Two-dimensional Modal Logics with Difference Relations

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Extended Abstract

1 Introduction

Modal logic has a rich history originating in the classical considerations of philosophers such as Aristotle. Originally conceived as the logic of necessary and contingent truths (so-called *alethic logic*), further study has shown the same underlying semantics to encapsulate a vast range of other natural linguistic phenomena; such disparate phenomena as *spatial* and *temporal* reasoning, *epistemic* and *doxastic* reasoning — relating to one's knowledge and belief, and *deontic* reasoning of one's obligations and permissions [5, 16].

While there are many applications of modal logics working in isolation, it is often their interactions in which we are most interested. The notion of *products* of propositional modal logics, introduced by Segerberg [35] and Shehtman [36], provides a conceptually appealing approach to combining modal logics. Products of modal logics are connected to several other 'many-dimensional' logical formalisms, such as finite variable fragments of classical, intuitionistic, modal, and temporal first-order logic, as well as temporal epistemic logics [8], and extensions of description logics with 'dynamic' features [3, 42, 43, 44]. The product construction has been studied extensively since its inception [27, 10, 23], and has borne many applications in computer science and artificial intelligence [30, 4, 9].

1.1 Definitions

We consider the bimodal language whose formulas are defined by the following grammar:

$$\varphi ::= \ p_i \ \mid \ \neg\varphi \ \mid \ \varphi_1 \wedge \varphi_2 \ \mid \ \Diamond_h\varphi \ \mid \ \Diamond_v\varphi,$$

where p_i ranges over a a set of propositional variables. Given two Kripke frames $\mathfrak{F}_h = (W_h, R_h)$ and $\mathfrak{F}_v = (W_v, R_v)$, we define their *product frame* to be the 2-frame

$$\mathfrak{F}_h \times \mathfrak{F}_v \ := \ \big(W_h \times W_v, \bar{R}_h, \bar{R}_v\big),$$

where $W_h \times W_v = \{(x, y) : x \in W_h \text{ and } y \in W_v\}$ is the Cartesian product of W_h and W_v and, for all $x, x' \in W_h$ and $y, y' \in W_v$, we have that

$$\begin{aligned}(x, y)\bar{R}_h(x', y') \quad &\Longleftrightarrow \quad xR_hx' \text{ and } y = y',\\ (x, y)\bar{R}_v(x', y') \quad &\Longleftrightarrow \quad x = x' \text{ and } yR_vy'.\end{aligned}$$

The subscripts h and v betray the geometric intuition behind this construction, illustrated in Figure 1; with the $\bar{R}_h$ denoting the *'horizontal'* accessibility relation and the $\bar{R}_v$ denoting the *'vertical'* accessibility relation. We may then define the *product logic* of two Kripke complete modal logics L_h and L_v, by taking

$$L_h \times L_v \ := \{\varphi \ : \ \varphi \text{ is valid in } \mathfrak{F}_h \times \mathfrak{F}_v, \text{ where } \mathfrak{F}_i \text{ is a frame for } L_i,\ i = h, v\}.$$

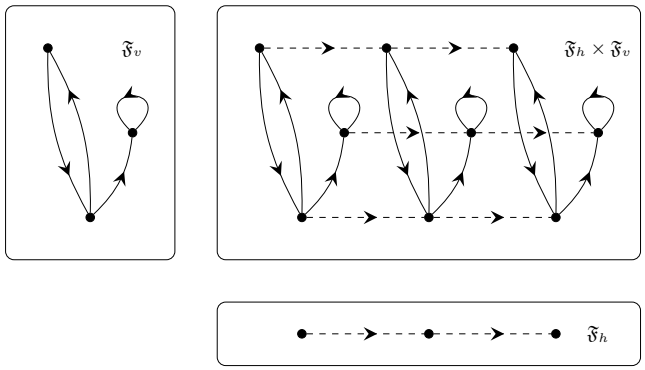

Figure 1: Illustration of the product construction.

1.2 Some known results

Unlike for unimodal logic, and for logics having multiple non-interacting modal operators — known as *fusions* — there is a scarcity of general results for logics with interacting modal operators, and for products of modal logics, in particular. Below we summarise some of the known results that are relevant to the thesis.

As concerns the decision problem, it is known that product logics of the form $L \times \mathbf{K}$ and $L \times \mathbf{S5}$ are typically decidable, where $\mathbf{K}$ and $\mathbf{S5}$ are the respective logics of all frames and all equivalence relations [10]. On the other hand, if both component logics are characterised by *transitive* frames of unbounded depth — as with $\mathbf{K4}$, the logic of all transitive relations — then the product logic is typically undecidable [26, 32, 13].

