Computational modeling

Lecture 9 : The Tacoma Bridge

Physics:

The *Tacoma* bridge

Programming:

Code explained

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Course : 4CCP1000

Today we write a model for …

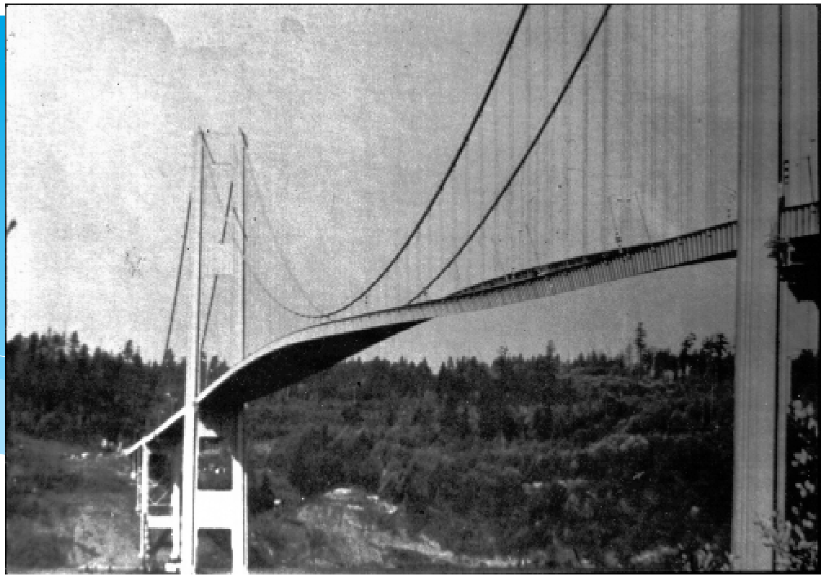
Tacoma Bridge

- Shortly after its construction in July, it was discovered that the bridge would sway and buckle dangerously in windy conditions.

- The failure of the bridge occurred when a torsional twisting mode occurred.

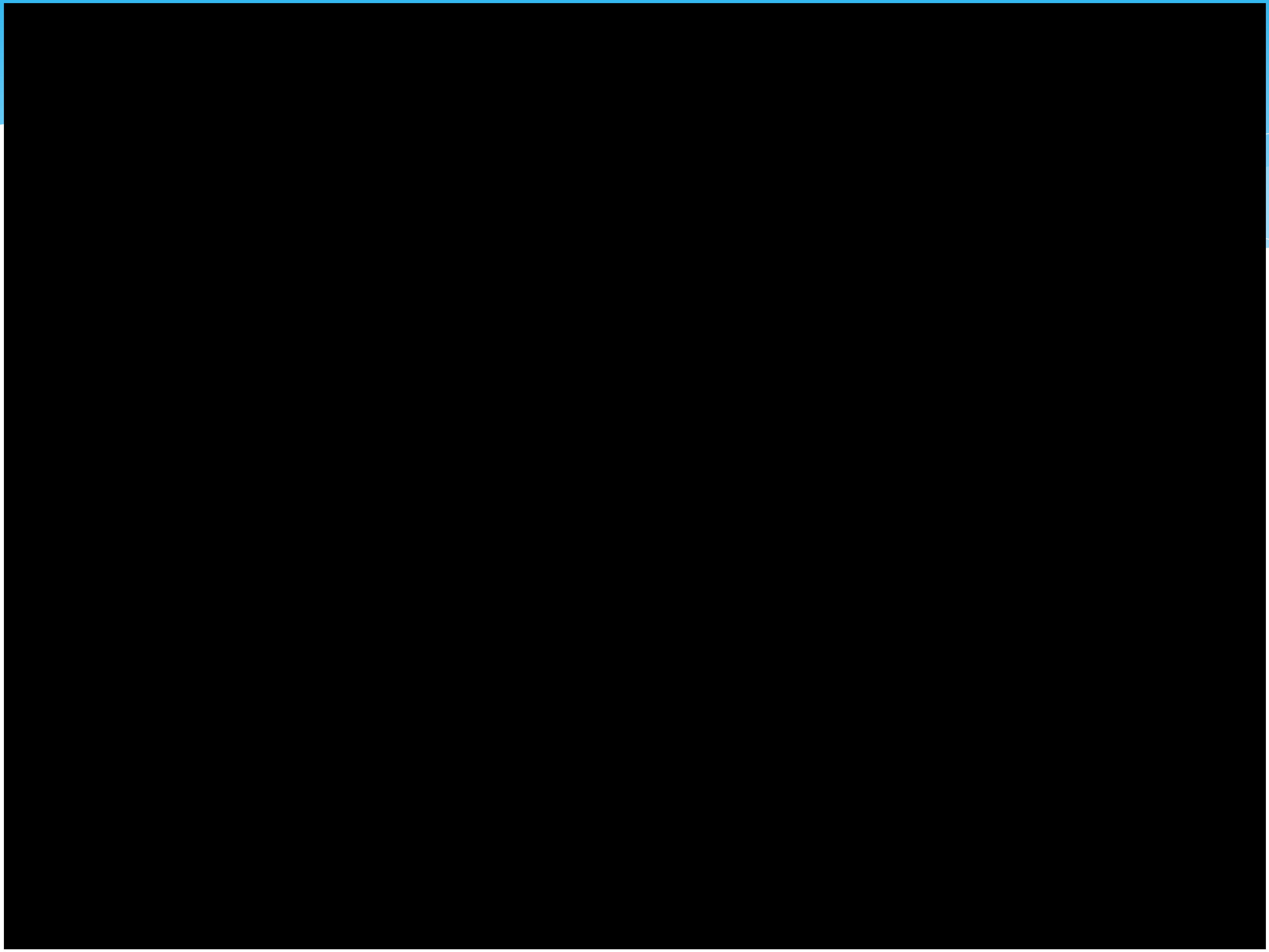
- The bridge's spectacular self-destruction is often used as an object lesson in the necessity to consider both aerodynamics and resonance effects

