Edge disjoint Hamilton cycles in sparse random graphs of minimum degree at least k

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Abstract

Let $G_{n,m,k}$ denote the space of simple graphs with n vertices, m edges and minimum degree at least k, each graph G being equiprobable. Let G have property $\mathcal{A}_k$ if G contains $\lfloor (k-1)/2 \rfloor$ edge disjoint Hamilton cycles, and, if k is even, a further edge disjoint matching of size $\lfloor n/2 \rfloor$. For $k \geq 3$, $\mathcal{A}_k$ occurs in $G_{n,m,k}$ with probability tending to 1 as $n \to \infty$, when $2m \geq c_k n$ for some suitable constant c_k.

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1 Introduction

Denote by $G_{n,m,k}$ the space of simple graphs with vertex set $[n] = \{1, 2, ..., n\}$, m edges and minimum degree at least k, each graph being equiprobable. We say a graph G has property $\mathcal{A}_k$, if G contains $\lfloor (k-1)/2 \rfloor$ edge disjoint Hamilton cycles, and, if k is even, a further edge disjoint matching of size $\lfloor n/2 \rfloor$. We prove the following theorem.

Theorem 1 *Let $k \geq 3$. There exists a constant $c_k \leq 2(k+1)^3$ such that if $2m \geq c_k n$ then* **whp** [1] *any $G \in G_{n,m,k}$ has property $\mathcal{A}_k$.*

In the paper *Hamilton cycles in random graphs of minimal degree at least k* [BFF], Bollobás, Fenner and Frieze establish a sharp threshold for a stronger property $\mathcal{A}_k^*$, in the case where $m/n \to \infty$. A graph G has property $\mathcal{A}_k^*$, if G contains $\lfloor k/2 \rfloor$ edge disjoint Hamilton cycles, and, if k is odd, a further edge disjoint matching of size $\lfloor n/2 \rfloor$.

Theorem 2 (BFF) *Let $2m = n\left(\frac{\log n}{k+1} + k \log\log n + d_n\right)$. If $G \in G_{n,m,k}$, then*

$$\lim_{n\to\infty} \mathbf{Pr}(G \in \mathcal{A}_k^*) = \begin{cases} 0 & \text{if } d_n \to -\infty, \qquad\qquad \text{slowly} \\ e^{-\theta_k(d)} & \text{if } d_n \to d, \\ 1 & \text{if } d_n \to +\infty, \end{cases}$$

were

$$\theta_k(d) = \frac{e^{-(k+1)d}}{(k+1)!\{(k-1)!\}^{k+1}(k+1)^{k(k+1)}}.$$

The primary obstruction to property $\mathcal{A}_k^*$ in $G_{n,m,k}$ is the presence of *k-spiders*. A k-spider is $k+1$ vertices of degree k having a common neighbour. In [BFF] it was shown that

$$\lim_{n\to\infty} \mathbf{Pr}(G \in G_{n,m,k} \text{ has a } k\text{-spider }) = 1 - e^{-\theta_k(d)}.$$

When $2m = cn$, where c is constant, k-spiders occur **whp** in $G_{n,m,k}$. Thus **whp** the property $\mathcal{A}_k^*$ does not occur, and property $\mathcal{A}_k$ is best possible.

2 Models for the space $G_{n,m,k}$

For any graph G in $G_{n,m,k}$ there are $m!$ ways to order, and 2^m ways to orient the edges, to give $2^m m!$ sequences of vertex labels of length $2m$. The set $\mathcal{S}(n, 2m, k)$ of sequences arising in this manner has the uniform measure induced by $G_{n,m,k}$. Let $\mathcal{M}(n, 2m, k)$ be

[1] **whp** *with high probability.* With probability tending to 1 as $n \to \infty$.

the space of equiprobable sequences $(a_i : a_i \in [n], i = 1, ..., 2m)$ specifying which of n labelled boxes contains each of $2m$ labelled balls; with the condition that the minimum occupancy of any box is at least k. Each element of $\mathcal{M}$ defines a multigraph with vertex set $[n]$. The set $\mathcal{S}$ is the subset of $\mathcal{M}$ whose underlying graphs are simple.

Denote by $\mathcal{O}(n, 2m, k)$ the space of sequences $(b_j : b_j \in [2m], j = 1, ..., n)$, giving the possible occupancies of the n boxes (the degrees of the vertices) arising from sequences in $\mathcal{M}(n, 2m, k)$, with the derived probability measure. A useful method of obtaining results about $\mathcal{O}(n, 2m, k)$ is to consider a larger space $\mathcal{P}(n, \lambda, k)$ in which each of the n boxes has independent occupancy given by a truncated Poisson random variable X with parameters λ and k. The space $\mathcal{O}(n, 2m, k)$ is obtained from $\mathcal{P}(n, \lambda, k)$ by conditioning on $\sum_{i=1}^n X_i = 2m$, as is explained in [BFU]. Let

$$\mathbf{Pr}(X = j) = \frac{\lambda^j}{j!} \frac{e^{-\lambda}}{\beta(\lambda, k)} \qquad j \in \{k, k+1, ...\}$$

where

$$\beta(\lambda, k) = 1 - e^{-\lambda} \left(1 + \lambda + \cdots + \tfrac{\lambda^{k-1}}{k-1!}\right).$$

It is natural to choose λ so as to maximize the probability of the conditioning event $\{\sum_i X_i = 2m\}$. This can be approximately achieved by making $\mathbf{E}X$ equal to c, the average vertex degree. Explicitly, we choose

$$\lambda \frac{\beta(\lambda, k-1)}{\beta(\lambda, k)} = c, \qquad (1)$$

so that $\mathbf{E}X = c$. The properties of (1) are given in Lemma A1 of the appendix.

If we consider an event $\mathcal{E}$ in $G_{n,m,k}$ which specifies the degree sequence $\mathbf{d}(S)$ of some subset S of vertices, the set of multigraph sequences with $\mathbf{d}(S)$ in $\mathcal{M}(n, 2m, k)$ is well defined and corresponds to well defined events in $\mathcal{O}(n, 2m, k)$ and $\mathcal{P}(n, \lambda, k)$. For such an *occupancy event*, $\mathcal{E}$, we can regard $\mathcal{E}$ as being defined in each of the spaces under consideration, rather than just in $G_{n,m,k}$.

