Cover time of a random graph with given degree sequence

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Abstract

In this paper we establish the cover time of a random graph $G(\mathbf{d})$ chosen uniformly at random from the set of graphs with vertex set $[n]$ and degree sequence $\mathbf{d}$. We show that under certain restrictions on $\mathbf{d}$, the cover time of $G(\mathbf{d})$ is **whp** asymptotic to $\frac{d-1}{d-2}\frac{\theta}{d}n\log n$. Here θ is the average degree and d is the *effective minimum degree.*

1 Introduction

Let $G = (V, E)$ be a connected graph with $|V| = n$ vertices and $|E| = m$ edges.

For a simple random walk $\mathcal{W}_v$ on G starting at a vertex v, let C_v be the expected time taken to visit every vertex of G. The *vertex cover time* $C(G)$ of G is defined as $C(G) = \max_{v\in V} C_v$. The vertex cover time of connected graphs has been extensively studied. It is a classic result of Aleliunas, Karp, Lipton, Lovász and Rackoff [2] that $C(G) \leq 2m(n-1)$. It was shown by Feige [8], [9], that for any connected graph G, the cover time satisfies $(1-o(1))n\log n \leq C(G) \leq (1+o(1))\frac{4}{27}n^3$. Between these two extremal examples, the cover time, both exact and asymptotic, has been determined for a number of different classes of graphs.

In this paper we study the cover time of random graphs $\mathcal{G}(\mathbf{d})$ picked uniformly at random (**uar**) from the set $\mathcal{G}(\mathbf{d})$ of simple graphs with vertex set $V = [n]$ and degree sequence $\mathbf{d} = (d_1, d_2, \ldots, d_n)$, where d_i is the degree of vertex $i \in V$. We make the following definitions:

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Let $V_j = \{i \in V : d_i = j\}$ and let $n_j = |V_j|$. Let $\sum_{i=1}^n d_i = 2m$ and let $\theta = 2m/n$ be the average degree. We use the notations d_i and $d(i)$ for the degree of vertex i.

Let $0 < \alpha \le 1$ be constant, $0 < c < 1/8$ be constant and let d be a positive integer. Let $\gamma = (\sqrt{\log n}/\theta)^{1/3}$. We suppose the degree sequence $\mathbf{d}$ satisfies the following conditions:

(i) Average degree $\theta = o(\sqrt{\log n})$.

(ii) Minimum degree $\delta \ge 3$.

(iii) For $\delta \le i < d$, $n_i = O(n^{ci/d})$.

(iv) $n_d = \alpha n + o(n)$. We call d the *effective minimum degree.*

(v) Maximum degree $\Delta = O(n^{c(d-1)/d})$.

(vi) Upper tail size $\displaystyle\sum_{j=\gamma\theta}^{\Delta} n_j = O(\Delta)$.

We call a degree sequence $\mathbf{d}$ which satisfies conditions (i)–(vi) *nice*, and apply the same adjective to $\mathcal{G}(\mathbf{d})$. Basically, nice graphs are sparse, with not too many high degree vertices. Any degree sequence with constant maximum degree, and for which $d = \delta$ is nice. The conditions hold in particular, for d-regular graphs, $d \ge 3$, $d = \delta = o(\sqrt{\log n})$, as condition (iii) is empty. The spaces of graphs we consider are somewhat more general. The condition nice, allows for example, bi-regular graphs where half the vertices are degree $d \ge 3$ and half of degree $a = o(\sqrt{\log n})$.

Conditions (i), (v), (vi) allow us to infer structural properties of $\mathcal{G}(\mathbf{d})$ via the configuration model, in a way that is explained in Section 3.1. The effective minimum degree condition (iv), ensures that some entry in the degree sequence occurs order n times. Condition (iii) is necessary for the analysis of the random walk, as Theorem 1 does not hold when $c > 1$, even if the maximum degree is constant. However, the value $c < 1/8$ in condition (iii) is somewhat arbitrary, as are the precise values in conditions (v), (vi).

It will follow from Lemma 7 that random graphs with a nice degree sequence are connected with high probability (**whp**). The following theorem gives the cover time of nice graphs.

Theorem 1. *Let $G(\boldsymbol{d})$ be chosen* **uar** *from $\mathcal{G}(\boldsymbol{d})$, where $\boldsymbol{d}$ is nice. Then* **whp**

$$C(G(\boldsymbol{d})) \sim \frac{d-1}{d-2}\frac{\theta}{d}\, n \log n. \tag{1}$$

In this paper, the notation **whp** means with probability $1 - n^{-\Omega(1)}$, and $A(n) \sim B(n)$ means $\lim_{n\to\infty} A(n)/B(n) = 1$.

We note that if $d \sim \theta$, i.e. the graph is pseudo-regular, then as long as condition (iii) holds,

$$C(G) \sim \frac{d-1}{d-2}\, n \log n.$$

This extends the result of [5] for random d-regular graphs.

Structure of the paper

The proof of Theorem 1 is based on an application of (7) below. Put simply, (7) says that, if we ignore which vertices the random walk visits during the mixing time, the probability a vertex v remains unvisited in the first t steps is asymptotic to $\exp(-\pi_v t/R_v)$. Here $\pi_v = d(v)/2m$ where $d(v)$ is the degree of vertex v and m is the number of edges. The variable R_v is the expected number of returns to v during the mixing time, for a walk starting at v. To estimate R_v in Section 4.2, we describe and prove the required **whp** graph properties in Section 3. Lemma 7, proved in the Appendix establishes that nice graphs have constant conductance **whp**; which implies connectivity as asserted in the introduction. The proof that (7) is valid **whp** for $\mathcal{G}(\mathbf{d})$ is similar to proofs in earlier papers and is given in the Appendix. The cover time $C(G)$ in (1) is established in Section 5 as follows. Firstly an upper bound of $(1+o(1))C(G)$ is proved in Section 5.1. In Section 5.2 a lower bound is determined by constructing a set of vertices S such that $\sum_{v \in S} \exp(-\pi_v t/R_v) \to \infty$ at $t = (1-o(1))C(G)$.

2 Estimating first visit probabilities

In this section G denotes a fixed connected graph with n vertices. A random walk $\mathcal{W}_u$ is started from a vertex u. Let $\mathcal{W}_u(t)$ be the vertex reached at step t, let P be the matrix of transition probabilities of the walk and let $P_u^{(t)}(v) = \mathbf{Pr}(\mathcal{W}_u(t) = v)$. We assume that the random walk $\mathcal{W}_u$ on G is ergodic with stationary distribution π, where $\pi_v = d(v)/(2m)$, and $d(v)$ is the degree of vertex v.

Let T be a positive integer such that for $t \geq T$

$$\max_{u,x \in V} |P_u^{(t)}(x) - \pi_x| \leq n^{-3}, \tag{2}$$

and let

$$\lambda = \frac{1}{KT} \tag{3}$$

for a sufficiently large constant K. The existence of such a T will follow from (20).

Considering a walk $\mathcal{W}_v$, starting at vertex v, let $r_t = \mathbf{Pr}(\mathcal{W}_v(t) = v)$ be the probability that the walk returns to v at step $t = 0, 1, ...,$ and let

$$R_T(z) = \sum_{j=0}^{T-1} r_j z^j. \tag{4}$$

Given vertices u, v, let $\mathcal{W}_u$ be a random walk starting at vertex u. For $t \geq T$ let $\boldsymbol{A}_v(t)$ be the event that $\mathcal{W}_u$ does not visit v in steps $T, T+1, \ldots, t$. Several versions of the following lemma have appeared previously (e.g. in [5], [6]). For completeness, a proof is given in Section 6.1 of the Appendix.

Lemma 2. *Let $v \in V$ satisfy the following conditions:*

(a) *For some constant $\psi > 0$, we have*

$$\min_{|z| \leq 1+\lambda} |R_T(z)| \geq \psi,$$

where $R_T(z)$ is from (4).

(b) *$T\pi_v = o(1)$ and $T\pi_v = \Omega(n^{-2})$ for all $v \in V$.*

Let

$$R_v = R_T(1). \tag{5}$$

Then there exists

$$p_v = \frac{\pi_v}{R_v(1 + O(T\pi_v))}, \tag{6}$$

such that for all $t \geq T$,

$$\mathbf{Pr}(\boldsymbol{A}_v(t)) = \frac{(1 + O(T\pi_v))}{(1 + p_v)^t} + O(T^2 \pi_v e^{-\lambda t/2}). \tag{7}$$

3 Graph properties

We make our **whp** calculations about properties of nice graphs in the configuration model, (see Bollobás [3]). Let $W = [2m]$ be the set of *configuration points* and for $i \in [n]$, let $W_i = [d_1 + \cdots + d_{i-1} + 1, d_1 + \cdots + d_i]$. Thus $W_i,\ i = 1, ..., n$ is a partition of W. For $u \in W_i$, define $\phi : [2m] \to [n]$ by $\phi(u) = i$. Thus, $|W_i| = d_i$, and $\phi(u)$ is the vertex corresponding to the configuration point u. Given a pairing F (i.e. a partition of W into m pairs $\{u, v\}$) we obtain a multi-graph G_F with vertex set $[n]$ and an edge $(\phi(u), \phi(v))$ for each $\{u, v\} \in F$.

