Hamilton Cycles in Random Lifts of Graphs

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Abstract

An *n-lift* of a graph K, is a graph with vertex set $V(K) \times [n]$ and for each edge $(i,j) \in E(K)$ there is a perfect matching between $\{i\} \times [n]$ and $\{j\} \times [n]$. If these matchings are chosen independently and uniformly at random then we say that we have a random n-lift. We show that there are constants h_1, h_2 such that if $h \geq h_1$ then a random n-lift of the complete graph K_h is hamiltonian **whp** and if $h \geq h_2$ then a random n-lift of the complete bipartite graph $K_{h,h}$ is hamiltonian **whp**.

1 Introduction

For a graph K, an *n-lift* G of K has vertex set $V(K) \times [n]$ where for each vertex $v \in V(K)$, $\{v\} \times [n]$ is called the *pillar* above v and will be denoted by Π_v. The edge set of a an n-lift G consists of a perfect matching between pillars Π_u and Π_w for each edge $(u,w) \in E(K)$. The set of n-lifts will be denoted $\mathcal{L}_n(K)$. In this paper we discuss random n-lifts, chosen uniformly from $\mathcal{L}_n(K)$. In this case, the matchings between pillars are chosen independently and uniformly at random.

Lifts of graphs were introduced by Amit and Linial in [1] where they proved that if K is a connected, simple graph with minimum degree $\delta \geq 3$, and G is chosen randomly from $\mathcal{L}_n(K)$ then G is $\delta-$connected **whp**, where the asymptotics are for $n \to \infty$. They continued the study of random lifts in [2] where they proved expansion properties of lifts. Together with Matoušek, they gave bounds on the independence number and chromatic number of random lifts in [3]. Linial and Rozenman [4] give a tight analysis for when a random n-lift has a perfect matching.

In this paper we discuss the probability that a random n-lift is hamiltonian. In particular we study the case where K is the complete graph K_h or the complete bipartite graph $K_{h,h}$. We use the notation $y \overset{r}{\in} Y$ for "y is chosen uniformly at random from Y".

Theorem 1. *There exists a constant h_1 such that if $h \geq h_1$ and $G \overset{r}{\in} \mathcal{L}_n(K_h)$ then G is hamiltonian* **whp**.

Theorem 2. *There exists a constant h_2 such that if $h \geq h_2$ and $G \overset{r}{\in} \mathcal{L}_n(K_{h,h})$ then G is hamiltonian* **whp**.

Theorem 1 is proved in the next section. Theorem 2 is proved in Section 3.

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2 Proof of Theorem 1

2.1 Structural Properties of $\mathcal{L}_n(K_h)$

The vertices of $\mathcal{L}_n(K_h)$ will be denoted by V and its edges wil be denoted E.

We will use the coloring argument of Fenner and Frieze [7] to show G is hamiltonian **whp**. For $G \in \mathcal{L}_n(K_h)$ we choose a set $H_1 = H_1(G) \subseteq E(G)$ as follows: Each vertex of G arbitrarily chooses 12 edges of G incident with it. Thus the number of distinct edges chosen is between $6hn$ and $12hn$ and the minimum degree of the graph induced by H_1 is at least 12. Next let $P_0 = P_0(G)$ be a specific longest path in G. Let $F(G) = P_0 \cup H_1$ be the fixed edges of G.

The analysis uses an *unspecified*, sufficiently small, positive constant $\beta < 1$.

Let $\mathcal{B} = \mathcal{B}(G)$ be the set of subsets of $E(G)$ of size $\beta\binom{h}{2}n$. We say that a subset of edges H is *acceptable* if $H = B \cup F$ for some $B \in \mathcal{B}(G)$. Let $\mathcal{H}(G)$ be the collection of acceptable subgraphs of G. For a lift G, each $B \in \mathcal{B}(G)$ defines a coloring of the edges of G in which the edges of $H = B \cup F$ are colored blue and the edges of $R = G\backslash H$ are colored red.

Let $S \subseteq V$ be of size s and let S_i be the intersection of $S \subseteq V$ with pillar Π_i for $i \in [h]$. The number of choices for S is $\binom{hn}{s}$ and by considering the number of choices for the S_i we see that

$$\sum_{s_1+\cdots+s_h=s} \prod_i \binom{n}{s_i} = \binom{hn}{s} \leq \left(\frac{hne}{s}\right)^s. \tag{1}$$

For a graph $G = (V, E)$ and $S \subseteq V$ let $N(S) = \{v \in V\backslash S : \exists u \in S \text{ such that } (u, v) \in E(G)\}$ be the disjoint neighborhood of S.