Lemma 3 *Let k be a fixed positive integer, and let $m = cn/2$, where $c \geq k$. Let $\mathcal{E}$ be an occupancy event, then for sufficiently large n,*

(i)

$$\mathbf{Pr}(\mathcal{E}; \mathcal{O}(n, 2m, k)) \leq (1 + o(1)) \sqrt{2\pi cn}\, \mathbf{Pr}(\mathcal{E}; \mathcal{P}(n, \lambda, k))$$

(ii)

$$\mathbf{Pr}(\mathcal{E}; G_{n,m,k}) \leq e^{O(c^2)}\, \mathbf{Pr}(\mathcal{E}; \mathcal{O}(n, 2m, k)).$$

For convenience we denote generic expressions of the form $O(\sqrt{c})\ e^{O(c^2)}$ by $A(c)$.

We will frequently use the model $\mathcal{P}(n,\lambda,k)$ to estimate the probability of an event $\mathcal{E}$, that a set S of vertices of a graph G in $G_{n,m,k}$ has degree sum T. In the model $\mathcal{P}(n,\lambda,k)$, the probability that a set S of s boxes (with occupancy $t_i : i = 1, ..., s$) has total occupancy T is

$$\begin{aligned}
\mathbf{Pr}(t_1 + ... + t_s = T; \mathcal{P}(n,\lambda,k)\ ) &= \sum_{\substack{t_1+...+t_s=T \\ t_i \geq k}} \left(\prod_{i=1}^{s} \frac{\lambda^{t_i}}{t_i!} \frac{e^{-\lambda}}{\beta(\lambda,k)} \right) \\
&= \frac{\lambda^T e^{-\lambda s}}{[\beta(\lambda,k)]^s} \frac{1}{T!} \sum_{\substack{t_1+...+t_s=T \\ t_i \geq k}} \binom{T}{t_1 ... t_s} \\
&\leq \frac{(\lambda s)^T}{T!} \frac{e^{-\lambda s}}{[\beta(\lambda,k)]^s}. \qquad (2)
\end{aligned}$$

Lemma 4 *Let k be a fixed positive integer, and let $2m = cn$ where $c \geq k$. For sufficiently large n,*

(i)

$$\frac{|G_{n,m,k}\ |}{|G_{n,m}|} \geq e^{-O(c^2)} \ \left[\left(\frac{c}{\lambda e} \right)^c e^{\lambda} \beta(\lambda,k) \right]^n .$$

(ii) Let $D = \frac{kc^{k-1}}{(k-1)!} e^{-(c-k)} e^{k^2/(c-k)}$, then

$$\mathbf{Pr}(\delta(G) \geq k; G_{n,m}) \geq e^{-nD}.$$

The proofs of Lemmas 3, 4 are provided in the appendix.

Although $|G_{n,m,k}\ |$ is bounded below by an exponentially small multiple of $|G_{n,m}|$, the constant D in the exponent also is small. For example, when $c = c_k,\ \ k = 3$, then $D = 1.4 \times 10^{-50}$. In particular, note that if $c \geq c_k$ then

$$D \leq e^{-8c/9}. \qquad (3)$$

This allows us to prove results concerning $G_{n,m,k}$ directly in $G_{n,m}$ in Lemma 8 by using the general estimate

$$\mathbf{Pr}(\mathcal{E};\ G_{n,m,k}) = \mathbf{Pr}(\mathcal{E} \mid \delta(G) \geq k) \leq \frac{\mathbf{Pr}(\mathcal{E};\ G_{n,m})}{\mathbf{Pr}(\delta(G) \geq k;\ G_{n,m})}. \qquad (4)$$

We will frequently use the following crude upper bound for the conditional probability that a sequence in $\mathcal{M}(n, 2m, k)$ corresponds to a multigraph in which there are q edges between A and B given that the sets of vertices A, B have degree a, b respectively, namely

$$\binom{m}{q}\left(\frac{a}{2m}\frac{b}{2m}\right)^q 2^{q\delta(A,B)} \tag{5}$$

where $\delta(A, B) = 0$ if $A = B$ and 1 if A and B are disjoint. We note that $\binom{m}{q} \le (me/q)^q$ and that the unconstrained maximum of $(me/q)^q$ with respect to q occurs at $q = me$.

3 Relevant properties of $G_{n,m,k}$

Let $P_{y_h} = y_0y_1...y_h$ be a longest path starting at y_0 (y_0 path) in G. A *Pósa rotation* $P_{y_h} \to P_{y_{i+1}}$, [Po],[Bo] gives the path $P_{y_{i+1}} = y_0y_1...y_iy_hy_{h-1}...y_{i+1}$ formed from Py_h by adding edge y_hy_i and erasing edge y_iy_{i+1}. We call y_hy_i, y_iy_{i+1} the *transformation edges* and $y_{i+1}y_{i+2}$ the *adjacent edge* of the rotation.

Pósa rotations of a longest x_0 path $Px_h = x_0Px_h$ with x_0 fixed, define a *rotation subgraph* $\mathcal{R} = \mathcal{R}(x_0)$ of G, as follows. Initially $\mathcal{R} = x_{h-1}x_h$, where x_h is an *active* endpoint. Perform all possible rotations based on x_h due to edges x_hx_i, adding the transformation and adjacent edges of each rotation to $\mathcal{R}$. Each x_{i+1} is now an active endpoint, whereas x_h is now *passive*. The final graph $\mathcal{R}^*$ is not necessarily unique, but we consider it to be fully explored, in the following sense. Let x be an active endpoint with $P_x = x_0...yz...x$ and where there is an edge xy in G. If z is a passive endpoint we add xy to $\mathcal{R}$ and consider this a transformation edge, else we add the transformation and adjacent edges corresponding to the rotation $P_x \to P_z$.
At any stage, we define a subgraph $\mathcal{T}$ of $\mathcal{R}$ which includes only transformation edges, where initially, $\mathcal{T} = x_h$.

