EMPIRICISM IN ARITHMETIC AND ANALYSIS

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Abstract

We discuss the philosophical status of the statement that $(9^n - 1)$ is divisible by 8 for various sizes of the number n. We argue that even this simple problem reveals deep tensions between truth and verification. Using Gillies' empiricist classification of theories into levels, we propose that statements in arithmetic should be classified into three different levels depending on the sizes of the numbers involved. We conclude by discussing the relationship between the real number system and the physical continuum.

1 The problem

In a recent paper, [5], Gillies rejects the Duhem-Quine thesis that the empirical status of the entire body of mathematics should be accepted on the basis of its applications in physical science, and proposes instead to classify different parts of mathematics into one of four levels according to their empirical status.[1] He puts the theory of large cardinals into the most abstract metaphysical category, and argues that theorems about them have no reference and hence no truth-value. This paper makes the more radical proposal that *finite* numbers of different sizes have different ontological statuses. In particular we argue that sufficiently large finite numbers only exist in a metaphysical sense: they play no role in science and our only access to them depends upon accepting the rules of Peano arithmetic.

In Section 6 we discuss the the continuum from a similar point of view. Following Poincaré, we argue that the physical and mathematical continua have different philosophical statuses, and provide a series of examples in which this difference is important.

[1] As I read the paper, the exact categories are a matter of convenience and have indeterminate boundaries.

We use the term 'metaphysical' without intending to suggest that metaphysical theories are lacking in interest. We regard them as an important way of clarifying our thoughts and understanding their internal logic. We do not pursue here the question about whether truth-values depend upon reference.

For the sake of definiteness we organize our discussion around a particular proposition; many others could be used to equal effect. One may routinely use induction to prove

$$(\boldsymbol{P_n}) \quad \text{there exists a positive integer } m \text{ such that } 9^n - 1 = 8 \times m$$

for all positive integers n. Nevertheless we will argue that the status of this proposition depends on the size of n.

The particular case $(\boldsymbol{P_3})$ becomes $9^3-1 = 729-1 = 8\times 91$. This may be verified by representing the numbers involved by tokens which may be moved around mentally or physically. This type of proof has an empirical feeling since moving around 729 tokens retains the possibility of error whether they are physical objects, signs on paper or imagined. Nevertheless with sufficient care one can *effectively eliminate* the potential for error. Statements of the above type are best regarded as being in Gillies' level zero or one: they are either observation statements or theories which can be decided for practical purposes by observation.

For larger values of n the idea that one could verify $(\boldsymbol{P_n})$ by such means becomes implausible. Let us consider, for example, the identity

$$\begin{aligned} 9^{100} - 1 &= 265613988875874769338781322035779626829233452653394495974574961739092490901302182994384699044000 \\ &= 8 \times 33201748609484346167347665254472453353654181581674311996821870217386561362662772874298087380500. \end{aligned} \tag{1}$$

obtained using a computer package (Maple V). It cannot be proved by manipulating individual tokens whether they are physical or imagined. It can, however, be verified in about a dozen pages by a hand computation involving repeated long multiplications. If one uses human computers the above is an empirical verification: it is quite possible that several people would all obtain different results when evaluating 9^{100}. Although there are systematic procedures for reducing such errors to a minimum, this type of problem was the main reason for Babbage's attempts to build a mechanical computer. Electronic computers are more reliable than humans, but computations done by them should be regarded as empirical verifications rather than proofs.

Although the above proof of $(\boldsymbol{P_{100}})$ has an empirical content, we describe a different and purely a priori proof which does not use the law of induction below. We concur with Gillies in putting statements such as $(\boldsymbol{P_{100}})$ into level two: they are scientific because they can be confirmed by a variety of methods and involve numbers which are not far removed in size from those used by physicists today.

2 The Ontology of Numbers

In his article Gillies adopts the definition of numbers as properties of sets. He follows Maddy in believing that some sets exist in the material world. Maddy cites the example of a carton containing three eggs, and argues that this is a physical set of three eggs and that we do not impose this fact upon it, [8]. Chemistry provides good examples of sets which exist without reference to human society. Thus ethyl acetate is composed of molecules each of which may be regarded as a set of 14 atoms.