With regard to their axiomatisations, product logics always validate the *commutativity* and *Church–Rosser* (*confluence*) axioms, describing certain interactions between the two modalities. Hence, in considering possible axiomatisations for $L_h \times L_h$, it is a good starting point to consider the *commutator* $[L_h, L_v]$: the smallest bimodal logic containing both L_h, L_v and the aforementioned interactions. The general result of Gabbay and Shehtman [11] says that if both component logics L_h and L_v are characterised by *Horn-definable* classes of frames, then they are *product matching*; which is to say that their product $L_h \times L_v$ coincides with their commutator $[L_h, L_v]$. It follows that, for product matching logics, the product logic is finitely axiomatisable whenever both component logics are such. However, not much is known about axiomatising product logics lying outside the scope of this result. It is known, however, that the logic $\mathbf{K} \times \mathbf{K4.3}$ is non-finitely axiomatisable [24], where $\mathbf{K4.3}$ denotes the (non-Horn definable) logic of all linear orders.

1.3 The Difference logic

Von Wright's 'logic of elsewhere' [40], henceforth denoted by **Diff**, is characterised by the class of all difference relations, in which each possible world is accessible from every other *distinct* possible world. The frames for **Diff** are, therefore, only modestly 'richer' in expressive power than the universal frames characterising **S5**. The logic **Diff** underlies the logic of the *difference operator* [6, 7], and is connected to *nominals* [14] and *graded modalities* [1], all of which find many applications in description logics [2]. It is also a simple example of a non-Horn definable modal logic. Moreover, **Diff** is known to be decidable and finitely axiomatisable,

in addition to possessing the finite model property. The coNP-completeness of its decision problem coincides with that of **S5**. Furthermore, the product logic **Diff** $\times$ **Diff** is related to the two-variable, substitution and equality-free fragment of first-order logic with counting quantifiers $\exists_{\leq m}x, \exists_{\leq m}y$, for $m = 0, 1$.

1.4 Problems tackled in the thesis

In the thesis we study axiomatisation and decision problems of modal product logics of the form $L \times$ **Diff** and related formalisms, such as one-variable modal and temporal logics with counting quantifiers and with various domain assumptions, and extensions that include a modal constant representing equality of the coordinates in the two-dimensional models. We make a case that, despite the similarities that **Diff** shares with **S5** — in the structure of their frames, their computational complexity and their axiomatisations — their respective interactions in two dimensions may differ considerably. This is in sharp contrast with the situation in the two-variable fragment of classical first-order logic, where the addition of 'elsewhere' quantifiers and/or equality has no influence on the complexity of the decision problem [18, 19, 29]. The contributions of the thesis may also serve as a case study to better steer investigation into more general principles governing the interactions between modal logics.

2 Thesis Summary

The thesis is divided into three parts, together with an appendix: Part I, comprising Chapters 2–4, serves as a general introduction and provides the necessary background and definitions for the following topics. In Part II, comprising Chapters 5–8, we consider problems relating to the axiomatisation and computational properties of various two-dimensional modal product logics involving the difference relation. Part III, that is Chapter 9–10, goes beyond the regular product construction to consider various *relativisations* of products, and products equipped with a *diagonal element*, mirroring the addition of an 'equality' operator connecting the two dimensions by their common elements. Throughout what follows, section and theorem numbers will be given as they appear in the thesis.

Chapter 5: Axiomatisation of Products

In Chapter 5, we tackle problems relating to the axiomatisation and finite frame problems[†] of logics of the form $L \times$ **Diff**. We show first that, unlike logics of the form $L \times$ **S5** that are finitely axiomatisable whenever L is Horn-definable, products taken with the non-Horn definable logic **Diff** tend not to admit such finite axiomatisations.

Theorem 5.1. *Let L be any bimodal logic such that*

$$\mathbf{K} \times \mathbf{wK5} \subseteq L \qquad \textit{and} \qquad (\mathbb{Z}, \mathbb{Z}^2) \times (\mathbb{Z}, \neq) \textit{ is a frame for } L.$$

Then L cannot be axiomatised using only finitely many variables.[‡].

[†]The *finite frame problem* for L is the problem of deciding whether an arbitrary finite frame is a frame for L? This is always reducible to the decision problem of L, and is trivial whenever L is finitely axiomatisable.

[‡]Here **wK5** denotes a sublogic of **Diff**, characterised by the class of all *weakly-Euclidean* frames whose binary relation R is such that $\forall x \forall y \forall z \big(xRy \wedge xRz \to yRz \vee y = z\big)$.

