Linear Operators and Their Spectra

Web Supplement, version 32

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10 November 2012

A large font version of this manuscript is available. Please specify the font size that you need using Latex notation.

I would welcome any information about errors in the text, to be sent to the address or email above.

When giving page references to LOTS, the number after the comma is for the line, counted down from the top if positive and up from the bottom if negative.

The text contains some new sections, containing material not mentioned in the original book.

The chapter numbers in this supplement coincide with those in LOTS, but this has at least one further chapter and some further sections.

An online version of 'E,B. Davies, Linear Operators and Their Spectra, Cambridge University Press, 2008' is available at

http://www.mth.kcl.ac.uk/staff/eb_davies/LOTS.html

Contents

Chapter 1

Elementary operator theory

1.2 Bounded linear operators

page 14. Solution of Problem 1.2.5 of LOTS

This depends on the fact that if $ab - ze$ is invertible for some non-zero $z \in \mathbf{C}$ then $ba - ze$ is invertible with inverse

$$z^{-1}\left(e + b(ze - ab)^{-1}a\right).$$

If a, b are bounded linear operators on some Banach space, then this identity implies a very close connection not only between the spectra of ab and ba, but also between their pseudospectra; see Section 9.2.

Problem 1.2.6 of LOTS states that $\lambda \in \mathbf{C}$ has the same geometric multiplicity as an eigenvalue of AB and as an eigenvalue of BA. The following theorem proves the same for algebraic multiplicities in a more general context.

Theorem 1.1 *Let $\lambda \in \mathbf{C}\backslash\{0\}$, and let $A : \mathcal{U} \to \mathcal{V}$ and $B : \mathcal{V} \to \mathcal{U}$ be linear maps. Given $n \in \mathbf{N}$, define the linear subspaces $\mathcal{M}_n$ and $\mathcal{N}_n$ by*

$$\begin{aligned} \mathcal{M}_n &= \{u \in \mathcal{U} : (BA - \lambda I)^n u = 0\}, \\ \mathcal{N}_n &= \{v \in \mathcal{V} : (AB - \lambda I)^n v = 0\}. \end{aligned}$$

Then $\dim(\mathcal{M}_n) = \dim(\mathcal{N}_n)$ *for all* $n \in \mathbf{N}$.

Proof If $u \in \mathcal{M}_n$ then

$$(AB - \lambda I)^n Au = A(BA - \lambda I)^n u = 0.$$

This proves the first half of

$$A(\mathcal{M}_n) \subseteq \mathcal{N}_n, \qquad B(\mathcal{N}_n) \subseteq \mathcal{M}_n. \tag{1.1}$$

The second half has a similar proof. If $u \in \mathcal{M}_n$ and $Au = 0$ then $0 = (BA - \lambda I)^n u = (-\lambda)^n u$ so $u = 0$. Therefore $A : \mathcal{M}_n \to \mathcal{N}_n$ is one-one. If $v \in \mathcal{N}_n$ then $0 = (AB - \lambda I)^n v = Ax + (-\lambda)^n v$ where $x = B(AB)^{n-1}v + \ldots + n(-\lambda)^{n-1}Bv$, so $x \in \mathcal{M}_n$ by repeated applications of (1.1). Therefore $v = -(-\lambda)^{-n}Ax$ and $A : \mathcal{M}_n \to \mathcal{N}_n$ is onto.

The two results in the last paragraph imply that $\mathcal{M}_n$ and $\mathcal{N}_n$ are linearly isomorphic, and therefore have the same dimension. □

Remark 1.2 Under the conditions of Theorem 1.1, put $\mathcal{W} = \mathcal{U} \oplus \mathcal{V}$ and define $D : \mathcal{W} \to \mathcal{W}$ by $D(u \oplus v) = (Bv) \oplus (Au)$, or equivalently, in matrix notation, by

$$D = \begin{pmatrix} 0 & B \\ A & 0 \end{pmatrix}.$$

Then

$$D^2 = \begin{pmatrix} BA & 0 \\ 0 & AB \end{pmatrix},$$

so $D^2|_{\mathcal{U}}$ and $D^2|_{\mathcal{V}}$ have the same non-zero eigenvalues with the same algebraic multiplicities. □

page 17. Solution of Problem 1.2.14 of LOTS

This can be deduced from the following theorem by putting $\mathcal{B} = \mathcal{C}$ and replacing A by $A - \lambda I$ for any chosen $\lambda \in \mathbf{C}$.

Theorem 1.3 *The operator $A : \mathcal{B} \to \mathcal{C}$ is invertible if and only if $A^* : \mathcal{C}^* \to \mathcal{B}^*$ is invertible.*

Proof If A is invertible then there exists $B : \mathcal{C} \to \mathcal{B}$ such that $AB = BA = I$. This implies that $B^*A^* = A^*B^* = I$, so A^* is invertible with inverse B^*.

Conversely suppose that $A^*C = CA^* = I$ for some $C : \mathcal{B}^* \to \mathcal{C}^*$. If $D = C^* : \mathcal{C}^{**} \to \mathcal{B}^{**}$, then we deduce that $DA^{**}f = f$ for all $f \in \mathcal{B}^{**}$. Moreover A^{**} is an extension of A if $\mathcal{B}$ is regarded as a subspace of $\mathcal{B}^{**}$ in the usual manner. Therefore $DAf = f$ for all $f \in \mathcal{B}$. The bound $\|f\| \leq \|D\| \, \|Af\|$ now implies that A is one-one with closed range in $\mathcal{C}$. If its range is not equal to $\mathcal{C}$ then the Hahn-Banach theorem implies that there exists a non-zero $\phi \in \mathcal{C}^*$ such that $\langle \phi, Af \rangle = 0$ for all $f \in \mathcal{B}$. We deduce that $A^*\phi = 0$ so $\phi = CA^*\phi = 0$. The contradiction implies that the range of A equals $\mathcal{C}$ and hence, using the inverse mapping theorem, that A is invertible. □

page 18. Inverses of triangular matrices

The following algebraic result is related to Problem 1.2.9 of LOTS. It has very similar versions in which ‘upper’ is replaced by ‘lower’ or the finite-dimensionality of $\mathcal{U}$ is replaced by the same assumption for $\mathcal{V}$.

Theorem 1.4 *Let $\mathcal{U}$ and $\mathcal{V}$ be two vector spaces over the field $\mathbf{F}$ and let $X : \mathcal{U} \oplus \mathcal{V} \to \mathcal{U} \oplus \mathcal{V}$ have an upper triangular block matrix*

$$X = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}.$$

(This is equivalent to the condition $A(\mathcal{U}) \subseteq \mathcal{U}$.) If $\mathcal{U}$ is finite-dimensional then X is invertible if and only if A and D are invertible, and in this case its inverse is

$$X^{-1} = \begin{pmatrix} A^{-1} & -A^{-1}BD^{-1} \\ 0 & D^{-1} \end{pmatrix}.$$

One also has

$$\mathrm{Spec}(X) = \mathrm{Spec}(A) \cup \mathrm{Spec}(D).$$

Proof This is obtained by manipulations of the eight equations that are equivalent to $XY = YX = I$, where

$$Y = \begin{pmatrix} E & F \\ G & H \end{pmatrix}.$$

The finite-dimensionality of $\mathcal{U}$ enables one to deduce $AE = I$ from $EA = I$. The final statement follows by considering the invertibility of $X - \lambda I$, where $\lambda \in \mathbf{C}$ is arbitrary. □

Corollary 1.5 *An $n \times n$ upper triangular matrix A is invertible if and only if its diagonal entries are all non-zero. If this is the case its inverse is also upper triangular. Hence, the set $\mathcal{T}$ of all upper triangular $n \times n$ matrices is an inverse-closed subalgebra of the algebra of all $n \times n$ matrices.*

The second statement of the corollary is a special case of the following proposition.

Proposition 1.6 *Let $\mathcal{A}$ be a subalgebra of the algebra of all $n \times n$ matrices over a field $\mathbf{F}$, then $\mathcal{A}$ is inverse-closed.*

Proof If $A \in \mathcal{A}$ is invertible then its minimum polynomial p has a non-zero constant coefficient $c \in \mathbf{F}$. Therefore $p(x) = xq(x) + c$ for all $x \in \mathbf{F}$, where q is also a polynomial.This yields

$$A^{-1} = -c^{-1}q(A),$$

which implies the stated result immediately. □

page 18. Consequences of results on the zeros of polynomials

The following new material goes at the end of Section 1.2. It assumes that the reader is familiar with the theory of the Jordan canonical form.

There is an elementary proof of the fact that the set $\mathcal{U}$ of $n \times n$ matrices that have n distinct eigenvalues (and hence are diagonalizable) is open and dense in the set $\mathcal{M}$ of all $n \times n$ matrices, but the following is deeper. The theorem and the lemma following it are classical.

Theorem 1.7 *The set $\mathcal{C} = \mathcal{M} \backslash \mathcal{U}$ of all $n \times n$ matrices whose characteristic polynomials have at least one repeated root is closed and has zero Lebesgue measure.*

Proof If $A \in \mathcal{M}$ then

$$p(\lambda) := \det(\lambda I - A) = \lambda^n + \sum_{r=0}^{n-1} b_r \lambda^r$$

where each b_r is a polynomial in the coefficients of A. If $\lambda_1, \ldots, \lambda_n$ are the roots of p then the discriminant

$$\delta = \prod_{r<s} (\lambda_r - \lambda_s)^2$$

which is the square of the Vandermonde determinant, is a symmetric function of the roots of p, so by a standard theorem about symmetric polynomials there exists a polynomial q in n variables such that

$$\delta = q(b_0, \ldots, b_{n-1}) = f(A)$$

where f is a polynomial in the n^2 coefficients of the matrix A. Since

$$\mathcal{C} = \{A : f(A) = 0\},$$

the proof is completed by the following lemma. $\square$

An analytic variety is the set of common zeros of one or more analytic functions of several (real or complex) variables. Such a variety may have regular and singular points, the former being points near which the zero set is an analytic manifold. An algebraic variety is obtained by assuming that the analytic functions above are all polynomials in several variables. The real case of the following lemma has an identical proof. Both are immediate consequences of the detailed structure theorem available for an analytic variety, which is far from elementary in the real case. The complex case is explained on p.93 of B Buffoni and J Toland, Analytic Theory of Bifurcation, Princeton Univ. Press, 2003.