- Luckily : only one car was on the bridge at this time. The passenger was injured but alive. The only casualty was his dog.



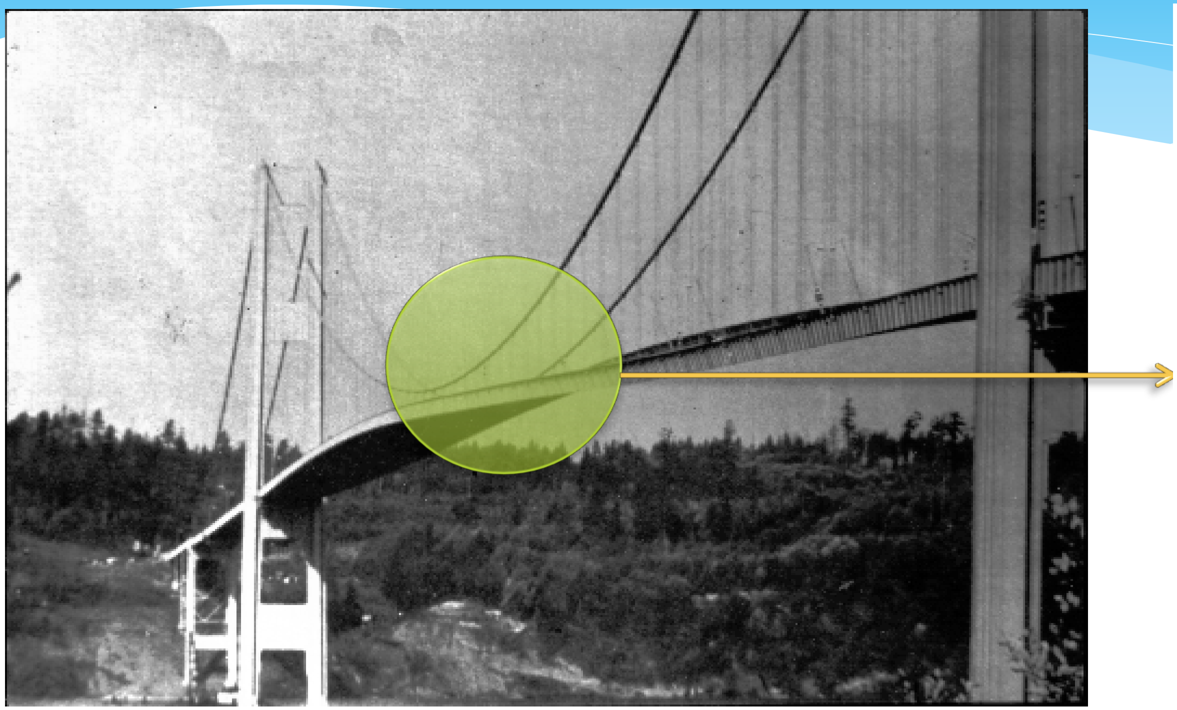
* Reference: Numerical analysis / Timothy Sauer