Choosing a pairing F uniformly at random from among all possible pairings of the points of W produces a random multi-graph G_F. Let

$$\mathcal{F}(2m) = \frac{(2m)!}{m!2^m}. \tag{8}$$

Thus $\mathcal{F}(2m)$ counts the number of distinct pairings F of the $2m$ points in W. Moreover the number of pairings corresponding to each simple graph $G \in \mathcal{G}(\mathbf{d})$ is the same, so that simple graphs are equiprobable in the space of multi-graphs. Let $\nu = \sum_i d_i(d_i-1)/(2m)$. Assuming that $\Delta = o(m^{1/3})$, (see e.g. [10]), the probability that G_F is simple is given by

$$P_S = \mathbf{Pr}(G_F \text{ is simple}) \sim e^{-\frac{\nu}{2}-\frac{\nu^2}{4}}. \tag{9}$$

Our assumption that conditions (i)–(vi) hold for $\mathbf{d}$, imply that $\Delta = o(m^{1/3})$. Also as $\gamma = (\sqrt{\log n}/\theta)^{1/3}$, then $\nu = o(\sqrt{\log n})$ follows from

$$\nu \leq \frac{1}{\theta n}\left(\sum_{j=3}^{\gamma\theta} n_j j^2 + \sum_{j=\gamma\theta}^{\Delta} n_j j^2\right) \leq \frac{1}{\theta n}(n\gamma^2\theta^2 + O(\Delta^3)) = o(\sqrt{\log n}).$$

If $\nu = o(\sqrt{\log n})$, then P_S in (9) is at least $e^{-o(\log n)}$. On the other hand, statements about graph structure we make in this paper using the configuration model fail with probability at most $n^{-\Omega(1)}$, which means they hold **whp** for simple graphs.

3.1 Structural properties of $G(\mathbf{d})$

In this section we establish the **whp** properties of nice graphs needed to estimate R_v in (5) for all $v \in V$.

Let C be a large constant, and let

$$\omega = C \log\log n. \tag{10}$$

A cycle or path is *small*, if it has at most $2\omega+1$ vertices, otherwise it is *large*. Let

$$\ell = B \log^2 n \tag{11}$$

for some large constant B. A vertex v is *light* if it has degree at most ℓ, otherwise it is *heavy*. A cycle or path is *light* if all vertices are light. A light vertex v is *small* if it has degree at most $d-1$.

Lemma 3. *Let $\boldsymbol{d}$ be a nice degree sequence and let $G(\boldsymbol{d})$ be chosen uniformly at random from the $\mathcal{G}(\boldsymbol{d})$. There exists $\epsilon > 0$ constant such that with with probability $1 - O(n^{-\epsilon})$,*

(a) *No vertex disjoint pair of small light cycles are joined by a small light path.*

(b) *No light vertex is in two small light cycles.*

(c) *No small cycle contains a heavy vertex or small vertex, or is connected to a heavy or small vertex by a small path.*

(d) *No pair of small or heavy vertices is connected by a small path.*

Proof We note a useful inequality. For integer $x > 0$, let $\mathcal{F}(2x) = \frac{(2x)!}{2^x x!}$, as defined in (8), then

$$\frac{\mathcal{F}(\theta n - 2x)}{\mathcal{F}(\theta n)} = \frac{(\theta n - 2x)!}{\left(\frac{\theta n}{2} - x\right)! 2^{\frac{\theta n}{2} - x}} \frac{\left(\frac{\theta n}{2}\right)! 2^{\frac{\theta n}{2}}}{(\theta n)!} = \left(\prod_{i=1}^{x} \theta n - 2i + 1\right)^{-1} \leq \left(\frac{1}{\theta n - 2x + 1}\right)^x. \quad (12)$$

(a) Let S denote the sum over a, b, c of the expected number of subgraphs consisting of small light vertex cycles of length a, b joined by a small light vertex path of length $c + 1$. Then

$$S \leq \sum_{a=3}^{2\omega+1} \sum_{b=3}^{2\omega+1} \sum_{c=0}^{2\omega+1} \binom{n}{a}\binom{n}{b}\binom{n}{c} \frac{(a-1)!}{2} \frac{(b-1)!}{2} c! ab \ell^{2(a+b+c+1)} \frac{\mathcal{F}(\theta n - 2(a+b+c+1))}{\mathcal{F}(\theta n)} \quad (13)$$

Explanation. Choose a vertices for one cycle, b vertices for the other and c vertices for the path. Each light vertex has most $\ell(\ell - 1)$ ways to connect to its neighbours on a given path or cycle. This explains the exponent of ℓ. Choosing $x = (a + b + c + 1) \leq 6\omega + 4$ in (12), we find S is bounded by

$$\begin{aligned} S &\leq \sum_{a=3}^{2\omega+1} \sum_{b=3}^{2\omega+1} \sum_{c=0}^{2\omega+1} n^a n^b n^c \ell^{2(a+b+c+1)} \left(\frac{1}{\theta n - (12\omega + 8)}\right)^{a+b+c+1} \\ &\leq \frac{\ell^2}{\theta n - (12\omega + 8)} \sum_a \sum_b \sum_c \left(\frac{n\ell^2}{\theta n - (12\omega + 8)}\right)^{a+b+c} \\ &= O\left(\frac{\omega^3 \ell^{12\omega+8}}{\theta n}\right) = o(1). \end{aligned} \quad (14)$$

(b) The proof for this part is similar to **(a)**.

(c) Note that, in condition (vi), the value of $\gamma\theta < \ell$ and thus the number, H, of heavy vertices is $O(\Delta) = O(n^{c(d-1)/d})$. Similarly, from condition (iii), the number of small vertices is $O(n^{c(d-1)/d})$. The expected number S of cycles of length $3 \leq a \leq 2\omega + 1$ with $a - k$ light vertices and $k \geq 1$ heavy vertices can be bounded by the expected number of configuration pairings of cycles of this type. Thus

$$S \leq \sum_{a=3}^{2\omega+1} \sum_{k \geq 1} \binom{n}{a-k}\binom{H}{k} (a-1)! \ell^{2(a-k)} \Delta^{2k} \frac{\mathcal{F}(\theta n - 2a)}{\mathcal{F}(\theta n)}.$$

Thus, using (12), we have

$$
\begin{aligned}
S &= O(1) \sum_a \sum_{k\geq 1} \binom{a}{k} n^{-k} \Delta^{3k} \ell^{2a} \\
&= O(1) \sum_a \ell^{2a} \frac{a\Delta^3}{n} \\
&= O(\omega^2) \frac{\ell^{4\omega+2}\Delta^3}{n} = o(1).
\end{aligned}
$$

We next count the expected number S of cycles of lengths $3 \leq a \leq 2\omega+1$ containing only light vertices, which are joined to a heavy vertex by a light vertex path of length $0 \leq b-1 \leq 2\omega$. This can be bounded by

$$
\begin{aligned}
S &\leq \sum_{a=3}^{2\omega+1} \sum_{b=1}^{2\omega+1} \binom{n}{a} \binom{n}{b-1} \binom{H}{1} (a-1)!(b-1)!\ell^{2a+2(b-1)+1} a\Delta \frac{\mathcal{F}(\theta n - 2(a+b))}{\mathcal{F}(\theta n)} \\
&= O(\omega^2) \frac{\ell^{8\omega+4}\Delta^2}{n} = o(1).
\end{aligned}
$$

(d) There are $H = O(\Delta)$ small or heavy vertices. The expected number S of small light paths length connecting such vertices is

$$
\begin{aligned}
S &\leq \sum_{a=0}^{2\omega+1} \binom{n}{a} \binom{H}{2} a! \ell^{2a} \Delta^2 \frac{\mathcal{F}(\theta n - 2(a+1))}{\mathcal{F}(\theta n)} \\
&= O\left(\frac{\omega \ell^{4\omega+2}\Delta^4}{n} \right).
\end{aligned}
$$

□

For a vertex v, let G_v be the subgraph induced by the set of vertices within a distance ω of v. As any paths or cycles contained in G_v are of length at most $2\omega+1$ and hence small, the following lemma is a corollary of Lemma 3.

Lemma 4. *Let $G(\boldsymbol{d})$ be nice. Assuming the conditions (a)-(d) of Lemma 3 hold, then*

(a) *If G_v contains a small or heavy vertex, G_v is a tree.*

(b) *If G_v is not a tree, then G_v contains exactly one small cycle, and all vertices of G_v are light.*

(c) *There are $O(\ell^\omega n^{ci/d})$ vertices v such that G_v contains a small vertex of degree i.*

(d) *There are $O(\ell^\omega n^{2c(d-1)/d})$ vertices v such that G_v contains a heavy vertex.*

Proof If G_v contains a small or heavy vertex then it is a tree, and all other vertices are light. Thus $|G_v| = O(\ell^\omega)$ for small vertices, and there are $O(n^{ci/d})$ small vertices of degree i. If G_v contains a heavy vertex then $|G_v| = O(\Delta \ell^\omega)$. □

Lemma 5. *Let $\boldsymbol{d}$ be a nice degree sequence and let $G(\boldsymbol{d})$ be chosen uniformly at random from the $\mathcal{G}(\boldsymbol{d})$. For any $\epsilon > 0$ constant, with probability $1 - O(n^{-\epsilon})$, there are at most $n^{4\epsilon}$ vertices v such that G_v contains a cycle.*

Proof The expected number of vertices on small light cycles is at most

$$\begin{aligned} S &\leq \sum_{a=3}^{2\omega+1} \binom{n}{a} \frac{(a-1)!}{2} \ell^{2a} \frac{\mathcal{F}(\theta n - 2a)}{\mathcal{F}(\theta n)} \\ &= O\left(\omega \ell^{4\omega+2}\right). \end{aligned}$$

The probability there are more than n^ϵ vertices on small light cycles is $o(n^{-\epsilon/2})$, for any $\epsilon > 0$. If G_v contains only light vertices, then $|G_v| = O(\ell^\omega)$, and thus (**whp**) there are at most $n^{2\epsilon}$ vertices v such that G_v contains a small light cycle. □

A vertex v is *d-compliant*, if G_v is a tree, and all vertices of G_v have degree at least d. A vertex v is *d-tree-like* to depth h if the graph induced by the vertices at distance at most h from v form a d-regular tree, (i.e. all vertices on levels $0, 1, ..., h-1$ have degree d).