While this approach is reasonable for small numbers, we consider that one must be rather cautious about adopting the same definition for large numbers. The problem is that large sets do not exist in the physically relevant sense. Thus while a cat is composed of atoms, the set of atoms in a cat is not a well-defined entity. The cat breathes in and out and there is no precise point at which the carbon dioxide molecules in its lungs should be regarded as no longer being a part of it. Nor is there a precise point at which the food in its stomach becomes a part of it. Even a block of silver has the same problem: atoms constantly evaporate off the surface while others combine with it and diffuse into it. We believe that no material object can be said to contain a precise number of atoms if that number is greater than 10^{30}. One may of course consider an *ideal* silver cube containing exactly 10^{10} atoms along each edge and therefore 10^{30} atoms altogether, but this cube is then a mental construction and not something in the material world. If one wishes to retain the definition of numbers as properties of sets which exist in the physical world, it seems that one must accept that there is no empirical distinction between two large numbers which only differ in their last few digits when written in the usual decimal notation.

For even bigger numbers the situation shifts again. The number of massive elementary particles in the universe is believed to be less than $M = 10^{100}$. Sets containing more than M particles therefore cannot exist in the material world. If one regards all sets of particles as candidates for material entities then 2^M is an upper bound on the number of different material entities. It is a matter of fact that physicists do not make use of numbers vastly bigger than this, and it is difficult to argue that they have *any* empirical status.

In his article Gillies does not directly address the distinction between Peano arithmetic and decimal arithmetic, although he does mention it briefly, [5, p 45]. In the next section we argue that there are well-defined statements about huge numbers which are completely outside the scope of decimal arithmetic. Such numbers have *never* been of use to scientists, and we consider that they only exist by convention. Statements about them should be regarded as in level three: they are metaphysical statements which are too far from observation to be confirmed or disconfirmed even indirectly.

3 Inductive Proofs

In this section we consider proofs of $(\boldsymbol{P_n})$ which are of a more theoretical nature. We start with the identity

$$9^{100} - 1 = (9-1) \times (9^{99} + 9^{98} + ... + 9 + 1). \tag{2}$$

which establishes $(\boldsymbol{P_{100}})$. Actually it does far more. The symbol 9 need not refer to the number nine in this context, or indeed to any number, since the identity (2) holds for any value of the symbol 9 in any commutative ring. The truth of the distributive identity

$$a(b+c) = ab + ac \tag{3}$$

for integers may be demonstrated in two ways. In the first one considers tokens placed in rectangular arrays and observes directly that the distributive identity is true for small arrays. One then declares that it is obviously also true however large the arrays are. This argument depends upon the belief that one's visual imagination is infallible.

The other method of proving (3) is by means of Peano's axioms, and a formal argument in which addition and multiplication are defined from first principles. One may justify Peano's axioms as being intuitively and infallibly obvious, or one may define arithmetic to be a certain type of formal mathematical structure, a ring satisfying a certain list of axioms including the inductive property. The existence and uniqueness of this structure is then regarded as a separate question, as is the relationship between the inductive axiom and the notion of counting. Whichever route one follows it is seen that the validity of (2) is not as simple as it seems. Either there is a deep theoretical input or one is basing the proof upon an intuitive understanding of why rules such as (3) are valid.

The symbols ... in the equation (2) refer either to the use of an inductive step or to omitted terms, which could be written down in this case. It is of some interest that (2) is more convincing if the terms are left to be filled in by the imagination than it would be if they were present. In the latter case one would have to read the entire identity carefully to check that all of the expected terms were indeed present. This proof that 8 is a factor of $9^{100} - 1$ is not open to serious doubt. In addition (2) provides a simple expression for the constant m in $(\boldsymbol{P_{100}})$.

The issues involved in the use of induction are clearer if one considers a much larger value of n. Let us define N to be the number produced by the following computer program (or strictly speaking pseudocode).