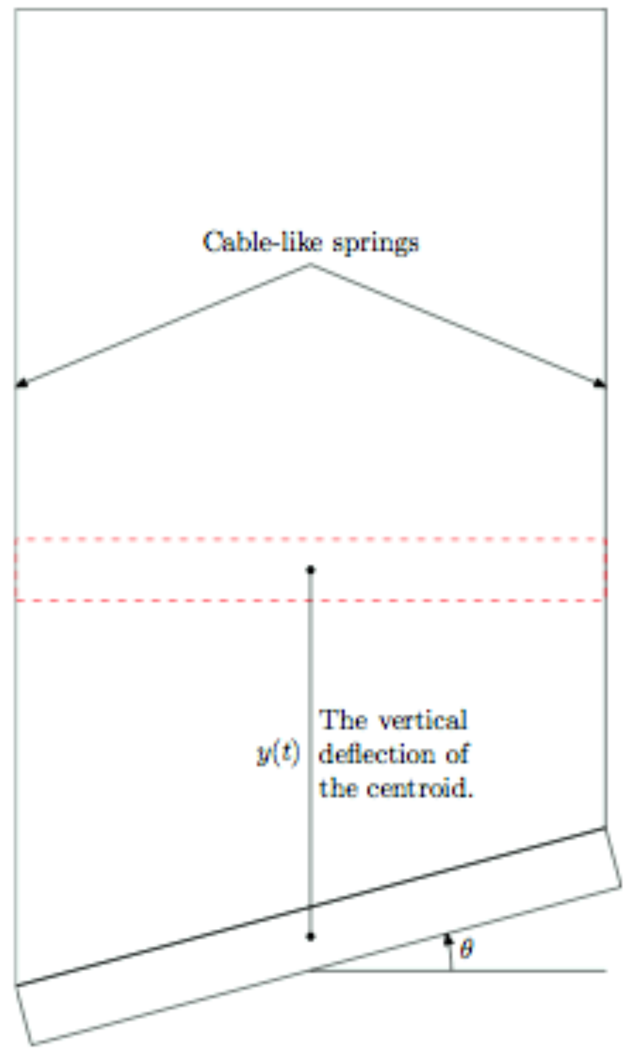
Tacoma Bridge



Tacoma bridge

* Simple model, we take a slice of the bridge, suspended by two cables :





* From the movie, can you identify which are the relevant parameters ?

* What type of oscillations do we need to describe ?

Simple model, two variables

- Consider a **roadway of width 2*l** suspended by two cables

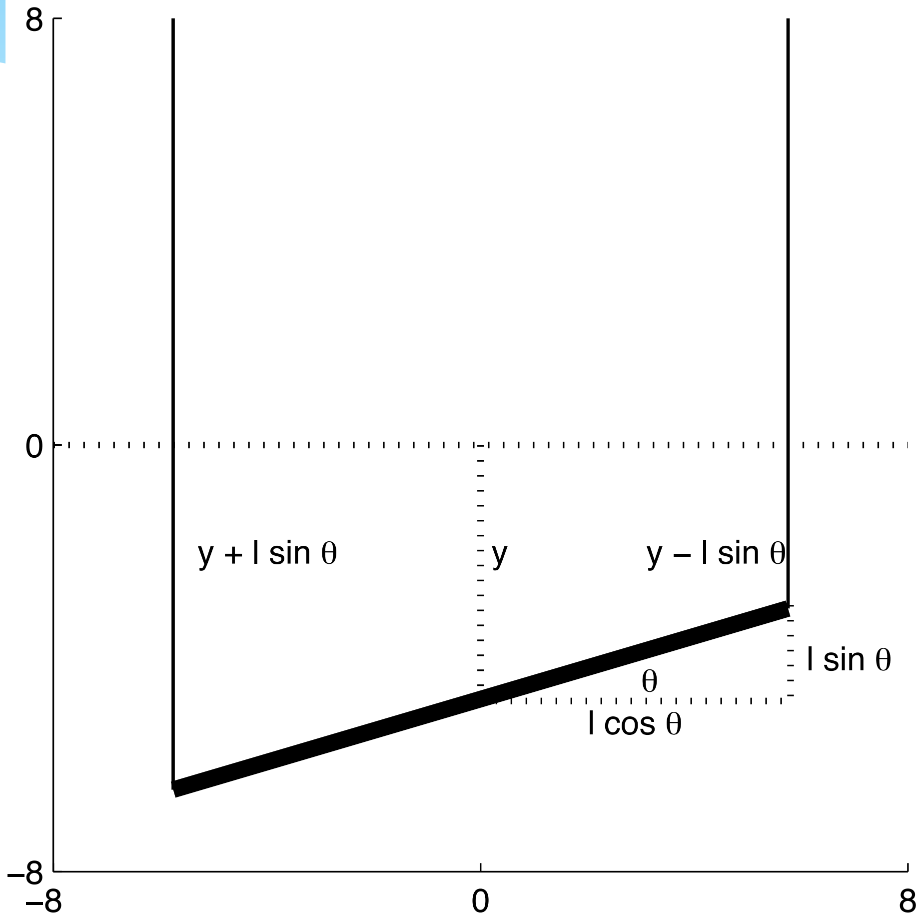
- We model the dynamic of the bridge with two functions, $y(t)$ and $\theta(t)$.