Lemma 1.8 *If $f : \mathbf{C}^m \to \mathbf{C}$ is an entire function and not identically zero then*

$$N = \{z \in \mathbf{C}^m : f(z) = 0\}$$

is a closed subset of $\mathbf{C}^m$ with zero Lebesgue measure.

Proof If $m = 1$ the set N is discrete, so the lemma is elementary. We assume inductively that the lemma holds for $m = n - 1$ and prove that it then holds for $m = n$.

Let $U \subseteq \mathbf{C}^{n-1}$ be the set of $\hat{z} = (z_1, \ldots, z_{n-1})$ such that $g(z_n) = f(\hat{z}, z_n)$ is not identically zero. Then U is open in $\mathbf{C}^{n-1}$ and $\{z_n : g(z_n) = 0\}$ is discrete and therefore of zero measure. This shows that $N' = N \cap (U \times \mathbf{C})$ is a null set. Now let $S \subseteq \mathbf{C}^{n-1}$ be the set of $\hat{z} = (z_1, \ldots, z_{n-1})$ such that $g(z_n) = f(\hat{z}, z_n)$ is identically zero. Choose $a \in \mathbf{C}$ such that $h(\hat{z}) = f(\hat{z}, a)$ is not identically zero. Such an a exists because f is not identically zero by hypothesis. Then $T = \{\hat{z} : h(\hat{z}) = 0\}$ is a null set in $\mathbf{C}^{n-1}$ by the inductive hypothesis. The formula $S \subseteq T$ implies that $S \times \mathbf{C}$ is a null set in $\mathbf{C}^n$. We conclude that $N = (S \times \mathbf{C}) \cup N'$ is a null set. □

The following is one of many applications of the same circle of ideas.

Theorem 1.9 *Let C be a convex set of $n \times n$ matrices and let D be the set of all $n \times n$ matrices that have n distinct eigenvalues. If $C \cap D$ is not empty then it is dense in C.*

Proof Let $A \in C$ and $B \in C \cap D$ and put $A(s) = (1 - s)A + sB$. Then

$$p(\lambda) = \det(\lambda I - A(s)) = \lambda^n + \sum_{r=0}^{n-1} b_r(s)\lambda^r$$

where each b_r is a polynomial in s. The discriminant $\delta(s)$, defined as before, is also a polynomial in s and it is non-zero for $s = 1$ by the hypothesis on B. Therefore $\delta(s) \neq 0$ for all s that do not lie in the finite set of roots of δ, and this includes all small enough positive s, for which $A(s) \in C$ by the convexity of C. □

1.4 Differentiation of vector-valued functions

Problem 1.4.6 of LOTS omits the assumption that $A(t)$ is invertible for all $t \in [a, b]$. The following extension of this problem is proved by induction on n and will be used later.

Problem 1.10 If $\mathcal{B}$ is a Banach algebra with identity and $A : [a, b] \to \mathcal{B}$ is n times continuously norm differentiable and $A(t)$ is invertible for every $t \in [a, b]$ then $t \to A(t)^{-1}$ is n times norm continuously differentiable on $[a, b]$. □

1.6 Banach algebras and the Sylvester Equation

Much of the theory of this chapter can be developed at a Banach algebra level. In particular Gel'fand's representation theorem for commutative Banach algebras is used in Lemma 2.4. of LOTS and Theorem 8.2.7 of LOTS. We will not write out the details of the Gel'fand theory but explain how it may be used to provide a nice solution of the Sylvester equation for bounded operators on a Banach space. This equation is of great importance in a variety of fields and has been studied intensively from many different points of view. See 'R Bhatia, P Rosenthal, How and why to solve the operator equation $AX - XB = Y$, Bull. London Math. Soc. 29 (1) (1997) 1-21'. The paper 'E B Davies, Algebraic Aspects of Spectral Theory, preprint, 2010' provides an algebraic version of the theorem, in which the underlying field $\mathbf{F}$ may be arbitrary; in control theory one might wish to let $\mathbf{F}$ be the field of all rational functions on the complex plane.

Let a, b, c be bounded operators on the Banach space $\mathcal{B}$ and let $\mathcal{A}$ denote the algebra of bounded operators on $\mathcal{B}$. The problem is to find $x \in \mathcal{A}$ that solves the Sylvester equation $ax - xb = c$.

Theorem 1.11 *If* $\mathrm{Spec}(a) \cap \mathrm{Spec}(b) = \emptyset$ *then the Sylvester equation is soluble and the solution is unique.*

Proof We recast the equation in the form $L(x) - M(x) = c$ where $L, M : \mathcal{A} \to \mathcal{A}$ are defined by $L(x) = ax$ and $M(x) = xb$. The problem is to prove that $L - M$ is invertible. We claim that $\mathrm{Spec}(L) \subseteq \mathrm{Spec}(a)$ and that $\mathrm{Spec}(M) \subseteq \mathrm{Spec}(b)$. These have similar proofs and we only treat the first.

If $z \notin \mathrm{Spec}(a)$ then there exists $r \in \mathcal{A}$ such that $r(a - ze) = (a - ze)r = e$ where e is the identity operator on $\mathcal{B}$. If $R : \mathcal{A} \to \mathcal{A}$ is defined by $R(x) = rx$ then $R(L - zI) = (L - zI)R = I$, and this proves that $z \notin \mathrm{Spec}(L)$.

We conclude that L, M are commuting bounding operators on $\mathcal{A}$ with disjoint spectra. Let $\mathcal{D}$ be the Banach algebra of all bounded linear operators on $\mathcal{A}$, let $\mathcal{C}$ be a maximal commutative subalgebra of $\mathcal{D}$ containing L and M, and let $\widehat{\ }: \mathcal{C} \to C(\Omega)$ be the Gel'fand representation of $\mathcal{C}$. Then

$$\begin{aligned}
\mathrm{Spec}(L - M) &= \{(L - M)\widehat{\ }(\omega) : \omega \in \Omega\} \\
&\subseteq \{\widehat{L}(\omega) : \omega \in \Omega\} - \{\widehat{M}(\omega) : \omega \in \Omega\} \\
&= \mathrm{Spec}(L) - \mathrm{Spec}(M).
\end{aligned}$$

Therefore $0 \notin \mathrm{Spec}(L - M)$ and $L - M$ is invertible. $\square$

Chapter 2

Function spaces

2.3 Approximation and regularization

The Stone-Weierstrass theorem does not provide an algorithm for approximating a continuous function by polynomials, but one can often do this by interpolation. Using a uniformly distributed set of interpolation points is not, however, to be recommended. The Lagrange interpolation theorem is as follows.

Theorem 2.1 (Lagrange) *Let $a \leq a_1 < a_2 < \ldots < a_n \leq b$ and let f be an n times differentiable function on $[a, b]$. Suppose also that $f^{(n)}$ is bounded on $[a, b]$. Then there exists a unique interpolating polynomial of degree at most $n-1$, i.e. a polynomial p of that degree satisfying $p(a_r) = f(a_r)$ for all $r \in \{1, \ldots, a_n\}$. Moreover*

$$|f(x) - p(x)| \leq \frac{|q(x)|}{n!} \sup\{|f^{(n)}(\xi)| : \xi \in (a, b)\} \tag{2.1}$$

for all $x \in [a, b]$, where $q(x) = \prod_{r=1}^{n}(x - a_r)$.

Proof The polynomial

$$p(x) = \sum_{r=1}^{n} f(x_r) p_r(x)$$

interpolates as required if

$$p_r(x) = \prod_{\{s:s \neq r\}} \frac{x - a_s}{a_r - a_s}.$$

The proof uses the identities $p_r(a_r) = 1$ and $p_r(a_s) = 0$ if $r \neq s$. If p, $\tilde{p}$ are two interpolating polynomials then $p - \tilde{p}$ is a polynomial of degree at most $n-1$ that vanishes at every a_r. However, such a polynomial has at most $n-1$ roots unless it vanishes identically.

The proof of the bound (2.1) is trivial if $x = a_r$ for some r so we suppose that this is not the case. We now define

$$g(s) = f(s) - p(s) - kq(s) \tag{2.2}$$

for all $s \in [a, b]$ where k is chosen so that $g(x) = 0$. The function g is n times differentiable with at least $n + 1$ distinct zeros, so by applying Rolle's theorem n times, g' is n times differentiable with at least n distinct zeros. Repeating this argument inductively, $g^{(n)}$ has at least one zero. We call it $\xi \in (a, b)$ and, after putting $s = x$ in (2.2), deduce that $f^{(n)}(\xi) - k\,n! = 0$. Solving for k yields

$$f(x) - p(x) = \frac{q(x)f^{(n)}(\xi)}{n!}$$

and then the bound of the theorem. □

The following corollary has an easy proof, but it is better to approach Chebychev polynomial approximation by using the Fourier cosine series expansion of the even periodic function $f(\cos(\theta))$.[1]

Corollary 2.2 *Let $[a, b] = [-1, 1]$. If p is the polynomial that interpolates f at the roots $a_1, \ldots, a_n$ of the nth Chebychev polynomial T_n, then*

$$|f(x) - p(x)| \le \frac{1}{2^{n-1}n!} \sup\{|f^{(n)}(\xi)| : \xi \in (a, b)\} \tag{2.3}$$

for all $x \in [-1, 1]$.