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n := 1;
for r from 1 to 100 do
n := n^n + 1;
end;
N:=n;
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This produces a sequence of numbers which we label with subscripts: $n_0 = 1$, $n_1 = 2$, $n_2 = 5$ and $n_3 = 5^5 + 1 = 3126$. The number $n_4 = 3126^{3126} + 1$ is still just small enough to be computable using current PC's: it has 10,926 digits and can be printed out on about ten pages of A4 paper. The next number is too large for any computer constructible in this universe to evaluate (in the usual decimal notation): only an insignificant fraction of the digits in the answer could be stored even if one allocated a trillion digits to every atom in the universe. The failure of the program is not detectable at the compilation stage since the syntax of the program is correct and the program itself only involves small numbers. The difficulty only appears as a run time error or memory overflow. Nevertheless with enormous confidence Platonists would refer to the 100th number in the sequence as if its existence posed no problems of principle.

One may justify the existence of a huge number such as N on the basis that a huge iterative procedure can be carried out *in principle*, even though it obviously cannot be in practice. An alternative attitude is that the pseudocode constitutes a valid sequence of formal expressions and therefore defines an expression N within the rules of Peano arithmetic: the symbol N does not need to *refer* to anything other than the expression used to define it. A perusal of the literature on mathematical Platonism suggests that there is no knock-down argument for deciding between these alternatives. The Platonic existence of the set of all numbers would have the valuable consequence that a formally correct proof about numbers could not lead to a contradiction. We all believe this, but Gödel's work in 1931 shows that intuitive beliefs about numbers may be totally incorrect.

The statement that $(9-1)$ is a factor of $(9^N - 1)$ cannot be verified by moving around tokens or by the use of decimal arithmetic, and there may be no proof except for the one which uses the general version of Peano's postulates. The proof of $(\boldsymbol{P_N})$ makes no use of the particular values which the symbols 9 and N happen to have, and there is no alternative way of verifying the truth of the statement. This suggests that the truth of the statement has no empirical content.

4 The Status of Induction

The principle of induction is usually regarded as being the statement of an obvious property of any counting system. For small numbers, representable by collections of tokens, we accept this view. Poincaré carefully avoided stating that the principle was true, claiming rather that it is necessarily imposed on us, because it is only the affirmation of a property of the mind itself, [10, p 13]. Elsewhere he argued that we accept the principle not on logical grounds but by an *a priori* synthetic judgement, [11, p 173]. If one considers Peano arithmetic as a formal system no justification of the principle is needed: it is just one of the rules of the game. The situation is quite different in decimal arithmetic. If one considers this to be an activity in its own right in which the numbers (i.e. finite strings of digits) have no reference to counting then it appears that there is no justification for using induction. This section resolves the apparent paradox.

The conclusion to be drawn from what follows is that the internal *structure* of decimal arithmetic is purely a priori. Its empirical *status* follows from the fact that it appears to be a reliable model for the relevant properties of large sets in the physical world. Of course, this physical interpretation was the central consideration when decimal numbers were invented, so it is not proper to regard it as a secondary feature. The argument below demonstrates that decimal numbers have an essential property for any successful model of counting.

We next describe a proof of the law of induction for decimal numbers. Let us consider a proposition $\mathcal{P}(n)$ such that $\mathcal{P}(0)$, and if $\mathcal{P}(n)$ then $\mathcal{P}(n+1)$. We will establish $\mathcal{P}(n)$ for all decimal numbers n without commitment to the beliefs that there is a completed infinite set of such numbers or that they refer indirectly to a counting process. We only permit the use of the syntax for manipulating decimal numbers, finite arguments in logic and the substitution rule. I will only consider decimal numbers with 100 or fewer digits, but the same methods can be applied to larger decimals without important changes. The first proof presented is not the best, but it introduces the fundamental idea.