In Section 5.3, we discuss products falling outside the aforementioned interval of logics and show that **Alt** and L are product matching, whenever L is any *canonical* logic, and where **Alt** is the logic of all functional relations (Theorem 5.13). In particular, $\mathbf{Alt} \times \mathbf{Diff} = [\mathbf{Alt}, \mathbf{Diff}]$ and $\mathbf{Alt} \times \mathbf{K4.3} = [\mathbf{Alt}, \mathbf{K4.3}]$, and hence, both products are finitely axiomatisable. These are, therefore, the first known examples of products that coincide with the respective commutators, whose components are not both Horn-definable.

In Section 5.4, we turn our attention to $\mathbf{S5} \times \mathbf{Diff}$ and provide a *full structural characterisation* of the finite frames for $\mathbf{S5} \times \mathbf{Diff}$ (Theorem 5.22), thereby providing a polynomial time algorithm for its finite frame problem, despite the lack of any finite axiomatisation (Theorem 5.1) and the coNExpTime-completeness of its decision problem.

Theorem 5.23. *The finite frame problem for* $\mathbf{S5} \times \mathbf{Diff}$ *is decidable in polynomial time.*

We conclude, in Section 5.5, with a discussion of how the characterisation theorem may be generalised to describe the finite frames for $\mathbf{Diff} \times \mathbf{Diff}$.

Chapter 6: Finite Model Properties

As product logics are 'semantically' defined, there can be other 'non-standard' (that is, *non-product*) frames validating all the same theorems. This is why it is meaningful to consider different versions of the finite model property (fmp) in the context of products of modal logics. A product logic $L_h \times L_v$ is said to have:

- the *fmp* if every $\varphi \notin L_h \times L_v$ is refutable in some finite frame for $L_h \times L_v$;
- the *product fmp* if every $\varphi \notin L_h \times L_v$ is refutable in a finite product frame for $L_h \times L_v$;
- the *square product fmp* if every $\varphi \notin L_h \times L_v$ is refutable in a finite product frame $\mathfrak{F}_h \times \mathfrak{F}_v$ for $L_h \times L_v$, where $\mathfrak{F}_h$ and $\mathfrak{F}_v$ have the same cardinality.

In Chapter 6 we discuss these properties relating to products of the form $L \times \mathbf{Diff}$.

In Section 6.1, we generalise a result of [19], showing that the two-variable fragment of first-order logic with counting quantifiers lacks the finite model property.

Theorem 6.3. *Let L be any bimodal logic such that:*

$$[\mathbf{wK5}, \mathbf{wK5}] \subseteq L \qquad \textit{and} \qquad (\omega, \neq) \times (\omega, \neq) \textit{ is a frame for } L.$$

Then L does not possess the fmp.

In particular, it follows that neither $[\mathbf{Diff}, \mathbf{Diff}]$ nor $\mathbf{Diff} \times \mathbf{Diff}$ can be characterised by their finite frames. We note that $\mathbf{Diff} \times \mathbf{Diff}$ is, therefore, one of the simplest examples of a non-finitely axiomatisable logic without the fmp, that is nonetheless decidable — the decidability of which follows from a straightforward reduction to the two-variable fragment of first-order logic with counting quantifiers.

On the other hand, in Section 6.2 we show the following, by a variation on the well-known quasimodel technique [10]:

Theorem 6.6. $\mathbf{S5} \times \mathbf{Diff}$ *has the exponential product fmp.*

However, there is a fine line: It is shown that any logic extending $\mathbf{S5} \times \mathbf{Diff}$ does not enjoy the square product fmp (Theorem 6.1). These results show that, despite their similarities, the products of **S5** and **Diff** behave very differently with regard to their finite model properties, as summarised below in Table 1.

	$\mathbf{S5} \times \mathbf{S5}$	$\mathbf{S5} \times \mathbf{Diff}$	$\mathbf{Diff} \times \mathbf{Diff}$
fmp	✓	✓	✗
product fmp	✓	✓	✗
square product fmp	✓	✗	✗

Table 1: Comparison of finite model properties for products of **S5** and **Diff**.

Chapter 7: Computational Complexity of Commutators

In Chapter 7, we discuss some decision problems of commutators involving **Diff**. No techniques are yet known for handling the decision problems of commutators that do not coincide with their respective products. In particular, the decidability and complexity of the decision problems for both $[\mathbf{Diff}, \mathbf{Diff}]$ and $[\mathbf{S5}, \mathbf{Diff}]$ remained open — by Theorem 5.1, neither of these logics coincide with their respective products.