A vertex v is *d-tree-regular*, if it is d-tree-like to depth h, d-compliant to depth ω and all vertices of G_v are light. For such a vertex v, the first h levels of the BFS tree, really are a d-regular tree, and the remaining $\omega - h$ levels can be pruned to a d-regular tree. We choose the following value for h, which depends on θ.

$$h = \frac{1}{\log d} \log\left(\frac{\log n}{(\log\log n)\log\theta}\right) \tag{15}$$

The exact value of h is not so important. The main thing is that $d^h \to \infty$ in Lemma 9, but not too fast in Lemma 6.

Lemma 6. *Let $\boldsymbol{d}$ be a nice degree sequence and let $G(\boldsymbol{d})$ be chosen uniformly at random from the $\mathcal{G}(\boldsymbol{d})$. There exists $\epsilon > 0$ constant such that with with probability $1 - O(n^{-\epsilon})$, there are $n^{1-O(1/\log\log n)}$ d-tree-regular vertices.*

Proof Recall that $n_d = |V_d| = \alpha n + o(n)$ for some constant $\alpha > 0$. We assume from Lemmas 4 and 5 that all but $O(n^\epsilon) + O(\ell^\omega \Delta^2)$ vertices of degree d are d-compliant, or have a heavy vertex within distance ω.

Let $N_2 = 1 + d(d-1)^h$. If v has degree d and is d-tree-like to depth h, then the tree of this depth rooted at v contains less than N_2 vertices. We bound the probability P that a vertex

v of degree $d(v) = d$ is d-tree-like, by bounding the probability of success of the construction of a d-regular tree of depth h in the configuration model.

$$P = \mathbf{Pr}(\text{vertex } v \text{ is } d\text{-tree-like}) = \prod_{i=1}^{N_2-1} \frac{d(n_d - i)}{\theta n - 2i + 1} \geq \left(d \frac{n_d - N_2}{\theta n} \right)^{N_2}. \tag{16}$$

Let M count the number of d-tree-like vertices, then $\mathbf{E}[M] = \mu = n_d P$, and for the value of h given in (15) we have that

$$\mu = \mathbf{E}[M] = n^{1-O(1/\log\log n)}. \tag{17}$$

To estimate $\mathbf{Var}[M]$, let I_v be the indicator that vertex v is d-tree-like. We have

$$\mathbf{E}[M^2] = \mu + \sum_{v \in V_d} \sum_{w \in V_d, w \neq v} \mathbf{E}[I_v I_w], \tag{18}$$

and

$$\mathbf{E}[I_v I_w] = \mathbf{Pr}(v, w \text{ are } d\text{-tree-like}, G_v \cap G_w = \emptyset) + \mathbf{Pr}(v, w \text{ are } d\text{-tree-like}, G_v \cap G_w \neq \emptyset).$$

Now

$$\mathbf{Pr}(v, w \text{ are } d\text{-tree-like}, G_v \cap G_w = \emptyset) = \prod_{i=1}^{2N_2-2} \frac{d(n_d - i - 1)}{\theta n - 2i + 1} \leq P^2. \tag{19}$$

For any vertex v, the number of vertices w such that $G_v \cap G_w \neq \emptyset$ is bounded from above by $N_2 + dN_2^2$. Using this and (19), we can bound (18) from above by $\mu + \mu^2 + \mu(N_2 + dN_2^2)$.

By the Chebychev Inequality, for some constant $0 < \tilde{\epsilon} < 1$,

$$\mathbf{Pr}\left(|M - \mu| > \mu^{\frac{1}{2}+\tilde{\epsilon}} \right) \leq \frac{\mathbf{Var}[M]}{\mu^{1+2\tilde{\epsilon}}} = \frac{\mathbf{E}[M^2] - \mathbf{E}[M]^2}{\mu^{1+2\tilde{\epsilon}}} \leq \frac{\mu + \mu N_2 + \mu d N_2^2}{\mu^{1+2\tilde{\epsilon}}} = O(n^{-\epsilon}).$$

The lemma now follows from (17). □

4 Random walk properties

4.1 Mixing time

Given a graph G, the *conductance* $\Phi(G)$ of a random walk $\mathcal{W}_u$ on G is defined by

$$\Phi(G) = \min_{\pi(S) \leq 1/2} \frac{e(S : \overline{S})}{d(S)}$$

where $d(S) = \sum_{v \in S} d(v)$, $\pi(S) = d(S)/2m$, and $e(A : B)$ denotes the number of edges with one endpoint in A and the other in B. The lemma below follows by applying (9) to Lemma 12 proved in Section 6.2 of the Appendix.

Lemma 7. *Let $\boldsymbol{d}$ be a nice degree sequence and let $G(\boldsymbol{d})$ be chosen uniformly at random from the $\mathcal{G}(\boldsymbol{d})$, then with probability $1 - O(n^{-1/9})$*

$$\Phi(G) \geq \frac{1}{100}.$$

Note that $\Phi(G) \geq 1/100$ in Lemma 7 implies $G(\mathbf{d})$ is connected.

We note a result from Sinclair [11], that

$$|P_u^{(t)}(x) - \pi_x| \leq (\pi_x/\pi_u)^{1/2}(1 - \Phi^2/2)^t. \tag{20}$$

Referring to Lemma 7 and (20), if we choose A sufficiently large and

$$T = A \log n \tag{21}$$

then (2) holds. There is a technical point here, in that the result (20) assumes that the walk is lazy. A lazy walk moves to a neighbour with probability 1/2 at any step. This assumption halves the conductance, and doubles the value of $R_T(1)$. Asymptotically, the cover time is also doubled by the inclusion of the lazy steps. The trajectory, and hence cover time of the underlying (non-lazy) walk can be recovered by removing the lazy steps. We will ignore the assumption in (20) for the rest of the paper; and continue as though there are no lazy steps.

4.2 Expected number of returns in the mixing time

Escape probability. Let $v \in V$, and $B \subseteq V$, and assume $v \notin B$. For a walk ${\mathcal{W}_v}^B$ starting at v, let $P_v(B)$ be the probability that the walk reaches B without return to v; the *escape probability* from v to B. The value of $P_v(B)$ is given by

$$P_v(B) = \frac{1}{d(v)R_{\text{eff}}(v, B)}, \tag{22}$$

where $R_{\text{eff}}(v, B)$ is the *effective resistance between v and B*, treating the edges as having unit resistance. If we treat B as an absorbing state, then $f_v(B) = 1 - P_v(B)$ is the probability of a first return to v by ${\mathcal{W}_v}^B$ before absorption at B; and $R_v(B) = 1/(1 - f_v(B)) = 1/P_v(B)$ is the expected number of returns to v before absorption at B.

The attractiveness of formula (22) is that by Rayleigh's monotonicity law, deleting edges of the graph does not decrease the effective resistance between v and B. Thus provided we do not delete any edges incident with v, such pruning cannot increase $P_v(B)$. See [7] for details of Rayleigh's monotonicity law, and a proof of (22).

For a vertex v, we defined G_v as the subgraph induced by the set of vertices within a distance ω of v. Denote by Γ_v° those vertices of G_v at distance exactly ω from v. The following lemma relates R_v in (5) of Lemma 2 to $R_v^* = R_v(\Gamma_v^\circ)$ obtained from (22) as described above.

Lemma 8. *Let $G(\boldsymbol{d})$ be nice, and assume the conditions of Lemma 3 and Lemma 7 hold. Let $\mathcal{W}_v^*$ denote a walk on G_v starting at v with Γ_v° made into an absorbing state. Let $R_v^* = \sum_{t=0}^\infty r_t^*$, where r_t^* is the probability that $\mathcal{W}_v^*$ is at vertex v at time t. Let R_v be given by (5), then*

$$R_v = R_v^* + o\left(\frac{1}{\log n}\right).$$

For completeness the proof of Lemma 8 is given in Section 6.3 of the Appendix. A similar proof is given in e.g. [5]. The precise value of R_v^* is given by (22). The next lemma gives some approximate bounds.