- Θ = torsion angle
- y = vertical elevation
- Rest position : $\theta=0$, $y=0$

- Hooke's law : restoring force in the cables will be proportional to **y**

- There are two suspension cables, stretched from equilibrium by :

$$y - l \sin\theta \quad \text{and} \quad y + l \sin\theta$$



Modeling the bridge oscillations

- Differential equations:

d * y', d is a coefficient

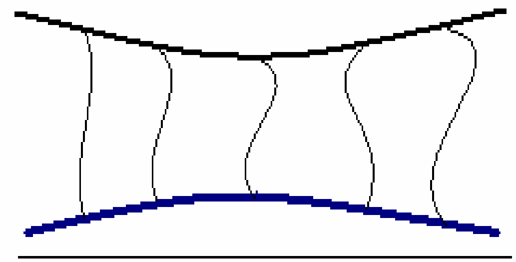
$$
\begin{aligned}
y'' &= -dy' - \left[\frac{K}{m}(y - l\sin\theta) + \frac{K}{m}(y + l\sin\theta)\right] \\
\theta'' &= -d\theta' + \frac{3\cos\theta}{l}\left[\frac{K}{m}(y - l\sin\theta) - \frac{K}{m}(y + l\sin\theta)\right]
\end{aligned}
$$

- K is the Hook's constant (F=Kx, where x is displacement, as in a spring)
- d : dissipative constant [d*y' and d*θ' : dissipative terms]

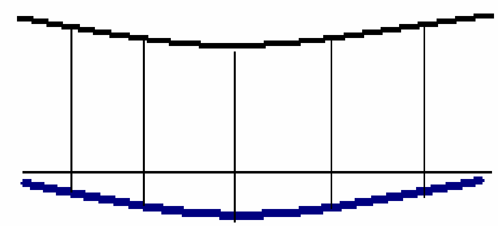
Cables described by springs ?

* **Modelling the cable as spring, is it realistic ?**

* **Minor modification**: Hook's spring law was used here, but indeed the cables of the bridge are not like a spring. They do not have a strong resistance once compressed.



Cables slack because roadbed is above rest position



Cables taut, roadbed below rest position

* We replace the Hook's terms with a similar term, but which only has resistance in one direction :

$$f(y) = (K/a)(e^{ay} - 1)$$

Tacoma bridge equations

* We obtain the final differential equations:

$$\begin{aligned} y'' &= -dy' - \frac{K}{ma}\left[e^{y-l\sin\theta} - 1 + e^{y+l\sin\theta} - 1\right] \\ \theta'' &= -d\theta' + \frac{3\cos\theta}{l}\frac{K}{ma}\left[e^{y-l\sin\theta} - e^{y+l\sin\theta}\right]. \end{aligned}$$

* We start at equilibrium (no wind, bridge stable) : $(y,\theta) = (0,0)$

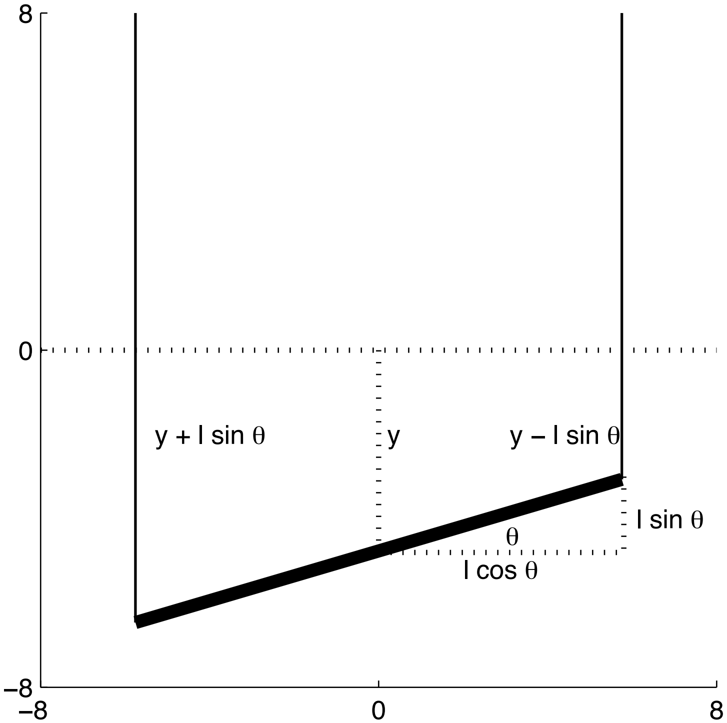
* Now we turn on the wind, this adds the following term to the right hand side of the y(t) equation (the wind is modeled by a periodic Force) :

$$A\sin\omega t$$

* <u>**Practice session**</u>: modify the pendulum code to treat these two coupled second order differential equation. Understand what is the reason why the Tacoma bridge collapsed.

