The union of two random permutations does not have a directed Hamilton cycle

Colin Cooper*

August 3, 2000

Abstract

We prove that the union of two random permutations does not usually contain a directed Hamilton cycle. This answers a question posed in [5]

1 Introduction

Let $\pi \in S_n$ be a permutation of the set $[n]$. The digraph $D(\pi) = \{[n], A_\pi\}$ has vertex set $[n]$ and arc set $A_\pi = \{(i, \pi(i)), i = 1, ..., n\}$. Thus $D(a)$ is the union of disjoint directed cycles. We define the digraph $D(\pi_1, ..., \pi_k) = \{[n], A_{\pi_1} \cup \cdots \cup A_{\pi_k}\}$ where the union of the arc sets is multi-set union, so that repeated arcs are retained. The bipartite graph $B(\pi) = \{[n], [n], A_\pi\}$ and its generalization to k permutations is similarly defined.

We can think of k-tuples of permutations, as defining a probability space, $\boldsymbol{P}_k = \{(\pi_1, ..., \pi_k) : \pi_j \in S_n, j = 1, ..., k\}$, with the uniform measure. Frieze [5] has proved that:

Theorem 1 *If π_1, π_2, π_3 are chosen independently and uniformly at random then*

$$\lim_{n\to\infty} \Pr(D(\pi_1, \pi_2, \pi_3) \text{ has a directed Hamilton cycle }) = 1.$$

This theorem immediately implies the result for $\boldsymbol{P}_k,\ k \geq 3$. The case of $D(\pi_1, \pi_2)$ was left unresolved. For $D \in \boldsymbol{P}_2$ the number $Z(D)$ of directed Hamilton cycles contained in D has expectation

$$\mathbf{E}(Z) = (n-1)! \sum_{k=0}^{n} \binom{n}{k} \frac{(n-k)!}{n!} \frac{k!}{n!} = \frac{n+1}{n},$$

so that $\mathbf{E}(Z) \to 1$. In [5] Frieze conjectures the following theorem:

*School of Mathematical Sciences, University of North London, London N7 8DB, UK. Research supported by the STORM Research Group. E-mail: `c.cooper@unl.ac.uk`.

Theorem 2 *Let $(a,b) \in \boldsymbol{P}_2$. Say $(a,b) \in \boldsymbol{HC}$ if $D(a,b)$ has a directed Hamilton cycle, then*

$$\lim_{n\to\infty} \frac{|\boldsymbol{HC}|}{n!n!} = 0.$$

A digraph $D = ([n], A)$ is r-regular if $d^-(v) = d^+(v) = r$ for every vertex $v \in [n]$. The usual method of generating simple r-regular digraphs uniformly at random, is to use a configuration model $\boldsymbol{C}_k$ and reject those digraphs with loops or parallel edges. It was proved in [3] that for $r \geq 3$ almost every simple r-regular digraph contains a directed Hamilton cycle. For $r = 2$ the expected number of directed Hamilton cycles tends to zero.

The configurations $\boldsymbol{C}_2$ are permutations of length $2n$. The measure of simple 2-regular digraphs derived from $\boldsymbol{C}_2$ is uniform and asymptotic to $2^{2n}e^{5/2}/(2n)!$. The measure on 2-regular digraphs derived from $\boldsymbol{P}_2$ is not uniform, but has value $m(D)/(n!n!)$ where $m(D)$ is the size of the set of pairs $(a,b) \in \boldsymbol{P}_2$ with the same underlying digraph D. In fact the measures on digraphs in the two spaces are very close. By exploiting this, Cooper [1, 2] used proofs in $\boldsymbol{P}_2$ to infer strong connectivity and 2-cyclic properties in the uniform model.

2 Proof of Theorem 2

Given $(a,b) \in \boldsymbol{P}_2$ let $B(a,b)$ be the bipartite graph of (a,b) and let $\boldsymbol{Q}(a,b) = \{(\alpha,\beta) : B(a,b) = B(\alpha,\beta)\}$. Let the components of $B(a,b)$ be $C_1, ..., C_k$. Each component C_l consists of a cycle alternating between the left and right vertex sets. Thus component C_l partitions into two perfect matchings σ_l, ϕ_l. Let $T_l = \{(\sigma_l, \phi_l), (\phi_l, \sigma_l)\}$, then

$$\boldsymbol{Q}(a,b) = T_1 \times T_2 \times \cdots \times T_k.$$

If m_2 is the number of components of $B(a,b)$ which are cycles of length 2, then $|\boldsymbol{Q}(a,b)| = 2^{k-m_2} \leq 2^k$.