For $G \overset{r}{\in} \mathcal{L}_n(K_h)$ and sets $S \subseteq \Pi_i$ and $T \subseteq \Pi_j$, $|S| = s, |T| = t$,

$$\mathbf{Pr}(N(S) \cap \Pi_j \subseteq T) = \frac{t(t-1)\dots(t-s+1)}{n(n-1)\dots(n-s+1)} \leq \left(\frac{t}{n}\right)^s. \tag{2}$$

Throughout this section all statements hold for n and h sufficiently large.

Lemma 1. *For $G \overset{r}{\in} \mathcal{L}_n(K_h)$,*

$$\mathbf{Pr}\left(\exists S \subseteq V : |S| \leq \frac{n}{10h} \text{ and } S \text{ contains at least } 2|S| \text{ edges}\right) = o(1)$$

Proof Using (1) we see that the expected number of sets S of size s that contain at least $2s$ edges is no more than

$$\begin{aligned}
\phi(s) &= \sum_{s_1+\cdots+s_h=s} \prod_i \binom{n}{s_i} \binom{\binom{s}{2}}{2s} \left(\frac{1}{n-2s}\right)^{2s} \\
&\leq \binom{hn}{s} \left(\frac{s^2 e}{4s}\right)^{2s} \left(\frac{1}{n(1-\frac{1}{5h})}\right)^{2s} \\
&\leq \left(\frac{hne}{s}\right)^s \left(\frac{se}{4}\right)^{2s} \left(\frac{2}{n}\right)^{2s} \\
&\leq \left(\frac{e^3 hs}{4n}\right)^s
\end{aligned}$$

Then

$$\sum_{s=5}^{n/10h} \phi(s) = o(1).$$

□

Lemma 2. *If* $G \overset{r}{\in} \mathcal{L}_n(K_h)$ *and* $H \overset{r}{\in} \mathcal{H}(G)$, *then* **whp** H *satisfies*

$$S \subseteq V, |S| \leq hn/4 \text{ implies } |N_H(S)| \geq 2|S|. \tag{3}$$

Proof Assume first that $|S| \leq n/10h$ and let $U = S \cup N(S)$. Let a be the number of edges contained in S and let b be the number of edges from S to $N(S)$. The degree sum of S in H_1 is at least $12|S|$ and so $2a + b \geq 12|S|$. But then U contains at least $a + b \geq 6|S|$ edges and we can assume by Lemma 1 that $|U| > 3|S|$. This completes the argument for $|S| \leq n/10h$.

Let H' be defined by including an edge of G in H' independently with probability β' where $\beta' < \beta$. Then $|H'|$ is a binomial random variable whose expected value is less than $\beta\binom{h}{2}n$. The Chernoff bound implies that for a monotone increasing property of lifts $\mathcal{Q}$, if $H' \in \mathcal{Q}$ **whp**, then $H \in \mathcal{Q}$ **whp**.

For $n/10h < |S| \leq hn/4$, let $T = N(S)$ and $t = |T|$. Using (1) and (2), the expected number Z of sets S with $|N_{H'}(S)| < 2|S|$ is bounded as follows: In the first line of the following display, the notation $j \succ i$ denotes $s_j + t_j > s_i + t_i$ or $s_j + t_j = s_i + t_i$ and $j > i$.