Proof Since the leading coefficient of T_n is 2^{n-1} we have

$$|q(\xi)| = 2^{-(n-1)}|T_n(\xi)| \le 2^{-(n-1)}$$

for all $\xi \in (-1, 1)$. □

2.4 Absolutely convergent Fourier series

The Weiner space $\mathcal{A}$ is defined here but some of its simpler properties should have been listed. These include the following, all of which are proved in any advanced text on Fourier analysis.

1. $\mathcal{A}$ contains the algebra of all smooth periodic functions on $[0, 2\pi]$;

[1] For a detailed exposition of many of the traps that people have fallen into with polynomial approximation, and much more about the numerical implementation of various algorithms, see L. N. Trefethen, Approximation Theory and Approximation Practice, book in preparation, 2011.

2. Membership of $\mathcal{A}$ is a local property: $f \in \mathcal{A}$ whenever the following holds. For all $x \in [0, \pi]$ there exist $\varepsilon > 0$ and $g \in \mathcal{A}$ such that $f = g$ in the ε-neighbourhood of x.

3. $f \in \mathcal{A}$ if f is periodic and there exist $\alpha > \frac{1}{2}$ and $c > 0$ such that
$$|f(x) - f(y)| \le c|x - y|^{\alpha}$$
for all x, $y \in [0, 2\pi]$.

4. The function f considered in Theorem 3.3.11 of LOTS is continuous and periodic but cannot lie in $\mathcal{A}$ because its Fourier series does not converge uniformly to f.

Our proof of Wiener's Theorem 2.4.2 of LOTS does not extend to the non-commutative context, but the following weaker version is useful. Let $\mathcal{A}$ be a Banach algebra with identity and let $\mathcal{B}$ denote the space of all $f : \mathbf{Z} \to \mathcal{A}$ that decrease in norm super-polynomially as $n \to \pm\infty$. Then $\mathcal{B}$ is a (generally non-commutative) algebra with identity under convolution:
$$(f * g)_n = \sum_{m \in \mathbf{Z}} f_{n-m} g_m.$$

Theorem 2.3 *The Fourier map $\mathcal{F} : \mathcal{B} \to C_{\mathrm{per}}([-\pi, \pi], \mathcal{A})$ defined by*
$$(\mathcal{F}f)(\theta) = \sum_{n \in \mathbf{Z}} f_n \mathrm{e}^{-in\theta}.$$
is an algebra homomorphism if multiplication of functions in $C_{\mathrm{per}}([-\pi, \pi], \mathcal{A})$ is defined pointwise. Moreover f is invertible as an element of $\mathcal{B}$ if and only if $(\mathcal{F}f)(\theta)$ is invertible in $\mathcal{A}$ for every $\theta \in [-\pi, \pi]$.

Proof We start by proving that $\mathcal{F}$ is one-one. If $f \in \mathcal{B}$, $\mathcal{F}f = 0$ and $\phi \in \mathcal{A}^*$ then
$$0 = \langle (\mathcal{F}f)(\theta), \phi \rangle = \sum_{n \in \mathbf{Z}} \langle f_n, \phi \rangle \mathrm{e}^{-in\theta}$$
for all $\theta \in [-\pi, \pi]$. Since the scalar Fourier transform is one-one on $\ell^1(\mathbf{Z})$, we deduce that $\langle f_n, \phi \rangle = 0$ for all $n \in \mathbf{Z}$ and all $\phi \in \mathcal{A}^*$. The Hahn-Banach theorem now implies that $f_n = 0$ for all $n \in \mathbf{Z}$. It follows routinely that $g = \mathcal{F}f$ is solved for f by calculating the Fourier coefficients of g, provided one knows that $f \in \mathcal{B}$ exists.

Most of the statements in the theorem are routine. Suppose that $(\mathcal{F}f)(\theta)$ is invertible in $\mathcal{A}$ for every $\theta \in [-\pi, \pi]$. The function $g : [-\pi, \pi] \to \mathcal{A}$ defined by $g(\theta) = ((\mathcal{F}f)(\theta))^{-1}$ is norm infinitely differentiable by Problem 1.10. Repeated integration by parts implies that the Fourier coefficients of g decrease super-polynomially in norm, and hence that $g = \mathcal{F}h$ where $h \in \mathcal{B}$ is the sequence of Fourier coefficients of g. We deduce that $\mathcal{F}(f*h) = \mathcal{F}(h*f) = 1$ in $C_{\mathrm{per}}([-\pi, \pi], \mathcal{A})$. Therefore $f * h = h * f = 1$ in $\mathcal{B}$. $\square$

Chapter 3

Fourier transforms and bases

3.3 Bases of Banach spaces

page 80. Completeness of eigenvectors

I have written almost nothing about this subject, in spite of the large Soviet literature proving the completeness of the (generalized) eigenvectors of compact operators on a Hilbert space subject to certain trace class conditions. See Chapter 7, 8 and 10 of I. Gohberg, A. Goldberg and M. A. Kaashoek, Classes of Linear Operators, vol. 1, Birkhäuser, Basel, 1990.

page 83. An exactly soluble wild operator

An example of a wild biorthogonal pair is given in Section 14.5 of LOTS, but the following example is closer to the spirit of this chapter. It is closely related to a class of evolution equations that may be solved by means of an integral representation involving a carefully chosen contour in the complex plane; see A S Fokas and B Pelloni.

The spectral properties of the operator $Lf(x) = \frac{\mathrm{d}^3 f}{\mathrm{d}x^3}$ acting in $L^2(0,1)$ depend heavily on the boundary conditions imposed. Thus for periodic boundary conditions, L is a skew-adjoint operator with a complete orthonormal set of eigenfunctions.

Theorem 3.1 (D A Smith, 2010) *If L is the above operator, subject to the boundary conditions $f(0) = f'(0) = f(1) = 0$, then the eigenvalues of L are of the form $\lambda = z^3$, where z are the zeros of the entire function*

$$\Delta(z) = \mathrm{e}^{iz} + \omega \mathrm{e}^{i\omega z} + \omega^2 \mathrm{e}^{i\omega^2 z}$$

and $\omega = \mathrm{e}^{2\pi i/3}$. There are infinitely many such zeros diverging to infinity in three asymptotic directions. The corresponding eigenfunctions are complete but form a wild sequence in $L^2(0,1)$.

D A Smith (Ph D thesis, Reading University, 2010) has obtained a closed formula for the eigenfunctions and has obtained the exact exponential asymptotics of the norms of the spectral projections.

Chapter 4

Intermediate Operator Theory

4.3 Fredholm Operators

We consider semi-Fredholm operators in Section 4.6 below, but here we continue with the Fredholm theory.

Theorem 4.1 *Let $\mathcal{B} = \mathcal{B}_0 \oplus \mathcal{B}_1$ where $\mathcal{B}_1$ is finite-dimensional. Let $L : \mathcal{B} \to \mathcal{B}$ be associated with the operator-valued matrix $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ where $A : \mathcal{B}_0 \to \mathcal{B}_0$ is bounded and B, C, D are finite rank operators. Then L is Fredholm if and only if A is Fredholm, and in this case* $\mathrm{Ind}(L) = \mathrm{Ind}(A)$.

Proof If $L_0 = \left(\begin{smallmatrix} A & 0 \\ 0 & D \end{smallmatrix}\right)$ then L is Fredholm if and only if L_0 is Fredholm by Corollary 4.3.8 of LOTS. Also L_0 is Fredholm if and only if A is Fredholm by an elementary argument.

Writing $L_t = L_0 + tK$ where $t \in \mathbf{R}$ and $K = L - L_0$ is compact, Theorem 4.3.11 of LOTS implies that $\mathrm{Ind}(L_t)$ does not depend on t. Therefore $\mathrm{Ind}(L) = \mathrm{Ind}(L_0)$. The identity $\mathrm{Ind}(L_0) = \mathrm{Ind}(A)$ is elementary. $\square$

Corollary 4.2 *Let $A : \ell^2(\mathbf{N}) \to \ell^2(\mathbf{N})$ be defined by*

$$(Af)_n = \sum_{r=a}^{b} c_{n,r} f_{n+r}$$

where we adopt the convention that $f_n = 0$ if $n \leq 0$. Suppose also that $\lim_{n\to\infty} c_{n,r} = c_r$ for all r. Then A is a Fredholm operator and its index equals that of the Toeplitz operator A_∞, where

$$(A_\infty f)_n = \sum_{r=a}^{b} c_r f_{n+r}$$

Proof Let A_s be the truncation of A to $\ell^2(\mathbf{N}_s)$, where $\mathbf{N}_s$ is the set of integers n such that $n \geq s$. Then $\|A_s - A_\infty\|$ converges to 0 as $s \to \infty$, so $\mathrm{Ind}(A_s)$ is Fredholm and $\mathrm{Ind}(A_s) = \mathrm{Ind}(A_\infty)$ for all large enough s by Theorem 4.3.11 of LOTS. Also $\mathrm{Ind}(A_s) = \mathrm{Ind}(A)$ for all s by Theorem 4.1. □

page 119. Theorem 4.3.9 of LOTS only proves the statement given (that if A is a Fredholm operator then so is A^*) in one direction. If $\mathcal{B}$ is reflexive this implies the converse. The general case is harder and follows from Banach's closed range theorem below. See Theorem 4.11 below.

Theorem 4.3 *Let A be a Fredholm operator on the Banach space $\mathcal{B}$. Then A has zero index if and only if there exists K in the space $\mathcal{K}(\mathcal{B})$ of all compact operators such that $A + K$ is invertible. The set of all Fredholm operators with zero index is closed under multiplication.*

Proof We follow the notation of Theorem 4.3.5 of LOTS. If $\mathrm{Ind}(A) = 0$ then $\mathcal{B}_1$ and $\mathcal{C}_1$ have the same finite dimension. If $C : \mathcal{B}_1 \to \mathcal{C}_1$ is invertible then $A' = \left(\begin{smallmatrix} A & 0 \\ 0 & C \end{smallmatrix}\right)$ is invertible and $A - A'$ is finite rank, and hence compact. Conversely if $A+K$ is invertible for some $K \in \mathcal{K}(\mathcal{B})$, then $\mathrm{Ind}(A+K) = 0$. Now $\mathrm{Ind}(A+tK)$ is independent of $t \in \mathbf{R}$ by Theorem 4.3.11, so $\mathrm{Ind}(A) = 0$.

