Two-dimensional Modal Logics with Difference Relations

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Extended Abstract

1 Introduction

Modal logic has a rich history originating in the classical considerations of philosophers such as Aristotle. Originally conceived as the logic of necessary and contingent truths (so-called alethic logic), further study has shown the same underlying semantics to encapsulate a vast range of other natural linguistic phenomena; such disparate phenomena as spatial and temporal reasoning, epistemic and doxastic reasoning — relating to one’s knowledge and belief, and deontic reasoning of one’s obligations and permissions [5, 16].

While there are many applications of modal logics working in isolation, it is often their interactions in which we are most interested. The notion of products of propositional modal logics, introduced by Segerberg [35] and Shehtman [36], provides a conceptually appealing approach to combining modal logics. Products of modal logics are connected to several other ‘many-dimensional’ logical formalisms, such as finite variable fragments of classical, intuitionistic, modal, and temporal first-order logic, as well as temporal epistemic logics [8], and extensions of description logics with ‘dynamic’ features [3, 42, 43, 44]. The product construction has been studied extensively since its inception [27, 10, 23], and has borne many applications in computer science and artificial intelligence [30, 4, 9].

1.1 Definitions

We consider the bimodal language whose formulas are defined by the following grammar:

\[ \varphi ::= p_i \mid \neg \varphi \mid \varphi_1 \land \varphi_2 \mid \Diamond_h \varphi \mid \Diamond_v \varphi, \]

where \( p_i \) ranges over a set of propositional variables. Given two Kripke frames \( \mathcal{F}_h = (W_h, R_h) \) and \( \mathcal{F}_v = (W_v, R_v) \), we define their product frame to be the 2-frame

\[ \mathcal{F}_h \times \mathcal{F}_v := (W_h \times W_v, \bar{R}_h, \bar{R}_v), \]

where \( W_h \times W_v = \{(x, y) : x \in W_h \text{ and } y \in W_v \} \) is the Cartesian product of \( W_h \) and \( W_v \). and, for all \( x, x' \in W_h \) and \( y, y' \in W_v \), we have that

\[ (x, y) \bar{R}_h(x', y') \iff xR_h x' \text{ and } y = y', \]

\[ (x, y) \bar{R}_v(x', y') \iff x = x' \text{ and } yR_v y'. \]

The subscripts \( h \) and \( v \) betray the geometric intuition behind this construction, illustrated in Figure 1 with the \( \bar{R}_h \) denoting the ‘horizontal’ accessibility relation and the \( \bar{R}_v \) denoting the ‘vertical’ accessibility relation. We may then define the product logic of two Kripke complete modal logics \( L_h \) and \( L_v \), by taking

\[ L_h \times L_v := \{\varphi : \varphi \text{ is valid in } \mathcal{F}_h \times \mathcal{F}_v, \text{ where } \mathcal{F}_i \text{ is a frame for } L_i, i = h, v\}. \]
1.2 Some known results

Unlike for unimodal logic, and for logics having multiple non-interacting modal operators — known as fusions — there is a scarcity of general results for logics with interacting modal operators, and for products of modal logics, in particular. Below we summarise some of the known results that are relevant to the thesis.

As concerns the decision problem, it is known that product logics of the form $L \times K$ and $L \times S5$ are typically decidable, where $K$ and $S5$ are the respective logics of all frames and all equivalence relations [10]. On the other hand, if both component logics are characterised by transitive frames of unbounded depth — as with $K4$, the logic of all transitive relations — then the product logic is typically undecidable [26, 32, 13].

With regard to their axiomatisations, product logics always validate the commutativity and Church–Rosser (confluence) axioms, describing certain interactions between the two modalities. Hence, in considering possible axiomatisations for $L_h \times L_h$, it is a good starting point to consider the commutator $[L_h, L_v]$: the smallest bimodal logic containing both $L_h$, $L_v$ and the aforementioned interactions. The general result of Gabbay and Shehtman [11] says that if both component logics $L_h$ and $L_v$ are characterised by Horn-definable classes of frames, then they are product matching; which is to say that their product $L_h \times L_v$ coincides with their commutator $[L_h, L_v]$. It follows that, for product matching logics, the product logic is finitely axiomatisable whenever both component logics are such. However, not much is known about axiomatising product logics lying outside the scope of this result. It is known, however, that the logic $K \times K4.3$ is non-finitely axiomatisable [24], where $K4.3$ denotes the (non-Horn definable) logic of all linear orders.