In Section 7.1, we introduce a novel approach to obtaining decidability results for the decision problems for each of these logics, together with elementary upper bounds on their respective complexities. We achieve these results with the aid of a recursive satisfiability-preserving reduction of each commutator to their corresponding product logic.

Theorem 7.1. *The decision problems for both* $[\mathbf{S5}, \mathbf{Diff}]$ *and* $[\mathbf{Diff}, \mathbf{Diff}]$ *are decidable.*

By a variation on this technique, we are able to show that, unlike $[\mathbf{Diff}, \mathbf{Diff}]$ (Theorem 6.3), the commutator $[\mathbf{S5}, \mathbf{Diff}]$ *is* characterised by its finite frames (Theorem 7.12).

Chapter 8: Computational Complexity of Products

Product frames are always 'grid-like' by definition. Hence, in those cases where coordinate-wise 'universal' and 'next-time' operators are both available, it becomes straightforward to obtain lower bounds by using reductions from various complex grid-based problems, such as tiling problems or Turing machine problems. For example, it is easy to see that the decision problem for $\mathbf{K}_u \times \mathbf{K}_u$ is undecidable [25], where $\mathbf{K}_u$ is the logic $\mathbf{K}$ of all frames enriched with the *universal modality*.

Previous lower bound proofs [26, 32, 13] on product logics over *transitive* frames overcome the lack of next-time operators by 'diagonally' encoding the $\omega \times \omega$-grid in product models. Here, we develop a novel technique, making direct use of the grid-like structure of product frames, to obtain undecidability results for a host of product logics using reductions from various (*Minsky*) *counter machine* problems.

In particular, in Section 8.2 we introduce the technique for cases when a 'horizontal' next-time operator is still available. We show that $\mathbf{K}_u \times \mathbf{Diff}$ is undecidable but still recursively enumerable (Corollary 8.4). Furthermore, a variation on this technique shows the product of **Diff** and **K** enriched with the *common knowledge* (*transitive closure*) operator is not even analytic. The same is true of $\mathbf{PTL}_{\bigcirc\Box} \times \mathbf{Diff}$, in which $\mathbf{PTL}_{\bigcirc\Box}$ denotes the bimodal *propositional temporal logic* over the natural numbers, with both 'next-time' and 'eventually' operators (Corollary 8.6).

In Section 8.3 we sharpen this technique and discuss products of the form $L \times \mathbf{Diff}$, where L is a unimodal logic (without next-time, with the sole temporal operator 'eventually')

characterised by some class *linear orders*. In particular, we prove that $\mathbf{K4.3} \times \mathbf{Diff}$ is undecidable, while for logics L characterised by various *discrete* linear orders, $L \times \mathbf{Diff}$ is highly undecidable (Corollaries 8.13, 8.16 and 8.36).

Finally, in Section 8.4 we show that the results of Section 8.3 are genuine generalisations of the undecidability results obtained in [26, 32], by giving a polynomial reduction from the decision problem for $L \times \mathbf{Diff}$ to that of $L \times \mathbf{K4.3}$, whenever L is Kripke complete. Note that despite the shared coNP-completeness of the decision problems for both **K4.3** and **Diff**, one cannot hope for a general reverse reduction, since $\mathbf{Diff} \times \mathbf{Diff}$ is decidable, while $\mathbf{Diff} \times \mathbf{K4.3}$ is undecidable.

Chapter 9: First-order Modal Logics and Relativised Products

In Chapter 9, we explore a variation on the standard product construction, motivated by connections between modal product logics of the form $L \times \mathbf{S5}$ and $L \times \mathbf{Diff}$, and various one-variable fragments of first-order modal logics. Owing to philosophical debate as to how we should interpret statements involving both modal operators and first-order quantifiers, investigation into such first-order modal logics has motivated a range of possible semantics, including those in which the domain of interpretation is permitted to either *expand* or *contract*, relative to the direction of the modal accessibility relation. This motivates the consideration of *relativised product logics*, characterised by subframes of product frames that, similarly, expand or contract with respect to a given dimension.

It is easy to see that the decision problems of both expanding and contracting relativised products are reducible to that of the standard products. Here, we investigate whether they can be genuinely simpler. We show, first, that when L is characterised by any class of strict linear orders, the decision problem for $L \times^{dec} \mathbf{Diff}$ over *decreasing* domains has the same complexity as that of its 'constant domain' counterpart $L \times \mathbf{Diff}$ (Theorems 9.6 & 9.8).