The second statement of the theorem is an immediate corollary of Theorem 4.3.7 of LOTS. □

Theorem 4.4 *Let A_1, A_2 be Fredholm operators on the Banach space $\mathcal{B}$. Then $A_1 A_2$ is a Fredholm operator and*

$$\mathrm{Ind}(A_1A_2) = \mathrm{Ind}(A_1) + \mathrm{Ind}(A_2).$$

Proof Let $L : \ell^2(\mathbf{N}) \to \ell^2(\mathbf{N})$ be the left shift operator $(Lf)_n = f_{n+1}$ and let R be the right shift $R = L^*$. A direct calculation establishes that $\mathrm{Ind}(L^m) = m$ and $\mathrm{Ind}(R^m) = -m$ for all $m \geq 0$. Now let $\mathcal{B}' = \mathcal{B} \oplus \ell^2(\mathbf{N}) \oplus \ell^2(\mathbf{N})$. If $\mathrm{Ind}(A_r) = p_r$ for $r = 1, 2$ then there exist $m_r \geq 0$ and $n_r \geq 0$ such that $p_r + m_r - n_r = 0$. Therefore $B_r = A_r \oplus L^{m_r} \oplus R^{n_r} : \mathcal{B}' \to \mathcal{B}'$ has zero index for $r = 1, 2$. Theorem 4.3 now yields

$$\begin{aligned} 0 &= \mathrm{Ind}(B_1B_2) \\ &= \mathrm{Ind}(A_1A_2) + (m_1 + m_2) - (n_1 + n_2) \\ &= \mathrm{Ind}(A_1A_2) - p_1 - p_2 \\ &= \mathrm{Ind}(A_1A_2) - \mathrm{Ind}(A_1) - \mathrm{Ind}(A_2). \end{aligned}$$

□

4.4 Finding the essential spectrum

page 134,2. Delete indexMartinez .

4.5 The stable spectrum

In this new section we determine the part of the spectrum of a bounded operator A on a Banach space $\mathcal{B}$ that is stable under compact perturbations of the operator. We call this part $\mathrm{Stab}(A)$.[1] If A is self-adjoint then it is easy to prove that $\mathrm{Stab}(A)$ coincides with the essential spectrum $\mathrm{Ess}(A)$ as we have defined it, but in general this is far from being the case, and one only has

$$\mathrm{Ess}(A) \subseteq \mathrm{Stab}(A) \subseteq \mathrm{Spec}(A).$$

Lemma 4.5 *If A is a bounded operator then*

$$\mathbf{C} = \mathrm{Ess}(A) \cup \bigcup_{n\in\mathbf{Z}} U_n(A) \tag{4.1}$$

where $\mathrm{Ess}(A)$ is closed, $U_n(A)$ are all open, all the sets on the right-hand side are disjoint and

$$U_n(A) = \{\lambda \in \mathbf{C} : A - \lambda I \textit{ is Fredholm and } \mathrm{Ind}(A - \lambda I) = n\} \tag{4.2}$$

where $\mathrm{Ind}(X)$ denotes the index of X for all X.

Proof The sets $U_n(A)$ are open by Theorem 4.3.11 of LOTS. The fact that $\mathrm{Ess}(A)$ is closed follows from Theorem 4.3.7 of LOTS or directly from (4.1). ☐

Theorem 4.6 (Schechter) *Let A be a bounded operator on $\mathcal{B}$ and put*

$$\mathrm{Stab}(A) = \mathrm{Ess}(A) \cup \bigcup_{n\neq 0} U_n(A) = \mathbf{C}\backslash U_0(A). \tag{4.3}$$

Then $\mathrm{Stab}(A)$ is closed and

$$\mathrm{Stab}(A + K) = \mathrm{Stab}(A) \subseteq \mathrm{Spec}(A). \tag{4.4}$$

for every compact operator K. Indeed

$$\mathrm{Stab}(A) = \bigcap\{\mathrm{Spec}(A + K) : K \textit{ is compact}\}.$$

[1]There are many different definitions of the essential spectrum of a bounded operator, all of which coincide for self-adjoint operators. What we refer to as the stable spectrum of A is called $\sigma_{e4}(A)$ in Section IX.1 of D E Edmunds and W D Evans, Spectral Theory and Differential Operators, OUP, 1987. It was introduced in M Schechter, Invariance of the essential spectrum, Bull. Amer. Math. Soc. 71 (1965) 365-367 and J. Math. Anal. Appl., On the essential spectrum of an arbitrary operator, I, 13 (1966) 205-215, where it was denoted $\sigma_{em}(A)$. See also K Gustafson and J Weidmann, On the essential spectrum, J. Math. Anal. Appl. 25 (1969), 121-127. Theorem 4.6 was proved by Schechter; see BAMS 1965 above.

Proof The second identity in (4.3) implies that $\mathrm{Stab}(A)$ is closed. It also implies the inclusion in (4.4). Corollary 4.3.8 of LOTS implies that $\mathrm{Ess}(A+K) = \mathrm{Ess}(A)$. By applying Theorem 4.3.11 of LOTS to the one-parameter family $t \to A + tK$, one sees that $U_n(A+K) = U_n(A)$ for all $n \in \mathbf{Z}$. These imply the equality in (4.4).

The instability under arbitrarily small finite rank perturbations of any point $\lambda \in \mathrm{Spec}(A) \cap U_0(A)$ is proved by using Theorem 4.3.5 of LOTS. Replacing A by $A - \lambda I$, we may reduce to the case in which $\lambda = 0$. In the notation of Theorem 4.3.5, $\mathcal{B}_1$ and $\mathcal{C}_1$ have the same finite dimension, so there exists a finite rank operator K mapping $\mathcal{B}_1$ one-one onto $\mathcal{C}_1$. If $\varepsilon > 0$ and one associates $B : \mathcal{B} \to \mathcal{C}$ with the matrix $\left(\begin{smallmatrix} A_0 & 0 \\ 0 & \varepsilon K \end{smallmatrix}\right)$, where A_0 is defined as in Theorem 4.3.5, then B is an invertible finite rank perturbation of A, so $0 \notin \mathrm{Spec}(B)$. $\square$

Our next theorem describes the part of $\mathrm{Spec}(A)$ in the set

$$U_0(A) = \{\lambda \in \mathbf{C} : A - \lambda I \text{ is Fredholm and } \mathrm{Ind}(A - \lambda I) = 0\}$$

Note that $\mathrm{Spec}(A)$, $\mathrm{Ess}(A)$ and $U_n(A)$ are all invariant under compact perturbations of A.

Theorem 4.7 *Let A be a bounded operator on the Banach space $\mathcal{B}$ and let V be a connected component of the open set $U_0(A)$. Then one of the following two cases holds.*

1. $V \subseteq \mathrm{Spec}(A)$*;*
2. $V \cap \mathrm{Spec}(A)$ *is at most countable. Every point $\lambda \in V \cap \mathrm{Spec}(A)$ is an isolated point and the corresponding spectral projection has finite rank.*

Case 2 is generic in the following sense. If $\mathcal{K}(\mathcal{B})$ denotes the set of all compact operators on $\mathcal{B}$ then Case 2 holds for $A + K$, provided K lies in a certain dense open subset of $\mathcal{K}(\mathcal{B})$.

Proof If Case 1 is false then there exists $a \in U_0(A)$ for which $A - aI$ is invertible. Replacing A by $A - aI$ everywhere, there is no loss in assuming that $a = 0$ and that A itself is invertible. The following facts are immediate consequences of the formula

$$A^{-1} - \lambda^{-1} I = \lambda^{-1} A^{-1} (\lambda I - A)$$

in which we assume that $\lambda \neq 0$. $\lambda^{-1} \in \mathrm{Spec}(A^{-1})$ if and only if $\lambda \in \mathrm{Spec}(A)$. $\lambda^{-1} \in \mathrm{Ess}(A^{-1})$ if and only if $\lambda \in \mathrm{Ess}(A)$. Assuming that $\lambda \notin \mathrm{Ess}(A)$, $\mathrm{Ind}(A^{-1} - \lambda^{-1} I) = \mathrm{Ind}(A - \lambda I)$. $\lambda^{-1} \in U_0(A^{-1})$ if and only if $\lambda \in U_0(A)$.

Since $0 \in V$ we deduce that V^{-1} is the unbounded component of $U_0(A^{-1})$. The properties claimed in Case 2 are now obtained by applying Theorem 4.3.18 of LOTS.

If $A+K_0$ is in Case 2 then there exists $\lambda \in V$ such that $A+K_0 = \lambda I$ is invertible. The same applies to all K close enough to K_0 by a perturbation argument. Therefore such $A+K$ also fall into Case 2. This implies that the set of K for which $A+K$ falls into Case 2 is open in $\mathcal{K}(\mathcal{B})$. It remains to prove that it is dense. The instability of Case 1 under arbitrarily small finite rank perturbations was shown in the proof of Theorem 4.6. $\square$

Theorem 4.8 *Let $\mathcal{B} = \bigoplus_{r=1}^n \mathcal{B}_r$ and let $A \in \mathcal{L}(\mathcal{B})$ be associated with the operator-valued matrix $\{A_{r,s}\}_{1\le r,s\le n}$ where $A_{r,s} : \mathcal{B}_s \to \mathcal{B}_r$ are bounded for all r, s and compact if $r \neq s$. Then*

$$\mathrm{Ess}(A) = \bigcup_{r=1}^n \mathrm{Ess}(A_{r,r})$$

and

$$\mathrm{Stab}(A) = \bigcup_{r=1}^n \mathrm{Ess}(A_{r,r}) \cup \bigcup_M \{U_{m_1}(A_{1,1}) \cap \ldots \cap U_{m_n}(A_{n,n})\} \tag{4.5}$$

where $U_n(\cdot)$ are defined in (4.2) and $M = \{(m_1,\ldots,m_n) : m_1 + \ldots + m_n \neq 0\}$. If $\mathcal{B}_s$ is finite-dimensional for some s then one may omit that index in the formula (4.5).