Lemma 9. *For a vertex $v \in V$, let $\mathcal{W}_v^*$ be a walk on G_v, starting at v, and with Γ_v° made into an absorbing state. Let $P_v(\Gamma_v^\circ)$ be the escape probability of a walk, and let $R_v^* = 1/P_v(\Gamma_v^\circ)$.*

(a) *If v is d-tree-regular, then $R_v^* = \frac{d-1}{d-2}(1+o(1))$.*

(b) *If v is d-compliant then $R_v^* \le \frac{d-1}{d-2}(1+o(1))$.*

(c) *If G_v is a tree, $R_v^* \le \frac{\delta-1}{\delta-2}(1+o(1))$.*

(d) *If G_v contains a single cycle, and all vertices of G_v have degree at least d, then $R_v^* \le \frac{d(d-1)}{(d-2)^2}(1+o(1))$.*

Proof **(a)**

For a biased random walk on the half-line $(0, 1, ..., k)$, starting at vertex i, with absorbing states $0, k$, and with transition probabilities at vertices $(1, \ldots, k-1)$ of $q = \mathbf{Pr}(\text{move left})$, $p = \mathbf{Pr}(\text{move right})$; then

$$\mathbf{Pr}(\text{absorption at } k) = \frac{1-(q/p)^i}{1-(q/p)^k}. \tag{23}$$

We first project $\mathcal{W}_v^*$ onto $(0, 1, \ldots, h)$ with $p = \frac{d-1}{d}$ and $q = \frac{1}{d}$. As v is d-tree-like, the probability $Q(h)$ of escaping from v to level h of the d-regular tree of depth h rooted at v is

$$Q(h) = \frac{1 - \frac{1}{d-1}}{1 - \left(\frac{1}{d-1}\right)^h}.$$

Thus for h given by (15), $(d-1)^h \to \infty$ and

$$P_v(\Gamma_v^\circ) \le Q(h) = (1+o(1))\frac{d-2}{d-1}.$$

On the other hand G_v is d-compliant so, by pruning, contains a d-regular subtree, and

$$P_v(\Gamma_v^\circ) \geq \frac{1-\frac{1}{d-1}}{1-\left(\frac{1}{d-1}\right)^\omega} = (1+o(1))\frac{d-2}{d-1}.$$

(b) We can find a lower bound on the escape probability as follows. Retain all edges incident with v. Working outward from the neighbours of v, prune all internal vertices of G_v down to degree d, to obtain a subtree Λ_v of G_v in which v has degree $d(v)$ as in G_v. Let Λ_v° be its leaves, and $P_v(\Lambda_v^\circ)$ the escape probability from v to Λ_v° in Λ_v. Then by considering effective resistance, in (22)

$$P_v(\Gamma_v^\circ) \geq P_v(\Lambda_v^\circ) = (1+o(1))\frac{d-2}{d-1}.$$

(c) If G_v is a tree, but has some vertex w of degree $\delta \leq d(w) < d$, then, we can prune the internal vertices of $G_v - \{v\}$ to a δ-regular tree. By arguments similar to (b), $P_v(\Gamma_v^\circ) \geq (1+o(1))(\delta-2)/(\delta-1)$.

(d) If G_v contains a unique cycle, and all vertices in G_v have degree at least d, the arguments in (a) can be modified to fit this case. By assumption, there are at most two cycle edges incident with v, and $d(v) \geq d$ so

$$P_v(\Gamma_v^\circ) \geq \frac{d-2}{d}\frac{d-2}{d-1} + \frac{2}{d}\Phi,$$

where $\Phi \geq 0$ is the probability of no return to v given a cycle edge, or an edge on a path to a cycle was taken at v. □

At this point, a brief summary may be useful.

- G_v is a vertex induced subgraph of G. Up to absorption at Γ_v°, the boundary of G_v, a walk starting from v in G is identically coupled with a walk on G_v.
- The escape probability $P_v(\Gamma_v^\circ)$ from v of the walk $\mathcal{W}_v^*$ has a precise value. For d-tree-regular vertices v it can be approximated by $P_v(\Gamma_v^\circ) = (d-2)/(d-1)(1+O(1/d^h))$. Our choice of h (see (15)) ensures the error term is $o(1)$.
- By choosing $\omega = C \log\log n$ as in (10), and C sufficiently large, $1/R_v$ can be written as

$$\frac{1}{R_v} = P_v(\Gamma_v^\circ) + o\left(\frac{1}{\log n}\right). \tag{24}$$

  The $o(1/\log n)$ accuracy is needed in the proof of the lower bound on the cover time.

5 Cover time of $G(\mathbf{d})$

5.1 Upper bound on cover time

Let $T_G(u)$ be the time taken by the random walk $\mathcal{W}_u$ to visit every vertex of a connected graph G. Let U_t be the number of vertices of G which have not been visited by $\mathcal{W}_u$ at step t. We note the following:

$$C_u = \mathbf{E}[T_G(u)] = \sum_{t>0} \mathbf{Pr}(T_G(u) \geq t), \tag{25}$$

$$\mathbf{Pr}(T_G(u) \geq t) = \mathbf{Pr}(T_G(u) > t-1) = \mathbf{Pr}(U_{t-1} > 0) \leq \min\{1, \mathbf{E}[U_{t-1}]\}. \tag{26}$$

Recall from (7) that $\boldsymbol{A}_s(v)$ is the event that vertex v has not been visited during steps $T, T+1, ..., s$. It follows from (25), (26) that

$$C_u \leq t + 1 + \sum_{s \geq t} \mathbf{E}[U_s] \leq t + 1 + \sum_v \sum_{s \geq t} \mathbf{Pr}(\boldsymbol{A}_s(v)). \tag{27}$$

Let $t_0 = \left(\frac{d-1}{d-2}\frac{\theta}{d}\right) n \log n$ and $t_1 = (1+\epsilon)\, t_0$, were $\epsilon = o(1)$ is sufficiently large that all inequalities claimed below hold. We assume that Lemma 7 holds, and also the high probability claims of Section 3. Thus Lemma 8 and Lemma 9 give values of R_v for all $v \in V$. In Section 6.4 of the Appendix, we establish that Condition (a) of Lemma 2 holds. The maximum degree of any vertex is $n^a, a < 1$, and $T = A \log n$ (see (21)), so Condition (b) of Lemma 2 that $T\pi_v = o(1)$, holds trivially.

Recall from (6) that $p_v = (1 + O(T\pi_v))d(v)/(\theta n R_v)$. Thus by (7), the probability that $\mathcal{W}_u$ has not visited v during $[T, t]$ is given by

$$\begin{aligned} \mathbf{Pr}(\boldsymbol{A}_t(v)) &= (1+o(1))e^{-tp_v} + O(T^2 \pi_v e^{-\lambda t/2}) \quad (28)\\ &= (1+o(1))e^{-tp_v}. \quad (29) \end{aligned}$$

Thus

$$\begin{aligned} \sum_{t \geq t_1} (1+o(1))e^{-tp_v} &= (1+o(1))e^{-t_1 p_v} \sum_{(t-t_1) \geq 0} e^{-(t-t_1)p_v} \\ &= \frac{(1+o(1))}{1-e^{-p_v}} e^{-t_1 p_v} \\ &= O(1) \frac{\theta n R_v}{d(v)} \exp\left\{-(1+\Theta(\epsilon)) \frac{d(v)}{d} \frac{d-1}{d-2} \frac{\log n}{R_v}\right\}. \quad (30) \end{aligned}$$

We consider the following partition of V:
(i) $V_A = \bigcup_v \{G_v \text{ contains a small vertex}\}$.

(ii) $V_B = \bigcup_{i\geq d}\{d(v) = i : v \text{ is } d\text{-compliant}\}$.
(iii)$V_C = \bigcup_{i\geq d}\{d(v) = i : G_v \text{ contains a cycle}\}$.

Case (i): G_v contains a small vertex.
By Lemma 4(c) there are $O(\ell^\omega n^{ci//d})$ vertices v for which G_v contains a vertex of degree $i < d$, By Lemma 9(c), $R_v \leq (1+o(1))\frac{\delta-1}{\delta-2}$. Also G_v can contain at most one small vertex of degree $i < d$, so $d(v) \geq i$. Thus (30) is bounded by

$$O(\theta n)n^{-(1+o(1))\frac{i}{d}\frac{d-1}{d-2}\frac{\delta-2}{\delta-1}} \leq O(\theta n)n^{-(1+o(1))\frac{i}{d}\frac{\delta(\delta-2)}{(\delta-1)^2}}.$$

The term $\delta(\delta-2)/(\delta-1)^2 \geq 3/4$, whereas $c < 1/8$. Thus

$$\sum_{\delta\leq i<d}\sum_{v\in V_i}\sum_{t\geq t_1}(1+o(1))e^{-tp_v} \leq O(\theta n)\sum_{\delta\leq i<d} n^{ci/d}n^{-(1+o(1))3i/4d} \;=\; o(t_1).$$

Case (ii): $d \leq d(v)$, v is d-compliant.
Note that this includes the d-tree-regular case. For $v \in V_B$ (30) is bounded by $O(\theta)n^{-\Theta(\epsilon)}$. Therefore

$$\sum_{v\in V_B}\sum_{t\geq t_1}(1+o(1))e^{-tp_v} \leq \sum_{v\in V_B} O(\theta)n^{-\Theta(\epsilon)} = O(\theta n)n^{-\Theta(\epsilon)} = o(t_1).$$

Case (iii): $d \leq d(v)$, G_v contains a cycle.
These vertices $v \in V_C$, R_v is given by Lemma 9(d). Thus (30) is bounded by $O(\theta n)n^{-(1+\Theta(\epsilon))\frac{d(d-1)}{(d-2)^2}}$. By Lemma 4, $|V_C| \leq n^{\epsilon'}$ where $\epsilon' > 0$ arbitrarily small, and so we choose $2\epsilon' < d(d-1)/(d-2)^2$. Hence

$$\begin{aligned}\sum_{v\in V_C}\sum_{t\geq t_1}(1+o(1))e^{-tp_v} &= \sum_{v\in V_C} O(\theta n)n^{-(1+\Theta(\epsilon))\frac{d(d-1)}{(d-2)^2}}\\ &= O(\theta n)n^{\epsilon'}n^{-(1+\Theta(\epsilon))\frac{d(d-1)}{(d-2)^2}}\\ &= o(t_1).\end{aligned}$$

In each of the cases above, the term $\sum_v\sum_{s\geq t}\mathbf{Pr}(\boldsymbol{A}_s(v)) = o(t_1)$. Thus, from (27), $C_u \leq (1+o(1))t_1$ as required. This completes the proof of the upper bound on cover time of $G(\mathbf{d})$.
□

5.2 Lower bound on cover time

Let $t_2 = (1-\epsilon)t_0$, were $\epsilon = o(1)$ is sufficiently large that all inequalities claimed below hold. To establish the lower bound, we exhibit a set of vertices S for which, the probability the set

S is covered by a walk $\mathcal{W}_u$ at time t_2, tends to zero. Hence $T_G(u) > t_2$, **whp** which implies that $C(G) \geq t_0 - o(t_0)$.