$$\begin{aligned}
Z &\leq \sum_{s=n/10h}^{hn/4} \sum_{t=0}^{2s-1} \sum_{s_1+\cdots+s_h=s} \sum_{t_1+\cdots+t_h=t} \prod_i \binom{n}{s_i} \prod_j \binom{n}{t_j} \prod_{i=1}^{h} \prod_{j \succ i} \left(\beta' \frac{s_j+t_j}{n} + (1-\beta')\right)^{s_i+t_i} \\
&\leq \sum_{s=n/10h}^{hn/4} \sum_{t=0}^{2s-1} \sum_{s_1+\cdots+s_h=s} \prod_i \binom{n}{s_i} \sum_{t_1+\cdots+t_h=t} \prod_j \binom{n}{t_j} \prod_{i=1}^{h} \prod_{j \neq i} \left(\beta' \frac{s_j+t_j}{n} + (1-\beta')\right)^{(s_i+t_i)/2} \\
&= \sum_{s=n/10h}^{hn/4} \sum_{t=0}^{2s-1} \sum_{s_1+\cdots+s_h=s} \prod_i \binom{n}{s_i} \sum_{t_1+\cdots+t_h=t} \prod_j \binom{n}{t_j} \prod_{j=1}^{h} \left(\beta' \frac{s_j+t_j}{n} + (1-\beta')\right)^{(s+t-(s_j+t_j))/2} \\
&\leq \sum_{s=n/10hn}^{hn/4} \sum_{t=0}^{2s-1} \sum_{s_1+\cdots+s_h=s} \prod_i \binom{n}{s_i} \sum_{t_1+\cdots+t_h=t} \prod_j \binom{n}{t_j} \left(\sum_{j=1}^{h} \left(\beta' \frac{s_j+t_j}{(h-1)n} + (1-\beta')\right)\right)^{(h-1)(s+t)/2} \\
&= \sum_{s=n/10h}^{hn/4} \sum_{t=0}^{2s-1} \sum_{s_1+\cdots+s_h=s} \prod_i \binom{n}{s_i} \sum_{t_1+\cdots+t_h=t} \prod_j \binom{n}{t_j} \left(\beta' \frac{s+t}{(h-1)n} + (1-\beta')\right)^{(h-1)(s+t)/2} \\
&\leq \sum_{s=n/10h}^{hn/4} \sum_{t=0}^{2s-1} \left(\frac{neh}{s}\right)^s \left(\frac{neh}{t}\right)^t \left(1 - \beta'\left(1 - \frac{s+t}{(h-1)n}\right)\right)^{(h-1)(s+t)/2} \\
&\leq \sum_{s=n/10h}^{hn/4} \sum_{t=0}^{2s-1} \left(\frac{neh}{s}\right)^s \left(\frac{neh}{t}\right)^t \exp\left\{-\beta'\left(1 - \frac{s+t}{(h-1)n}\right)(h-1)(s+t)/2\right\} \\
&\leq \sum_{s=n/10h}^{hn/4} \left(\frac{neh}{s}\right)^{3s} \exp\left\{-\frac{\beta' hs}{10}\right\} \\
&\leq e^{-\beta n/199}.
\end{aligned} \tag{4}$$

□

Lemma 3. *If $G \overset{r}{\in} \mathcal{L}_n(K_h)$ and $H \overset{r}{\in} \mathcal{H}(G)$, then* **whp** *$H$ is connected.*

Proof If H is not connected, Lemma 2 implies that **whp** H is the union of a constant number of components of size at least $hn/4$. We will again work under the assumption that edges are included in H' independently with probability β' where $\beta' < \beta$.

Assume without loss of generality that $|S| \leq hn/2$. The expected number of sets S of size $|S| \in [hn/4, hn/2]$ with no edges between S and its complement is no more than

$$\begin{aligned}
&\sum_{s=hn/4}^{hn/2} \left(\sum_{s_1+\ldots s_h=s} \prod_i \binom{n}{s_i} \right) \prod_{i=1}^{h} \prod_{j \succ i} \left(\beta' \left(\frac{s_j}{n} \right) + (1-\beta') \right)^{s_i} \\
\leq\ & \sum_{s=hn/4}^{hn/2} \left(\frac{neh}{s} \right)^s \left(\beta' \left(\frac{s}{(h-1)n} \right) + (1-\beta') \right)^{(h-1)s/2} \\
\leq\ & \sum_{s=hn/4}^{hn/2} \left(\frac{neh}{s} \right)^s \exp\left\{ -\frac{\beta' s}{2}(h/2-1) \right\} \\
\leq\ & e^{-\beta h^2 n/5} \qquad (5)
\end{aligned}$$

□

Let $P = (v_0, \ldots, v_k)$ be a longest path in graph H. A *Pósa rotation* of P [10] with v_0 fixed gives another longest path $P' = (v_0, \ldots v_i v_k \ldots v_{i+1})$ created by adding edge (v_k, v_i) and deleting edge (v_i, v_{i+1}). Let $END_H(v_0, P)$ be the set of endpoints obtained by a sequence of Pósa rotations starting with P, keeping v_0 fixed and using an edge (v_k, v_i) of H.