Proof We first observe that A has the same essential spectrum as B, where $B_{r,s} = A_{r,s}$ if $r=s$ and $B_{r,s}=0$ otherwise. Also

$$\mathrm{Ess}(B) = \bigcup_{r=1}^n \mathrm{Ess}(A_{r,r}).$$

Equivalently $B-\lambda I$ is Fredholm if and only if $A_{r,r}-\lambda I$ is Fredholm for all $r \in \{1,\ldots,n\}$. If $B-\lambda I$ is Fredholm, we put $m_r = \mathrm{Ind}(A_{r,r}-\lambda I)$ for all r, or equivalently suppose that $\lambda \in U_{m_1}(A_{1,1}) \cap \ldots \cap U_{m_n}(A_{n,n})$. Then

$$\begin{aligned}\mathrm{Ind}(A-\lambda I) &= \mathrm{Ind}(B-\lambda I)\\ &= \sum_{r=1}^n \mathrm{Ind}(A_{r,r}-\lambda I)\\ &= \sum_{r=1}^n m_r.\end{aligned}$$

Therefore $\lambda \in \mathrm{Stab}(A)$ if and only if $\sum_{r=1}^n m_r \neq 0$.

If $\mathcal{B}_s$ is finite-dimensional then $\mathrm{Ind}(A_{s,s}-\lambda I) = 0$ for all $\lambda \in \mathbf{C}$. Therefore the only relevant value of m_s is 0 and $\sum_{r=1}^n m_r \neq 0$ if and only if $\sum_{\{r:r\neq s\}} m_r \neq 0$. $\square$

4.6 Operators with closed ranges

Subsection 4.6.1 An extension of Banach's closed range theorem below to unbounded closed operators may be found in Kato 'Perturbation Theory for Linear Operators', Theorem 5.13 and Corollary 5.14, and also in Yosida 'Functional Analysis', section VII.5.

Theorem 4.9 (Banach) *Let $A : X \to Y$ be a bounded linear operator, where X, Y are any Banach spaces. Then the following are equivalent.*

1. $\mathrm{Ran}(A)$ *is a closed subspace of* Y*;*
2. $\mathrm{Ran}(A^*)$ *is a weak* closed subspace of* X^**;*
3. $\mathrm{Ran}(A^*)$ *is a norm closed subspace of* X^**.*

The proof is achieved in a series of steps.

Step 1 If $B : X \to Y$ is one-one with closed range R, then B^* has range X^*.

Proof One may write $B = iC$ where $C : X \to R$ is one-one onto and $i : R \to Y$ is the natural injection. Therefore $B^* = C^* i^*$ and

$$\mathrm{Ran}(B^*) = C^*(\mathrm{Ran}(i^*)) = C^*(R^*) = X^*.$$

In the last step we use the fact that $C^* : R^* \to X^*$ is invertible with $(C^*)^{-1} = (C^{-1})^*$. $\square$

Step 2 Item 1 implies Items 2 and 3.

Proof Let $A : X \to Y$ be bounded with kernel K and closed range R. If $i : X \to X/K$ is the standard quotient map then $A = Bi$ where $B : X/K \to Y$ is one-one with range R. Step 1 now yields

$$A^*(Y^*) = i^* B^*(Y^*) = i^*((X/K)^*) = K^\perp,$$

where

$$K^\perp = \{\phi \in X^* : \langle x, \phi \rangle = 0 \text{ for all } x \in K\}$$

is a weak* closed subspace of X^*. $\square$

Step 3 If $B : X \to Y$ is one-one and B^* has weak* closed range $L \subseteq X^*$ then B has closed range.

Proof If $L \neq X^*$ then by applying the Hahn-Banach theorem to L as a closed subspace of X^* subject to the weak* topology, there exists a non-zero $x \in X$ satisfying $\langle x, \phi \rangle = 0$ for all $\phi \in L$. This is equivalent to $\langle x, B^*\psi \rangle = 0$ for all $\psi \in Y^*$ and thus to $\langle Bx, \psi \rangle = 0$ for all $\psi \in Y^*$. We deduce that $Bx = 0$ and hence that $x = 0$. The contradiction implies that $L = X^*$.

The identity $\mathrm{Ran}(B^*) = X^*$ implies that B^{**} is one-one. On applying the fact that Item 1 implies Item 2 to B^* we deduce that $B^{**} : X^{**} \to Y^{**}$ has norm closed range. The inverse mapping theorem now implies that there exists $c > 0$ such that $\|B^{**}\xi\| \geq c\|\xi\|$ for all $\xi \in X^{**}$. Restricting to X, which is canonically and isometrically embedded as a subspace of X^{**}, we obtain $\|Bx\| \geq c\|x\|$ for all $x \in X$. This implies that $\mathrm{Ran}(B)$ is closed. □

Step 4 Item 2 implies Item 1.

Proof We assume that $A : X \to Y$ has kernel K and that $A^* : Y^* \to X^*$ has weak* closed range. If $i : X \to X/K$ is the canonical quotient map then $B : X/K \to Y$, defined by $A = Bi$, is one-one and satisfies $\mathrm{Ran}(B) = \mathrm{Ran}(A)$. We have to prove that Step 3 is applicable to B in order to complete the proof.

Since $A^* = i^* B^*$ and $i^* : (X/K)^* \to X^*$ is one-one, the range of B^* is the inverse image under i^* of the range of A^*. As the inverse image under a weak* continuous map of a weak* closed subspace, the range of B^* must be weak* closed, as required for Step 3. □

Step 5 Item 3 implies Item 2.

Proof Let R be the norm closure of $\mathrm{Ran}(A)$ and let $i : R \to Y$ be the natural injection. Then $A = iB$ where $B : X \to R$ equals A. Step 1 implies that $i^* : Y^* \to R^*$ is surjective and this implies that $A^* = B^* i^*$ has the same range as B^*. Therefore the range of A^* is weak* closed if and only if the range of B^* is weak* closed. The fact that $B : X \to R$ has dense range implies that B^* is one-one. We now focus attention on B.

Put $L = \mathrm{Ran}(A^*) = \mathrm{Ran}(B^*) \subseteq X^*$. We define

$$\begin{aligned} L_c &= L \cap \{\phi \in X^* : \|\phi\| \leq c\} \\ &= \{\phi = B^*\psi : \psi \in R^* \text{ and } \|\phi\| \leq c\} \\ &= B^*(S) \end{aligned}$$

where

$$\begin{aligned} S &= \{\psi \in R^* : \|B^*\psi\| \leq c\} \\ &= \bigcap_{\{x \in X : \|x\| \leq 1\}} \{\psi \in R^* : |\langle \psi, Bx \rangle| \leq c\}, \end{aligned}$$

which is weak* closed.

We next observe that $B^* : R^* \to L$ is one-one onto, where L is norm closed by assumption. The inverse mapping theorem now implies that there exists $b > 0$ such that $\|B^*\psi\| \geq b\|\psi\|$ for all $\psi \in R^*$. Therefore

$$S \subseteq \{\psi \in R^* : \|\psi\| \leq b^{-1}c\},$$

which is weak* compact. Therefore S is weak* compact. Since B^* is weak* continuous $L_c = B^*(S)$ is weak* compact and therefore weak* closed. It now follows that L is weak* closed by the Krein-Šmulian theorem. (See N Dunford and J T Schwartz, 'Linear Operators, Part 1', Theorem V.5.7, Interscience, 1958). □

Example 4.10 Let $X = C[0,1]$ and let $B : X \to X$ be defined by $(Bf)(x) = xf(x)$. Then B is one-one but Step 3 of Theorem 4.9 is not applicable to B because the range of B^* is not closed or even weak* dense in X^*. Moreover B^{**} is not one-one. □

Theorem 4.11 *If $A : X \to Y$ is a bounded linear operator then A is Fredholm if and only if A^* is Fredholm. If $X = Y$ then*

$$\mathrm{Ess}(A) = \mathrm{Ess}(A^*).$$

Proof Theorem 4.3.9 of LOTS contains a proof that A^* is Fredholm if A is Fredholm; the assumption that $X = Y$ is irrelevant. A second proof can be based on the ideas below.

If A^* is Fredholm then A^* has closed range by Theorem 4.3.4. Theorem 4.9 now implies that A^* has weak* closed range and that A has closed range. Let $K_1 = \mathrm{Ker}(A)$, $R_1 = \mathrm{Ran}(A)$, $K_2 = \mathrm{Ker}(A^*)$ and $R_2 = \mathrm{Ran}(A^*)$. Standard applications of the Hahn-Banach Theorem now yield $K_1^* \sim X^*/R_2$ and $(Y/R_1)^* \sim K_2$, where $\sim$ denotes a canonical isometric isomorphism. Since we are assuming that A^* is Fredholm, we deduce that K_1 and Y/R_1 are finite-dimensional. Therefore A is Fredholm.

The second statement follows immediately, because $\lambda \notin \mathrm{Ess}(A)$ if and only if $A - \lambda I$ is Fredholm, by definition. Similarly for A^*. □

Subsection 4.6.2 Semi-Fredholm operators provide a second topic relating to operators with closed ranges. Given a Banach space $\mathcal{B}$ we say that $A : \mathcal{B} \to \mathcal{B}$ is semi-Fredholm if $\mathrm{Ker}(A)$ is finite-dimensional and $\mathrm{Ran}(A)$ is closed with infinite co-dimension. In the following we will assume that $\mathrm{Ker}(A) = \{0\}$; the general case may be treated by using the techniques developed for Fredholm operators in LOTS.