Lemma 5 *Let G be a non-Hamiltonian graph of minimum degree at least 3 and let P_0 be a longest path in G. One of the following two possibilities must hold:*

(a) *G is connected and the rotation subgraph $\mathcal{R}$ of P_0 contains at least two cycles.*

(b) *G is not connected and the vertices of P_0 form a single component.*

Proof We consider a final graph $\mathcal{R} = \mathcal{R}^*$. Let U be the set of endpoints of P_0 obtained while constructing $\mathcal{R}^*$. We note that in $\mathcal{R}$ each vertex of U is on a cycle. This follows because $U \subseteq V(\mathcal{T})$ and $\mathcal{T}$ has minimum degree at least 2.

Suppose that $\mathcal{T}$ consists of exactly one cycle C. Note that $U \subseteq V(C)$ and every edge of C is incident with a vertex of U. There are two cases to consider.

Case I. $N(U) \cap U = \emptyset$. The vertices of U alternate with vertices of $N(U)$ on C.
Thus the cycle C is of length $2|U|$. Each vertex of U in C has at least one further vertex of $N(U)$ attached as a pendant leaf of C. This follows as the minimum degree in $\mathcal{R}$ of any vertex in U is at least 3, but there is only one cycle. Thus $|U \cup N(U)| \geq 3|U|$, contradicting the Pósa condition that $|N(U)| \leq 2|U| - 1$.

Case II. Two vertices of U are adjacent on C.
We claim that every vertex on the cycle is an endpoint. We proceed inductively.
Orient C, and let $(u,v) \in C,\ u,v \in U$. Consider P_u. If $P_u = x_0...vx...u$, then $x \in U$. Hence $(v,x) \in C$ and $v,x \in U$. If $P_u = x_0...vu$ then $P_u = x_0...a_1b_1...a_2b_2...vu$ where there are edges $\{a_1,u\},\ \{a_2,u\}$ and paths $ua_1b_1,\ ua_2b_2$. At best $C = ...b_1a_1uv...$, and there is a chordal path ua_2b_2 of C as $b_2 \in U$. This contradicts the unicyclicity assumption.

Suppose now that $U = V(C)$. Consider the initial longest path $P_0 = x_0x_1...x_h$. Let $b \in U$ be the first occurrence of an endpoint vertex in P_0 beyond x_0. Thus either $P_0 = x_0b...x_h$ or $P_0 = x_0Qab...x_h$.

In the first case there exists a sequence of transformations $P_{x_h}, ..., P_w$ such that x_0w is an edge of G. This implies there is a cycle through the vertices of P_0. As we suppose G is non-Hamiltonian, G cannot be connected else P_0 is not a longest path. Thus (b) holds here.

In the second case there must be an endpoint w, an edge wa and a transformation $P_w \to P_b$ using wa. As x_0Qa, the initial segment of P_0 is never broken, the vertex $a \notin C$. We conclude that $\mathcal{R}$ is not unicyclic as wab is a chordal path of C. $\square$

Lemma 6 *Let $s_0 = n\ (3/c)^3 e^{-(2k+12)}$ and $S \subset [n],\ |S| < s_0$. There exists a c_k such that for $c \geq c_k$* **whp** *no $G \in G_{n,m,k}$ has any subset of vertices S which induces at least $3|S|/2$ edges.*

Proof Let $|S| = s$, fix the degree sum T of S. The expected number of vertex sets S inducing at least $3s/2$ edges is at most

$$\gamma(s) = A(c)\sqrt{n}\binom{n}{s} \sum_{T \geq ks} \frac{(\lambda s)^T}{T!} \frac{e^{-\lambda s}}{[\beta(\lambda,k)]^s} \binom{m}{3s/2} \left(\frac{T}{2m}\right)^{3s},$$

where the right hand side of this expression follows from (2) and (5) and Lemma 3. Thus, as $c - k \leq \lambda \leq c$ we can apply (10) as follows.

$$\gamma(s) \leq A(c)\sqrt{n}\left(\frac{ne}{s}\right)^s \frac{e^{ks}}{[\beta(\lambda,k)]^s} \left(\frac{e}{3scn}\right)^{3s/2} \left[e^{-cs} \sum_{T \geq ks} \frac{(cs)^T}{T!} T^{3s}\right]$$

$$\leq A(c)\sqrt{n}\frac{e^{(k+1)s}}{[\beta(\lambda,k)]^s}\left(\frac{n}{s}\right)^s\left(\frac{e}{3scn}\right)^{3s/2}(cse^{3/k})^{3s}$$
$$\leq A(c)\sqrt{n}\left(\frac{s}{n}\frac{c^3}{3^3}e^{(2k+5+18/k+2k^2/(c-k))}\right)^{s/2},$$

where any constants have been absorbed into $A(c)$. Provided $k \geq 3$ and $c > 2k^2 + k$,

$$\sum_{4\leq s\leq s_0}\gamma(s) = o\left(\frac{1}{n}\right)$$

□

Lemma 7 *Let $S \subseteq [n]$, $|S| = s$, $1 \leq s \leq n$. There exists a constant c_k such that if $c \geq c_k$ then* **whp** *no $G \in G_{n,m,k}$ has any set of vertices S such that*

(i) the degree sum T of S satisfies $ks \leq T \leq cs/4k$,

(ii) $|N(S)| \leq k|S|$,

(iii) the subgraph F consisting of the edges induced by S and by $S \times N(S)$, is connected and contains at least two cycles.