Each vertex $v_j \in END_H(v_0, P)$ can then be used as the initial vertex of another set of longest paths $END_H(v_j, P)$, this time using v_j as the fixed vertex, but again only adding edges from H. Let $END_H(P) = \{v_0\} \cup END_H(v_0, P)$.

The Pósa condition

$$|N(END(v,P))| \leq 2\,|END(v,P)| - 1$$

for $v \in END_H(P)$ together with Lemma 2 implies the following.

Lemma 4. *If $G \overset{r}{\in} \mathcal{L}_n(K_h)$ and $H \overset{r}{\in} \mathcal{H}(G)$, then* **whp** *$|END_H(v,P)| \geq hn/4$ for all $v \in END_H(P)$, $P = P(G)$.*

We say next that an *ordered* pair of pillars (Π_k, Π_l) is *good* w.r.t. a longest path P if

$$|\{u \in \Pi_k \cap END_H(P) : |\{v \in \Pi_l \cap END_H(u,P) : (u,v) \notin E(H)\}| \geq n/500\}| \geq n/500. \qquad (6)$$

In words, Π_k contains at least $n/500$ vertices $u \in END_H(P)$ for which there at least $n/500$ vertices $v \in \Pi_l \cap END_H(u,P)$ such that the edge $(u,v) \notin E(H)$.

Lemma 5. *If* (3) *holds then G has at least $\binom{h}{2}/3000$ good pillar pairs.*

Proof We show first that for $u \in END_H$ there are at least $h/7 - 1$ pillars for which

$$|\{v \in \Pi_l \cap END_H(u,P) : (u,v) \notin E(H)\}| \geq n/8 \qquad (7)$$

holds. Let $u \in END_H$ and suppose that there are m pillars for which (7) fails. The total number of vertices in $END(u,H)$ must be at least $hn/4$ by Lemma 4 which gives the inequality

$$mn/8 + (h-m)n \geq hn/4$$

so that $m \leq 6h/7$. We get $h/7 - 1$ "good" pillars, because we have to discount the pillar containing u.

Next, we say that a non-edge $(x,y) \notin E(H)$ must be *avoided* if $x \in END_H$ and $y \in END(u,H)$. We have just shown that for each $u \in END_H$, there are at least $hn/57$ edges incident with u that must be avoided. As $|END_H| \geq |END(u,H)|$ and each non-edge is counted at most twice, the total number of non-edges in G that must be avoided is at least $\frac{1}{2}hn/4 \cdot hn/57$.

Assume now that there are $\delta\binom{h}{2}$ pillar pairs that contain at least $n^2/250$ edges that must be avoided. We then get the inequality

$$\delta\binom{h}{2}n^2 + (1-\delta)\binom{h}{2}n^2/250 \geq h^2n^2/456$$

which gives $\delta > 1/3000$.

Let (Π_k, Π_l) be a pillar pair that contains at least $n^2/250$ edges that must be avoided. To show that (Π_k, Π_l) is good, let

$$|\{u \in END \cap \Pi_k : |\{v \in END(u,H) \cap \Pi_l,\ (u,v) \notin E(H)| \geq n/500\}| = \gamma n. \tag{8}$$

We then get the inequality

$$\gamma n^2 + (1-\gamma)n^2/500 \geq n^2/250$$

so $\gamma > 1/500$. □

2.2 The Proof

For a lift G, let $\mathcal{D}(G)$ be the subset of $\mathcal{H}(G)$ in which H is connected and satisfies (3) for $|S| > n/10h$ and let $\mathcal{D} = \cup_G \mathcal{D}(G)$. Let $\mathcal{A}$ be the subset of $\mathcal{L}_n(K_h)$ such that for $G \in \mathcal{A}$ and H chosen randomly from $\mathcal{H}(G)$,

$$\mathbf{Pr}(H \in \mathcal{D}(G)) \geq 1 - \alpha.$$

where $\alpha = e^{-\beta n/200}$.

Let $\mathcal{C}$ be the subset of $\mathcal{L}_n(K_h)$ that is not hamiltonian and let $\mathcal{F} = \mathcal{A} \cap \mathcal{C}$. To show that $\mathbf{Pr}(\mathcal{C}) \to 0$, we will first show that $|\mathcal{A}| = (1-o(1))\,|\mathcal{L}_n(K_h)|$ and then use the coloring argument of Fenner and Frieze [7] to show that $\mathbf{Pr}(\mathcal{F}) \to 0$.