Tacoma bridge

$$\begin{aligned} y'' &= -dy' - \frac{K}{ma}\left[e^{y-l\sin\theta} - 1 + e^{y+l\sin\theta} - 1\right] \\ \theta'' &= -d\theta' + \frac{3\cos\theta}{l}\frac{K}{ma}\left[e^{y-l\sin\theta} - e^{y+l\sin\theta}\right]. \end{aligned}$$



* Initial condition : $$\begin{aligned} (y,\theta) &= (0,0) \\ (y',\theta') &= (0,0) \end{aligned}$$

* Now we turn on the wind, this adds to the right hand side of the y(t) equation :

$$A\sin\omega t$$

Reminder : coupled first order equations, **quantum mechanics**

* We discussed in the last lecture that this equation can be decomposed in two coupled first order equations:

$$\boxed{\frac{d^2 y(x)}{dx^2} = -Ey(x)}$$

$$\overbrace{y(0) = 0 \quad y'(0) = 0}^{\text{Initial conditions}}$$

* Simple idea: let's define a new variable:

* We get the equations :

$$\begin{cases} \dfrac{dy}{dx} = z & z(0) = 1 \\ \dfrac{dz}{dx} = -Ey & y(0) = 0 \end{cases}$$

System of first order equations, **Tacoma bridge**

- We go from :

$$\boxed{\begin{aligned} y'' &= -dy' - \frac{K}{ma}\left[e^{y-l\sin\theta} - 1 + e^{y+l\sin\theta} - 1\right] \\ \theta'' &= -d\theta' + \frac{3\cos\theta}{l}\frac{K}{ma}\left[e^{y-l\sin\theta} - e^{y+l\sin\theta}\right]. \end{aligned}}$$

- to

d * z, d is a coefficient

$$\left\{\begin{aligned} \frac{dy}{dx} &= z \\ \frac{dz}{dx} &= -dz - \frac{K}{ma}\left(e^{y-lsin\theta} + e^{y+lsin\theta} - 2\right) \\ \frac{d\theta}{dx} &= \gamma \\ \frac{d\gamma}{dx} &= -d\gamma + \frac{3Kcos\theta}{lma}\left(e^{y-lsin\theta} - e^{y+lsin\theta}\right) \end{aligned}\right.$$

System of first order equations, **Functions F1,F2,F3,F4**

* We go from :

$$\boxed{\begin{aligned} y'' &= -dy' - \frac{K}{ma}\left[e^{y-l\sin\theta} - 1 + e^{y+l\sin\theta} - 1\right] \\ \theta'' &= -d\theta' + \frac{3\cos\theta}{l}\frac{K}{ma}\left[e^{y-l\sin\theta} - e^{y+l\sin\theta}\right]. \end{aligned}}$$

* To :

$$\begin{cases} \dfrac{dy}{dx} = F_1(y,z,\theta,\gamma) \\ \dfrac{dz}{dx} = F_2(y,z,\theta,\gamma) \\ \dfrac{d\theta}{dx} = F_3(y,z,\theta,\gamma) \\ \dfrac{d\gamma}{dx} = F_4(y,z,\theta,\gamma) \end{cases}$$

With :

$$\boxed{\begin{aligned} &F_1(y,z,\theta,\gamma) = z \\ &F_2(y,z,\theta,\gamma) = -dz - \frac{K}{ma}\left(e^{y-lsin\theta} + e^{y+lsin\theta} - 2\right) \\ &F_3(y,z,\theta,\gamma) = \gamma \\ &F_4(y,z,\theta,\gamma) = -d\gamma + \frac{3Kcos\theta}{lma}\left(e^{y-lsin\theta} - e^{y+lsin\theta}\right) \end{aligned}}$$

Wind force

* The wind add a contribution to y''(x) :

$$\frac{dy}{dx} = F_1(y,z,\theta,\gamma)$$
$$\frac{dz}{dx} = F_2(y,z,\theta,\gamma)$$
$$\frac{d\theta}{dx} = F_3(y,z,\theta,\gamma)$$
$$\frac{d\gamma}{dx} = F_4(y,z,\theta,\gamma)$$

$$\frac{dy}{dx} = F_1(y,z,\theta,\gamma)$$
$$\frac{dz}{dx} = F_2(y,z,\theta,\gamma) + Asin(\omega x)$$
$$\frac{d\theta}{dx} = F_3(y,z,\theta,\gamma)$$
$$\frac{d\gamma}{dx} = F_4(y,z,\theta,\gamma)$$