1.3 The Difference logic

Von Wright’s ‘logic of elsewhere’ [40], henceforth denoted by Diff, is characterised by the class of all difference relations, in which each possible world is accessible from every other distinct possible world. The frames for Diff are, therefore, only modestly ‘richer’ in expressive power than the universal frames characterising S5. The logic Diff underlies the logic of the difference operator [6, 7], and is connected to nominals [14] and graded modalities [1], all of which find many applications in description logics [2]. It is also a simple example of a non-Horn definable modal logic. Moreover, Diff is known to be decidable and finitely axiomatisable,
in addition to possessing the finite model property. The coNP-completeness of its decision problem coincides with that of S5. Furthermore, the product logic Diff × Diff is related to the two-variable, substitution and equality-free fragment of first-order logic with counting quantifiers $\exists_{\leq m} x, \exists_{\leq m} y$, for $m = 0, 1$.

### 1.4 Problems tackled in the thesis

In the thesis we study axiomatisation and decision problems of modal product logics of the form $L \times \text{Diff}$ and related formalisms, such as one-variable modal and temporal logics with counting quantifiers and with various domain assumptions, and extensions that include a modal constant representing equality of the coordinates in the two-dimensional models. We make a case that, despite the similarities that Diff shares with S5 — in the structure of their frames, their computational complexity and their axiomatisations — their respective interactions in two dimensions may differ considerably. This is in sharp contrast with the situation in the two-variable fragment of classical first-order logic, where the addition of ‘elsewhere’ quantifiers and/or equality has no influence on the complexity of the decision problem [18, 19, 29]. The contributions of the thesis may also serve as a case study to better steer investigation into more general principles governing the interactions between modal logics.

### 2 Thesis Summary

The thesis is divided into three parts, together with an appendix: Part I, comprising Chapters 2–4, serves as a general introduction and provides the necessary background and definitions for the following topics. In Part II, comprising Chapters 5–8, we consider problems relating to the axiomatisation and computational properties of various two-dimensional modal product logics involving the difference relation. Part III, that is Chapter 9–10, goes beyond the regular product construction to consider various relativisations of products, and products equipped with a diagonal element, mirroring the addition of an ‘equality’ operator connecting the two dimensions by their common elements. Throughout what follows, section and theorem numbers will be given as they appear in the thesis.

#### Chapter 5: Axiomatisation of Products

In Chapter 5, we tackle problems relating to the axiomatisation and finite frame problem† of logics of the form $L \times \text{Diff}$. We show first that, unlike logics of the form $L \times \text{S5}$ that are finitely axiomatisable whenever $L$ is Horn-definable, products taken with the non-Horn definable logic Diff tend not to admit such finite axiomatisations.

**Theorem 5.1.** Let $L$ be any bimodal logic such that

$$ K \times \text{wK5} \subseteq L \quad \text{and} \quad (\mathbb{Z}, \mathbb{Z}^2) \times (\mathbb{Z}, \neq) \text{ is a frame for } L. $$

Then $L$ cannot be axiomatised using only finitely many variables‡.

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†The finite frame problem for $L$ is the problem of deciding whether an arbitrary finite frame is a frame for $L$? This is always reducible to the decision problem of $L$, and is trivial whenever $L$ is finitely axiomatisable.

‡Here wK5 denotes a sublogic of Diff, characterised by the class of all weakly-Euclidean frames whose binary relation $R$ is such that $\forall x \forall y \forall z (x Ry \land x Rz \to y Rz \lor y = z)$. 