Lemma 4.12 *Let $A : [0,1] \to \mathcal{L}(\mathcal{B})$ be a norm continuous operator-valued function such that $\mathrm{Ker}(A_t) = \{0\}$ for all $t \in [0,1]$. Then the following are equivalent.*

1. *$\mathrm{Ran}(A_t)$ is closed for all $t \in [0,1]$;*
2. *There exists a constant $c > 0$ such that $\|A_t f\| \geq c\|f\|$ for all $f \in \mathcal{B}$ and all $t \in [0,1]$.*

Proof The implication $2 \Rightarrow 1$ is elementary, so we concentrate on $1 \Rightarrow 2$. Let

$$\alpha_t = \max\{c : \|A_t f\| \geq c\|f\| \text{ for all } f \in \mathcal{B}\}.$$

The inverse mapping theorem implies that $\alpha_t > 0$. If $s,\, t \in [0,1]$ then

$$\alpha_t \geq \alpha_s - \|A_t - A_s\|.$$

This implies that

$$|\alpha_t - \alpha_s| \leq \|A_t - A_s\|$$

for all $s,\, t \in [0,1]$, and hence that α_s depends continuously on s. The lemma follows. □

Theorem 4.13 *Let $A_{(\cdot)}$ be a family of operators with all the properties listed in Lemma 4.12. Then exactly one of the following occurs.*

1. *A_t is Fredholm for all $t \in [0,1]$ and $\mathrm{Ind}(A_t)$ is independent of t;*

2. *the codimension of $\mathrm{Ran}(A_t)$ is infinite for all $t \in [0,1]$.*

Proof The set $U = \{s \in [0,1] : \mathrm{Codim}(\mathrm{Ran}(A_s)) < \infty\}$ is open (relative to $[0,1]$) by Theorem 4.3.11 of LOTS. If we can prove that

$$V = \{s \in [0,1] : \mathrm{Codim}(\mathrm{Ran}(A_s)) = \infty\},$$

is also open then by the connectedness of $[0,1]$ it follows that $U = [0,1]$ or $V = [0,1]$.

In order to prove that V is open, let $s \in V$, and let $\mathcal{L}$ be a subspace of dimension $n < \infty$ in $\mathcal{B}$ satisfying $\mathrm{Ran}(A) \cap \mathcal{L} = \{0\}$. Under these conditions it is easy to prove that $\mathcal{C} = \mathrm{Ran}(A) + \mathcal{L}$ is a closed subspace of $\mathcal{B}$. Let $B : \mathcal{B} \oplus \mathcal{L}$ to $\mathcal{B}$ be defined by $B(f \oplus v) = Af + v$. Then B is bounded and it maps $\mathcal{B} \oplus \mathcal{L}$ one-one onto $\mathcal{C}$. The inverse mapping theorem now implies that $\|Bg\| \geq c\|g\|$ for some $c > 0$ and all $g \in \mathcal{B} \oplus \mathcal{L}$. If one defines $B_t : \mathcal{B} \oplus \mathcal{L}$ to $\mathcal{B}$ by $B_t(f \oplus v) = a_t f + v$, then the proof of Lemma 4.12 implies that there exists $\delta > 0$ such that $\|B_t g\| \geq \frac{c}{2}\|g\|$ for all $g \in \mathcal{B} \oplus \mathcal{L}$, provided $|t - s| < \delta$. This implies that A_t has closed range and that $\mathrm{Ran}(A_t) \cap \mathcal{L} = \{0\}$ for all such t. Since n can be taken arbitrarily large, the codimension of $\mathrm{Ran}(A_t)$ is infinite provided $|t - s| < \delta$. Therefore the δ-neighbourhood of s is contained in V. □

Corollary 4.14 *Let $A_{(\cdot)}$ be a family of operators with all the properties listed in Lemma 4.12. If A_t is invertible for some $t \in [0,1]$ then it is invertible for all such t.*

Proof The assumptions imply that A_t is Fredholm with index 0 and Theorem 4.13 implies that the same holds for all $t \in [0,1]$. But $\mathrm{Ker}(A_t) = \{0\}$ for every such t so the codimension of $\mathrm{Ran}(A_t)$ is 0 for all such t. In other words $\mathrm{Ran}(A_t) = \mathcal{B}$ and A_t is invertible for all such t. □

4.7 Unbounded Fredholm operators

I should have mentioned in LOTS that Fredholm operators lie at the core of the K-theory of Atiyah and Singer, developed in the 1960s. The Atiyah-Singer index theorem in effect uses the index of a system of differential operators to study the geometry of a related compact manifold. We will not pursue this, but it demonstrates the importance of the present subject matter. The following explains the application of Fredholm theory to unbounded linear operators.

If X is a compact Riemannian manifold, these facts may be applied to a differential operator A of order n whose domain is the Sobolev space $\mathcal{D} = W^{n,2}(X)$. If one puts $Jf = (1-\Delta)^{-n/2}$, where Δ is the Laplace-Beltrami operator on X, then $\mathrm{Ran}(J) = W^{n,2}(X)$ and $B = AJ$ is a pseudodifferential operator of zero order.

Let A be an unbounded closed operator with domain $\mathcal{D}$ acting in the Banach space $\mathcal{B}$. Let A' be the same operator, but regard as acting on the Banach space $\mathcal{D}'$, which is the vector space $\mathcal{D}$ provided with the graph norm $|||f||| = \|f\| + \|Af\|$, or any equivalent norm. Let $I' : \mathcal{D}' \to \mathcal{B}$ denote the bounded restriction of the identity operator I. One says that the unbounded operator $A - \lambda I$ is Fredholm if $A' - \lambda I' : \mathcal{D}' \to \mathcal{B}$ is Fredholm. One defines $\mathrm{Ess}(A)$ to be the set of $\lambda \in \mathbf{C}$ for which $A' - \lambda I'$ is not Fredholm. If $\lambda \notin \mathrm{Ess}(A)$ then one puts

$$\mathrm{Ind}(A - \lambda I) = \dim(\mathrm{Ker}(A - \lambda I)) - \dim(\mathrm{Coker}(A - \lambda I))$$

as in the bounded case.

The following example can be extended to an analysis of the spectrum of an unbounded operator on a graph, some of whose edges are discrete while others are continuous.

Example 4.15 Let $\mathcal{B} = L^2(-\infty,0) \oplus \ell^2(\mathbf{N})$ and let $\mathcal{D}$ be the closed subspace of $W^{2,2}(-\infty,0) \oplus \ell^2(\mathbf{N})$ consisting of all $f \oplus g$ that satisfy the 'continuity' condition $f'(0) = g_1 - g_0$, where we adopt the convention $g_0 = f(0)$. Let $A : \mathcal{D} \to \mathcal{B}$ be the bounded linear operator

$$(A(f \oplus g))(x) = \begin{cases} -f''(x) + f(x) & \text{if } x \le 0, \\ \alpha_x g_{x-1} + \beta_x g_x + \gamma_x g_{x+1} & \text{if } x \in \mathbf{N}, \end{cases}$$

where α, β, γ are bounded complex-valued sequences. We will consider whether A is a Fredholm operator; the answer to this question does not depend on whether we regard A as bounded or unbounded in the above senses. $\square$

Theorem 4.16 *Let T be the truncation T of A to $\ell^2(\mathbf{N})$. Then the bounded operator $A : \mathcal{D} \to \mathcal{B}$ is Fredholm if and only if $0 \notin \mathrm{Ess}(T)$, and in this case* $\mathrm{Ind}(A) = \mathrm{Ind}(T)$.

Proof Let $U : \mathcal{E} = W_0^{2,2}(-\infty,0) \oplus \ell^2(\mathbf{N}) \to \mathcal{D}$ be the bounded invertible operator defined by $U(h \oplus g) = f \oplus g$ where

$$f(x) = h(x) + \delta \mathrm{e}^x$$

for all $x \leq 0$, and $2\delta = g_1 - h'(0)$. The operator $B = AU : \mathcal{E} \to \mathcal{B}$ is a rank one perturbation of $\left(\begin{smallmatrix} S & 0 \\ 0 & T \end{smallmatrix}\right)$ where $Sf = -f'' + f$, which is invertible as a bounded operator from $W_0^{2,2}(-\infty, 0)$ to $L^2(-\infty, 0)$, with zero index. Corollary 4.3.8 of LOTS implies that B (and hence A) is Fredholm if and only if T is Fredholm, and

$$\mathrm{Ind}(A) = \mathrm{Ind}(B) = \mathrm{Ind}\left(\begin{smallmatrix} S & 0 \\ 0 & T \end{smallmatrix}\right) = \mathrm{Ind}(S) + \mathrm{Ind}(T) = \mathrm{Ind}(T).$$

□

4.8 Real Operators

One defines a conjugation C on a Banach space $\mathcal{B}$ to be a bounded conjugate linear map such that $C^2 = I$. If $\mathcal{B}$ is a space of functions $f : X \to \mathbf{C}$, the map $(Cf)(x) = \overline{f(x)}$ is called the standard conjugation. This assumes, of course, that $f \in \mathcal{B}$ implies $\overline{f} \in \mathcal{B}$, but this is the case for most of the function spaces in this book. A bounded (complex linear) operator $A : \mathcal{B} \to \mathcal{B}$ is said to be real if $AC = CA$.

The following theorem has a straightforward adaptation to unbounded real operators A, in which case one imposes the further condition that $\mathrm{Dom}(A)$ is invariant under C.