Proof Let the pair $(S, N(S))$ satisfy the conditions of (i),(ii) and (iii). Let $T = 2q+p$ where there are q edges induced by S and p edges induced by $S \times N(S)$. Denote $|N(S)|$ by r. Let R be the degree sum of $N(S)$, where $R \geq \max\{p, kr\}$. Because F is connected and contains at least two cycles we have that

$$p + q \geq r + s + 1,$$

and thus

$$T \geq q + r + s + 1.$$

For fixed T, r, s the expected number of pairs $(S, N(S))$ satisfying conditions (i),(ii),(iii) is at most $\gamma(r, s, T)$ where from (5),

$$\gamma(r,s,T) = A(c)\sqrt{n}\binom{n}{s}\binom{n}{r}\frac{(\lambda s)^T}{T!}\frac{e^{-\lambda s}}{[\beta(\lambda,k)]^s}\left(\sum_{R\geq p}\frac{(\lambda r)^R}{R!}\frac{e^{-\lambda r}}{[\beta(\lambda,k)]^r}\right)$$
$$\times \binom{m}{p}\binom{m}{q}\left[\left(\frac{T}{2m}\right)\left(\frac{T}{2m}\right)\right]^q\left[\left(\frac{T}{2m}\right)\left(\frac{R}{2m}\right)\right]^p.$$

Thus, as $c - k \leq \lambda \leq c$ an upper bound on $\gamma(r, s, T)$ is given by

$$\frac{A(c)\sqrt{n}}{[\beta(\lambda,k)]^s}\left(\frac{s}{n}\right)^{T-(r+s+q)}e^{(k+1)s-cs+T}\left(\frac{ces}{2q}\right)^q\left(\frac{e}{p}\right)^p\left(\frac{se^{k+1}}{\beta(\lambda,k)\,r}\right)^r\left[e^{-cr}\sum_{R\geq p}\frac{(cr)^R}{R!}R^p\right].$$

From (10) of Lemma A3,

$$e^{-cr}\sum_{R\geq p}\frac{(cr)^R}{R!}R^p \leq 2(cre)^p,$$

which is monotone increasing in r. The function $(se^{k+1}/(\beta(\lambda,k)\ r))^r$ is monotone increasing in $r=|N(S)|$ over the range $0\leq |N(S)|\leq k|S|$, and the constrained maximum occurs at $r=ks$ giving

$$\gamma(r,s,T)\leq \frac{A(c)}{[\beta(\lambda,k)]^s}(\frac{s}{\sqrt{n}})\frac{e^{(k+1)^2 s}}{k^{ks}}e^{-cs}e^T\left(\frac{ces}{2q}\right)^q\left(\frac{kce^2s}{p}\right)^p.$$

Now, $p=T-2q$ and $1-x>\exp\{-x/(1-x)\}$ so, $p^{-p}\leq T^{-T}(eT)^{2q}$ and thus

$$\begin{aligned}
e^T\left(\frac{ces}{2q}\right)^q\left(\frac{kce^2s}{p}\right)^p &\leq \left(\frac{kce^3s}{T}\right)^T\left(\frac{T^2}{2k^2ce^2s}\frac{e}{q}\right)^q\\
&\leq \exp\left\{\frac{T^2}{2k^2ce^2s}\right\}\left(kce^2s\frac{e}{T}\right)^T\\
&\leq \left[\exp\left\{\frac{1}{8k^3e^2}\right\}(4k^2e^3)\right]^{\frac{cs}{4k}}.
\end{aligned}$$

Thus provided $k\geq 3$ and $c\geq 2(k+1)^3$ say,

$$\begin{aligned}
\sum_{r,s,T}\gamma(r,s,T) &\leq \frac{cA(c)}{\sqrt{n}}\sum_{s=1}^n s^3\left[\frac{(5k^2e^3)^{c/4k}}{k^k}e^{-c+(k+1)^2+\frac{k^2}{c-k}}\right]^s\\
&\leq \frac{cA(c)}{\sqrt{n}}\sum_{s=1}^n s^3e^{-cs/5}.
\end{aligned}$$

□

Lemma 8 *Let* $\alpha=\frac{1}{\sqrt{2(k+1)}}$. *Let* $s_0=n(3/c)^3\exp\{-(2k+12)\},\ s_1=\alpha n$.

Let $S\subset[n],\ |S|=s, s_0/(k+1)\leq s\leq s_1$.
If $c\geq c_k$ *then* **whp** *no* $G\in G_{n,m,k}$ *has any* S *such that* $|N(S)|\leq k|S|$.

Proof Let $\mathcal{E}$ denote the event that there exists a set S, such that $|N(S)|\leq k|S|$.

We work in $G_{n,p}$, where $p=c/n$. We assume that $\Gamma_S=G[S\cup N(S)]$ is connected or we can consider each connected subgraph separately. There are $(k+1)s-1$ edges in a spanning tree of Γ_S and suppose there are q other edges incident with S in Γ_S. The expected number $\gamma(q)$ of such Γ_S is at most,

$$\gamma(q)\ \leq\ \binom{n}{s}\binom{n-s}{ks}((k+1)s)^{(k+1)s-2}\binom{\binom{s}{2}+ks^2}{q}$$

$$
\begin{aligned}
&\times \quad \left(\frac{c}{n}\right)^{q+(k+1)s-1}\left(1-\frac{c}{n}\right)^{s(n-(k+1)s)+[\binom{s}{2}+ks^2-(q+(k+1)s-1)]} \\
&\leq \quad A(c)\frac{n}{s^2}\left(\frac{c^{k+1}e^{k+1}(k+1)^{k+1}e^{-c}}{k^k}\right)^s\left(\frac{s^2(k+\frac{1}{2})e^{1+c/n}c}{nq}\right)^q e^{(c-2k)s^2/(2n)}.
\end{aligned}
$$

Now, $(xe/q)^q \leq e^x$, so

$$
\begin{aligned}
\max_q \gamma(q) &\leq A(c)\frac{n}{s^2}\left(\frac{c^{k+1}e^{k+1}(k+1)^{k+1}}{k^k}e^{-c}e^{\frac{(k+1)cs}{n}}\right)^s \\
&\leq A(c)\exp\left\{-cs\left(\left(1-\frac{1}{c}\log\frac{c^{k+1}e^{k+1}(k+1)^{k+1}}{k^k}\right)-\frac{(k+1)s}{n}\right)\right\} \qquad (6) \\
&\leq A(c)e^{-2Dn}.
\end{aligned}
$$

Provided $\lambda < 1$ the function $(1-\lambda)s - \frac{(k+1)s^2}{n}$ has an unconstrained maximum at $s = \frac{(1-\lambda)n}{2(k+1)}$. For the value of λ given in (6) above, this is in the range (s_0, s_1). The right hand side of (6) is minimized at s_0, and the final inequality follows from (3). Thus from Lemmas 3 and 4, and (4)

$$\mathbf{Pr}(\mathcal{E}; G_{n,m,k}) \leq \sqrt{n}e^{-Dn}$$

□

Lemma 9 *For $c \geq c_k$, $G \in G_{n,m,k}$ satisfies the following conditions* **whp**.