In Section 5.3, we discuss products falling outside the aforementioned interval of logics and show that Alt and $L$ are product matching, whenever $L$ is any canonical logic, and where Alt is the logic of all functional relations (Theorem 5.13). In particular, $\text{Alt} \times \text{Diff} = [\text{Alt}, \text{Diff}]$ and $\text{Alt} \times \text{K4.3} = [\text{Alt}, \text{K4.3}]$, and hence, both products are finitely axiomatisable. These are, therefore, the first known examples of products that coincide with the respective commutators, whose components are not both Horn-definable.

In Section 5.4, we turn our attention to $\text{S5} \times \text{Diff}$ and provide a full structural characterisation of the finite frames for $\text{S5} \times \text{Diff}$ (Theorem 5.22), thereby providing a polynomial time algorithm for its finite frame problem, despite the lack of any finite axiomatisation (Theorem 5.1) and the coNExpTime-completeness of its decision problem.

**Theorem 5.23.** The finite frame problem for $\text{S5} \times \text{Diff}$ is decidable in polynomial time.

We conclude, in Section 5.5, with a discussion of how the characterisation theorem may be generalised to describe the finite frames for $\text{Diff} \times \text{Diff}$.

### Chapter 6: Finite Model Properties

As product logics are ‘semantically’ defined, there can be other ‘non-standard’ (that is, non-product) frames validating all the same theorems. This is why it is meaningful to consider different versions of the finite model property (fmp) in the context of products of modal logics. A product logic $L_h \times L_v$ is said to have:

- the fmp if every $\varphi \not\in L_h \times L_v$ is refutable in some finite frame for $L_h \times L_v$;
- the product fmp if every $\varphi \not\in L_h \times L_v$ is refutable in a finite product frame for $L_h \times L_v$;
- the square product fmp if every $\varphi \not\in L_h \times L_v$ is refutable in a finite product frame $\mathcal{F}_h \times \mathcal{F}_v$ for $L_h \times L_v$, where $\mathcal{F}_h$ and $\mathcal{F}_v$ have the same cardinality.

In Chapter 6 we discuss these properties relating to products of the form $L \times \text{Diff}$.

In Section 6.1, we generalise a result of [19], showing that the two-variable fragment of first-order logic with counting quantifiers lacks the finite model property.

**Theorem 6.3.** Let $L$ be any bimodal logic such that:

$$[\text{wK5, wK5}] \subseteq L \quad \text{and} \quad (\omega, \neq) \times (\omega, \neq) \text{ is a frame for } L.$$  

Then $L$ does not possess the fmp.

In particular, it follows that neither $[\text{Diff, Diff}]$ nor $\text{Diff} \times \text{Diff}$ can be characterised by their finite frames. We note that $\text{Diff} \times \text{Diff}$ is, therefore, one of the simplest examples of a non-finitely axiomatisable logic without the fmp, that is nonetheless decidable — the decidability of which follows from a straightforward reduction to the two-variable fragment of first-order logic with counting quantifiers.

On the other hand, in Section 6.2 we show the following, by a variation on the well-known quasimodel technique [10]:

**Theorem 6.6.** $\text{S5} \times \text{Diff}$ has the exponential product fmp.

However, there is a fine line: It is shown that any logic extending $\text{S5} \times \text{Diff}$ does not enjoy the square product fmp (Theorem 6.1). These results show that, despite their similarities, the products of $\text{S5}$ and $\text{Diff}$ behave very differently with regard to their finite model properties, as summarised below in Table 1.
Table 1: Comparison of finite model properties for products of $S5$ and $\text{Diff}$.

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<td>$\text{product fmp}$</td>
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<td>$\text{square product fmp}$</td>
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Chapter 7: Computational Complexity of Commutators

In Chapter 7, we discuss some decision problems of commutators involving $\text{Diff}$. No techniques are yet known for handling the decision problems of commutators that do not coincide with their respective products. In particular, the decidability and complexity of the decision problems for both $[\text{Diff}, \text{Diff}]$ and $[S5, \text{Diff}]$ remained open — by Theorem 5.1 neither of these logics coincide with their respective products.