Theorem 4.17 *If A is a real linear operator on $\mathcal{B}$ then* $\mathrm{Spec}(A)$ *and* $\mathrm{Ess}(A)$ *are closed under complex conjugation. If $\lambda \notin \mathrm{Ess}(A)$ then*

$$\mathrm{Ind}(A - \lambda I) = \mathrm{Ind}(A - \overline{\lambda} I). \tag{4.6}$$

Proof Each of the statements follows directly from the following facts, whose proofs are elementary.

1. $$\mathrm{Ker}(A - \overline{\lambda} I) = C\left(\mathrm{Ker}(A - \lambda I)\right).$$

   Hence the two kernels have the same dimension.

2. $$\mathrm{Ran}(A - \overline{\lambda} I) = C\left(\mathrm{Ran}(A - \lambda I)\right).$$

   Hence one range is closed if and only if the other range is closed.

3. If $\mathcal{M}$ is a finite-dimensional subspace such that $\mathcal{M} \cap \mathrm{Ran}(A - \lambda I) = \{0\}$ and $\mathcal{M} + \mathrm{Ran}(A - \lambda I) = \mathcal{B}$, then $\mathcal{N} = C(\mathcal{M})$ has the same properties with respect to $\mathrm{Ran}(A - \overline{\lambda} I)$. Moreover $\mathcal{M}$ and $\mathcal{N}$ have the same dimension.

□

The following is a limited version of a perturbation result of general importance.

Theorem 4.18 *Let A and B be bounded real operators on the Banach space $\mathcal{B}$ and suppose that λ is an isolated real eigenvalue of A with algebraic multiplicity 1. The $A + sB$ has an isolated eigenvalue λ_s, with algebraic multiplicity 1, near λ for all small enough $s \in \mathbf{R}$. Moreover $\lambda_s \in \mathbf{R}$ for all such s.*

Proof Let γ be a circle with centre λ and assume that it is sufficiently small so that λ is the only point of $\mathrm{Spec}(A)$ on or inside γ. Everything except the final statement of the theorem is given in Theorem 1.5.6 of LOTS. Since there is only one eigenvalue of $A + sB$ inside γ for all small enough $s \in \mathbf{R}$, Item 1 of the proof of Theorem 1 implies that it is real. □

Generically, as s increases the eigenvalue λ_s moves along the real axis until it meets another eigenvalue, after which the eigenvalues emerge as a complex conjugate pair. At the critical value of s the eigenvalue has algebraic multiplicity 2 but geometric multiplicity 1. The eigenvalue itself has a square root singularity as s passes through the critical value. These phenomena are all seen in Example 8.9.

Chapter 5

Operators on Hilbert space

5.7 The compactness of $f(Q)g(P)$

pages 160-162. Strictly speaking this section treats certain bounded operators $A(f, g)$ defined using Fourier transform techniques. Whether these operators equal $f(Q)g(P)$ as defined is a separate question, but if f and g are bounded in addition to satisfying the stated conditions there is no problem.

5.8 The $\mathcal{C}_p$ spaces

The $\mathcal{C}_p$ spaces have many applications in the theory of Schrödinger operators, and the theory in covered in great detail in many places. We assume throughout that $\mathcal{H}$ is a Hilbert space and that $1 \le p < \infty$. The following list of results is extracted from Chapters 1 and 2 of [Simon2005A].

Proposition 5.1

1. *If A is a bounded operator on $\mathcal{H}$ then A, A^*, $|A|$ and $|A^*|$ are all compact if any one of them is. We assume that this condition holds below.*
2. *The singular values $s_n(A)$ and $s_n(A^*)$ coincide if they are enumerated in decreasing order and repeated according to their multiplicities.*
3. *We say that $A \in \mathcal{C}_p$ if $\|A\|_p = \left[\sum_n s_n(A)^p\right]^{1/p}$ converges. The function $\|\cdot\|_p$ is a norm which makes $\mathcal{C}_p$ into a Banach space.*
4. *If $A \in \mathcal{C}_p$ and B, C are bounded then $BAC \in \mathcal{C}_p$ and $\|BAC\|_p \le \|B\|\,\|A\|_p\,\|C\|$.*
5. *Let $\mathcal{O}$ denote the family of all finite or countable orthonormal sequences in $\mathcal{H}$. Then*
$$\|A\|_p^p = \sup_{\phi,\psi\in\mathcal{O}} \left\{ \sum_n |\langle A\phi_n, \psi_n\rangle|^p \right\},$$

the right hand side being finite if and only if $A \in \mathcal{C}_p$.

6. *Let $A \in \mathcal{C}_p$ and let $\{\lambda_n\}$ be the finite or countable sequence of non-zero eigenvalues of A, each eigenvalue being repeated according to its algebraic multiplicity. Then*
$$\|A\|_p^p \geq \sum_n |\lambda_n|^p.$$
The proof of this uses item 5 with $\psi = \phi$ and Lemma 5.2.

Lemma 5.2 (Schur) *Let T be a bounded operator on $\mathcal{H}$. Then there exists a finite or countable orthonormal sequence $\{\phi_n\}$ in $\mathcal{H}$ such that*

1. *$\mathcal{L} = \mathrm{lin}\{\phi_n\}$ is an invariant subspace for T;*
2. *The restriction of T to $\mathcal{L}$ has an upper triangular matrix;*
3. *$\{\langle T\phi_n, \phi_n\rangle\}$ is the sequence of all non-zero isolated eigenvalues with finite algebraic multiplicity of T, each repeated according to its algebraic multiplicity.*

Proof We start by observing that there are at most countably many isolated eigenvalues with finite algebraic multiplicities. Each one is associated with an finite-dimensional invariant subspace within which T can be written in its Jordan form. This leads to the representation of T as an upper triangular matrix with respect to a finite or countable sequence of vectors e_n in which $T_{n,n}$ are the eigenvalues of T repeated according to their algebraic multiplicities, $T_{n,n+1}$ equals 0 or 1 for each n and all other $T_{r,s}$ vanish. The linear span $\mathcal{L}$ of the e_n may or may not be dense in $\mathcal{H}$ but it is always invariant under the action of T. We now apply the Gram-Schmidt method to the sequence $\{e_n\}$ to produce an orthonormal sequence $\{\phi_n\}$. The matrix of T with respect to this sequence is still upper triangular and its diagonal entries are unchanged. $\square$

The following theorem of Hansmann may be used to bound the complex eigenvalues of non-self-adjoint Schrödinger operators and the zeros of a class of analytic functions on the unit disc.[1] This contrasts with an earlier work which uses known properties of the zeros of analytic functions on the unit disc to obtains bounds on the distribution of the discrete eigenvalues of small perturbations of unitary operators.[2]

Theorem 5.3 (Hansmann) *Let A be a bounded operator on $\mathcal{H}$, let $B \in \mathcal{C}_p$ and let $\{\lambda_n\}$ be the sequence of all isolated eigenvalues with finite algebraic multiplicity*

[1]See M. Hansmann, An eigenvalue estimate and its application to non-selfadjoint Jacobi and Schrödinger operators, Lett. Math. Phys. 98(1) (2011) 79-95 and M. Hansmann, G. Katriel, From spectral theory to bounds on zeros of holomorphic functions, arXiv:1103.1487v2

[2]S. Favorov and L. Golinskii, A Blaschke-type condition for analytic and subharmonic functions and application to contraction operators, pp. 37-47 in "Linear and Complex Analysis: Dedicated to V. P. Havin on the Occasion of His 75th Birthday", Translation Amer. Math. Soc. 226, 2009.

of $A+B$, each repeated according to its multiplicity. Then

$$\sum_n \left[\operatorname{dist}(\lambda_n, \overline{\operatorname{Num}(A)})\right]^p \leq \|B\|_p^p. \tag{5.1}$$

Proof Since B is compact, A and $A+B$ have the same essential spectrum, which is contained in $\overline{\operatorname{Num}(A)}$. Therefore the part of the spectrum of $A+B$ outside $\overline{\operatorname{Num}(A)}$ consists entirely of isolated eigenvalues with finite algebraic multiplicity, which can only accumulate in $\operatorname{Ess}(A) \subseteq \overline{\operatorname{Num}(A)}$. Other eigenvalues of $A+B$ do not contribute to the sum in (5.1).

Adopting the notation of Lemma 5.2 with $T = A+B$, we have

$$\begin{aligned}
\sum_n \left[\operatorname{dist}(\lambda_n, \overline{\operatorname{Num}(A)})\right]^p &\leq \sum_n |\lambda_n - \langle A\phi_n, \phi_n\rangle|^p \\
&= \sum_n |\langle T\phi_n, \phi_n\rangle - \langle A\phi_n, \phi_n\rangle|^p \\
&= \sum_n |\langle B\phi_n, \phi_n\rangle|^p \\
&\leq \|B\|_p^p.
\end{aligned}$$

The last inequality uses item 5 of Proposition 5.1 with $\phi = \psi$. □

Corollary 5.4 *Let A be a contraction on $\mathcal{H}$, let $B \in \mathcal{C}_p$ and let $\{\lambda_n\}$ be the sequence of all eigenvalues λ of $A+B$ such that $|\lambda| > 1$, each repeated according to its algebraic multiplicity. Then*

$$\sum_n (|\lambda_n| - 1)^p \leq \|B\|_p^p.$$

Example 5.5 One cannot replace $\overline{\operatorname{Num}(A)}$ by $\operatorname{Spec}(A)$ in the statement of Theorem 5.3. Let $\mathcal{H} = \ell^2(Z)$ and define the unitary operator A on $\mathcal{H}$ by $(Af)_n = f_{n+1}$ for all $n \in \mathbf{Z}$. Then define the rank 1 perturbation B by

$$(Bf)_n = \begin{cases} -f_1 & \text{if } n = 0, \\ 0 & \text{otherwise.} \end{cases}$$

It may be seen that $A+B$ decomposes as the direct sum of two operators on $\ell^2(\mathbf{Z}_+)$ and $\ell^2(Z_-)$, and then that every $\lambda \in \mathbf{C}$ such that $|\lambda| < 1$ is an eigenvalue of $A+B$. Therefore the number of such eigenvalues is uncountable and the (modified) sum in (5.1) is infinite. □

Chapter 6

One-parameter semigroups

6.1 Basic properties of semigroups

page 165. Problem 6.1.4 Replace Z by A twice.

