = 1, 2, \ldots, r, j = 1, 2, \ldots s-1$ (corresponding to the vertices in cliques with v) and with $s-1$ parallel edges between h and each $w_{i,j}$. Each vertex $w = w_{i,j}$ is the root of $r-1$ copies of an infinite tree isomorphic to $\mathcal{T}_G^*$ defined in Section A3.2.

The probability P_γ of a return to h of a walk on $\mathcal{T}_\Gamma$ starting at h is given by

$$P_\gamma = \sum_{k=0}^{\infty} \rho^k (1-\hat{\rho})^k \hat{\rho} = \frac{\hat{\rho}}{1-\rho(1-\hat{\rho})} \tag{36}$$

where $\rho = \frac{1}{(r-1)(s-1)}$ (see (32) and (33)) is the probability of a return to the root w of a $\mathcal{T}_G^*$ and $\hat{\rho} = \frac{s-1}{s-1+(r-1)(s-1)} = \frac{1}{r}$ is the probability of moving from a $w_{i,j}$ to the root h in a single step. Plugging these values into (36) gives

$$P_\gamma = \frac{s-1}{r(s-1)-1}.$$

Therefore, using arguments similar to those in Lemma 7 of [6] we see that

$$R_\gamma = \frac{1}{1-P_\gamma} + o(1) = \frac{r(s-1)-1}{(r-1)(s-1)-1} + o(1). \tag{37}$$

For tree-like vertices v, using (12), the value of p_γ in (11) is given by

$$p_\gamma = (1+o(1))\frac{s-1}{n}\frac{(r-1)(s-1)-1}{r(s-1)-1}. \tag{38}$$

A3.4 Returns in $\Gamma(e)$

Let $\mathcal{T}_\Gamma'$ be an infinite multi-graph consisting of a root h (corresponding to γ) joined to s distinct vertices $w_i, i = 1, 2, \ldots s$ (corresponding to the vertices in clique of edge e) and with $s-1$ parallel edges between h and each $w_{i,}$. Each vertex $w = w_i$ is the root of $r-1$ copies of an infinite tree isomorphic to $\mathcal{T}_G^*$ defined in Section A3.2.

We find $R_\gamma' = (1+o(1))R_\gamma$, that $d(\gamma) = s(s-1)$ and that $d(\Gamma(e)) = rn(s-1) + s(s-1)$.

For tree-like vertices v, using (13), and assuming $s = o(n)$, the value of p_γ in (11) is given by

$$p_\gamma = (1+o(1))\frac{s}{rn}\frac{(r-1)(s-1)-1}{r(s-1)-1}. \tag{39}$$

A4 Technical condition (9) of Lemma 5

We will only verify this for tree-like vertices. Observe first that if $R = R_v, R_\gamma$ satisfies $R \le 2-\epsilon$ for some constant $\epsilon > 0$ then it is easy to verify this condition. Indeed, for $|z| \le 1+\lambda$,

$$|R_T(z)| \ge r_0 - (1+\lambda)^T \sum_{t=1}^{T} r_t = 1 - (1+\lambda)^T (R-1) \ge 1 - (1+\lambda)^T (1-\epsilon) > \epsilon/2.$$

A4.1 Case of $G(H)$

We write

$$R_v = 1 + \frac{1}{(r-1)(s-1)-1} + o(1)$$

and see that we only need to consider $r = 2, s = 3$.

For any z,

$$|R_T(z) - R_T(1)| \le \sum_{j=1}^{T} r_j |z^j - 1|. \tag{40}$$

Now $R_v \sim 2$ in our case, so we only need to show that the RHS of (40) is strictly less than 2.

Next observe that $\pi_v = 1/n$ for $v \in V$ and that (6) implies that

$$S_0 = \sum_{i=\omega}^{T} r_j |z^j - 1| \leq 2 \sum_{i=\omega}^{T} r_i \leq 2 \sum_{i=\omega}^{T} (\lambda_2^i + \pi_v) = o(1) \tag{41}$$

since for the second eigenvalue λ_2 of the matrix P of transition probabilities we have $\lambda_2 \leq \zeta < 1$ for some constant ζ.

Now consider $j < \omega$. Fix $0 \leq \theta < 2\pi$ and let $z = e^{i\theta}$, then

$$|z^j - 1| = (2(1 - \cos j\theta))^{1/2} = 2|\sin j\theta/2|.$$

Then

$$S_1 = \sum_{j=1}^{\omega-1} r_j |z^j - 1| = \sum_{j=1}^{\omega-1} r_j \left(2(1 - \cos j\theta)\right)^{1/2} = 2 \sum_{j=1}^{\omega-1} r_j |\sin j\theta/2|.$$

Note that $r_1 = 0, r_2 = \frac{1}{4}$ and $r_3 = \frac{1}{16}$. Suppose first that $\theta \notin I = [\frac{3\pi}{8}, \frac{5\pi}{8}] \cup [\frac{11\pi}{8}, \frac{13\pi}{8}]$. Then $|\sin\theta| \leq \sin\frac{3\pi}{8}$ and so

$$S_1 \leq 2 \sum_{j=1}^{\omega-1} r_j - r_2 \left(1 - \sin\frac{3\pi}{8}\right). \tag{42}$$

On the other hand, if $\theta \in I$ then $|\sin 3\theta/2| \leq \sin\frac{7\pi}{16}$ and then

$$S_1 \leq 2 \sum_{j=1}^{\omega-1} r_j - r_3 \left(1 - \sin\frac{7\pi}{16}\right). \tag{43}$$

From (6),

$$R_T(1) = 1 + \sum_{j=1}^{\omega-1} r_j + O(\lambda_2^{\omega}).$$

Thus, as $R_T(1) = 2 + o(1)$, (41), (42), (43) imply that $S_0 + S_1 \leq 2(R_T(1) - 1) - 1/20 = 39/20 - o(1)$.

This confirms that the RHS of (40) is less than 2 and that technical condition (9) holds.

A4.2 Case of $\Gamma(v)$

From (37) we write

$$R_\gamma = 1 + \frac{s - 1}{(r - 1)(s - 1) - 1} + o(1).$$

Once again we see that we only need to consider $r = 2, s = 3$.

We then use the same argument as in Section A4.1. Equation (42) holds and this time $r_2 = 1/2, r_3 = 0, r_4 = 1/3$. If $\theta \in I$ then $|\sin 2\theta| \leq \sin\frac{\pi}{4}$ and then

$$S_1 \leq 2 \sum_{j=1}^{\sigma-1} r_j - r_4 \left(1 - \sin\frac{\pi}{4}\right). \tag{44}$$

We use (44) in place of (43) to prove that technical condition (9) holds.

A4.3 Case of $\Gamma(e)$

It was noted in Section A3.4 that the value of R'_γ in $\Gamma(e)$ satisfies $R'_\gamma = (1 + o(1))R_\gamma$ in $\Gamma(v)$, as given in (37). Thus the results of the above section apply.