$\mathbf{S5} \times \mathbf{Diff}$ has the exponential product model property

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Abstract

It is known that the two-variable fragment of first-order logic with equality is not only coNExpTime-complete but it enjoys the exponential finite model property (fmp) [6,3]. When extended with counting quantifiers, it retains its coNExpTime decidability [4,7,8], albeit at the expense of losing its fmp. Here we investigate some logics in between these two in the modal setting. We show that the product modal logic $\mathbf{S5} \times \mathbf{Diff}$ (where $\mathbf{Diff}$ is the logic of difference frames) enjoys the exponential fmp w.r.t. to product frames. The result is optimal in the sense that extending $\mathbf{S5} \times \mathbf{Diff}$ with a modal 'diagonal' constant (interpreted as equality in the first-order setting) results in a logic lacking even the 'abstract' fmp [5].

Keywords: modal product logics, difference operator, exponential model property, quasimodels

1 Introduction

The product construction for modal logics is connected to many other logical formalisms and has been studied extensively (see [2,1] for surveys and references). The product of two Kripke frames $\mathfrak{F}_i = (W_i, R_i)$, for $i = 1, 2$, is taken to be the bimodal frame $\mathfrak{F}_1 \times \mathfrak{F}_2 = (W_1 \times W_2, R_h, R_v)$ where $W_1 \times W_2$ is the Cartesian product of the W_i and, for all $x, x' \in W_1$, $y, y' \in W_2$,

$$(x, y) R_h (x', y') \iff x R_1 x' \text{ and } y = y',$$
$$(x, y) R_v (x', y') \iff y R_2 y' \text{ and } x = x'.$$

For any two Kripke complete modal logics L_1 and L_2 (formulated in the language having $\Box_h$ and $\Box_v$, respectively), their product is defined to be

$$L_1 \times L_2 = \{\text{bimodal formula } \phi : \mathfrak{F}_1 \times \mathfrak{F}_2 \models \phi \text{ for all } \mathfrak{F}_i \text{ with } \mathfrak{F}_i \models L_i,\ i = 1, 2\}.$$

Here we study products with components among $\mathbf{S5}$ (the logic of universal frames) and $\mathbf{Diff}$ (the logic of *difference frames* $(W, \neq)$). While $\mathbf{Diff}$ clearly includes $\mathbf{S5}$, it is more expressive: with the difference operator one can express counting up to 2, and so the uniqueness of properties. The validity problem for each of the logics $L_1 \times L_2$, for $L_i \in \{\mathbf{S5}, \mathbf{Diff}\}$, is coNExpTime-complete, owing to a reduction to the two-variable fragment of first-order logic with counting quantifiers [4,7,8]. However, while the two-variable fragment (without counting) enjoys the exponential finite model property [6,3] (and so $\mathbf{S5} \times \mathbf{S5}$ has the exponential model property w.r.t. product models), $\mathbf{Diff} \times \mathbf{Diff}$ lacks even the *abstract* finite model property [5]. Here we show that the intermediate logic $\mathbf{S5} \times \mathbf{Diff}$ has the exponential product model property, akin to $\mathbf{S5} \times \mathbf{S5}$. Our result is optimal in the sense that extending $\mathbf{S5} \times \mathbf{Diff}$ with a modal 'diagonal' constant (interpreted as equality in the first-order setting) results in a logic lacking even the abstract fmp [5].

It remains open whether $\mathbf{K} \times \mathbf{Diff}$ has the product fmp (with a 'smallish' bound on the size of the product models). In particular, we do not know whether the techniques presented here can be combined with the techniques used in [10,1] proving the product fmp for $\mathbf{K} \times \mathbf{K}$ and $\mathbf{K} \times \mathbf{S5}$. Note that as the difference operator is term definable in the bimodal logic $\mathbf{Lin}_{FP}$ of all strict linear orders, decidability of $\mathbf{K} \times \mathbf{Diff}$ follows from that of $\mathbf{K} \times \mathbf{Lin}_{FP}$ [9,1]. However, all known upper bounds, even for $\mathbf{K} \times \mathbf{K4.3}$, are non-elementary.

2 Main results

Theorem 2.1 $\mathbf{S5} \times \mathbf{Diff}$ *has the exponential product model property.*

We prove this theorem by employing a version of the method of *quasimodels* [10,1]. We begin by adapting the usual notions to the $\mathbf{S5} \times \mathbf{Diff}$ case. Then, for any given formula ϕ, we demonstrate a procedure by which large quasimodels for ϕ may be reduced to a more desirable size, exponential in $n = |\mathrm{sub}(\phi)|$, where $\mathrm{sub}(\phi)$ is the set of subformulas of ϕ. The new ideas are on pages 4–5.