In Section 7.1, we introduce a novel approach to obtaining decidability results for the decision problems for each of these logics, together with elementary upper bounds on their respective complexities. We achieve these results with the aid of a recursive satisfiability-preserving reduction of each commutator to their corresponding product logic.

**Theorem 7.1.** The decision problems for both $[S5, \text{Diff}]$ and $[\text{Diff}, \text{Diff}]$ are decidable.

By a variation on this technique, we are able to show that, unlike $[\text{Diff}, \text{Diff}]$ (Theorem 6.3), the commutator $[S5, \text{Diff}]$ is characterised by its finite frames (Theorem 7.12).

Chapter 8: Computational Complexity of Products

Product frames are always ‘grid-like’ by definition. Hence, in those cases where coordinate-wise ‘universal’ and ‘next-time’ operators are both available, it becomes straightforward to obtain lower bounds by using reductions from various complex grid-based problems, such as tiling problems or Turing machine problems. For example, it is easy to see that the decision problem for $K_u \times K_u$ is undecidable [25], where $K_u$ is the logic $K$ of all frames enriched with the universal modality.

Previous lower bound proofs [26, 32, 13] on product logics over transitive frames overcome the lack of next-time operators by ‘diagonally’ encoding the $\omega \times \omega$-grid in product models. Here, we develop a novel technique, making direct use of the grid-like structure of product frames, to obtain undecidability results for a host of product logics using reductions from various (Minsky) counter machine problems.

In particular, in Section 8.2 we introduce the technique for cases when a ‘horizontal’ next-time operator is still available. We show that $K_u \times \text{Diff}$ is undecidable but still recursively enumerable (Corollary 8.4). Furthermore, a variation on this technique shows the product of $\text{Diff}$ and $K$ enriched with the common knowledge (transitive closure) operator is not even analytic. The same is true of $\text{PTL}_{\bowtie} \times \text{Diff}$, in which $\text{PTL}_{\bowtie}$ denotes the bimodal propositional temporal logic over the natural numbers, with both ‘next-time’ and ‘eventually’ operators (Corollary 8.6).

In Section 8.3 we sharpen this technique and discuss products of the form $L \times \text{Diff}$, where $L$ is a unimodal logic (without next-time, with the sole temporal operator ‘eventually’).
characterised by some class linear orders. In particular, we prove that $\mathbf{K4.3} \times \text{Diff}$ is undecidable, while for logics $L$ characterised by various discrete linear orders, $L \times \text{Diff}$ is highly undecidable (Corollaries 8.13, 8.16 and 8.36).

Finally, in Section 8.4 we show that the results of Section 8.3 are genuine generalisations of the undecidability results obtained in [26, 32], by giving a polynomial reduction from the decision problem for $L \times \text{Diff}$ to that of $L \times \mathbf{K4.3}$, whenever $L$ is Kripke complete. Note that despite the shared coNP-completeness of the decision problems for both $\mathbf{K4.3}$ and Diff, one cannot hope for a general reverse reduction, since Diff $\times$ Diff is decidable, while Diff $\times$ K4.3 is undecidable.

Chapter 9: First-order Modal Logics and Relativised Products

In Chapter 9, we explore a variation on the standard product construction, motivated by connections between modal product logics of the form $L \times \mathbf{S5}$ and $L \times \text{Diff}$, and various one-variable fragments of first-order modal logics. Owing to philosophical debate as to how we should interpret statements involving both modal operators and first-order quantifiers, investigation into such first-order modal logics has motivated a range of possible semantics, including those in which the domain of interpretation is permitted to either expand or contract, relative to the direction of the modal accessibility relation. This motivates the consideration of relativised product logics, characterised by subframes of product frames that, similarly, expand or contract with respect to a given dimension.

It is easy to see that the decision problems of both expanding and contracting relativised products are reducible to that of the standard products. Here, we investigate whether they can be genuinely simpler. We show, first, that when $L$ is characterised by any class of strict linear orders, the decision problem for $L \times^{\text{dec}} \text{Diff}$ over decreasing domains has the same complexity as that of its ‘constant domain’ counterpart $L \times \text{Diff}$ (Theorems 9.6 & 9.8).