Lemma 6. $|\mathcal{A}| = (1-o(1))\,|\mathcal{L}_n(K_h)|$

Proof If $G \overset{r}{\in} \mathcal{L}_n(K_h)$ and $H \overset{r}{\in} \mathcal{H}(G)$ then

$$\begin{aligned}
\mathbf{Pr}(H \in \mathcal{D}) &= \sum_{G \in \mathcal{L}_n(K_h)} \mathbf{Pr}(H \in \mathcal{D}|G)\mathbf{Pr}(G) \\
&= \sum_{G \in \mathcal{A}} \mathbf{Pr}(H \in \mathcal{D}|G)\mathbf{Pr}(G) + \sum_{G \notin \mathcal{A}} \mathbf{Pr}(H \in \mathcal{D}|G)\mathbf{Pr}(G) \\
&\leq \mathbf{Pr}(\mathcal{A}) + (1-\alpha)(1-\mathbf{Pr}(\mathcal{A})) \\
&= 1 - \alpha + \alpha\mathbf{Pr}(\mathcal{A})
\end{aligned} \tag{9}$$

and (4) and (5) imply that

$$\mathbf{Pr}(H \in \mathcal{D}) \geq 1 - \alpha^2. \tag{10}$$

Putting (9) and (10) together, we get

$$1 - \alpha + \alpha \mathbf{Pr}(\mathcal{A}) \geq 1 - \alpha^2.$$

so that

$$\mathbf{Pr}(\mathcal{A}) \geq 1 - \alpha.$$

□

To get an upper bound on the number of graphs $G \in \mathcal{L}_n(K_h)$ such that $G \in \mathcal{F}$, we construct a 0-1 matrix $A = ||a_{i,j}||$. Row index i corresponds to a graph $G_i \in \mathcal{L}_n(K_h)$ and index j ranges over all acceptable subgraphs $H \in \mathcal{H}(G_i)$. Subgraph j of G_i will be denoted by $H_{i,j}$. Let

$$a_{i,j} = 1 \text{ if } \begin{cases} (i) & S \subseteq V, |S| \leq hn/4 \text{ implies } \left|N_{H_{i,j}}(S)\right| \geq 2|S| \\ (ii) & H_{i,j} \text{ is connected} \\ (iii) & H_{i,j} \supseteq P_0(G_i) \\ (iv) & G_i \text{ is not Hamiltonian} \\ (v) & |E_{H_{i,j}}(\Pi_k, \Pi_l)| \in [(1 \pm n^{-1/3})\beta n], \ \forall\ k \neq l \in [h] \end{cases} \tag{11}$$

Note that (ii), (iii) and (iv) imply

$$\nexists \text{ longest path } P \text{ of } H_{i,j},\ (u,v) \in E(R_{i,j}) \ : u \in END_{H_{i,j}}(P),\ v \in END_{H_{i,j}}(u,P) \tag{12}$$

Now let

$$N_1 = \sum_i \sum_j a_{i,j}$$

be the number of ones in A.

Lemma 7. *If $G_i \in \mathcal{F}$ then*

$$\sum_j a_{i,j} \geq (1 - o(1)) \binom{\binom{h}{2}n - 13hn}{(1-\beta)\binom{h}{2}n - 13hn}.$$

Proof $G_i \in \mathcal{F}$ and $H_{i,j} \overset{r}{\in} \mathcal{H}(G_i)$ implies that $H_{i,j}$ satisfies $(i), (ii), (iii)$ and (iv) **whp**. Now $B_1, B_2 \in \mathcal{B}(G)$ may give rise to the same subgraph H if the edges not in $B_1 \cap B_2$ are all in F. So we count the number of ways to select R as a lower bound on $|\mathcal{H}(G_i)|$. We have $|H| \leq \beta\binom{h}{2}n + 13hn$ since there are at most $13hn$ edges in P_0 and H_1. Then the number of choices for R is at least the number of ways to select a set of $(1-\beta)\binom{h}{2}n - 13hn$ edges from the $\binom{h}{2}n - 13hn$ not in F. Condition (v) holds through the Chernoff bound. □

It follows immediately from Lemma 7 that

$$N_1 \geq (1 - o(1)) \binom{\binom{h}{2}n - 13hn}{(1-\beta)\binom{h}{2}n - 13hn} |\mathcal{F}|. \tag{13}$$