In what follows, a *type* for ϕ will be any Boolean saturated subset of the set $\mathrm{sub}(\phi)$. A **Diff**-*quasistate* for ϕ is a pair (T, S) such that T is a finite set of types, S is a binary relation on T containing $\neq$, and such that for all $x \in T$, $\Diamond_v \alpha \in \mathrm{sub}(\phi)$, $\Diamond_v \alpha \in x$ iff $\exists y \in T;\ xSy$ and $\alpha \in y$. We define a *basic structure* for ϕ to be a pair $(W, \mathbf{q})$ where W is a non-empty set and $\mathbf{q}$ is a function associating each $w \in W$ with a quasistate $\mathbf{q}(w) = (T_w, S_w)$. A *run through* $(W, \mathbf{q})$ is a function f associating each $w \in W$ with a type $f(w) \in T_w$.

An $\mathbf{S5} \times \mathbf{Diff}$-*quasimodel for* ϕ (or, hereafter, simply a *quasimodel* for ϕ) is a tuple $\mathfrak{Q} = (W, \mathbf{q}, \mathfrak{R}, \langle\cdot\rangle)$ such that

(qm1) $(W, \mathbf{q})$ is a basic structure for ϕ, $\mathfrak{R}$ is a non-empty index set, $\langle\cdot\rangle$ is a function associating each index $r \in \mathfrak{R}$ with a run $\langle r\rangle$ through $(W, \mathbf{q})$, and $\phi \in \langle r_0\rangle(w_0)$ for some $w_0 \in W$ and $r_0 \in \mathfrak{R}$.

(qm2) *(coherence)* For all $r \in \mathfrak{R}$, $w \in W$ and $\Diamond_h \alpha \in \mathrm{sub}(\phi)$

$$\exists v \in W;\ \alpha \in \langle r\rangle(v) \quad \Longrightarrow \quad \Diamond_h \alpha \in \langle r\rangle(w).$$

(qm3) *(saturation)* For all $r \in \mathfrak{R}$, $w \in W$ and $\Diamond_h \alpha \in \mathrm{sub}(\phi)$

$$\Diamond_h \alpha \in \langle r\rangle(w) \quad \Longrightarrow \quad \exists v \in W;\ \alpha \in \langle r\rangle(v).$$

(qm4) For all $r \in \mathfrak{R}$, $w \in W$ and $t \in T_w$, if $\langle r \rangle(w) S_w t$ then there is some $r' \in \mathfrak{R}$ such that $r \neq r'$ and $\langle r' \rangle(w) = t$.

(qm5) For all $w \in W$ and $r, r' \in \mathfrak{R}$, if $r \neq r'$ then $\langle r \rangle(w) S_w \langle r' \rangle(w)$.

Note that distinct indices in $\mathfrak{R}$ may correspond to identical runs through $(W, \mathbf{q})$. Also, whenever a type $t \in T_w$ is such that $\neg(t S_w t)$, then there is a *unique* $r \in \mathfrak{R}$ with $\langle r \rangle(w) = t$. So we need to be careful when constructing an $\mathbf{S5} \times \mathbf{Diff}$-quasimodel step-by-step, or from smaller pieces.

Lemma 2.2 *ϕ is $\mathbf{S5} \times \mathbf{Diff}$-satisfiable iff there is a quasimodel for ϕ.*

Proof. Suppose that ϕ is $\mathbf{S5} \times \mathbf{Diff}$-satisfiable, then there is some product model $\mathfrak{M} = (\mathfrak{F}_1 \times \mathfrak{F}_2, \mathfrak{V})$ such that $\mathfrak{M}, (r_h, r_v) \models \phi$, where $\mathfrak{F}_1 = (W_1, W_1 \times W_1)$ and $\mathfrak{F}_2 = (W_2, \neq)$ is a difference frame. With every pair $(x, y) \in W_1 \times W_2$ we associate the type $t(x, y) = \{\alpha \in \mathrm{sub}(\phi) : \mathfrak{M}, (x, y) \models \alpha\}$, and define the basic structure $(W_1, \mathbf{q})$ by taking $\mathbf{q}(x) = (T_x, S_x)$, where

$$T_x = \{t(x, y) : y \in W_2\} \quad \text{and} \quad t(x, y) S_x t(x, y') \quad \iff \quad y \neq y'.$$

for all $x \in W_1$. For each $y \in W_2$ we associate the run $\langle y \rangle$ by taking $\langle y \rangle(x) = t(x, y)$, for every $x \in W_1$. It is straightforward to check that $(W_1, \mathbf{q}, W_2, \langle \cdot \rangle)$ is a quasimodel for ϕ.