We construct S as follows. Let S_d be the set of d-tree-regular vertices. Lemma 6 tells us that $|S_d| = n^{1-o(1)}$. Let $\omega = C \log\log n$ for some large C, as in (10). Let S be a maximal subset of S_d such that the distance between any two elements of S is least $2\omega + 1$. Thus $|S| = \Omega(n^{1-o(1)}/\ell^{2\omega})$.

Let $S(t)$ denote the subset of S which is still un-visited after step t of $\mathcal{W}_u$. Let $v \in S$, then

$$\mathbf{Pr}(\boldsymbol{A}_v(t_2)) = (1+o(1))e^{-t_2 p_v(1-O(p_v))} + o(n^{-2}).$$

Hence

$$\begin{aligned} \mathbf{E}(|S(t_2)|) &\geq (1+o(1))|S|e^{-(1-\epsilon)t_0 p_v} - O(T) \qquad (31)\\ &= \Omega\left(\frac{n^{\epsilon/2-o(1)}}{\ell^{2\omega}}\right) \to \infty. \qquad (32) \end{aligned}$$

The term $O(T)$ above, counts vertices of S visited during the first T steps of the walk. Let $Y_{v,t}$ be the indicator for the event $\boldsymbol{A}_t(v)$. Let $Z = \{v, w\} \subset S$. We will show (below) that that for $v, w \in S$

$$\mathbf{E}(Y_{v,t_2} Y_{w,t_2}) = \frac{1 + O(T\pi_v)}{(1+p_Z)^{t_2}} + o(n^{-2}), \qquad (33)$$

where

$$p_Z = p_v + p_w + o\left(\frac{d}{\theta n \log n}\right). \qquad (34)$$

Thus

$$\mathbf{E}(Y_{v,t_2} Y_{w,t_2}) = (1+o(1))\mathbf{E}(Y_{v,t_2})\mathbf{E}(Y_{w,t_2})$$

which implies

$$\mathbf{E}(|S(t_2)|(|S(t_2)|-1)) \sim \mathbf{E}(|S(t_2)|)(\mathbf{E}(|S(t_2)|)-1). \qquad (35)$$

It follows from (32) and (35), that

$$\mathbf{Pr}(S(t_2) \neq \emptyset) \geq \frac{\mathbf{E}(|S(t_2)|)^2}{\mathbf{E}(|S(t_2)|^2)} = \frac{1}{\frac{\mathbf{E}(|S(t_2)|(|S(t_2)|-1))}{\mathbf{E}(|S(t_2)|)^2} + \mathbf{E}(|S(t_2)|)^{-1}} = 1 - o(1).$$

Proof of (33)-(34)**.** Let $\widehat{G}$ be obtained from G by merging v, w into a single vertex Z. Let ρ be the expected number of passages between v, w in T steps. By construction, as G_w is a tree, whenever the walk arrives at Γ_w° after leaving v it will have to traverse a unique path of length ω to reach w. Using (23) and arguments similar to Lemma 9, we find $\rho = O(T^2/(d-1)^\omega) = o(1/\log n)$. Thus Lemma 8 is valid for $\widehat{G}$.

There is a natural measure-preserving map from the set of walks in G which start at u and do not visit v or w, to the corresponding set of walks in $\widehat{G}$ which do not visit Z. Thus the probability that $\mathcal{W}_u$ does not visit v or w in steps $T...t$ is asymptotically equal to the probability that a random walk $\widehat{\mathcal{W}}_u$ in $\widehat{G}$ which also starts at u does not visit Z in steps $T...t$. The detailed argument is given in [6].

We apply Lemma 2 to $\widehat{G}$. The value of $\pi_Z = 2d/\theta n$. The vertex Z has degree $2d$ and G_Z is otherwise d-tree-regular, as G_v, G_w are vertex disjoint. The derivation of R_Z^*, can be made as follows. The escape probability from Z to Γ_Z° is given by

$$P_Z(\Gamma_Z^\circ) = \frac{1}{2}(P_v(\Gamma_v^\circ) + P_w(\Gamma_w^\circ)),$$

as with probability 1/2 the walk takes a v-edge at Z and escapes to Γ_v° etc. Thus, as claimed in (34),

$$p_Z = \frac{\pi_Z}{R_Z}(1 + O(T\pi_Z)) = \frac{2d}{\theta n}(P_Z(\Gamma_Z^\circ) + o(1/\log n)) = p_v + p_w + o(d/\theta n \log n).$$

□

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6 Appendix

6.1 Proof of Lemma 2

Generating function formulation

Let $d_t = \max_{u,x\in V} |P_u^{(t)}(x) - \pi_x|$, and let T be such that, for $t \geq T$

$$\max_{u,x\in V} |P_u^{(t)}(x) - \pi_x| \leq n^{-3}. \tag{36}$$

It follows from e.g. Aldous and Fill [1] that $d_{s+t} \leq 2d_s d_t$ and so for $k \geq 1$,

$$\max_{u,x\in V} |P_u^{(kT)}(x) - \pi_x| \leq \frac{2^{k-1}}{n^{3k}}. \tag{37}$$

Fix two vertices u, v. Let $h_t = \mathbf{Pr}(\mathcal{W}_u(t) = v)$ be the probability that the walk $\mathcal{W}_u$ visits v at step t. Let

$$H(z) = \sum_{t=T}^{\infty} h_t z^t \tag{38}$$

generate h_t for $t \geq T$.

Next, considering the walk $\mathcal{W}_v$, starting at v, let $r_t = \mathbf{Pr}(\mathcal{W}_v(t) = v)$ be the probability that this walk returns to v at step $t = 0, 1, ...$ Let

$$R(z) = \sum_{t=0}^{\infty} r_t z^t$$

generate r_t. Our definition of return includes the term $r_0 = 1$.

For $t \geq T$ let $f_t = f_t(u \to v)$ be the probability that the first visit of the walk $\mathcal{W}_u$ to v in the period $[T, T+1, \ldots]$ occurs at step t. Let

$$F(z) = \sum_{t=T}^{\infty} f_t z^t$$

generate f_t. Then we have

$$H(z) = F(z)R(z). \tag{39}$$

First visit time lemma

For $R(z)$ let

$$R_T(z) = \sum_{j=0}^{T-1} r_j z^j. \tag{40}$$

Let

$$\lambda = \frac{1}{KT} \tag{41}$$

for some sufficiently large constant K.

Lemma 10. *Suppose that*

(a) *For some constant $\psi > 0$, we have*

$$\min_{|z| \leq 1+\lambda} |R_T(z)| \geq \psi.$$

(b) *$T\pi_v = o(1)$ and $T\pi_v = \Omega(n^{-2})$.*

There exists

$$p_v = \frac{\pi_v}{R_T(1)(1 + O(T\pi_v))}, \tag{42}$$

where $R_T(1)$ is from (40), such that for all $t \geq T$,

$$f_t(u \to v) = (1 + O(T\pi_v)) \frac{p_v}{(1 + p_v)^{t+1}} + O(T\pi_v e^{-\lambda t/2}). \tag{43}$$

Proof Write

$$R(z) = R_T(z) + \widehat{R}_T(z) + \frac{\pi_v z^T}{1 - z}, \tag{44}$$

where $R_T(z)$ is given by (40) and

$$\widehat{R}_T(z) = \sum_{t \geq T} (r_t - \pi_v) z^t$$

generates the error in using the stationary distribution π_v for r_t when $t \geq T$. Similarly,

$$H(z) = \widehat{H}_T(z) + \frac{\pi_v z^T}{1 - z}. \tag{45}$$

Equation (37) implies that the radii of convergence of both $\widehat{R}_T$ and $\widehat{H}_T$ exceed $1+2\lambda$. Moreover, for $Z = H, R$ and $|z| \leq 1 + \lambda$,

$$|\widehat{Z}(z)| = o(n^{-2}). \tag{46}$$

Using (44), (45) we rewrite $F(z) = H(z)/R(z)$ from (39) as $F(z) = B(z)/A(z)$ where

$$A(z) \;=\; \pi_v z^T + (1 - z)(R_T(z) + \widehat{R}_T(z)), \tag{47}$$

$$B(z) \;=\; \pi_v z^T + (1 - z)\widehat{H}_T(z). \tag{48}$$

For real $z \geq 1$ and $Z = H, R$, we have

$$Z_T(1) \leq Z_T(z) \leq Z_T(1) z^T.$$

Let $z = 1 + \beta \pi_v$, where $\beta = O(1)$. Since $T\pi_v = o(1)$ we have

$$Z_T(z) = Z_T(1)(1 + O(T\pi_v)).$$

$T\pi_v = o(1)$ and $T\pi_v = \Omega(n^{-2})$ and $R_T(1) \geq 1$ implies that

$$A(z) = \pi_v(1 - \beta R_T(1) + O(T\pi_v))$$

It follows that $A(z)$ has a real zero at z_0, where

$$z_0 = 1 + \frac{\pi_v}{R_T(1)(1 + O(T\pi_v))} = 1 + p_v, \tag{49}$$

say. We also see that

$$A'(z_0) = -R_T(1)(1 + O(T\pi_v)) \neq 0 \tag{50}$$

and thus z_0 is a simple zero (see e.g. [4] p193). The value of $B(z)$ at z_0 is

$$B(z_0) = \pi_v \left(1 + O(T\pi_v)\right) \neq 0. \tag{51}$$

Thus,

$$\frac{B(z_0)}{A'(z_0)} = -(1 + O(T\pi_v)) p_v. \tag{52}$$

Thus (see e.g. [4] p195) the principal part of the Laurent expansion of $F(z)$ at z_0 is

$$f(z) = \frac{B(z_0)/A'(z_0)}{z - z_0}. \tag{53}$$

To approximate the coefficients of the generating function $F(z)$, we now use a standard technique for the asymptotic expansion of power series (see e.g.[12] Th 5.2.1).