We now obtain an upper bound on N_1. Let

$$\mathcal{X} = \{H :\ \exists i, j \text{ for which } H_{i,j} = H \text{ and } a_{i,j} = 1\}$$

The following bound follows from the definition and a concentration inequality for sampling without replacement, see Hoeffding [9], Theorem 4:

$$|\mathcal{X}| \leq \binom{\binom{h}{2}n}{13hn} \left((1+o(1)) \binom{n}{\beta n}^2 (\beta n)! \right)^{\binom{h}{2}}. \tag{14}$$

For a fixed $H \in \mathcal{X}$ let

$$\mathcal{G}_H = \{G_i : H_{i,j} = H \text{ and } a_{i,j} = 1\}.$$

Thus,

$$N_1 = \sum_{H \in \mathcal{X}} |\mathcal{G}_H|.$$

Lemma 8.

$$H \in \mathcal{X} \textit{ implies } |\mathcal{G}_H| \leq e^{-ch^2 n} \left(((1-\beta+O(n^{-1/3}))n)! \right)^{\binom{h}{2}} \tag{15}$$

for some absolute constant $c > 0$.

Proof We begin with H and count the number of ways to add back the edges of R to form a lift $G_i \in \mathcal{G}_H$. The number of edges in $R(k,l)$ between two pillars of G_i is no more than $(1-\beta+O(n^{-1/3}))n$. Thus there are at most $((1-\beta+O(n^{-1/3}))n)!$ possible matchings to add back between each pair of pillars.

When adding back new edges to H we must avoid edges (u,v) where $u \in END_H$ and $v \in END(u,H)$ so that $a_{i,j}=1$ in the resulting graph. For a good pillar pair (Π_k, Π_l) as defined in (6), there are at least $n/500$ vertices $x \in \Pi_k$, each adjacent at least $n/500$ vertices $y \in \Pi_l$ that give rise to an edge (x,y) that must be avoided. The probability that we avoid all such edges between a good pillar pair is at most

$$\prod_{i=0}^{n/500-1} \left(1 - \frac{n/500-i}{n-i}\right) \leq e^{-n/250,000}$$

As there are at least $\binom{h}{2}/3000$ good pillar pairs, the probability that a set of new edges avoids all required edges in G_i is at most $(e^{-n/250,000})^{\binom{h}{2}/3000}$. □

It follows from (13), (14) and (15) that $\frac{|\mathcal{F}|}{|\mathcal{L}_n(K_h)|}$ is bounded above by

$$\begin{aligned}
& \frac{e^{-ch^2 n} \left((1-\beta+O(n^{-1/3})n)!\right)^{\binom{h}{2}} \binom{\binom{h}{2}n}{13hn} \left((1+o(1))\binom{n}{\beta n}^2 (\beta n)!\right)^{\binom{h}{2}}}{(1-o(1))(n!)^{\binom{h}{2}} \binom{\binom{h}{2}n-13hn}{(1-\beta)\binom{h}{2}n-13hn}} \\
\leq \quad & \frac{e^{-ch^2 n/2} \binom{n}{\beta n}^{\binom{h}{2}}}{\binom{\binom{h}{2}n}{\beta\binom{h}{2}n} \beta^{14hn}} \\
\leq \quad & e^{-ch^2 n/2 + 14hn \ln(1/\beta)} \\
= \quad & o(1)
\end{aligned}$$

where the second line uses $\binom{a-x}{b-x} \geq (\frac{b-x}{a-x})^x \binom{a}{b}$. □

3 Proof of Theorem 2

3.1 Structural Properties of $\mathcal{L}_n(K_{h,h})$

Let V_1, V_2 be the bipartition of $K_{h,h}$ and let W_1, W_2 be the bipartition of the lifts of $K_{h,h}$ that it induces.

We now prove similar properties to those in Section 2.1. Let H_1, P_0 be sets of edges defined as in Section 2.1 and let $F = P_0 \cup H_1$. Again we use an unspecified, suitably small constant $\beta < 1$, let B be a set of $\beta\binom{h}{2}n$ edges in G and $\mathcal{B}(G)$ the collection of subgraphs B. A set of edges H in G is acceptable if $H = B \cup F$ for some $B \in \mathcal{B}(G)$. Let $\mathcal{H}(G)$ be the collection of acceptable subgraphs of G and let $R = G \backslash H$.