Lemma 1 *Let $\phi, \sigma \in S_n$. The number of components of $B(\phi,\sigma)$ is the number of cycles in the permutation $\phi^{-1}\sigma$.*

Proof Let $C_1, ..., C_k$ be the components of $B(\phi,\sigma)$. Let $V_1, ..., V_k$ be the partition of the left vertex set $[n]$ induced by the components. Let $i, j \in [n]$ be (not necessarily distinct) vertices of the left vertex set with edges $(i, \sigma(i))$, $(j, \phi(j))$ where $\sigma(i) = \phi(j) = l$. Reversing the direction of $(j, \phi(j))$ and replacing the path $(i, \sigma(i)), (\sigma(i), \phi^{-1}(\sigma(i)))$ by (i, j) gives the digraph D of the permutation $\phi^{-1}\sigma$. The cycle X_l of $D(\phi^{-1}\sigma)$ with vertex set V_l corresponds uniquely to the component C_l of $B(\phi,\sigma)$. $\square$

Let $M(a)$ denote the number of cycles in the permutation a. Let μ, ν^2 be the mean and variance of M. Then $\mu, \nu^2 \sim \log n$ and $(M-\mu)/\nu$ is asymptotically distributed with the standard normal distribution (see [4]). Thus for fixed ϕ and random σ,

$$\begin{aligned} Pr(B(\phi,\sigma) \text{ has at least } \mu + x\nu \text{ components } \mid \phi) &= Pr(M(\phi^{-1}\sigma) \geq \mu + x\nu) \\ &\leq \frac{2}{x} e^{-x^2/2}. \end{aligned}$$

Let $\omega \to \infty$, $\omega = o(\sqrt{\log n})$, and let

$$A_\phi = \{(\phi,\sigma) : B(\phi,\sigma) \text{ has at most } \log n + \omega\sqrt{\log n} \text{ components. } \}.$$

Let $\boldsymbol{R} = \cup_\phi A_\phi$, then $|\boldsymbol{R}| = |\boldsymbol{P}|(1-o(1))$.

Let H_1 be the permutations a in S_n whose digraph $D(a)$ is a directed Hamilton cycle.

Lemma 2 *$D(\alpha,\beta)$ has a directed Hamilton cycle, if and only if there exists $a \in H_1$ such that $(\alpha,\beta) \in \boldsymbol{Q}(a,b)$.*

Proof

$$D(a,b) = D(\alpha,\beta) \iff B(a,b) = B(\alpha,\beta) \iff (\alpha,\beta) \in \boldsymbol{Q}(a,b).$$

Certainly if $a \in H_1$ then $D(a)$ is a directed Hamilton cycle of $D(a,b)$. If $D(\alpha,\beta)$ has a directed Hamilton cycle h then $h = D(a)$ for some $a \in S_n$ and $D(\alpha,\beta) - h = D(b)$ for some $b \in S_n$ so that $(\alpha,\beta) \in \boldsymbol{Q}(a,b)$. $\square$

Now

$$\boldsymbol{R} \cap \boldsymbol{HC} \subseteq \bigcup_{a \in H_1} \bigcup_{(a,b) \in A_a} \boldsymbol{Q}(a,b)$$

so that

$$\begin{aligned} \frac{|\boldsymbol{R} \cap \boldsymbol{HC}|}{n!n!} &\leq \frac{(n-1)!n!2^{\log n + \omega\sqrt{\log n}}}{n!n!} \\ &\leq \frac{1}{n^{3/10}}. \end{aligned}$$

Thus

$$\frac{|\boldsymbol{HC}|}{n!n!} \leq \frac{1}{n^{3/10}} + \frac{2}{\omega} e^{-\omega^2/2}.$$

This concludes the proof of Theorem 2.

References

[1] C. Cooper, *A note on the connectivity of 2-regular digraphs.* Random Structures and Algorithms 4 (1993) 469-472.

[2] C. Cooper, *On the 2-cyclic property in 2-regular digraphs.* Random Structures and Algorithms 6 (1995) 439-448.

[3] C. Cooper, A.M. Frieze and M.J. Molloy, *Hamilton cycles in random regular digraphs.* Combinatorics, Probability and Computing 3 (1994) 39-50.

[4] W. Feller, *An Introduction to Probability Theory and its Applications.* (1960) Wiley.

[5] A.M. Frieze, *Hamilton cycles in the union of random permutations.* Submitted to Random Structures and Algorithms (1999).