Conversely, suppose that $\mathfrak{Q} = (W, \mathbf{q}, \mathfrak{R}, \langle \cdot \rangle)$ is a quasimodel for ϕ, from which we define the model $\mathfrak{M} = (\mathfrak{F}, \mathfrak{V})$, by taking

$$\mathfrak{F} = (W, W^2) \times (\mathfrak{R}, \neq) \tag{1}$$

and $\mathfrak{V}(p) = \{(w, r) : p \in \langle r \rangle(w)\}$, for all $p \in \mathrm{sub}(\phi)$. It is straightforward to check that ϕ is satisfiable in $\mathfrak{M}$. $\square$

It remains to show that large quasimodels for ϕ can be reduced to smaller quasimodels bounded in size by a single exponential function in n. For this, suppose $\mathfrak{Q} = (W, \mathbf{q}, \mathfrak{R}, \langle \cdot \rangle)$ is an arbitrarily large quasimodel for ϕ. By **(qm1)** there is some $w_0 \in W$ and some $r_0 \in \mathfrak{R}$ such that $\phi \in \langle r_0 \rangle(w_0)$. By **(qm4)** we may fix, for each $t \in T_{w_0}$, some $s_t \in \mathfrak{R}$ such that $\langle s_t \rangle(w_0) = t$, and take $\mathfrak{S} = \{s_t : t \in T_{w_0}\}$. Now, courtesy of **(qm3)**, for all $t \in T_{w_0}$ and $\Diamond_h \alpha \in t$ we may fix some $v_{(t,\alpha)} \in W$ such that $\alpha \in \langle s_t \rangle(v_{(t,\alpha)})$. We then define a new basic structure $(W_1, \mathbf{q}_1)$ by taking

$$\begin{aligned} W_1 &= \{w_0\} \cup \{v_{(t,\alpha)} \in W : t \in T_{w_0} \text{ and } \Diamond_h \alpha \in t\}, \\ \mathbf{q}_1(w) &= \mathbf{q}(w) \quad \text{for } w \in W_1, \end{aligned}$$

through which each $\langle s_t \rangle$ is coherent and saturated. However $\mathfrak{S}$ need not be plentiful to accomodate **(qm4)**. To remedy this, we extend $\mathfrak{S}$ to a 'small' subset $\mathfrak{R}_1$ of $\mathfrak{R}$ by choosing, for all $w \in W_1$ and $t \in T_w$, some $r_{(w,t)} \in \mathfrak{R}$ such that $\langle r_{(w,t)} \rangle(w) = t$, and additional $r'_{(w,t)} \in \mathfrak{R}$, whenever $t S_w t$, such that $r_{(w,t)} \neq r'_{(w,t)}$ and $\langle r'_{(w,t)} \rangle(w) = t$. We then define $\langle \cdot \rangle_1$ for all $r \in \mathfrak{R}_1$ by taking $\langle r \rangle_1(w) = \langle r \rangle(w)$, $w \in W_1$. It is readily observed that

$$|W_1| \leq 1 + 2^n \cdot n \qquad \text{and} \qquad |\mathfrak{R}_1| \leq (1 + 2^n \cdot n) \cdot 2^{n+1}. \tag{2}$$

From here on we diverge from the techniques of [10,1] that were used to show the product fmp for various products with **S5**. We begin by introducing, not one but *many*, 'copies' of each quasistate $\mathbf{q}(u)$, $u \in W_1$; one for each $r \in \mathfrak{R}_1$: We take

$$W_2 = W_1 \cup \{(u,r) : u \in W_1 \text{ and } r \in \mathfrak{R}_1\},$$

and define $\mathbf{q}_2$ and $\langle s \rangle_2$, for all $s \in \mathfrak{R}_1$, on the 'original' points as before, and on the 'copies' by $\mathbf{q}_2((u,r)) = \mathbf{q}_1(u)$ and $\langle s \rangle_2((u,r)) = \langle s \rangle_1(u)$.

It is easy to see that $\mathfrak{Q}_2 = (W_2, \mathbf{q}_2, \mathfrak{R}_1, \langle\cdot\rangle_2)$ satisfies all conditions **(qm1)**–**(qm5)** except **(qm3)**. Furthermore, we note that

$$|W_2| \leq |W_1| \cdot (1 + |\mathfrak{R}_1|). \tag{3}$$

We will now define runs $\langle r \rangle^*$ through $(W_2, \mathbf{q}_2)$, for each $r \in \mathfrak{R}_1$, such that all of **(qm1)**–**(qm5)** are satisfied. The idea is not to introduce new runs by copy-and-pasting old ones (as this might interfere with certain types being uniquely bound to a run), but keeping the number of runs through each type the same, and producing 'new' saturated runs by interchanging values between 'old' ones. To this end, for each $t \in T_{w_0}$, we let

$$\Omega_t = \{(r,\alpha) : r \in \mathfrak{R}_1,\ \langle r \rangle(w_0) = t \text{ and } \Diamond_h \alpha \in t\}.$$