(i) If $S \subset [n]$, $\alpha n \leq |S| \leq (1-\alpha)n$ then there are at least $4k|S|$ edges from S to $[n]-S$,

(ii) If $L(3c) = \{e \in E(G) : e$ is incident with a vertex of degree at least $3c\}$ then $|L(3c)| \leq ne^{-c/6}$

(iii) Let $L(k) = \{e \in E(G) : e$ is incident with a vertex of degree $k\}$ then $|L(k)| \leq ne^{-c/6}$.

Proof In all three cases we estimate the probability of failure in $G_{n,m}$ or $G_{n,p}, p = c/n$ and show it is at most e^{-2nD}. This estimate can then be inflated by $O(n^{1/2}e^{nD})$ as was done in Lemma 8.

(i) We assume w.l.o.g. that $s = |S| \leq n/2$. In $G_{n,p}$ the number of edges between S and $[n]-S$ has binomial distribution $B(s(n-s),p)$. This has mean $s(n-s)p$ and if $4ks = (1-\epsilon)s(n-s)p$ then $\epsilon \geq 2/3$ (using $c \geq 2(k+1)^3$ and $s \leq n/2$.) Applying the Chernoff-Hoeffding bound

$$\mathbf{Pr}(B(N,\theta) \leq (1-\epsilon)N\theta) \leq e^{-\epsilon^2 N\theta/3}$$

we obtain

$$\begin{aligned}\mathbf{Pr}(\text{(i) fails in } G_{n,p}) &\leq \sum_{s=\alpha n}^{n/2} \binom{n}{s} e^{-sc/9} \\ &\leq \sum_{s=\alpha n}^{n/2} \left(\frac{ne}{s} e^{-c/9}\right)^s \\ &\leq n(e\sqrt{2}(k+1)e^{-2(k+1)^3/9})^{n/(\sqrt{2}(k+1))} \\ &\leq e^{-2nD}.\end{aligned}$$

(ii) The probability that an edge $e \in G_{n,m}$ is incident with a vertex of degree at least $3c$ is at most 3^{-c}. Thus $\mathbf{E}(|L(3c)|) \leq 3^{-c}n$ in $G_{n,m}$. Changing one edge of $G_{n,m}$ changes $|L(3c)|$ by at most $12c$. Applying the Azuma-Hoeffding martingale tail inequality we get

$$\begin{aligned}\mathbf{Pr}(\text{(ii) fails in } G_{n,m}) &\leq \exp\{-n(e^{-c/6} - 3^{-c})^2/(144c^2)\} \\ &\leq \exp -2nD.\end{aligned}$$

(iii) The probability that edge $e \in G_{n,m}$ is incident with a vertex of degree k is at most $\exp -2c/3$. We proceed as in (ii). □

4 The proof of Theorem 1

We will write k as $2l+1$ if k is odd, and as $k = 2l+2$ otherwise. The property $\mathcal{A}_k$ requires the existence of l edge disjoint Hamilton cycles $H_i : i = 1, ..., l$, and, if k is even, a further edge disjoint (near) perfect matching H_0.

We prove the existence **whp** of these structures in a sequential manner. Initially $i = 0$, and the set of excluded edges $\mathcal{Q}(0)$ is empty. If k is even, we first prove **whp** the existence of a Hamilton cycle by the methods described below, and use the edges of this cycle to obtain a (near) perfect matching H_0. The edges of the matching H_0 are added to $\mathcal{Q}$. At the start of iteration $i = 1, ..., l$, the set $\mathcal{Q}(i)$ contains those edges to be excluded from the cycle H_i, by virtue of appearing in $H_0, ..., H_{i-1}$.Thus

$$\mathcal{Q}(i) = \cup_{j=0}^{i-1} E(H_j).$$

To prove the existence of H_i, we follow the method of Fenner and Frieze [FF]. A set $\mathcal{T}$ of edges of G is said to be *deletable* if

D(a) $\mathcal{T}$ is not incident with any vertex of degree k or degree at least $3c$,

D(b) $\mathcal{T}$ avoids a specified longest path $P_0 = x_0 P x_h$ in $G - \mathcal{Q}$, D(c) $\mathcal{T}$ avoids the specified set $\mathcal{Q}$,

D(d) $\mathcal{T}$ is a matching.

Let $\mathcal{N}(G)$ be the set of edges of G which $\mathcal{T}$ must avoid in order to satisfy the conditions D(a),(b),(c) above and let $H = G - \mathcal{T} - \mathcal{Q}(i)$. Let $END(x_0; H)$ be a rotation endpoint set of the fixed longest path $x_0 P x_h$ in the subgraph H, and $\mathcal{R}(x_0)$ the associated rotation subgraph.

Lemma 10 *Let $\mathcal{B}$ be the subset of graphs in $G_{n,m,k}$ which satisfy the conditions of Lemma's 6, 7, 8 and 9, then*

(i) $|\mathcal{B}| = (1 - o(1))|G_{n,m,k}|$,

(ii) If $G \in \mathcal{B}$ *then* $|END(x_0; H)| \geq \alpha n$,

(iii) $G \in \mathcal{B}$ *implies* H *is connected.*

Proof (i) This is a consequence of Lemmas 6-9.

(ii) Let $END(x_0; H) = S$. We assume $|S| < \alpha n$.