We prove below that $F(z) = f(z) + g(z)$, where $g(z)$ is analytic in $C_\lambda = \{|z| \leq 1 + \lambda\}$ and that $M = \max_{z \in C_\lambda} |g(z)| = O(T\pi_v)$.

Let $a_t = [z^t]g(z)$, then (see e.g.[4] p143), $a_t = g^{(t)}(0)/t!$. By the Cauchy Inequality (see e.g. [4] p130) we see that $|g^{(t)}(0)| \leq Mt!/(1+\lambda)^t$ and thus

$$|a_t| \leq \frac{M}{(1+\lambda)^t} = O(T\pi_v e^{-t\lambda/2}).$$

As $[z^t]F(z) = [z^t]f(z) + [z^t]g(z)$ and $[z^t]1/(z - z_0) = -1/z_0^{t+1}$ we have

$$[z^t]F(z) = \frac{-B(z_0)/A'(z_0)}{z_0^{t+1}} + O(T\pi_v e^{-t\lambda/2}). \tag{54}$$

Thus, we obtain

$$[z^t]F(z) = (1 + O(T\pi_v))\frac{p_v}{(1+p_v)^{t+1}} + O(T\pi_v e^{-t\lambda/2}),$$

which completes the proof of (43).

Now $M = \max_{z \in C_\lambda} |g(z)| \leq \max |f(z)| + \max |F(z)| = O(T\pi_v) + \max |F(z)|$, where $F(z) = B(z)/A(z)$. On C_λ we have, using (46)-(48),

$$|F(z)| \leq \frac{O(\pi_v)}{\lambda |R_T(z)| - O(T\pi_v)} = O(T\pi_v).$$

We now prove that z_0 is the only zero of $A(z)$ inside the circle C_λ and this implies that $F(z) - f(z)$ is analytic inside C_λ. We use Rouché's Theorem (see e.g. [4]), the statement of which is as follows: *Let two functions $\phi(z)$ and $\gamma(z)$ be analytic inside and on a simple closed contour C. Suppose that $|\phi(z)| > |\gamma(z)|$ at each point of C, then $\phi(z)$ and $\phi(z) + \gamma(z)$ have the same number of zeroes, counting multiplicities, inside C.*

Let the functions $\phi(z), \gamma(z)$ be given by $\phi(z) = (1-z)R_T(z)$ and $\gamma(z) = \pi_v z^T + (1-z)\widehat{R}_T(z)$.

$$|\gamma(z)|/|\phi(z)| \leq \frac{\pi_v(1+\lambda)^T}{\lambda\psi} + \frac{|\widehat{R}_T(z)|}{\psi} = o(1).$$

As $\phi(z) + \gamma(z) = A(z)$ we conclude that $A(z)$ has only one zero inside the circle C_λ. This is the simple zero at z_0. $\square$

Corollary 11. *For $t \geq T$ let $\boldsymbol{A}_t(v)$ be the event that $\mathcal{W}_u$ does not visit v in steps $T, T+1, \ldots, t$. Then, under the assumptions of Lemma 10,*

$$\mathbf{Pr}(\boldsymbol{A}_t(v)) = \frac{(1+O(T\pi_v))}{(1+p_v)^t} + O(T^2\pi_v e^{-\lambda t/2}).$$

Proof We use Lemma 10 and

$$\mathbf{Pr}(\boldsymbol{A}_t(v)) = \sum_{\tau > t} f_\tau(u \to v).$$

□

6.2 Proof of conductance bound in Lemma 7

By the conductance of a configuration C, we mean the conductance of a random walk on the underlying multi-graph $M(C)$. It is however, the configurations we sample **uar** in the proof of Lemma 12.

Lemma 12. *Let $\boldsymbol{d} = (d_1, d_2, ..., d_n)$ be a sequence of natural numbers, satisfying $\min d_i \geq 3$ and $\theta \leq n^{1/4}$. With probability $1 - o(n^{-1/9})$ the conductance Φ of a* **uar** *sampled configuration $C(\boldsymbol{d})$ satisfies $\Phi \geq 0.01$.*

Proof Let $F(a) = a!/((a/2)!2^{(a/2)})$. With this notation,

$$\frac{F(b)F(a-b)}{F(a)} = \frac{\binom{a/2}{b/2}}{\binom{a}{b}} = O(1)\left(\frac{b}{a}\right)^{b/2}\left(1-\frac{b}{a}\right)^{(a-b)/2}. \tag{55}$$

For any $S \subseteq V$ let $d(S)$ denote the sum of the degrees of the vertices of S. A set S is *small* if $d(S) \leq (\theta n)^{1/4}$. A set is *large* if $(\theta n)^{1/4} \leq d(S) \leq \theta n/2$. Let $8/9 < \beta < 1$ be a positive constant.

SMALL SETS $(\delta|S| \leq d(S) \leq (\theta n)^{1/4})$.
Let $N(s, \beta)$ be the expected number of small sets S of size s with at least $\beta d(S)$ induced edges.

$$N(s,\beta) = \sum_S \binom{d(S)}{\beta d(S)} \frac{F(\beta d(S))F(\theta n - \beta d(S))}{F(\theta n)}. \tag{56}$$

Noting that $\binom{L}{k} \leq (Le/k)^k$ and using (55) with $\delta s \leq d(S) \leq (\theta n)^{1/4}$ and $\delta \geq 3$ we find

$$\begin{aligned}
N(s,\beta) &\leq O(1)\sum_S \left(\frac{d(S)e}{\beta d(S)}\right)^{\beta d(S)} \left(\frac{\beta d(S)}{\theta n}\right)^{\beta d(S)/2} \left(1-\frac{\beta d(S)}{\theta n}\right)^{(\theta n - \beta d(S))/2} \\
&\leq O(1)\sum_S \left(\frac{e}{\beta}\right)^{\beta d(S)} \left(\frac{\beta d(S)}{\theta n}\right)^{\beta d(S)/2} \\
&\leq O(1)\left(\frac{ne}{s}\left(\frac{e^2}{\beta(\theta n)^{3/4}}\right)^{3\beta/2}\right)^s \\
&= O((e^4 n^{-(9\beta/8-1)})^s).
\end{aligned}$$

Thus

$$\sum_{\substack{|S|=s \\ S \text{ Small}}} N(s,\beta) = O(n^{-(9\beta/8-1)}).$$

LARGE SETS$((\theta n)^{1/4} \leq d(S) \leq \theta n/2)$.
Let $N(s,\beta)$ be the expected number of large sets S of size s inducing at least $\beta d(S)$ edges. As before, $N(s,\beta)$ is given by (56). Let $d(S) = \alpha\theta n$ where $0 < \alpha \leq 1/2$. Let $\varepsilon = 1-\beta$. We note the following approximation:

$$\binom{d(s)}{\beta d(S)} = \binom{\alpha\theta n}{\beta\alpha\theta n} = \frac{O(1)}{\sqrt{\varepsilon\beta\alpha\theta n}} \frac{1}{\beta^{\beta\alpha\theta n}\varepsilon^{\varepsilon\alpha\theta n}}.$$

Thus

$$N(s,\beta) \leq \sum_S \frac{O(1)}{\sqrt{\varepsilon\beta\alpha\theta n}} \left(\frac{(\alpha\beta)^{\alpha\beta}(1-\alpha\beta)^{1-\alpha\beta}}{(\varepsilon^\varepsilon\beta^\beta)^{2\alpha}}\right)^{\frac{\theta n}{2}} = \sum_S f(S). \tag{57}$$

Let $s = cn$. We henceforth assume that we choose the value $\alpha = \alpha^*$ which maximizes $f(S)$ for $|S| = cn$. With this convention we can write

$$N(cn,\beta) \leq \frac{O(1)}{\sqrt{\varepsilon\beta c(1-c)\alpha\theta n^2}} \left(\left(\frac{(\alpha\beta)^{\alpha\beta}(1-\alpha\beta)^{1-\alpha\beta}}{(\varepsilon^\varepsilon\beta^\beta)^{2\alpha}}\right)^{\frac{\theta}{2}} \frac{1}{c^c(1-c)^{1-c}}\right)^n. \tag{58}$$

We split the proof for large sets into two parts: Those sets for which $\alpha \leq 1/\theta$ and those for which $1/\theta \leq \alpha \leq 1/2$.

Case of $\alpha \leq 1/\theta$.
We need to remove the dependence on c in the right hand side of the expression (58) for $N(cn,\beta)$. We first deal with the square root term. Since $\frac{1}{n} \leq c \leq \frac{(n-1)}{n}$, we have that $c(1-c) \geq \frac{n-1}{n^2}$ and so

$$c(1-c)\alpha\theta n^2 \geq \frac{n-1}{n^2}(\theta n)^{1/4} n \geq (\theta n)^{1/4}/2.$$

Therefore, as β, ε are positive constants,

$$\frac{1}{\sqrt{\varepsilon\beta c(1-c)\alpha\theta n^2}} = \frac{O(1)}{(\theta n)^{1/8}}.$$

We next consider the main term of (58). For $0 \le x \le 1/2$, the function

$$g(x) = x^x(1-x)^{1-x}$$

satisfies, $g(0) = 1$ and is monotonically decreasing with minimum $g(1/2) = 1/2$.