By definition, for each $(r,\alpha) \in \Omega_t$, there is $v_{(t,\alpha)} \in W_1$ such that $\alpha \in \langle s_t \rangle_2(v_{(t,\alpha)})$. For each $r \in \mathfrak{R}_1 - \mathfrak{S}$, we define $\langle r \rangle^*$ by taking, for all $w \in W_2$,

$$\langle r \rangle^*(w) = \begin{cases} \langle s_t \rangle_2(w) & \text{if } (r,\alpha) \in \Omega_t \text{ and } w = (v_{(t,\alpha)}, r), \\ \langle r \rangle_2(w) & \text{otherwise.} \end{cases}$$

And, for each $s_t \in \mathfrak{S}$, we define $\langle s_t \rangle^*$ by taking, for all $w \in W_2$,

$$\langle s_t \rangle^*(w) = \begin{cases} \langle r \rangle_2(w) & \text{if } w = (v_{(t,\alpha)}, r) \text{ for some } (r,\alpha) \in \Omega_t, \\ \langle s_t \rangle_2(w) & \text{otherwise.} \end{cases}$$

This is to say that we interchange the values of $\langle r \rangle_2$ and $\langle s_t \rangle_2$ at the 'copy' $(v_{(t,\alpha)}, r)$ whenever $(r,\alpha) \in \Omega_t$. Now take $\mathfrak{Q}' = (W_2, \mathbf{q}_2, \mathfrak{R}_1, \langle\cdot\rangle^*)$.

By (2) and (3), the size of $\mathfrak{Q}'$ is exponential in n, so it suffices to show that $\mathfrak{Q}'$ is a quasimodel for ϕ. Clearly $\mathfrak{Q}'$ retains property **(qm1)**. Since we only interchange values between two runs when they meet at w_0, we never sacrifice coherence. For saturation: on the one hand, the saturated runs $\langle s_t \rangle_2$ for $s_t \in \mathfrak{S}$ remain saturated after several interchanges, as in $\langle s_t \rangle^*$ we always keep the values at 'original' non-copy quasistates intact. It remains to show that runs $\langle r \rangle^*$ for $r \in \mathfrak{R}_1 - \mathfrak{S}$ all become saturated as a result of interchanging values. So suppose that $r \in \mathfrak{R}_1 - \mathfrak{S}$, $w \in W_2$ and $\Diamond_h \alpha \in \langle r \rangle^*(w)$, and let $t = \langle r \rangle^*(w_0) = \langle r \rangle(w_0)$. Then by coherence and saturation of both $\langle r \rangle$ and $\langle s_t \rangle$, we have $\Diamond_h \alpha \in \langle s_t \rangle(w_0) = t$, and so $(r,\alpha) \in \Omega_t$ and $\alpha \in \langle s_t \rangle(v_{(t,\alpha)}) = \langle s_t \rangle_2((v_{(t,\alpha)}, r)) = \langle r \rangle^*((v_{(t,\alpha)}, r))$ (see Figure 1). Finally, conditions **(qm4)** and **(qm5)** for $\mathfrak{Q}'$ follow from the observation that they held in $\mathfrak{Q}_2$, and when defining $\langle\cdot\rangle^*$ from $\langle\cdot\rangle_2$ by swapping values, the number of runs passing through each type remains unchanged.

Theorem 2.1 now follows readily from our construction of product models from quasimodels in (1), above.

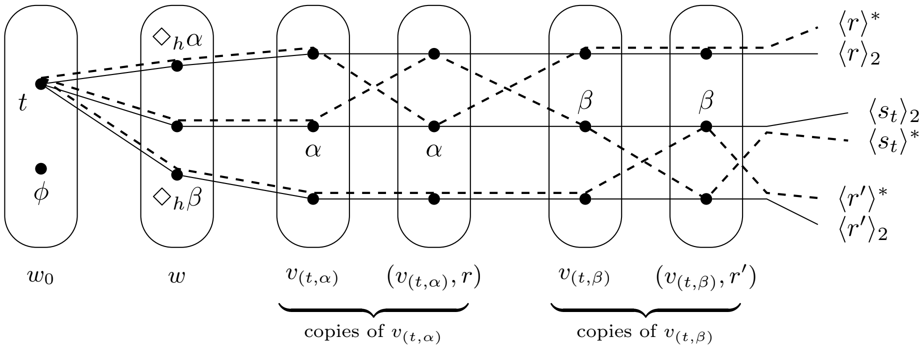

Fig. 1. Result of swapping the values of $\langle\cdot\rangle_2$

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