In Section 9.5, we turn our attention to relativised products with *expanding* domains. We employ the techniques described in Section 8.3, with the aid of *unreliable* counter machine problems, to obtain lower complexity bounds for various expanding product logics. In particular, we show that over expanding domain models, the decision problem for $L \times^{exp} \mathbf{Diff}$ is non-elementary, whenever L is characterised by any class of strict linear orders containing $(\omega, <)$ and is even non-primitive recursive, for logics characterised by expanding products whose first component is some finite strict linear order (Theorems 9.9 & 9.21, respectively).

These lower bounds are notably weaker than those respective lower bounds obtained in Section 8.3, owing to the lower complexity of unreliable counter machines compared with their reliable counterparts. However, we demonstrate several cases where these bounds are optimal, via model-level reductions to known results. These results are summarised, below, in Table 2.

Chapter 10: Products with a 'Diagonal' Operator

A well-know variation on the standard translation identifies the product logic $\mathbf{S5} \times \mathbf{S5}$ with the two-variable, equality- and substitution-free fragment of first-order logic. In this modal setting, equality can be modelled with an additional modal constant, denoted by δ, interpreted in square frames as the identity ('*diagonal*') relation. In the *cylindric algebra* algebraisation of two-variable first-order logic [22], this diagonal element is denoted by d_{01}. In Chapter 10, we consider the decision problems of products of *arbitrary* modal logics expanded with this

	$(\omega, <)$	all finite strict linear orders	all strict linear orders	all modally discrete strict linear orders
full products	Π^1_1-complete Theorem 8.14	Π^0_1-complete Theorem 8.19	Σ^0_1-complete Theorem 8.8	Π^1_1-hard $\boxed{?}$ Theorem 8.14
decreasing products	Π^1_1-complete Theorems 9.6 & 9.2	Π^0_1-complete Theorems 9.6 & 9.2	Σ^0_1-complete Theorems 9.8 & 9.2	Π^1_1-hard $\boxed{?}$ Theorem 9.6
expanding products	Σ^0_1-complete Theorems 9.6 & 9.2	decidable $\boxed{?}$ ACKERMANN-hard Theorems 9.21 & 9.23	non-ELEM, r.e. $\boxed{\text{decidable?}}$ Theorem 9.9	Σ^0_1-hard $\boxed{\text{r.e.?}}$ Theorem 9.6

Table 2: Complexity of vaious relativised products for several classes of linear orders with **Diff**.

additional feature. We show that, unlike the two-variable fragment whose validity problem remains coNExpTime-complete, both with and without equality [34, 28, 18], the addition of a dimension-joining diagonal constant can often lead to a considerable increase in the complexity of product logics.

In Section 10.2, we first establish a connection between δ-products and regular products. In particular, we show that the *global consequence problem* for certain product logics can be reduced to the decision problem for their respective δ-products (Theorem 10.3). This provides us with undecidable lower bounds for several δ-product logics whose δ-free counterparts are decidable. In particular, we show that the decision problems for the δ-products $\mathbf{K} \times^\delta \mathbf{K4}$ and $\mathbf{K} \times^\delta \mathbf{K}$ are undecidable (Corollary 10.4).

The latter result has a surprising consequence: In [12], the authors introduce the *simple square fragment* $\mathcal{SF}$ as the decidable fragment of first-order logic, whose formulas are constructed from atomic binary predicates $P_i(x, y)$, by freely applying Boolean connectives and relativised quantifiers of the form

$$\exists z\big(R(x, z) \wedge \varphi(z, y)\big) \qquad \text{and} \qquad \exists z\big(R(y, z) \wedge \varphi(x, z)\big),$$

where R is a built-in binary predicate. Unlike the decidable two-variable fragment of first-order logic, $\mathcal{SF}$ is situated within the undecidable three-variable fragment [37, 38]. This simple square fragment is readily seen to be the image of propositional bimodal formulas under the following variation of the standard translation $(\cdot)^*$,

$$p_j^* = P_j(x, y), \quad \text{for } p_i \in \text{PROP}, \qquad (\neg\psi)^* = \neg\psi^*, \qquad (\psi_1 \wedge \psi_2)^* = \psi_1^* \wedge \psi_2^*,$$
$$(\Diamond_h\psi)^* = \exists z\big(R(x, z) \wedge \psi^*(z/x, y)\big), \qquad (\Diamond_v\psi)^* = \exists z\big(R(y, z) \wedge \psi^*(x, z/y)\big),$$