At the start of iteration $i = 0, 1, ..., l$ the degree in $\mathcal{Q}(i)$ of any vertex $v \in [n]$ is $2(i - 1) + 1\{i \geq \delta \text{ and } k = 2l + 2\}$. We note that $|N_G(S)| < k|S|$, for if $|N_G(S)| \geq k|S|$ then $|N_{G-\mathcal{Q}(i)}(S)| \geq 3|S|$. We delete at most a matching from S in $G - \mathcal{Q}$, so this implies that $|N_H(S)| \geq 2|S|$ in contradiction of the Pósa condition. By Lemma 8, $|S| \leq s_0/(k + 1)$.

If (b) of Lemma 5 holds then $S = P_0$ and S induces a connected subgraph of minimum degree at least 3. This contradicts the condition of Lemma 6.

Assume now that (a) of Lemma 5 holds. As $\mathcal{R}(x_0)$ is a connected graph it satisfies Lemma 7(iii), and $|N(S)| < ks$ satisfies 7(ii). Thus we conclude that T the degree of S satisfies $T > cs/4k$. Let $|N(S)| = \theta s < ks$. By Lemma 6, the total number of edges in $S \cup N(S)$ satisfies

$$p + q \leq \frac{3}{2}(1 + \theta)s,$$

and the total number of edges in S satisfies

$$q \leq \frac{3}{2}s.$$

A simple optimization shows that $T = 2q + p$ is maximized at $(p, q) = (3\theta s/2, 3s/2)$. Thus $T \leq 3s + \frac{3}{2}\theta s$ so that $T < \frac{3}{2}(k+2)s$. However, as $T > \frac{cs}{4k}$, this implies $c < 6k(k+2)$, and contradicts the assumption that $c \geq c_k$.

(iii) Starting with any vertex v, we see that both $\mathcal{R}(v)$ and P_0, are connected subgraphs of H, containing at least αn vertices by (ii). The connectivity of H follows from Lemma 9. □

Let $\mathcal{E}$ be the subset of $G_{n,m,k}$ which does not have property $\mathcal{A}_k$. We will apply the edge colouring argument of Fenner and Frieze [FF] in an inductive manner to the set $\mathcal{E} \cap \mathcal{B}$ to prove that $\mathbf{Pr}(\mathcal{E}) \to 0$

Let $\mathcal{T}$ be a deletable set of edges of G of size $t = \lceil \log n \rceil$ avoiding the set $\mathcal{N}(G)$ of size s, and let αn be a lower bound on $|END(x_0)|$. By transforming P_0 in H using Pósa rotations there are at least $(\alpha n)^2/2$ longest paths aPb in H with distinct endpoint sets $\{a, b\}$. Thus if $G \in \mathcal{E}$ at least $(\alpha n)^2/2$ non-edges must be avoided when replacing a subgraph $\mathcal{T}'$, to form a graph $G' = G - \mathcal{T} + \mathcal{T}'$, $G' \in \mathcal{E}$. We will call such a replacement subgraph *addable*.

Let η be a lower bound on the number of ways of selecting a deletable $\mathcal{T}$ from G. Then

$$\begin{aligned}\eta &\geq \frac{1}{t!}\prod_{j=0}^{t-1}(m - s - 6cj)\\ &\geq \frac{1}{t!}m^t \exp\left\{-\frac{t(s+6ct)}{m-(s+6ct)}\right\},\end{aligned}$$

where

$$s = |\mathcal{N}(G)| \leq \left(\frac{1}{2}(k-1) + 2e^{-c/6}\right)n.$$

Let μ be an upper bound on the number of ways of choosing an addable edge set $\mathcal{T}'$, then

$$\mu \leq \binom{\binom{n}{2} - \binom{\alpha n}{2} - (m-t)}{t} \leq \frac{1}{t!}\binom{n}{2}^t e^{-\alpha^2 t}.$$

The argument of [FF] states that,

$$\mathbf{Pr}(\mathcal{E} \cap \mathcal{B}) \leq (1 + o(1))\ \frac{\mu |G_{n,m-t,k}|}{\eta |G_{n,m,k}|}.$$

Now

$$\begin{aligned}\frac{|G_{n,m-t,k}|}{|G_{n,m,k}|} &= \frac{\binom{\binom{n}{2}}{m-t}}{\binom{\binom{n}{2}}{m}}\ \frac{\mathbf{Pr}(\delta \geq k \text{ in } G_{n,m-t})}{\mathbf{Pr}(\delta \geq k \text{ in } G_{n,m})}\\ &\leq (1+o(1))\left(\frac{m}{\binom{n}{2}}\right)^t\end{aligned}$$

since the probability $\mathbf{Pr}(\delta \geq k \text{ in } G_{n,m-t})$ is non-decreasing as $t \to 0$.

We now find that

$$\mathbf{Pr}(\mathcal{E}) \leq O(1)\ \exp\left\{-O\left(t\left(\alpha^2 - \frac{s}{m-s}\right)\right)\right\}.$$

We require that

$$\alpha = \frac{1}{\sqrt{2}(k+1)} > \sqrt{\frac{(k-1)+4e^{-c/6}}{c-((k-1)+4e^{-c/6})}},$$

which is satisfied when $c \geq c_k = 2(k+1)^3$. The value of α is the same as in Lemma 8.

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6 Appendix

Numerical calculations show that even for moderate values of c, the value of λ rapidly converges to c provided k is small. Similarly $\beta(\lambda, k)$ tends rapidly to 1. For example, $1 - 10^{-10} \le \beta(30, 3) \le 1$. Thus we can effectively treat λ and c as equal, and ignore $\beta(\lambda, k)$. The properties of $\beta(\lambda, k)$ are given in the following lemma.