Throughout this section all statements hold for n and h sufficiently large. The proof is similar to that for K_h and so we will omit calculations that are almost identical to those of the previous sections.

The main difficulty with using a Posá type argument is that if a longest path P in G is even then it cannot be closed to a cycle, connectivity notwithstanding i.e. we gain nothing from avoiding choosing green edges to join v to $END(v)$. In this case, there are no edges to avoid. We therefore have to modify the argument. We follow Bollobás and Kohayakawa [6] who considerably simplified the argument of [8].

Lemma 9. *For $G \overset{r}{\in} \mathcal{L}_n(K_{h,h})$*

$$\mathbf{Pr}(\exists S \subseteq V : |S| \leq \frac{n}{20h} \text{ and } S \text{ contains at least } 2|S| \text{ edges}) = o(1)$$

□

Lemma 10. *If $G \overset{r}{\in} \mathcal{L}_n(K_{h,h})$ and $H \overset{r}{\in} \mathcal{H}(G)$, then* **whp** *$H$ satisfies*

$$S \subseteq W_i, |S| \leq hn/4 \text{ implies } |N_H(S)| \geq 2|S|. \tag{16}$$

□

Lemma 11. *If $G \overset{r}{\in} \mathcal{L}_n(K_{h,h})$ and $H \overset{r}{\in} \mathcal{H}(G)$ then* **whp** *H is connected.*

□

Lemma 12. *If K has a 2-factor and $G \in \mathcal{L}_n(K)$, then G has a 2-factor.*

Proof Let $C \subseteq V(K)$ be one of the cycles of a 2-factor of K and let $G[C]$ the subgraph of G induced by the pillars above the vertices of C. Let $v_1, \ldots, v_k$ be an ordering of the vertices of C such that (v_i, v_{i+1}) is an edge of C (where $v_1 = v_{k+1}$) and let Π_i be the pillar of G above $v_i \in C$. Let σ_i be the permutation that defines the matching from pillar Π_i to Π_{i+1} for each $\Pi_i \in G[C]$. For each $j \in \Pi_1$, define $\sigma(j) = \sigma_k\sigma_2 \cdots \sigma_1(j)$ to be the permutation on the vertices of Π_1 that results from following the permutations σ_1 through σ_k back to Π_1. Then a cycle of σ is a cycle of G so that the cycles of σ define a 2-factor of $G[C]$. This process can be repeated for all cycles of a 2-factor of K to obtain a 2-factor of $G \in \mathcal{L}_n(K)$. □

We now describe an extension-rotation process which attempts to transform the 2-factor F of Lemma 12 into a Hamilton cycle.

General Step: Given the current 2-factor (initially F) choose an edge $e = (x, y)$ of G which joins two distinct cycles C, C'. This is possible because G is connected **whp**. Let f be an edge of

C incident with x and f' be an edge of C' incident with y. Let P be the path $C \cup C' \cup \{e\} \setminus \{f, f'\}$. There are now several possibilities.

(a): There is an endpoint u say, of P which has a neighbour v in a cycle C'' disjoint from P. We *extend* P by replacing P, C'' by $P \cup C'' \cup \{(u, v)\} \setminus f''$ where f'' is an edge of C'' incident with v. We repeat this operation as long as we can. We then carry out (b) or (c).

(b) The endpoints u, v of P are connected by an edge in H. Adding (u, v) to P creates a 2-factor with at least one less cycle than at the start of the General Step and completes it.

(c) Carry out rotations on P until either (i) we construct a path Q with an endpoint x which is adjacent to a vertex y on cycle C outside Q or (ii) we satisfy the condition of (b). In the latter case we proceed as in (b) above. In the former case we extend Q by adding the edge (x, y) and deleting an edge of C incident with y.

We continue the above operations until we either obtain a Hamilton cycle or obtain a path $P_0 = P_0(G) = (v_0, v_1, \ldots, v_p)$ that cannot be extended or closed to a cycle via a sequence of rotations. Note that this path is necessarily of odd length.

We therefore let P_0 be a longest path of *odd* length which (i) cannot be extended by rotations and (ii) for which there are a set of vertex disjoint cycles covering the vertices not in P.

We use the Pósa condition (which still holds) and Lemma 10 to get the following.