where $R, P_0, P_1, \ldots$ are binary predicate symbols. Moreover, it is straightforward to check that φ is a theorem of $\mathbf{K} \times \mathbf{K}$ if and only if φ^* is a theorem of first-order logic. Since the decision problem for $\mathbf{K} \times \mathbf{K}$ is decidable [11], so too must be the simple square fragment $\mathcal{SF}$. However, by expanding $\mathcal{SF}$ to include equality, the fragment $\mathcal{SF}_\approx$ is readily seen to be the image of the bimodal language with the δ-constant under the extension of $(\cdot)^*$ that interprets $\delta^* = (x \approx y)$. It is similarly straightforward to check that φ is a theorem of $\mathbf{K} \times^\delta \mathbf{K}$ if and

only if φ^* is a theorem of first-order logic with equality. Hence, it follows from Corollary 10.4 that the simple square fragment *with equality* is undecidable. Thus $\mathcal{SF}_{\approx}$ joins the *Gödel class* [15, 17] as yet another example of an undecidable fragment of first-order logic, whose equality-free fragment is decidable.

It turns out that there are limitations to the approach of Section 10.2. Indeed, there are cases where such a reduction is either unhelpful or demonstrably non-existent. For example, while the global consequence problem for $\mathbf{K} \times \mathbf{S5}$ is reducible to the decision problem for $\mathbf{PDL} \times \mathbf{S5}$, where **PDL** denotes Propositional Dynamic Logic, and is thus decidable in coN2ExpTime [41, 33], the decision problem for $\mathbf{K} \times^{\delta} \mathbf{S5}$ is shown to be undecidable (Theorem 10.7). Furthermore, we show in Section 10.5 that the decision problem for $\mathbf{K} \times^{\delta} \mathbf{Alt}$ can be decided in coNExpTime, whereas the undecidability of the global consequence problem for $\mathbf{K} \times \mathbf{Alt}$ can be established by a straightforward reduction from the unconstrained tiling problem [39].

In Section 10.3, we introduce the notion of computation by means of *faulty approximations* as a novel variation on unreliable counter machines. Unlike lossy and incrementing counter machines, we show that computation by faulty approximation is *Turing-complete*. In Section 10.4.1, we exploit the greater flexibility of this new formalism to obtain undecidable lower bounds for a host of δ-products, using a variation on the techniques described in Chapter 8. Among which, we show that the decision problem for $\mathbf{K} \times^{\delta} \mathbf{S5}$ is undecidable, despite the decidability of both the decision problem and the global consequence problem for $\mathbf{K} \times \mathbf{S5}$. This marks a surprising 'jump' from the coNExpTime-complexity of $\mathbf{K} \times \mathbf{S5}$ [25].

In Section 10.4.2, we extend this technique to δ-products in which the first component is characterised by some class of linear orders. In particular, we show that the decision problems for $\mathbf{K4.3} \times^{\delta} \mathbf{K}$ and $\mathbf{K4.3} \times^{\delta} \mathbf{S5}$ are both undecidable (Theorem 10.12). Note that the complexity of the global consequence problem for $\mathbf{K4.3} \times \mathbf{K}$ remains open, while that of $\mathbf{K4.3} \times \mathbf{S5}$ is known to be decidable [31].

We conclude Chapter 10 by probing the limitations of the above counter machine approach and show that the δ-product $\mathbf{K} \times^{\delta} \mathbf{Alt}$ — lying outside the remit of the aforementioned results — has the exponential product fmp and is thus decidable (Theorem 10.17).

3 Conclusion

The thesis leaves open several questions throughout that serve as directions for future research endevours; some have, since publication of this thesis, been successfully addressed such as Questions 6.13–6.14, regarding the product fmp and computational complexity of the decision problem for $\mathbf{K} \times \mathbf{Diff}$. This was addressed in [20], in which $\mathbf{K} \times \mathbf{Diff}$ was shown to share the same coNExpTime-completeness as $\mathbf{K} \times \mathbf{S5}$. Indeed, a stronger result was shown: that the one-variable fragment of first-order modal logic **K** with *arbitrary* counting quantifiers, denoted $\mathbf{Q}^{\#}\mathbf{K}$, is coNExpTime-complete — the product logic $\mathbf{K} \times \mathbf{Diff}$ being a syntactic variant of the special case in which we allow only the counting quantifiers $\exists_{\leq 0} x$ and $\exists_{\leq 1} x$.