Lemma A 1 *Let $\lambda = \lambda(c)$ be defined by*

$$\lambda \frac{\beta(\lambda, k-1)}{\beta(\lambda, k)} = c, \tag{7}$$

where

$$\begin{aligned} \beta(\lambda, 0) &= 1, \\ \beta(\lambda, k) &= 1 - e^{-\lambda}(1 + \lambda + \cdots + \frac{\lambda^{k-1}}{k-1!}), \qquad k \ge 1. \end{aligned}$$

For $c > k$ the function $\lambda(c)$ is well defined and

(i) $c - k < \lambda < c$,

(ii) *(a)* $\frac{\beta(\lambda,k-1)}{\beta(\lambda,k)}$ *is a monotone decreasing function of λ, tending to 1 as $\lambda \to \infty$,*

(b) $\frac{\beta(\lambda,k-1)}{\beta(\lambda,k)}$ *is a monotone increasing function of k,*

(iii) $\left(\frac{c}{\lambda e}\right) e^{\lambda/c} > 1$,

(iv) *(a)* $\beta(\lambda, k)$ *is a monotone increasing function of λ tending to 1 as $\lambda \to \infty$.*

(b) $\frac{1}{\beta(\lambda,k)} \le e^{\frac{k^2}{c-k}}$.

(c) $\beta(\lambda, k) \ge e^{-D}$ *where* $D = \frac{kc^{k-1}}{(k-1)!} e^{-(c-k)} e^{k^2/(c-k)}$.

Proof (i),(ii) We note that

$$\frac{\beta(\lambda, k-1)}{\beta(\lambda, k)} = 1 + \frac{k}{\lambda} \frac{1}{\left(1 + \frac{\lambda}{k+1} + \frac{\lambda^2}{(k+1)(k+2)} + ... + \frac{\lambda^i}{(k+1)^{(i)}} + ...\right)}. \tag{8}$$

(iii) For simplicity denote $\beta(\lambda, i)$ by β_i, and $\frac{\lambda^{k-1}}{k-1!} e^{-\lambda}$ by ℓ then $\beta_{k-1} = \beta_k + \ell$ and

$$\left(\frac{c}{\lambda e}\right) e^{\lambda/c} = \left(e^{\left(\frac{\beta_k}{\beta_{k-1}} - 1\right)} \frac{\beta_{k-1}}{\beta_k}\right)$$

$$
\begin{aligned}
&= \left(e^{-\frac{\ell}{\beta_k+\ell}}\right)\left(1+\frac{\ell}{\beta_k}\right) \\
&> \left(1-\frac{\ell}{\beta_k+\ell}\right)\left(1+\frac{\ell}{\beta_k}\right) \\
&= 1.
\end{aligned}
$$

(iv)(b) This comes from (8) by iterating

$$\beta(\lambda, j-1) \leq (1+\frac{j}{\lambda})\beta(\lambda, j).$$

(iv)(c) This follows by applying (iv)(b) to the right hand side of

$$\beta(\lambda, k) \geq \exp\left\{-\frac{(1+\lambda+\cdots+\frac{\lambda^{k-1}}{(k-1)!})e^{-\lambda}}{\beta(\lambda, k)}\right\}.$$

□

Proof of Lemma 3

Let $X = X_i$ be a truncated Poisson random variable with parameters λ, k giving the occupancy of cell i, then $\mathbf{E}X = c$ where $cn = 2m$. Conditioning on $\sum_{i=1}^n X_i = 2m$ in $\mathcal{P}(n, \lambda, k)$, we obtain $\mathcal{O}(n, 2m, k)$. Specifically, if $\mathbf{x} \in \mathcal{P}(n, \lambda, k)$, $\mathbf{x} = (X_1, ..., X_n)$ and $\sum_{i=1}^n X_i = 2m$ so that $\mathbf{x} \in \mathcal{O}(n, 2m, k)$, then

$$\mathbf{Pr}(\mathbf{x}; \mathcal{O}(n, 2m, k)\ ) = \mathbf{Pr}([\mathbf{x} \mid \sum X_i = 2m]; \mathcal{P}(n, \lambda, k)\ ).$$

This is a generalization of the result that a multinomial random variable may be obtained from independent Poisson random variables by conditioning on their sum; and the details are described in, for example, [BFU]. Thus

$$\mathbf{Pr}(\mathcal{E}; \mathcal{O}(n, 2m, k)\ ) \leq \frac{\mathbf{Pr}(\mathcal{E}; \mathcal{P}(n, \lambda, k)\ )}{\mathbf{Pr}(\sum_{i=1}^n X_i = 2m; \mathcal{P}(n, \lambda, k)\ )}.$$

By the Local Limit Theorem, (see [BFU] and [Du]) $\mathbf{Pr}(\sum_{i=1}^n X_i = 2m)$ is asymptotic to $1/\sqrt{2\pi\sigma^2 n}$, where $\sigma^2 = \mathbf{Var}(X)$ is given by

$$\sigma^2 = \lambda^2 \frac{\beta(\lambda, k-2)}{\beta(\lambda, k)} - c^2 + c, \tag{9}$$

and by (7) and Lemma A1(ii)(b) we see that $\sigma^2 < c$.

An element $\mathbf{d} = (d_i : i = 1, ...n)$ of $\mathcal{O}(n, 2m, k)$, induces a set of configuration multigraphs $M(\mathbf{d}) \subset \mathcal{M}(n, 2m, k)$. The expected number of loops and multiple edges in a configuration multigraph arising from an element of $\mathcal{O}(n, 2m, k)$, is a function of the degree sequence $\mathbf{d}$. The maximum occupancy of any box in $\mathcal{O}(n, 2m, k)$ is $o(\log n)$ with probability of the complementary event $n^{-O(\log\log n)}$. Conditioning on maximum degree $o(\log n)$ and using the methods given in [Bo], the probability there are no loops or multiple edges in such a configuration multigraph is asymptotic to $\exp\{-\zeta/2 - \zeta^2/4\}$, where

$$\zeta = \frac{1}{2m}\sum_{i=1}^{n} d_i(d_i-1).$$

In $\mathcal{P}(n, \lambda, k)$, the random variable ζ is the sum of independent random variables and is sharply concentrated with expected value

$$\mathbf{E}\zeta = \lambda\frac{\beta(\lambda, k-2)}{\beta(\lambda, k-1)}.$$

Fix $\epsilon > 0$, small. Then with probability $1-e^{-\Omega(\epsilon^2 n)}$, we have $(1-\epsilon)\mathbf{E}\zeta \le \zeta \le (1+\epsilon)\mathbf{E}\zeta$ in $\mathcal{P}(n, \lambda, k)$ and $\mathcal{O}(n, 2m, k)$. If $\mathcal{S}$ is the subset of $\mathcal{M}(n, 2m, k)$ corresponding to $G_{n,m,k}$, then

$$\begin{aligned}\mathbf{Pr}(\mathcal{S}; \mathcal{M}(n, 2m, k)\ ) &= o(1) + \sum_{\mathbf{d}}(1+o(1))\exp\{-(\zeta(\mathbf{d})/2 + \zeta(\mathbf{d})^2/4)\mathbf{Pr}(M(\mathbf{d}); \mathcal{M}(n, 2m, k)\ )\} \\ &= (1+O(\epsilon))\exp\{-(\mathbf{E}\zeta/2 + (\mathbf{E}\zeta)^2/4)\}.\end{aligned}$$