Physical parameters

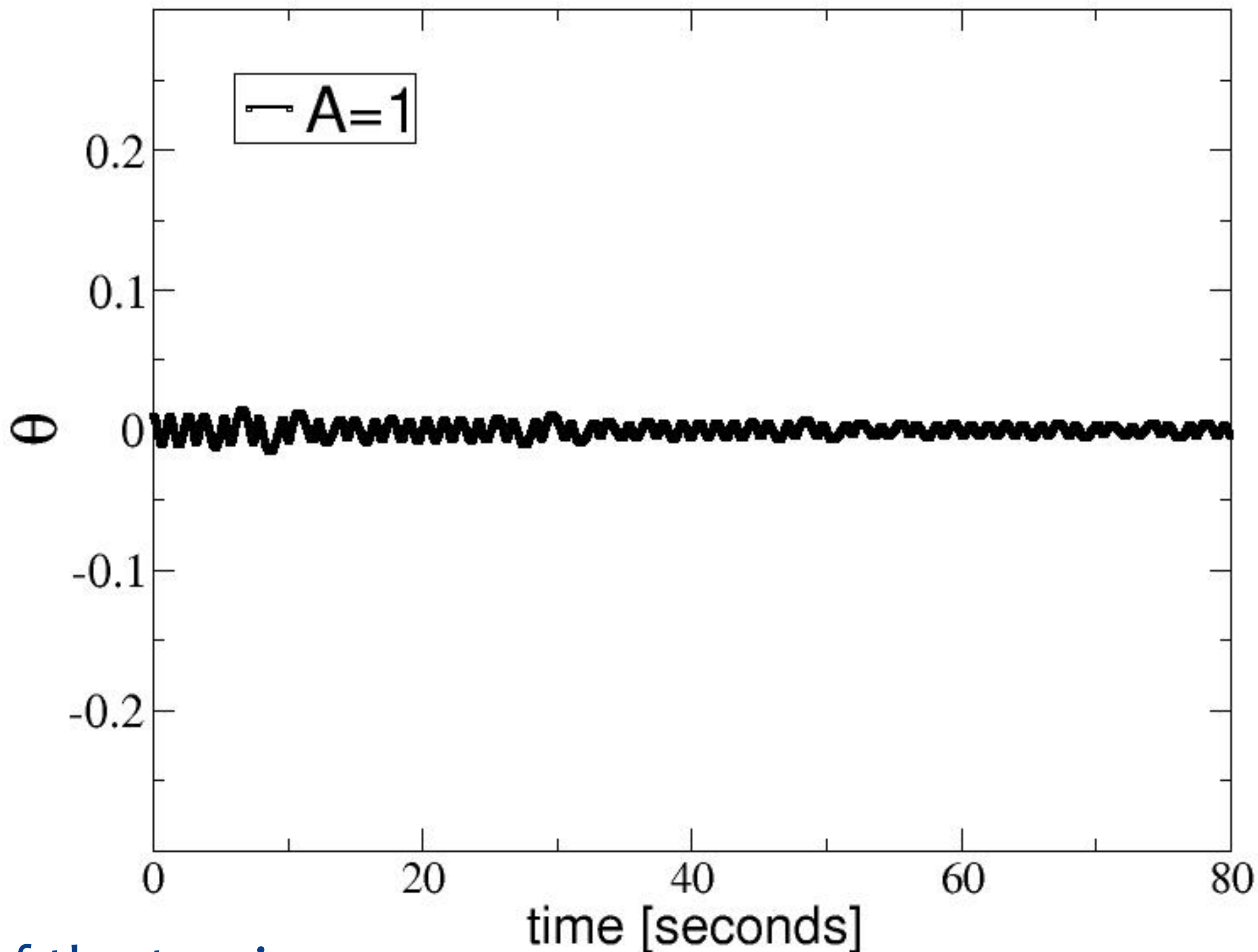
- M (mass of bridge) : **M = 2500 Kg**
- Spring constant : **K = 1000 Newtons**
- Hooke's coefficient : **a = 0.1**
- Roadway 12 meters wide : **l=12 m**
- Friction coefficient : **d = 0.01**
- Vertical force due to the wind caused the bridge to oscillate once every 2 seconds, so :