Another forthcoming result concerns the axiomatisation of $\mathbf{S5} \times \mathbf{Diff}$, shown to be non-finitely axiomatisable in Theorem 5.1. Based on the characterisation of its frames given in Section 5.4, we were able to construct an infinite *explicit* axiomatisation for $\mathbf{S5} \times \mathbf{Diff}$ [21]. This is the first explicit axiomatisation for a non-finitely axiomatisable product logic that is 'orthodox' in the sense that it uses only the 'traditional' rules of Substitution, Modus Ponens and Necessitation, and does not use any 'irreflexivity-type' rules. Owing to the more intricate

structure of the frames for $\mathbf{Diff} \times \mathbf{Diff}$ (see Section 5.5), an explicit 'orthodox' axiomatisation of the recursively enumerable product logic $\mathbf{Diff} \times \mathbf{Diff}$ remains subject to future work.

References

[1] Carlos Areces, Guillaume Hoffmann, and Alexandre Denis. Modal logics with counting. In Anuj Dawar and Ruy de Queiroz, editors, *Logic, Language, Information and Computation*, volume 6188 of *Lecture Notes in Artificial Intelligence*, pages 98–109. Springer, 2010.

[2] Franz Baader, Diego Calvanese, Deborah L. McGuinness, Daniele Nardi, and Peter F. Patel-Schneider. *The Description Logic Handbook: Theory, Implementation, and Applications*. Cambridge University Press, New York, NY, USA, 2003.

[3] Franz Baader and Armin Laux. Terminological logics with modal operators. In *Proceedings of the 14th International Joint Conference on Artificial Intelligence, IJCAI 95*, volume 1, pages 808–815. Morgan Kaufmann, 1995.

[4] Franz Baader and Hans Jürgen Ohlbach. A multi-dimensional terminological knowledge representation language. *Journal of Applied Non-Classical Logics*, 5(2):153–197, 1995.

[5] Patrick Blackburn, Maarten de Rijke, and Yde Venema. *Modal Logic*. Cambridge University Press, New York, NY, USA, 2001.

[6] Maarten de Rijke. The modal logic of inequality. *The Journal of Symbolic Logic*, 57(2):566–584, 1992.

[7] Maarten de Rijke. *Extending Modal Logic*. PhD thesis, Institute of Logic, Language and Computation, University van Amsterdam, 1993.

[8] Ronald Fagin, Joseph Y. Halpern, Yoram Moses, and Moshe Y. Vardi. *Reasoning about Knowledge*. MIT Press, 1995.

[9] Marcelo Finger and Mark Reynolds. Imperative history: Two-dimensional executable temporal logic. In Hans Jürgen Ohlbach and Uwe Reyle, editors, *Logic, Language and Reasoning*, volume 5 of *Trends in Logic*, pages 73–98. Springer, 1999.

[10] Dov M. Gabbay, Agi Kurucz, Frank Wolter, and Michael Zakharyaschev. *Many-Dimensional Modal Logics: Theory and Applications: Theory and Applications*, volume 148 of *Studies in Logic and the Foundations of Mathematics*. Elsevier, 2003.

[11] Dov M. Gabbay and Valentin Shehtman. Products of modal logics. Part I. *Logic Journal of the IGPL*, 6(1):73–146, 1998.

[12] Dov M. Gabbay and Valentin Shehtman. Products of modal logics. Part II: Relativised quantifiers in classical logic. *Logic Journal of the IGPL*, 8(2):165–210, 2000.

[13] David Gabelaia, Agi Kurucz, Frank Wolter, and Michael Zakharyaschev. Products of 'transitive' modal logics. *The Journal of Symbolic Logic*, 70(3):993–1021, 2005.

[14] George Gargov and Valentin Goranko. Modal logic with names. *Journal of Philosophical Logic*, 22(6):607–636, 1993.

[15] Kurt Gödel. Ein Interpretation des intuitionistischen Aussagenkalkülus. *Ergebnisse eines mathematischen Kolloquiums*, 4:34–40, 1933.

[16] Robert Goldblatt. Mathematical modal logic: A view of its evolution. *Journal of Applied Logic*, 1(5–6):309–392, 2003.

[17] Warren Goldfarb. The unsolvability of the Gödel class with identity. *The Journal of Symbolic Logic*, 49(4):1237–1252, 1984.

[18] Erich Grädel, Phokion G. Kolaitis, and Moshe Y. Vardi. On the decision problem for two-variable first-order logic. *Bulletin of symbolic logic*, 3(1):53–69, 1997.

[19] Erich Grädel, Martin Otto, and Eric Rosen. Two-variable logic with counting is decidable. In *Proceedings of the 12th Annual IEEE Symposium on Logic in Computer Science (LICS'97)*, pages 306–317. IEEE, 1997.

[20] Christopher Hampson. Decidable first-order modal logics with counting quantifiers. In András Máté Lev Beklemishev, Stéphane Demri, editor, *Advances in Modal Logic*, volume 11, pages 383–400. College Publications, 2016.