Lemma 13. *If* $G \overset{r}{\in} \mathcal{L}_n(K_{h,h})$ *and* $H \overset{r}{\in} \mathcal{H}(G)$*, then* **whp** $|END_H(v, P_0)| \geq hn/4$ *for all* $v \in END_H(P_0)$*,* $P_0 = P_0(G)$*.*

We say next that an *ordered* pair of pillars (Π_k, Π_l) is *good* w.r.t. a longest path P if $\Pi_k \in W_x$, $\Pi_l \in W_{3-x}$, $x = 1, 2$ and

$$|\{u \in \Pi_k \cap END_H(P) : |\{v \in \Pi_l \cap END_H(u, P) : (u, v) \notin E(H)\}| \geq n/500\}| \geq n/500. \quad (17)$$

In words, Π_k contains at least $n/500$ vertices $u \in END_H(P)$ for which there at least $n/500$ vertices $v \in \Pi_l \cap END_H(u, P)$ such that the edge $(u, v) \notin E(H)$.

Lemma 14. *If* (16) *holds then* G *has at least* $\binom{h}{2}/3000$ *good pillar pairs.*

Proof We first note that P_0 and the paths obtained by rotations are of odd length and so each has one endpoint in each of W_1, W_2.

Now we can argue as in Lemma 5 that for each $u \in W_x \cap END_H$, $x = 1, 2$ there are at least $h/7$ pillars $\Pi_j \in W_{3-x} \cap END(u, H)$ for which

$$|\{v \in \Pi_k \cap END_H(u, P) : (u, v) \notin E(H)\}| \geq n/8.$$

The rest of the proof is identical to that of Lemma 5. □

3.2 The Proof

Define the sets $\mathcal{A}, \mathcal{C}, \mathcal{F}$ as in the proof of Theorem 1. We have $|\mathcal{A}| \geq (1 - o(1)) |\mathcal{L}_n(K_{h,h})|$ using the argument in Lemma 6 with the results from Lemmas 10 and 11. Define also the matrix A and N_1 as in the proof of Theorem 1. The proofs of the following Lemmas are similar to the proofs of Lemmas 7 and 8.

Lemma 15. *If $G_i \in \mathcal{F}$ then*

$$\sum_j a_{i,j} \geq (1-o(1))\binom{h^2n-23hn}{(1-\beta)h^2n-23hn}.$$

It follows immediately from Lemma 15 that

$$N_1 \geq (1-o(1))\binom{h^2n-23hn}{(1-\beta)h^2n-23hn}|\mathcal{F}|. \tag{18}$$

We now obtain an upper bound on N_1. Let

$$\mathcal{X} = \{H : \exists i, j \text{ for which } G_{i,j} = H \text{ and } a_{i,j} = 1\}$$

It follows from the definition that

$$|\mathcal{X}| \leq \binom{h^2n}{23hn}\left((1+o(1))\binom{n}{\beta n}^2(\beta n)!\right)^{h^2}. \tag{19}$$

For a fixed $H \in \mathcal{H}$ let

$$\mathcal{G}_H = \{G_{i,j} : H_{i,j} = H \text{ and } a_{i,j} = 1\}.$$

Thus,

$$N_1 = \sum_{H\in\mathcal{X}} |\mathcal{G}_H|.$$

Lemma 16.

$$H \in \mathcal{X} \text{ implies } |\mathcal{G}_H| \leq e^{-ch^2n}\left((1-\beta+O(n^{-1/3})n)!\right)^{h^2}. \tag{20}$$

It follows from (18), (19) and (20) that $\frac{|\mathcal{F}|}{|\mathcal{L}_n(K_h)|}$ is bounded above by

$$\begin{aligned}
&\frac{e^{-ch^2n}\left((1-\beta+O(n^{-1/3})n)!\right)^{h^2}\binom{h^2n}{23hn}\left((1+o(1))\binom{n}{\beta n}^2(\beta n)!\right)^{h^2}}{(1-o(1))(n!)^{h^2}\binom{h^2n-23hn}{(1-\beta)h^2n-23hn}} \\
\leq\ & \frac{e^{-ch^2n/2}\binom{n}{\beta n}^{h^2}}{\binom{h^2n}{\beta h^2n}\beta^{24hn}} \\
\leq\ & e^{-ch^2n/2+24hn\ln(1/\beta)} \\
=\ & o(1).
\end{aligned}$$

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