Since $d(S) \ge 3s$, and $s = cn$, from $d(S) = \alpha\theta n$ we deduce that $c \le \alpha\theta/3$. As $\alpha \le 1/\theta$ then $c \le \alpha\theta/3 \le 1/3$. Therefore $g(c) \ge g(\alpha\theta/3)$, and we can replace c by $\alpha\theta/3$ in (58). Hence

$$\begin{aligned} N(cn,\beta) &= \frac{O(1)}{(\theta n)^{1/8}} \left(\frac{(\alpha\beta)^{\alpha\beta\theta/2}(1-\alpha\beta)^{1-\alpha\beta\theta/2}}{(\alpha\theta/3)^{\alpha\theta/3}(1-\alpha\theta/3)^{1-\alpha\theta/3}} \frac{(1-\alpha\beta)^{\theta/2-1}}{(\varepsilon^\varepsilon\beta^\beta)^{\alpha\theta}} \right)^n \\ &= \frac{O(1)}{(\theta n)^{1/8}} (\phi(\alpha,\beta,\theta))^n. \end{aligned}$$

We next maximize $\phi(\alpha,\beta,\theta)$. Let $h(x,y) = (yx)^x(1-yx)^{1-x}$ for $0 < x, y \le 1$. Considering $h(x,y)$ as a function of y, there is a unique maximum at $y = 1$, given by

$$\begin{aligned} \frac{\partial}{\partial y} \log(h(x,y)) &= x\left(\frac{1}{y} - \frac{1-x}{1-yx}\right) = 0, \\ \frac{\partial^2}{\partial y^2} \log(h(x,y)) &= -x\left(\frac{1}{y^2} + \frac{x(1-x)}{(1-yx)^2}\right) < 0. \end{aligned}$$

Therefore $h(x,y) < h(x,1) = g(x)$. So $h(\alpha\beta\theta/2, 2/\theta) < g(\alpha\beta\theta/2) < g(\alpha\theta/3)$. Hence

$$\phi(\alpha,\beta,\theta) \le \frac{(1-\alpha\beta)^{\theta/2-1}}{(\varepsilon^\varepsilon\beta^\beta)^{\alpha\theta}}.$$

We prove below, that

$$\frac{\partial}{\partial\theta}\left\{\frac{(1-\alpha\beta)^{\theta/2-1}}{(\varepsilon^\varepsilon\beta^\beta)^{\alpha\theta}}\right\} < 0. \tag{59}$$

Since $\theta \ge \delta \ge 3$, we have that

$$\frac{(1-\alpha\beta)^{\theta/2-1}}{(\varepsilon^\varepsilon\beta^\beta)^{\alpha\theta}} \le \frac{e^{-\alpha\beta/2}}{(\varepsilon^\varepsilon\beta^\beta)^{3\alpha}} \le \lambda^\alpha,$$

where $\lambda < 0.7$, provided $\beta \ge 0.99$.

Now since $\alpha\theta n \ge (\theta n)^{1/4}$ for large sets, and $\theta \le n^{1/4}$ by conditions of the lemma, we have that $\alpha n \ge n^{1/16}$. Thus

$$\begin{aligned} N(cn,\beta) &= \frac{O(1)}{(\theta n)^{1/8}} (\phi(\alpha,\beta,\theta))^n \\ &= O(\lambda^{n^{1/16}}). \end{aligned}$$

As $s = cn$ can take at most n values we have that $\sum N(cn, \beta) = O(n\lambda^{n^{1/16}})$.

Proof of (59)**.**

$$\frac{\partial}{\partial\theta}\left\{\frac{(1-\alpha\beta)^{\theta/2-1}}{(\varepsilon^\varepsilon\beta^\beta)^{\alpha\theta}}\right\} = \frac{1}{1-\alpha\beta}\left(\frac{(1-\alpha\beta)^{\frac{1}{2}}}{(\varepsilon^\varepsilon\beta^\beta)^{\alpha}}\right)^{\theta}\log\left(\frac{(1-\alpha\beta)^{\frac{1}{2}}}{(\varepsilon^\varepsilon\beta^\beta)^{\alpha}}\right).$$

Let

$$f(\alpha,\beta) = \frac{(1-\alpha\beta)}{(\varepsilon^\varepsilon\beta^\beta)^{2\alpha}}.$$

When $\alpha = 0$, $f(\alpha,\beta) = 1$. We prove that, for $\beta \geq 0.99$, $f(\alpha,\beta) < 1$ for $\alpha > 0$, which will establish the result. Note that

$$\frac{\partial}{\partial\alpha}\, f(\alpha,\beta) = \frac{-1}{(\varepsilon^\varepsilon\beta^\beta)^{2\alpha}}\left(\beta + (1-\alpha\beta)\log(\varepsilon^\varepsilon\beta^\beta)^2\right). \tag{60}$$

Consider

$$\begin{aligned}\frac{d}{d\beta}\left\{\log(\varepsilon^\varepsilon\beta^\beta)^2 + \beta\right\} &\equiv \frac{d}{d\beta}\left\{\log((1-\beta)^{1-\beta}\beta^\beta)^2 + \beta\right\}\\ &= 2\log\left(\frac{\beta}{1-\beta}\right) + 1.\end{aligned}$$

For $\beta > \frac{1}{2}$, the last line above is positive, and thus $\log(\varepsilon^\varepsilon\beta^\beta)^2 > -\beta$. It follows that (60) is negative, as required.

Case of $1/\theta \leq \alpha \ \leq 1/2$.
Continuing to evaluate $N(s,\beta)$ as before, and referring to $f(S)$ as given by the right hand side term of (57), let

$$A(\alpha) = \frac{(\alpha\beta)^{\alpha\beta}(1-\alpha\beta)^{1-\alpha\beta}}{(\varepsilon^\varepsilon\beta^\beta)^{2\alpha}}.$$

Thus

$$\begin{aligned}\log(A(\alpha)) &= (\alpha\beta)\log((\alpha\beta)) + (1-\alpha\beta)\log(1-\alpha\beta) - 2\alpha\log(\varepsilon^\varepsilon\beta^\beta),\\ \frac{\partial}{\partial\alpha}\log(A(\alpha)) &= \beta\log(\alpha\beta) - \beta\log(1-\alpha\beta) - 2\log(\varepsilon^\varepsilon\beta^\beta).\end{aligned}$$

Setting $\frac{\partial}{\partial\alpha}\log(A(\alpha)) = 0$ gives

$$\alpha = \frac{\varepsilon^{2\varepsilon/\beta}\beta}{1+\varepsilon^{2\varepsilon/\beta}\beta^2}.$$

Let α_0 be the solution to this when $\beta = 0.99$. Thus $\alpha_0 \approx 0.477$. Also,

$$\frac{\partial^2}{\partial\alpha^2}\log(A(\alpha)) = \beta\left(\frac{1}{\alpha} + \frac{\beta}{1-\alpha\beta}\right) > 0$$

hence the stationary point α_0 is a minima. As $\theta \geq 3$ and by inspection, $A(0.5) < A(1/3)$ then $A(\alpha_0) \leq A(1/\theta)$. We can use $\alpha^* = 1/\theta$ as the value of α maximizing $A(\alpha)$ in the range $1/\theta \leq \alpha \leq 1/2$. It follows that

$$\sum_{\substack{S Large\\ \alpha \geq 1/\theta}} f(S) = \left(\frac{1}{\sqrt{\theta n}}\right) 2^n (A(1/\theta))^{\frac{\theta n}{2}}$$

$$= O(1) 2^n \left(\frac{(\beta/\theta)^{\frac{\beta}{2}} (1-\beta/\theta)^{\frac{1}{2}(\theta-\beta)}}{\varepsilon^\varepsilon \beta^\beta}\right)^n .$$

Let

$$T(\theta) = \left(\frac{\beta}{\theta}\right)^\beta \left(1 - \frac{\beta}{\theta}\right)^{\theta-\beta},$$

then

$$\frac{\partial}{\partial \theta} \log(T(\theta)) = \log\left(\frac{\theta-\beta}{\theta}\right).$$

Thus $T(\theta)$ is monotone decreasing in θ, and so $T(\theta) \leq T(3)$. Finally

$$\sum N(s, \beta) \leq O(n) 2^n \left(\frac{(\beta/3)^{\frac{\beta}{2}} (1-\beta/3)^{\frac{1}{2}(3-\beta)}}{\varepsilon^\varepsilon \beta^\beta}\right)^n$$

$$= O\left(n \, (0.8)^n\right).$$

This completes the proof of the lemma. □

6.3 Proof of Lemma 8

For convenience, we restate the lemma.