[21] Christopher Hampson, Stanislav Kikot, Agi Kurucz, and Sérgio Marcelino. Axiomatisation problems of modal product logics with the difference operator. 2016. *(unpublished manuscript)*.

[22] Léon Henkin, James D. Monk, and Alfred Tarski. *Cylindric Algebras: Part I*, volume 64 of *Studies in Logic and the Foundations of Mathematics*. North-Holland Publishing Company, 1971.

[23] Agi Kurucz. Combining modal logics. In Patrick Blackburn, Johan van Benthem, and Frank Wolter, editors, *Handbook of Modal Logic*, volume 3 of *Studies in Logic and Practical Reasoning*, pages 869–924. Elsevier, 2007.

[24] Agi Kurucz and Sérgio Marcelino. Non-finitely axiomatisable two-dimensional modal logics. *The Journal of Symbolic Logic*, 77(3):970–986, 2012.

[25] Maarten Marx. Complexity of products of modal logics. *Journal of Logic and Computation*, 9(2):197–214, 1999.

[26] Maarten Marx and Mark Reynolds. Undecidability of compass logic. *Journal of Logic and Computation*, 9(6):897–914, 1999.

[27] Maarten Marx and Yde Venema. *Multi-dimensional modal logic*. Springer, 1997.

[28] Michael Mortimer. On languages with two variables. *Mathematical Logic Quarterly*, 21(1):135–140, 1975.

[29] Leszek Pacholski, Wieslaw Szwast, and Lidia Tendera. Complexity results for first-order two-variable logic with counting. *SIAM J. Comput.*, 29:1083–1117, 2000.

[30] John Reif and Aravinda Prasad Sistla. A multiprocess network logic with temporal and spatial modalities. In Josep Diaz, editor, *Automata, Languages and Programming*, volume 154 of *Lecture Notes in Computer Science*, pages 629–639. Springer, 1983.

[31] Mark Reynolds. A decidable temporal logic of parallelism. *Notre Dame Journal of Formal Logic*, 38(3):419–436, 1997.

[32] Mark Reynolds and Michael Zakharyaschev. On the products of linear modal logics. *Journal of Logic and Computation*, 11(6):909–931, 2001.

[33] Renate A. Schmidt and Dmitry Tishkovsky. Combining dynamic logic with doxastic modal logics. In Philippe Balbiani, Nobu-Yuki Suzuki, Frank Wolter, and Michael Zakharyaschev, editors, *Advances in Modal Logic*, volume 4, pages 371–391. King's College Publications, 2003.

[34] Dana Scott. A decision method for validity of sentences in two variables. *The Journal of Symbolic Logic*, 27:477, 1962.

[35] Krister Segerberg. Two-dimensional modal logic. *Journal of Philosophical Logic*, 2(1):77–96, 1973.

[36] Valentin Shehtman. Two-dimensional modal logics. *Mathematical Notes of the USSR Academy of Sciences*, 23(5):417–424, 1978. (Translated from Russian).

[37] János Surányi. Zur Reduktion des Entscheidungsproblems des logischen Funktionenkalküls. *Matematikai és Fizikai Lapok*, 50:51–74, 1943.

[38] János Surányi. Contributions to the reduction theory of the decision problem. *Acta Mathematica Academiae Scientiarum Hungarica*, 1(2–4):261–271, 1950.

[39] Peter van Emde Boas. The convenience of tilings. In Andrea Sorbi, editor, *Complexity, Logic, and Recursion Theory*, Lecture Notes in Pure and Applied Mathematics, pages 331–363. Marcel Dekker, Inc., 1997.

[40] Georg H. von Wright. A modal logic of place. In Ernest Sosa, editor, *The philosophy of Nicolas Rescher*, pages 65–73. Dordrecht, 1979.

[41] Frank Wolter. The product of converse **PDL** and polymodal **K**. *Journal of Logic and Computation*, 10(2):223–251, 2000.

[42] Frank Wolter and Michael Zakharyaschev. Temporalizing description logics. In Dov M. Gabbay and Maarten de Rijke, editors, *Frontiers of Combining Systems 2*, pages 104–109, 1998.

[43] Frank Wolter and Michael Zakharyaschev. Modal description logics: modalizing roles. *Fundamenta Informaticae*, 39(4):411–438, 1999.

[44] Frank Wolter and Michael Zakharyaschev. Dynamic description logic. In Michael Zakharyaschev, Krister Segerberg, Maarten de Rijke, and Heinrich Wansing, editors, *Advances in Modal Logic*, volume 2, pages 431–445. CSLI Publications, 2001.