  $\omega = 2\pi / 2 \sim 3$

We study the effect of the force amplitude of the wind : A

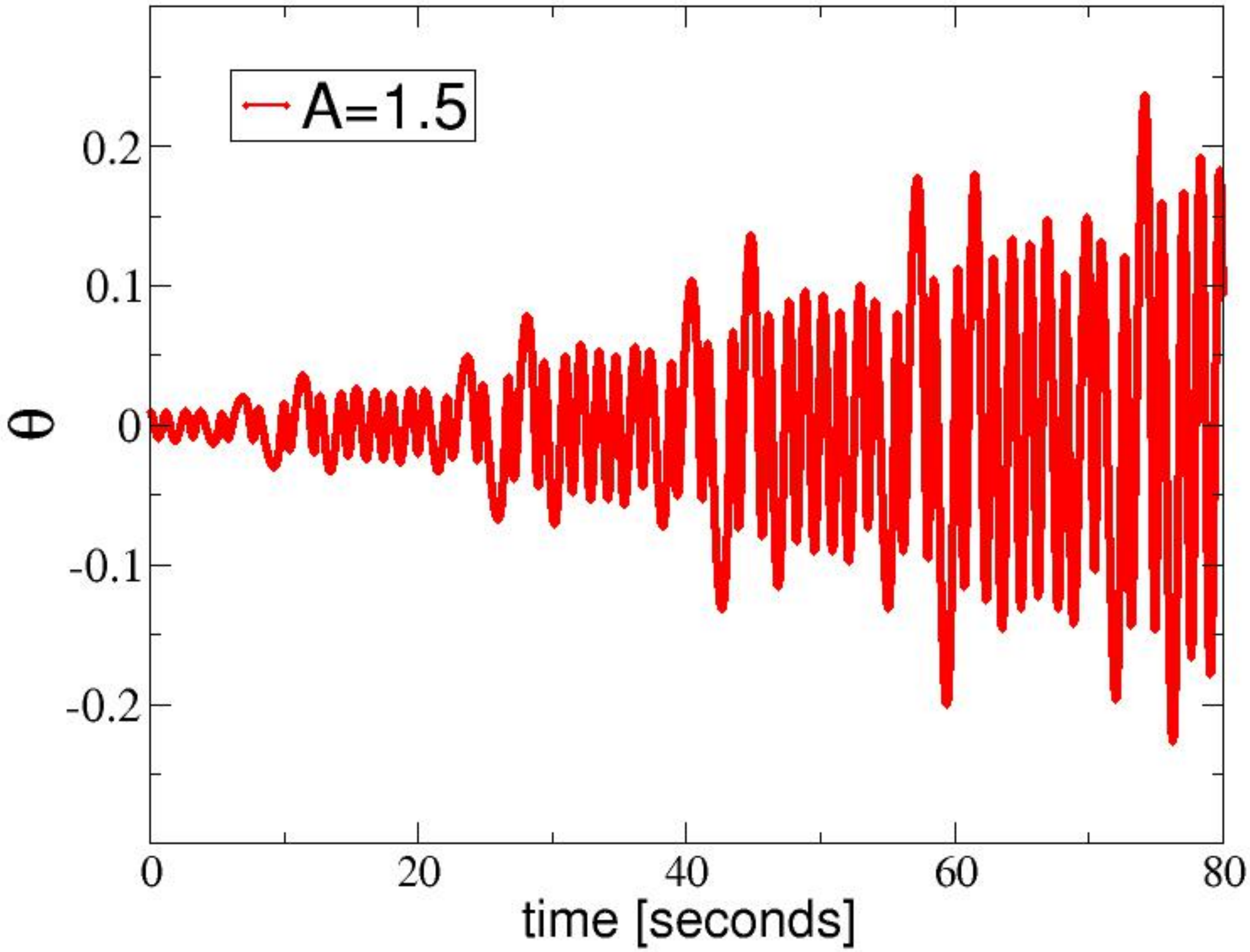
Moderate wind, A=1

<u>Moderate wind</u>, torsion is controlled, small oscillations



* Time dependence of the torsion

Strong wind, A=1.5



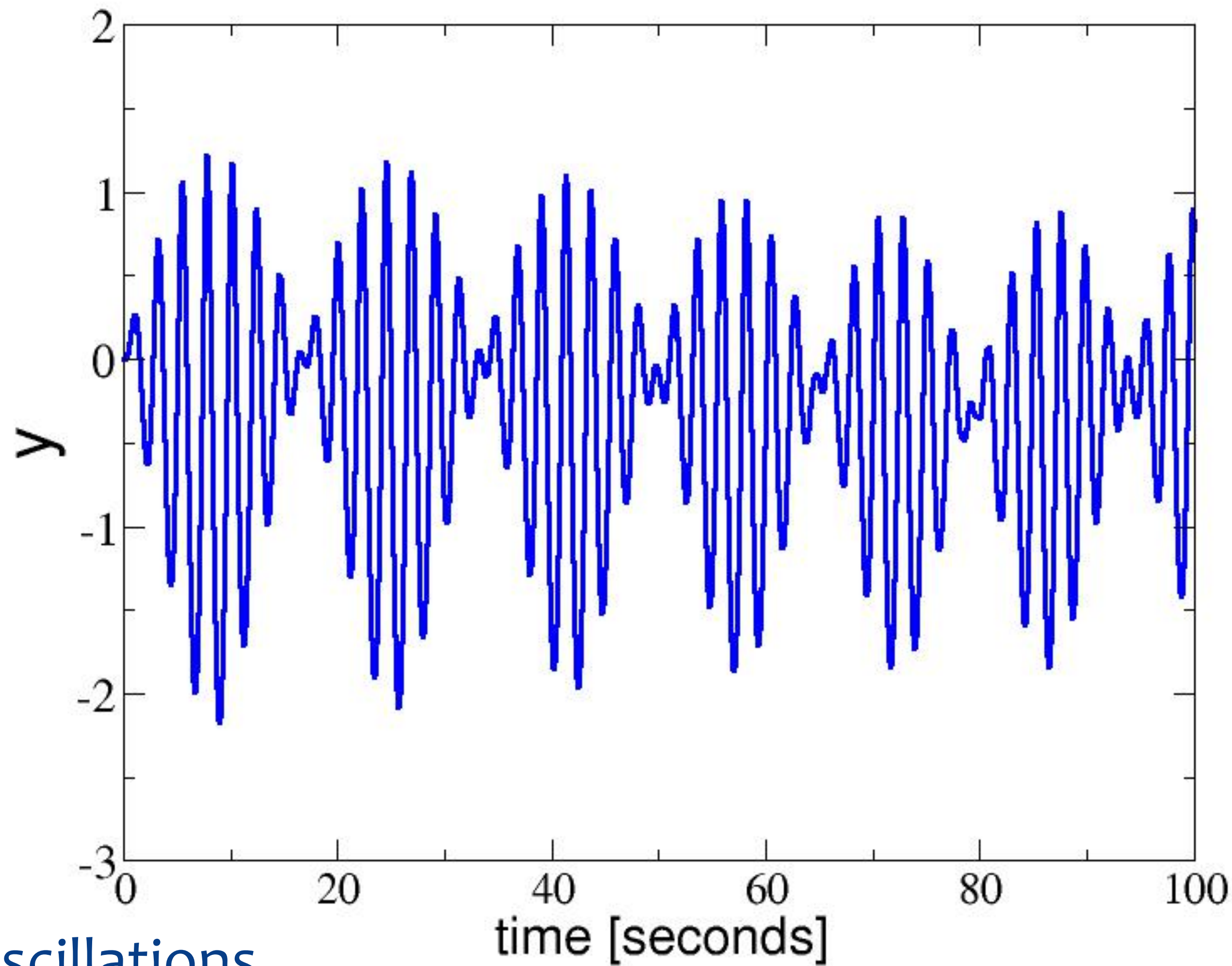
Stronger wind, torsion is uncontrolled, it gets larger and larger