Lemma 13. *Let $\mathcal{W}_v^*$ denote the walk on G_v starting at v with Γ_v° made into an absorbing state. Let $R_v^* = \sum_{t=0}^{\infty} r_t^*$ where r_t^* is the probability that $\mathcal{W}_v^*$ is at vertex v at time t. There exists a constant $\zeta \in (0,1)$ such that*

$$R_v = R_v^* + O(\zeta^\omega).$$

Proof We bound $|R_v - R_v^*|$ by using

$$R_v - R_v^* = \sum_{t=0}^{\omega}(r_t - r_t^*) + \sum_{t=\omega+1}^{T}(r_t - r_t^*) - \sum_{t=T+1}^{\infty} r_t^*. \tag{61}$$

<u>*Case $t \leq \omega$.*</u> When a particle starting from v is absorbed at Γ_v°, this is at at distance ω from v. Thus for $t < \omega$, $r_t^* = r_t$, and

$$\sum_{t=0}^{\omega}(r_t - r_t^*) = 0. \tag{62}$$

Case $\omega+1 \le t \le T$. Using (20) with $x = u = v$ and $\zeta = (1 - \Phi^2/2) < 1$, we have for $t \ge \omega$, that $r_t = \pi_v + O(\zeta^t)$. Since $\Delta = O(n^a), a < 1$, we have $T\pi_v = o(\zeta^\omega)$ and so

$$\sum_{t=\omega+1}^{T} |r_t - r_t^*| = \sum_{t=\omega+1}^{T} r_t \le \sum_{t=\omega+1}^{T} (\pi_v + \zeta^t) = O(\zeta^\omega). \tag{63}$$

Case $t \ge T+1$. It remains to estimate $\sum_{t=T+1}^{\infty} r_t^*$. We upper bound r_t^* by a probability σ_t as follows. Assume first that G_v is a tree. Consider an unbiased random walk $X_0^{(b)}, X_1^{(b)}, \ldots$ starting at $|b| < a \le \omega$ on the infinite line $(..., -a, ..., -1, 0, 1, ..., a, ...)$. $X_m^{(b)}$ is the sum of m independent ± 1 random variables. The central limit theorem implies that there exists a constant $c > 0$ such that

$$\mathbf{Pr}(|X_{ca^2}^{(0)}| < a) \le e^{-1/2}. \tag{64}$$

Now for any t and b with $|b| < a$, we have

$$\mathbf{Pr}(|X_\tau^{(b)}| < a, \tau = 0, ..., t) \le \mathbf{Pr}(|X_\tau^{(0)}| < a, \tau = 0, ..., t) \tag{65}$$

which is justified with the following game: We have two walks, A and B coupled to each other, with A starting at position 0 and B at position b, which, w.l.o.g, we shall assume is positive. The walk is a simple random walk which comes to a halt when either of the walks hits an absorbing state (that being, $-a$ or a). Since they are coupled, B will win iff they drift $(a-b)$ to the right from 0 and A will win iff they drift $-a$ to the left from 0. Given the symmetry of the walk, B has a higher chance of winning.

For $t > T$, we define σ_t by

$$\sigma_t = \mathbf{Pr}(|X_\tau^{(0)}| < a,\ \tau = 0, 1, \ldots, t) \le \left(e^{-1/2}\right)^{\lfloor t/(ca^2) \rfloor}. \tag{66}$$

The paths from v to Γ_v° in the tree satisfy $a \le \omega$, and so

$$\sum_{t=T+1}^{\infty} \sigma_t \le \sum_{t=T+1}^{\infty} e^{-t/(3c\omega^2)} \le \frac{e^{-T/(3c\omega^2)}}{1 - e^{-1/(3c\omega^2)}} = O(\omega^2 e^{-\Theta(\frac{\log n}{\omega^2})}) = O(\zeta^\omega)$$

We now turn to the case where G_v contains a unique light cycle C. Let x be the furthest vertex of C from v in G_v. This is the only possible place where the random walk is more likely to get closer to v at the next step. We can see this by considering the breadth first construction of G_v. Thus we can compare our walk with random walk on $[-a, a]$ where there is a unique value $x < a$ such that only at $\pm x$ is the walk more likely to move towards the origin and even then this probability is at most 2/3. Using results (64), (65) for the unbiased walk on the line, we have

$$\mathbf{Pr}(\exists \tau \le ca^2 :\ |X_\tau^{(b)}| \ge x) \ge 1 - e^{-1/2}.$$

The probability the particle walks from x to a without returning to the cycle is at least $1/3(a-x)$. Thus

$$\mathbf{Pr}(\exists\tau \le ca^2 :\ |X^{(b)}_{\tau+a-x}| \ge a) \ge (1-e^{-1/2})/3a \ge \frac{13}{100a},$$

and so

$$\sigma_t = \mathbf{Pr}(|X^{(0)}_\tau| < a,\ \tau = 0,1,\ldots,t) \le (1-13/(100a))^{\lfloor t/(2ca^2)\rfloor} \le e^{-t/(20ca^3)}. \tag{67}$$

As $a \le \omega$,

$$\sum_{t=T+1}^{\infty} \sigma_t \le \sum_{t=T+1}^{\infty} e^{-t/(20c\omega^3)} \le \frac{e^{-T/(20c\omega^3)}}{1-e^{-1/(20c\omega^3)}} = O\left(\omega^3 e^{-O\left(\frac{\log n}{\omega^3}\right)}\right) = O(\zeta^\omega)$$

□

6.4 Condition (a) of Lemma 2

Lemma 14. *For $|z| \le 1+\lambda$, there exists a constant $\psi > 0$ such that $|R_T(z)| \ge \psi$.*

Proof As in Lemma 8, we consider the walk $\mathcal{W}^*_v$ on G_v, starting from v, and with absorption at Γ°_v. For this walk, let β_t be the probability of a first return to v at step t, and let r^*_t be the probability of a return to v at step t.

Let $\beta(z) = \sum_{t=1}^{T} \beta_t z^t$, let $\alpha(z) = 1/(1-\beta(z))$, and write $\alpha(z) = \sum_{t=0}^{\infty} \alpha_t z^t$. Thus α_t is the probability of a return to v at time t for a walk $\mathcal{W}^\dagger_v$, all of whose excursions from v are length at most T. Observe that $\alpha_t \le r^*_t \le r_t$. We shall prove below that the radius of convergence of $\alpha(z)$ is at least $1+\Omega(1/\omega^3)$.

We can write

$$\begin{aligned} R_T(z) &= \alpha(z) + Q(z) \\ &= \frac{1}{1-\beta(z)} + Q(z), \end{aligned} \tag{68}$$

where $Q(z) = Q_1(z) + Q_2(z)$, and

$$\begin{aligned} Q_1(z) &= \sum_{t=0}^{T} (r_t - \alpha_t) z^t \\ Q_2(z) &= -\sum_{t=T+1}^{\infty} \alpha_t z^t. \end{aligned}$$

We note that $Q(0) = 0$, $\alpha(0) = 1$ and $\beta(0) = 0$.

We will show below that
$$|Q_2(z)| = o(1) \tag{69}$$
for $|z| \leq 1 + 2\lambda$ and thus the radius of convergence of $Q_2(z)$ (and hence $\alpha(z)$) is greater than $1 + \lambda$. This will imply that $|\beta(z)| < 1$ for $|z| \leq 1 + \lambda$, so that the expression (68) is well defined. For suppose there exists z_0 such that $|\beta(z_0)| \geq 1$. Then $\beta(|z_0|) \geq |\beta(z_0)| \geq 1$ and we can assume (by scaling) that $\beta(|z_0|) = 1$. We have $\beta(0) < 1$ and so we can assume that $\beta(|z|) < 1$ for $0 \leq |z| < |z_0|$. But as ρ approaches 1 from below, (68) is valid for $z = \rho|z_0|$ and then $|R_T(\rho|z_0|)| \to \infty$, contradiction.

Recall that $\lambda = 1/KT$. Clearly $\beta(1) \leq 1$ and so for $|z| \leq 1 + \lambda$
$$\beta(|z|) \leq \beta(1 + \lambda) \leq \beta(1)(1 + \lambda)^T \leq e^{1/K}.$$
Using $|1/(1 - \beta(z))| \geq 1/(1 + \beta(|z|))$ we obtain
$$|R_T(z)| \geq \frac{1}{1 + \beta(|z|)} - |Q(z)| \geq \frac{1}{1 + e^{1/K}} - |Q(z)|. \tag{70}$$
We now prove that $|Q(z)| = o(1)$ for $|z| \leq 1 + \lambda$ and the lemma will follow.

Turning our attention first to $Q_1(z)$, we have
$$|Q_1(z)| \leq (1 + \lambda)^T |Q_1(1)| \leq e^{2/K} \sum_{t=0}^{T} |r_t - \alpha_t| \tag{71}$$
From (62), (63) of the proof of Lemma 8, we see that $\sum_{t=0}^{T} |r_t - \alpha_t| = o(1)$, hence $|Q_1(z)| = o(1)$.

We now consider $Q_2(z)$. As in Lemma 8, let r_t^* be the probability that a walk $\mathcal{W}_v^*$ on G_v starting at v has not been absorbed at Γ_v° by step t. Then $\alpha_t \leq r_t^* \leq \sigma_t$, so
$$|Q_2(z)| \leq \sum_{t=T+1}^{\infty} \sigma_t |z|^t,$$
In the case where G_v is a tree we can use (66) to prove that the radius of convergence of $Q_2(z)$ is at least $e^{1/(3c\omega^2)} > 1 + 1/(3c\omega^2) > 1 + 2\lambda$, where $\omega = \log\log\log n$ is given in (10), and $\lambda = O(1/\log n)$. So for $|z| \leq 1 + \lambda$,
$$|Q_2(z)| \leq \sum_{t=T+1}^{\infty} e^{\lambda t - t/(3c\omega^2)} = o(1).$$

In the case that G_v contains a unique cycle, we can use (67) to see that the radius of convergence of $Q_2(z)$ is at least $e^{\frac{1}{20c\omega^3}} > 1 + 2\lambda$. So for $|z| \leq 1 + \lambda$,

$$|Q_2(z)| \leq \sum_{t=T+1}^{\infty} e^{\lambda t - t/(20c\omega^3)} = o(1).$$

□