Replacing n by $n+r$ for $r = 1, 2, \ldots, 9$ we see that $\mathcal{P}(n+1)$ implies $\mathcal{P}(n+2)$, $\mathcal{P}(n+2)$ implies $\mathcal{P}(n+3)$ and so on up to $\mathcal{P}(n+9)$ implies $\mathcal{P}(n+10)$. Combining these implications we deduce that $\mathcal{P}(n)$ implies $\mathcal{P}(n+10)$. Take this as the new starting point. Replacing n by $n+r$ for $r = 10, 20, \ldots, 90$ we see that $\mathcal{P}(n+10)$ implies $\mathcal{P}(n+20)$, $\mathcal{P}(n+20)$ implies $\mathcal{P}(n+30)$ and so on up to $\mathcal{P}(n+90)$ implies $\mathcal{P}(n+100)$. Combining these implications we deduce that $\mathcal{P}(n)$ implies $\mathcal{P}(n+100)$. Repeating this process one hundred times we get $\mathcal{P}(n)$ implies $\mathcal{P}(n+10^{100})$. Putting $n = 0$ we finally deduce $\mathcal{P}(10^{100})$. I have not written out the complete proof, but only referred to the 1000 steps needed to do so. A complete proof could be written out in less than 100 pages of text.

The above only proves $\mathcal{P}(10^{100})$, whereas one wants to show that $\mathcal{P}(n)$ is true for *all* numbers up to 10^{100}. One may write out the same sort of proof for any particular decimal number but this is not possible for all such numbers simultaneously. The following procedure is more complicated, but solves the general problem. Let $\mathcal{Q}(m)$ be the following proposition:

> If $\mathcal{P}(n)$ then $\mathcal{P}(n+r)$ for any number r which has no more than m digits.

The proof of $\mathcal{Q}(1)$ was already described in the last paragraph: r can then only take the values $1, 2, \ldots, 9$. The difficult step is to prove that $\mathcal{Q}(m)$ implies $\mathcal{Q}(m+1)$. Suppose $\mathcal{Q}(m)$ is assumed and the number r has no more that $m+1$ digits. Then with a little effort we can write

$$r = r_1 + r_2 + \ldots + r_{11}$$

where each r_i has no more that m digits. By combining the implications $\mathcal{P}(n)$ implies $\mathcal{P}(n+r_1)$, $\mathcal{P}(n)$ implies $\mathcal{P}(n+r_2)$, $\ldots \mathcal{P}(n)$ implies $\mathcal{P}(n+r_{11})$ in exactly the way already discussed we deduce that $\mathcal{P}(n)$ implies $\mathcal{P}(n+r)$. So $Q(m+1)$ is true.

We now have $\mathcal{Q}(1)$, and $\mathcal{Q}(m)$ implies $\mathcal{Q}(m+1)$; we wish to deduce $\mathcal{Q}(100)$. This can be done in 100 steps. However, the process can be much abbreviated because we can use the same method as we have already discussed, but applied to $\mathcal{Q}$ instead of $\mathcal{P}$. The end result is a proof which can be written out in complete detail in less than two pages. It provides a general theorem of induction for all numbers with up to 100 digits. If one is willing to invoke a third stage in the hierarchy, by defining a new proposition $\mathcal{R}$ in terms of $\mathcal{Q}$, then one can deal with far bigger numbers, with a proof taking only three pages.

5 Factorization and Irrational Numbers

The general theory of prime numbers is metaphysical in the sense that the existence of a unique prime factorization of huge numbers depends upon the general form of the induction hypothesis. On the other hand, the factorization of numbers with a few hundred digits has important commercial applications related to cryptography. In this application the decimal numbers in question are *not* interpreted as properties of large sets, and all of their digits are equally important. The primality testing algorithms used are probabilistic, and hence empirical. The difficulty of factorizing a number which is known to be the product of two primes increases rapidly with its number of digits, and appears to be practically impossible for sufficiently large numbers. See Fallis [4] for a discussion of this subject.

Let us consider the statement that $\sqrt{65}$ is an irrational number. This can be rephrased as the non-existence of positive integers m, n such that $m^2 = 65 \times n^2$. Since m, n may be arbitrarily large, this statement requires the full strength of Peano arithmetic and is therefore of only metaphysical status. It is certainly difficult to imagine any empirical way of distinguishing between the statements that $\sqrt{65}$ is irrational and that it could be expressed as a fraction with a truly huge numerator and denominator. The Pythagoreans were very disturbed about the fact that such numbers did not exist within their mental framework, i.e. that the arithmetic of rational numbers did not provide an adequate model for their physical intuition about distance. They did not have a better model and were forced to develop geometry along different lines because of this. From our point of view $\sqrt{65}$ has as strong a claim to empirical existence as almost any number.