→ **Tacoma's breakdown**

- Time dependence of the torsion

Strong wind, A=1.5

Note that the vertical amplitude are controlled, they are not dramatic, the Takoma bridge broke because of the torsion!



- Vertical amplitude oscillations

Code explained : F1,F2,F3,F4 functions

```
module library

real(8), parameter ::   d=0.01, K=1000.0, a=0.1, l=6.0, m=2500

contains

  function F1 ( y , z , t , g )
  implicit none
  real(8) :: F1
  real(8) :: y , z , t , g
    F1 = z
  end function

  function F2(y,z,t,g)
  implicit none
  real(8) :: F2
  real(8) :: y,z,t,g
    F2 = ……. [ FILL IN ] ……
  end function

  function F3(y,z,t,g)
  implicit none
  real(8) :: F3
  real(8) :: y,z,t,g
    F3 = g
  end function

  function F4(y,z,t,g)
  implicit none
  real(8) :: F4
  real(8) :: y,z,t,g
    F4 = ……… [FILL IN]………
  end function

end module
```

$$F_1(y,z,\theta,\gamma) = z$$

$$F_2(y,z,\theta,\gamma) = -dz - \frac{K}{ma}\left(e^{y-lsin\theta} + e^{y+lsin\theta} - 2\right)$$

$$F_3(y,z,\theta,\gamma) = \gamma$$

$$F_4(y,z,\theta,\gamma) = -d\gamma + \frac{3Kcos\theta}{lma}\left(e^{y-lsin\theta} - e^{y+lsin\theta}\right)$$

Code explained : Taylor's method

```
program tacoma
use library
implicit none
integer,parameter                 :: N=100000
integer                           :: j
real(8)                           :: h,tinitial,tfinal,Amp
real(8)                           :: x(0:N),y(0:N),z(0:N),t(0:N),g(0:N)

 tinitial=0.0    ;     tfinal=100.0    ;    h = (tfinal - tinitial) / dble(N)

 write(*,*) 'please enter Wind force amplitude'
 read(*,*)   Amp

 x(0)=0
 y(0)          =…[FILL IN]….
 z(0)          =…[FILL IN]….
 t(0)          =…[FILL IN]….
 g(0)          =…[FILL IN]….

 do j =  1, N
  x(j)         = j*h
  y(j)         = y(j-1)        + h *   F1 (y(j-1),z(j-1),t(j-1),g(j-1))
  z(j)         = z(j-1)        + h * ( F2 (y(j-1),z(j-1),t(j-1),g(j-1)) + Amp * … [FILL IN] )
  t(j)         = t(j-1)        + h *   F3 (y(j-1),z(j-1),t(j-1),g(j-1))
  g(j)         = g(j-1)        + h *   F4 (y(j-1),z(j-1),t(j-1),g(j-1))

  !   vertical amplitude y
  write(100,*) x(j), …………. [FILL IN]……….
  !   torsion theta
  write(101,*) x(j), …………. [FILL IN]……….

 enddo

end program
```

Initial conditions?

Wind?

Writing solutions into
Two different files?

1. define the initial and final time
2. define the initial conditions
3. we start with a small torsion t(0) = 0.01
4. we solve the four coupled differential equations with the Taylor's method
5. we write the vertical amplitude y(j) and the torsion t(j) obtained at time x(j) into the files : fort.100 and fort.101

Problem

- **Problem 1** : Oscillation modes of the Tacoma bridge