Now

$$\mathbf{Pr}(\mathcal{E}; G_{n,m,k}\ ) = \frac{\mathbf{Pr}(\mathcal{E}\cap\mathcal{S}; \mathcal{M}(n, 2m, k)\ )}{\mathbf{Pr}(\mathcal{S}; \mathcal{M}(n, 2m, k)\ )} \le \frac{\mathbf{Pr}(\mathcal{E}; \mathcal{O}(n, 2m, k)\ )}{\mathbf{Pr}(\mathcal{S}; \mathcal{M}(n, 2m, k)\ )}.$$

□

The following lemma, and its proof are due to B. Pittel [Pi], who uses this approach in, for example, [PW] and [Pi1]. The proof technique uses the Local Limit Theorem (see [Gn], [Du]) to avoid the direct application of a saddle point method. The origins of this technique can be traced back to A. I. Kinchin [Ki]. The use of the Local Limit Theorem in conjunction with generating functions for problems of this type was championed by V. F. Kolchin (see [Ko] for a wide ranging discussion).

Lemma A 2 (Pi)

$$\frac{|\mathcal{M}(n, 2m, k)\ |}{|\mathcal{M}(n, 2m)|} = (1+o(1))\sqrt{\frac{c}{\sigma^2}}\ \left[\left(\frac{c}{\lambda e}\right)^c e^{\lambda}\beta(\lambda, k)\right]^n.$$

Proof Let $f(z) = \sum_{j\ge k}\frac{z^j}{j!}$ so that $f(z) = e^z\beta(z, k)$, and let $[z^t]g(z)$ denote the coefficient of z^t in the power series of $g(z)$.

$$\begin{aligned}\frac{|\mathcal{M}(n, 2m, k)\ |}{|\mathcal{M}(n, 2m)|} &= \sum_{\substack{b_1,\dots,b_n\ge k \\ b_1+\cdots+b_n=2m}} \binom{2m}{b_1\cdots b_n}\left(\frac{1}{n}\right)^{2m} \\ &= \frac{(2m)!}{n^{2m}}[z^{2m}](f(z))^n.\end{aligned}$$

Let $x > 0$, so that

$$[z^{2m}](f(z))^n = \frac{(f(x))^n}{x^{2m}}[z^{2m}]\left(\frac{f(zx)}{f(x)}\right)^n.$$

Let $Y(x)$ be a random variable chosen so that

$$\mathbf{E}z^Y = \frac{f(zx)}{f(x)},$$

where, such a Y exists as $\frac{f(zx)}{f(x)}\Big|_{z=1} = 1$. Let $Y_1, ..., Y_n$ be independent copies of Y. Then

$$\begin{aligned}[z^{2m}]\left(\frac{f(zx)}{f(x)}\right)^n &= [z^{2m}]\mathbf{E}(z^{Y_1+\cdots+Y_n}) \\ &= \mathbf{Pr}(Y_1 + \cdots + Y_n = 2m).\end{aligned}$$

Now

$$\mathbf{E}Y = \frac{d}{dz}\mathbf{E}z^Y\Big|_{z=1} = \frac{xf'(x)}{f(x)} = x\frac{\beta(x,k-1)}{\beta(x,k)},$$

so that, if we choose $x = \lambda$, then $\mathbf{E}Y = c$. Similarly, $\mathbf{Var}(Y) = \sigma^2$ is given by (9). By the Local Limit Theorem,

$$\mathbf{Pr}(Y_1 + \cdots + Y_n = 2m) \sim \frac{1}{\sqrt{2\pi\sigma^2 n}}.$$

Hence

$$\frac{|\mathcal{M}(n,2m,k)\ |}{|\mathcal{M}(n,2m)|} = \frac{1+o(1)}{\sqrt{2\pi\sigma^2 n}}\frac{(2m)!}{n^{2m}}\frac{(f(\lambda))^n}{\lambda^{2m}},$$

and the result follows. □

Proof of Lemma 4

(i) We have for $j = 0$ and k that

$$|G_{n,2m,j}|2^m m! = (1+o(1))e^{-\theta_j/2-\theta_j^2/4}|\mathcal{M}(n,2m,j)|,$$

where $\theta_0 = c$, and $\theta_k = \lambda\beta(\lambda,k-2)/\beta(\lambda,k-1)$ was shown in the proof of Lemma 3.

(ii) Apply Lemma A1 (i),(iii) and (iv). □

Lemma A 3 *Let t, b be integer, where $t \geq b \geq 0$, and let $a > 0$, then*

$$\sum_{T \geq t} T^b \frac{a^T}{T!} \leq 2e^a(ae^{b/t})^b. \tag{10}$$

Proof Suppose $T \geq t \geq b$, then

$$\frac{T^b}{(T)_b} = \frac{T^b(T-b)!}{T!} \leq 2\left(1 - \frac{b}{T}\right)^{T-b} e^b \leq 2\exp\left\{\frac{b^2}{T}\right\}.$$

Thus

$$\begin{aligned}\sum_{T\geq t} T^b \frac{a^T}{T!} &\leq \sum_{T\geq t} 2e^{b^2/T} \left((T)_b \frac{a^T}{T!}\right) \\ &\leq 2e^{b^2/t}\, a^b e^a.\end{aligned}$$

□