It may be measured physically as the height of the $(8,9,9)$ triangle. It may be computed numerically by several different methods, yielding

$$8.0622577482985496524...$$

For instance it may be expressed as the limit of the sequence defined by $a_0 := 10$ and

$$a_{n+1} = a_n - \frac{a_n^2 - 65}{2a_n}$$

which converges at an explicitly controlled rate. We conclude that $\sqrt{65}$ has unarguable empirical status within the framework of decimal arithmetic, even though its irrationality is metaphysical. Some may consider that its existence has been achieved by means of a psychological trick, namely convincing oneself that one can sensibly refer to the infinite sequence of all its digits.[2] However, the existence of a practically implementable procedure for writing down a reasonable number of digits (say 100) is all that is relevant for empirical purposes.

We now turn to an argument in favour of the empirical status of rationality. In the fractional quantum Hall effect a very strong magnetic field is applied across a thin sheet of semiconductor at a very low temperature. It is found that the resistance is quantized as the field strength varies, and that in suitable units the permitted values are rational numbers. The almost periodic Schrödinger operator in one dimension is a very simple, rigorous mathematical model which exhibits similar properties as a certain parameter varies. It is known that the spectral type of the operator depends upon whether the parameter is rational or not: for rational values the operator has periodic coefficients and its spectrum is absolutely continuous, while for typical irrational values the spectrum is singular continuous or dense pure point. If one computes certain spectral quantities as a function of the parameter one finds a local minimum at all rational points: smaller denominators give rise to deeper and wider local minima. Of course any particular computation has only a finite resolution and so detects only the local minima associated with those rational numbers whose denominators are sufficiently small. The true graphs are extremely complex, both in this simplified model and in the experiments.

The obvious difficulty with the above results is that any rational (resp. irrational) number may be approximated arbitrarily closely by irrational (resp. rational) numbers, so it is hard to see how there can be a fundamental distinction between them in an experimental context. The answer is as follows. Every real number x may indeed be approximated arbitrarily closely by rationals, but there are many ways of doing this. Simply truncating the decimal expansion at the nth place is not always the most interesting method. Given a real number one can produce the

[2] The various 19th century constructions of real numbers all involved accepting the existence of infinite entities.

approximation sequence

$$a_n := \min\left\{\left|x - \frac{r}{s}\right| : r, s \in \mathbf{Z},\ 1 \le s \le n\right\}.$$

While it is the case than x is rational if and only if $a_n = 0$ for all large enough n, this is of no practical value if the relevant n is too large. One may classify real numbers into types according to the asymptotics of a_n as $n \to \infty$, but it is of more interest on physical grounds to ask what is the *apparent* asymptotic behaviour of a_n as n increases up to the largest values which of physical relevance. This is closely related to the apparent spectral behaviour of the operator up to a certain degree of resolution in the energy space, and also to the long time asymptotics of the evolution under the time-dependent Schrödinger equation, provided one does not look at this for an unphysically long time. We conclude that only those rational numbers whose denominators are sufficiently small, so that $a_n = 0$ for reasonably small n, have genuinely empirical status.

6 The Continuum

One may distinguish between two different attitudes towards the continuum. Physicists (and other applied scientists) use a system which differs in a subtle way from the real number field as constructed by Dedekind and Weierstrass.[3] Assuming a reasonable system of units, they do not need to refer to magnitudes larger than 10^{100} or smaller than 10^{-100}. Nor do they distinguish between two numbers which only differ in their hundredth significant digit. One reason is that no theories in physics are believed to be valid at space scales radically shorter than the Planck length.