In Section 9.5, we turn our attention to relativised products with expanding domains. We employ the techniques described in Section 8.3, with the aid of unreliable counter machine problems, to obtain lower complexity bounds for various expanding product logics. In particular, we show that over expanding domain models, the decision problem for $L \times^{\text{exp}} \text{Diff}$ is non-elementary, whenever $L$ is characterised by any class of strict linear orders containing $(\omega, <)$ and is even non-primitive recursive, for logics characterised by expanding products whose first component is some finite strict linear order (Theorems 9.9 & 9.21, respectively).

These lower bounds are notably weaker than those respective lower bounds obtained in Section 8.3, owing to the lower complexity of unreliable counter machines compared with their reliable counterparts. However, we demonstrate several cases where these bounds are optimal, via model-level reductions to known results. These results are summarised, below, in Table 2.

Chapter 10: Products with a ‘Diagonal’ Operator

A well-know variation on the standard translation identifies the product logic $\mathbf{S5} \times \mathbf{S5}$ with the two-variable, equality- and substitution-free fragment of first-order logic. In this modal setting, equality can be modelled with an additional modal constant, denoted by $\delta$, interpreted in square frames as the identity (‘diagonal’) relation. In the cylindric algebra algebraisation of two-variable first-order logic [22], this diagonal element is denoted by $d_{01}$. In Chapter 10, we consider the decision problems of products of arbitrary modal logics expanded with this
additional feature. We show that, unlike the two-variable fragment whose validity problem remains coNExpTime-complete, both with and without equality \[34, 28, 18\], the addition of a dimension-joining diagonal constant can often lead to a considerable increase in the complexity of product logics.

In Section 10.2, we first establish a connection between δ-products and regular products. In particular, we show that the global consequence problem for certain product logics can be reduced to the decision problem for their respective δ-products (Theorem 10.3). This provides us with undecidable lower bounds for several δ-product logics whose δ-free counterparts are decidable. In particular, we show that the decision problems for the δ-products \(K \times δ K4\) and \(K \times δ K\) are undecidable (Corollary 10.4).

The latter result has a surprising consequence: In \[12\], the authors introduce the simple square fragment \(SF\) as the decidable fragment of first-order logic, whose formulas are constructed from atomic binary predicates \(P_i(x,y)\), by freely applying Boolean connectives and relativised quantifiers of the form
\[
∃z(R(x,z) \land ϕ(z,y)) \quad \text{and} \quad ∃z(R(y,z) \land ϕ(x,z)),
\]
where \(R\) is a built-in binary predicate. Unlike the decidable two-variable fragment of first-order logic, \(SF\) is situated within the undecidable three-variable fragment \[37, 38\]. This simple square fragment is readily seen to be the image of propositional bimodal formulas under the following variation of the standard translation \((\cdot)^*\),
\[
p_j^* = P_j(x,y), \quad \text{for} \ p_i \in \text{PROP}, \quad (¬ϕ)^* = ¬ϕ^*, \quad (ψ_1 \land ψ_2)^* = ψ_1^* \land ψ_2^*,
\]
\[
(∖_k ψ)^* = ∃z(R(x,z) \land ϕ^*(z/x,y)) , \quad (∖_k ψ)^* = ∃z(R(y,z) \land ϕ^*(x,z/y)),
\]
where \(R, P_0, P_1, \ldots\) are binary predicate symbols. Moreover, it is straightforward to check that \(ϕ\) is a theorem of \(K \times K\) if and only if \(ϕ^*\) is a theorem of first-order logic. Since the decision problem for \(K \times K\) is decidable \[11\], so too must be the simple square fragment \(SF\). However, by expanding \(K \times K\) to include equality, the fragment \(SF_∞\) is readily seen to be the image of the bimodal language with the δ-constant under the extension of \((\cdot)^*\) that interprets \(δ^* = (x ≈ y)\). It is similarly straightforward to check that \(ϕ\) is a theorem of \(K \times δ K\) if and

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Table 2: Complexity of various relativised products for several classes of linear orders with \text{Diff}.
only if $\varphi^*$ is a theorem of first-order logic with equality. Hence, it follows from Corollary 10.4 that the simple square fragment with equality is undecidable. Thus $SF_\approx$ joins the Gödel class $[15, 17]$ as yet another example of an undecidable fragment of first-order logic, whose equality-free fragment is decidable.