Physicists may well utilize the idealizations of pure mathematics in order to simplify their description of nature, but that does not mean that they endow such idealizations with any direct physical reality. When mathematicians write $\lim_{n\to\infty} a_n = a$ they are referring to Cauchy's definition of limit:

$$\forall \varepsilon > 0 \,.\, \exists N_\varepsilon \,.\, \forall n \,.\, n \ge N_\varepsilon \Rightarrow |a_n - a| < \varepsilon.$$

In cases such as those below physicists know the precise value of the parameter (n could be the velocity of light c, the inverse of Planck's constant $\hbar$ or the inverse of the fine structure constant α) and are more interested in whether it is large enough for the difference between a_n and a to be negligible. One might try to reconcile the two points of view by means of error estimates of the form

$$|a_n - a| \le c n^{-\gamma} \text{ for all } n \ge N$$

[3] The difference between the mathematical and physical continua was described by Poincaré in 1902, [10, p22], but we give detailed examples to explain why it is even more important now than it appeared then.

but the same problem arises. With rare exceptions analysts focus their attention on the largest value of the constant γ and show little interest in c or N. Physicists show no interest in error bounds, preferring expressions such as

$$a_n \sim a + cn^{-\gamma}$$

for values of c and γ which can be computed, but with no error estimates. The problem is highlighted for expressions such as

$$a_n = 1 + n^{-1} + 10^{-8}\log(n) \quad (4)$$

which a physicist might well write in the form $a_n \sim 1 + n^{-1}$, remembering that n should be smaller than about 10^{100}, or even declaring that the final term may be dropped by a 'logarithmic renormalization'. Pure mathematicians, on the other hand would correctly insist that a_n diverges as $n \to \infty$.

We now provide some examples to support the claim that the difference in outlook between mathematicians and physicists is best understood in terms of their having a different understanding of the nature of real analysis and of the real number system; of course individuals may lie anywhere along the continuous spectrum of attitudes irrespective of the name they apply to themselves.

Statistical mechanics is the study of the properties of bulk materials whose linear dimensions lie between about 10^{-8} metres and 10^{7} metres. Much smaller 'mesoscopic bodies' containing around 100 atoms have properties intermediate between those of individual atoms moving according to the Schrödinger equation and those of bulk materials. For much larger bodies gravitational self-interactions affect their bulk properties to a degree dependent on their size or may lead to their collapse. In contrast the standard rigorous mathematical approach to phase transitions via statistical mechanics involves proving the discontinuity of certain thermodynamic functions for infinite volume systems of interacting particles, [12]. Setting the gravitational constant equal to zero is necessary to develop the mathematics and is analogous to disregarding the final term on the RHS of (4). The thermodynamic functions also exist for finite volume systems, but they are then analytic functions of the thermodynamic parameters, such as temperature. Of course finite samples of material a centimetre across do exhibit phase transitions in a physical sense and there are no infinite volume systems in the real world. The clean mathematical discontinuity does not occur in real systems, but is an approximation to what does occur, namely a very rapid change in the value of a thermodynamic function within an extremely narrow range of temperatures, even though the function remains smooth. The infinite volume mathematical theory involves two independent idealizations and is studied because it removes some of the complications of real, finite volume systems and enables mathematicians to prove theorems, even if those theorems lose some of the important physical distinctions, [13, p 9].

So prevalent is this procedure of 'passing to the thermodynamic limit' that Ingarden has been driven to write

> At present many physicists think that this assumption (considering only infinite systems with homogeneous density) is the only possible one for thermodynamical and statistical theories. Such an extreme point of view seems to be, however, absurd. [7, p 76]

The standard approach is only a model, and is not well adapted to the consideration of dynamical phenomena. These include fundamental descriptions of superheated liquids, of amorphous solids (glasses) and of the flow of heat between bodies at different temperatures. Physicists need to be constantly aware of the limitations of their models, and should not press them beyond the regime in which they have some plausibility; mathematicians have a much better excuse for concentrating on the theorems which they can prove about the models irrespective of their physical relevance.

The difference between the conceptual worlds of theoretical physicists and mathematicians is particularly clear in the study of quantum electrodynamics. This is rightly regarded by physicists as one of the best confirmed theories in fundamental physics, in spite of the fact that it involves the application of some extremely questionable mathematical procedures. A sustained attempt to turn it into a coherent and rigorous mathematical subject by a group of mathematicians and mathematical physicists between about 1965 and 1980 met with only limited success, [6]. It proved possible to construct a variety of self-interacting relativistic quantum field theories in two space-time dimensions, and a much more limited number in three dimensions. But in the physical case of four space-time dimensions the group were not able to prove the existence of any non-trivial models satisfying the Wightman axioms, and regarded the existence of a self-consistent mathematical theory of QED as a theoretical puzzle, [6, p 277]. This was regarded as being an entirely irrelevant issue by many more experimentally based physicists, since they considered that QED was already fully understood in their terms: the existence of a systematic procedure for extracting numbers which could be checked with experiments was all that mattered. In other words they regarded the asymptotic expansions, Feynman diagrams, multiple integrals and infinite renormalizations as procedures which they could borrow from mathematics without any consideration of whether their use was technically justified. The justification came from the very impressive experimental confirmation of the predictions and not from the possible existence of a rigorous model according to the mathematical canons of the time.

Yet another example of the way in which scientists misuse real analysis (by the standards of pure mathematicians!) arises in the study of fractal sets. The fractional dimension of a set in Euclidean space is defined by examining the limit of a certain expression as a scale parameter $\varepsilon > 0$ converges to zero. The Minkowski and Hausdorff dimensions have different technical definitions but are often equal. In applications to the study of physical objects which have a 'fractal' character the behaviour of the expression is determined for a finite range of values of ε and the best fit constant is declared to be the fractal dimension of the object concerned. It is found that predictions based upon this model are only valid for a modest range of scale parameters beyond those used to fix the constants in the model. From the point of view of pure mathematics the values measured have no relevance to any actual fractal dimension unless one has information about the rate of convergence with explicit error estimates. In the applications such estimates cannot exist, since by the time one gets down to atomic dimensions no trace of the examined phenomenon remains. Once again the applied scientists know but do not care about the fact that they are using a mathematical model whose relationship with reality is strictly limited.

As a final example we consider the eigenvalues of $n \times n$ matrices. It is well known that if one disregards a Lebesgue null set of matrices which have non-trivial Jordan forms, then every square matrix A is similar to a diagonal matrix whose entries are the eigenvalues of A. An enormous amount of pure mathematics has been based upon this fact. On the other hand if one does a statistical analysis of typical randomly generated non-self-adjoint matrices one finds that the condition number of the matrix which implements the diagonalization grows exponentially fast with the size of the matrix, [3]. This implies that the eigenvalues of typical non-self-adjoint 100×100 matrices are so unstable that they are essentially uncomputable. Trefethen and others have developed the theory of pseudospectra, which attempts to come to terms with such instabilities. This new theory emphasizes the quantitative side of matrix theory, and recognizes that the appearance of numbers bigger than 10^{20} is a sign that the theory has moved away from reality. There is no suggestion in any of this that the spectral analysis of the pure mathematicians is incorrect, but just a realization that it does not connect with anything which an empirical scientist might regard as relevant to the real world.

We consider the real number field of Dedekind and Weierstrass to be metaphysical because it adds features to the physical continuum which have no empirical justification. There are indeed other ways of making our physical intuitions mathematically precise. One is the non-standard analysis of Robinson and another is the constructive analysis of Bishop, [1]. Both have small groups of devotees and have been used to prove series of theorems which differ from what are obtained in classical analysis. It is arguable that classical analysis is dominant mainly because it was constructed more than fifty years before its rivals. Mathematicians have produced these idealized versions of the empirical continuum because of their need to have sharp formalisms if they are to prove theorems, and because the idealized systems are easier to grasp intuitively. By and large the properties which scientists actually need are valid in all of the formalizations, and the aspects in which the formalizations differ are of no empirical significance.

We have seen that the mathematics of pure mathematicians has many differences from that used by applied scientists. Pure mathematics has a more Platonic flavour because using the axiomatic method involves a commitment to the existence of various idealized objects and to the law of the excluded middle. Empirical scientists inhabit a more messy world in which models only have limited validity and one does not concern oneself with issues which are far beyond what is experimentally testable. The rich variety of mathematics cannot be described within a single philosophical system, and investigations of the extent to which *particular parts* of mathematics are empirical or metaphysical provides a fuller appreciation of the subject.

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