It turns out that there are limitations to the approach of Section 10.2. Indeed, there are cases where such a reduction is either unhelpful or demonstrably non-existent. For example, while the global consequence problem for $K \times S_5$ is reducible to the decision problem for $PDL \times S_5$, where $PDL$ denotes Propositional Dynamic Logic, and is thus decidable in $\text{coNExpTime}$ $[41, 33]$, the decision problem for $K \times S_5$ is shown to be undecidable (Theorem 10.7). Furthermore, we show in Section 10.5 that the decision problem for $K \times Alt$ can be decided in $\text{coNExpTime}$, whereas the undecidability of the global consequence problem for $K \times Alt$ can be established by a straightforward reduction from the unconstrained tiling problem $[39]$.

In Section 10.3, we introduce the notion of computation by means of faulty approximations as a novel variation on unreliable counter machines. Unlike lossy and incrementing counter machines, we show that computation by faulty approximation is Turing-complete. In Section 10.4.1, we exploit the greater flexibility of this new formalism to obtain undecidable lower bounds for a host of $\delta$-products, using a variation on the techniques described in Chapter 8. Among which, we show that the decision problem for $K \times S_5$ is undecidable, despite the decidability of both the decision problem and the global consequence problem for $K \times S_5$. This marks a surprising ‘jump’ from the $\text{coNExpTime}$-complexity of $K \times S_5$ $[25]$.

In Section 10.4.2, we extend this technique to $\delta$-products in which the first component is characterised by some class of linear orders. In particular, we show that the decision problems for $K_{4.3} \times K$ and $K_{4.3} \times S_5$ are both undecidable (Theorem 10.12). Note that the complexity of the global consequence problem for $K_{4.3} \times K$ remains open, while that of $K_{4.3} \times S_5$ is known to be decidable $[31]$.

We conclude Chapter 10 by probing the limitations of the above counter machine approach and show that the $\delta$-product $K \times Alt$ — lying outside the remit of the aforementioned results — has the exponential product fmp and is thus decidable (Theorem 10.17).

3 Conclusion

The thesis leaves open several questions throughout that serve as directions for future research endeavours; some have, since publication of this thesis, been successfully addressed such as Questions 6.13–6.14, regarding the product fmp and computational complexity of the decision problem for $K \times Diff$. This was addressed in $[20]$, in which $K \times Diff$ was shown to share the same $\text{coNExpTime}$-completeness as $K \times S_5$. Indeed, a stronger result was shown: that the one-variable fragment of first-order modal logic $K$ with arbitrary counting quantifiers, denoted $Q^\# K$, is $\text{coNExpTime}$-complete — the product logic $K \times Diff$ being a syntactic variant of the special case in which we allow only the counting quantifiers $\exists_{\leq 0} x$ and $\exists_{\leq 1} x$.

Another forthcoming result concerns the axiomatisation of $S_5 \times Diff$, shown to be non-finitely axiomatisable in Theorem 5.1. Based on the characterisation of its frames given in Section 5.4, we were able to construct an infinite explicit axiomatisation for $S_5 \times Diff$ $[21]$. This is the first explicit axiomatisation for a non-finitely axiomatisable product logic that is ‘orthodox’ in the sense that it uses only the ‘traditional’ rules of Substitution, Modus Ponens and Necessitation, and does not use any ‘irreflexivity-type’ rules. Owing to the more intricate
structure of the frames for $\text{Diff} \times \text{Diff}$ (see Section 5.5), an explicit ‘orthodox’ axiomatisation of the recursively enumerable product logic $\text{Diff} \times \text{Diff}$ remains subject to future work.

References