6.3 Some standard examples

page 183. Theorem 6.3.2

The proof of the second part of the theorem was omitted. The fact that T_t is positivity preserving under the extra hypothesis is elementary. Corollary 2.2.19 and the comments at the top of page 53 imply that each T_t is a contraction.

page 188,5. Complete proof with a box symbol.

Chapter 7

Special classes of semigroup

7.1 Norm continuity

page 191. Theorem 7.1.2 of LOTS

The bound on page 191,-4 of the proof only shows that T_t is norm continuous on the right at c. However one can also use it to prove norm continuity on the left at any point $b = t + c$ by regarding c as the variable. One then needs to note that if $0 < a < c < b$ then $\|ZT_c\| \leq \|ZT_b a\| \, \|T_{c-a}\| \leq k$ where k does not depend on c.

7.2 Trace class semigroups

page 194 On the line before Lemma 7.2.1 of LOTS Replace problem by lemma.

page 196,-7. The reference to (7.8) should be to the bound on line 9.

7.4 Differentiable and analytic vectors

page 201,-6. Replace Section 1.5 by Section 1.4.

7.5 Subordinated semigroups

page 207,7. The formula for $(\lambda - Z)^{-1} f$ depends on Theorem 8.2.1 of LOTS, on page 218, as well as on the use of the equation on p207,5. The simplest proof of p207,5 involves taking Fourier transforms of both sides, and then using Theorem 3.1.15 of LOTS on page 74 to invert the identity obtained.

Chapter 8

Resolvents and generators

8.2 Resolvents and semigroups

page 224,4. This should refer to page 178.

8.3 Classification of generators

page 230,-3. Theorem 8.3.1

This theorem can be extended in various ways, the following being typical. An alternative proof can be based on Theorem 8.3.11, which is closely related to the Legendre transform as developed in Section 10.2 of LOTS.

Theorem 8.1 *If Z is the generator of a one-parameter semigroup T_t on the Banach space $\mathcal{B}$ then the following are equivalent.*

1.
$$\|T_t\| \leq M(1+ct)\mathrm{e}^{at} \tag{8.1}$$
for all $t \geq 0$;

2.
$$\|\mathbf{R}(\lambda+a, Z)^n\| \leq \frac{M}{\lambda^n}\left(1+\frac{nc}{\lambda}\right)$$
for all $\lambda > 0$ and $n \geq 1$.

Proof We start by replacing T_t by $\mathrm{e}^{-at}T_t$, which reduces the proof to the case $a = 0$.

1⇒2 The formula

$$R(\lambda, Z)^n v = \frac{1}{\Gamma(n)} \int_0^\infty t^{n-1} \mathrm{e}^{-\lambda t} T_t v \,\mathrm{d}t$$

valid for all $\lambda > 0$, $n \geq 1$ and $v \in \mathcal{B}$, implies that

$$\begin{aligned}
\|R(\lambda, Z)^n v\| &\leq \frac{1}{\Gamma(n)} \int_0^\infty t^{n-1} \mathrm{e}^{-\lambda t} \|T_t v\| \,\mathrm{d}t \\
&\leq \frac{M\|v\|}{\Gamma(n)} \int_0^\infty t^{n-1} \mathrm{e}^{-\lambda t} (1 + ct) \,\mathrm{d}t \\
&= \frac{M\|v\|}{\Gamma(n)} \left(\frac{\Gamma(n)}{\lambda^n} + c \frac{\Gamma(n+1)}{\lambda^{n+1}} \right) \\
&= \frac{M\|v\|}{\lambda^n} \left(1 + \frac{nc}{\lambda} \right)
\end{aligned}$$

for all $\lambda > 0$ and $v \in \mathcal{B}$. The result follows.

2⇒1 We follow the same calculations and use the same notation as in the proof of Theorem 8.3.1. of LOTS, except for the need to modify (8.20) and its proof. The crucial bound is

$$\begin{aligned}
\|T_t^\lambda\| &= \|\mathrm{e}^{\lambda(-I+\lambda R_\lambda)t}\| \\
&\leq \mathrm{e}^{-\lambda t} \sum_{n=0}^\infty t^n \lambda^{2n} \|R_\lambda^n\| / n! \\
&\leq \mathrm{e}^{-\lambda t} M \sum_{n=0}^\infty t^n \lambda^n (1 + nc/\lambda) / n! \\
&= M(1 + ct).
\end{aligned}$$

□

Example 8.2 Prove an analogue of Theorem 8.1 when (8.1) is replaced by the condition that

$$\|T_t\| \leq M \mathrm{e}^{at} (1 + bt + ct^2)$$

for all $t \geq 0$. □

page 232,3. This uses Problem 6.1.2 page 165.

page 235,-1. The word is semigroups

8.5 Operator-valued multiplication operators

Within LOTS the phrase 'multiplication operator' is taken to refer to multiplication by complex-valued functions. In this section we consider a more general class of multiplication operators.

Theorem 8.3 *Let X be a set with a countably generated compact Hausdorff topology and let $\mathcal{B} = L^2(X, \mathcal{C}, \mathrm{d}x)$, where $\mathcal{C}$ is a Banach space and $\mathrm{d}x$ is a Borel measure with support equal to X. Let $A : \mathcal{B} \to \mathcal{B}$ be the bounded linear operator defined by $(Af)(x) = a(x)f(x)$ for all $f \in \mathcal{B}$, where $a : X \to \mathcal{L}(\mathcal{C})$ is a bounded norm continuous function. Then*

$$\mathrm{Spec}(A) = \bigcup_{x \in X} \mathrm{Spec}(a(x)). \tag{8.2}$$

If $\mathrm{d}x$ has no atoms then $\mathrm{Spec}(A) = \mathrm{Ess}(A)$.

Proof If the union in (8.2) is denoted by S and $\lambda \notin S$ then $\lambda - a(x)$ is invertible for every $x \in X$. Problem 5 above, with $n = 0$, implies that $(\lambda - a(x))^{-1}$ is a norm continuous (and hence bounded) function of x. The operator $B : \mathcal{B} \to \mathcal{B}$ defined by $(Bf)(x) = (\lambda - a(x))^{-1} f(x)$ is bounded and a direct calculation shows that $B(\lambda I - A) = (\lambda I - A)B = I$, so $\lambda \notin \mathrm{Spec}(A)$.

Conversely suppose that $\lambda \in S$, or more specifically $\lambda \in \mathrm{Spec}(a(u))$ for some $u \in X$. Lemma 1.2.13 of LOTS implies that one of the following holds.

1. There exists a sequence $f_n \in \mathcal{C}$ such that $\|f_n\| = 1$ for all n and $\|a(u)f_n - \lambda f_n\| < 1/n$;
2. There exists a sequence $f_n \in \mathcal{C}^*$ such that $\|f_n\| = 1$ for all n and $\|a(u)^* f_n - \lambda f_n\| < 1/n$;

We start with Case 1. If $\{u\}$ has positive measure c then $g_n = c^{-1/2} f_n \delta_u \in \mathcal{B}$ satisfies $\|g_n\| = 1$ and $\|Ag_n - \lambda g_n\| < 1/n$ for all n. Therefore $\lambda \in \mathrm{Spec}(A)$. If $\{u\}$ has zero measure then one defines $g_n \in \mathcal{B}$ by

$$g_n(x) = \begin{cases} |W_n|^{-1/2} f_n & \text{if } x \in W_n, \\ 0 & \text{otherwise,} \end{cases}$$

where W_n is an open subset of X and $|W_n| > 0$ is its measure, so that $\|g_n\| = 1$. We choose W_n to be a small enough neighbourhood of u to ensure that $\|Ag_n - \lambda g_n\| < 1/n$ and $|W_n| < 1/n$; this is possible by the norm continuity of the function a. It follows from the conditions on W_n that g_n converges weakly to 0 in $\mathcal{B}$ and that $\lim_{n\to\infty} \|Ag_n - \lambda g_n\| = 0$. Therefore λ lies in the essential spectrum of A by Lemma 4.3.15 of LOTS.

In Case 2 a similar argument may be applied to A^*, and the proof is completed by using Theorem 4.11. $\square$

The following theorem may be also formulated in terms of pseudospectra.

Theorem 8.4 *Let X be a set with a countably generated locally compact Hausdorff topology and let $\mathrm{d}x$ be a Borel measure on X with support equal to X. let $\mathcal{C}$ be a Banach space and let $a : X \to \mathcal{L}(\mathcal{C})$ be norm continuous and uniformly* *bounded on X. Let A be the bounded operator on $\mathcal{B} = L^2(X, \mathcal{C}, \mathrm{d}x)$ defined by $(Af)(x) = a(x)f(x)$. Then*

$$\mathrm{Spec}(A) = S_1 \cup S_2$$

where

$$S_1 = \bigcup_{x \in X} \mathrm{Spec}(a(x))$$

and S_2 is the set of $\lambda \notin S_1$ such that $\|(a(x) - \lambda I)^{-1}\|$ is an unbounded function on X.

Proof If $\lambda \notin S_1 \cup S_2$ then the formula $(Bf)(x) = (a(x) - \lambda I)^{-1} f(x)$ defines a bounded operator on $\mathcal{B}$ and one readily checks that $B(A - \lambda I) = (A - \lambda I)B = I$. Therefore $\lambda \notin \mathrm{Spec}(A)$.

If $\lambda \in S_1$ then $\lambda \in \mathrm{Spec}(A)$ by minor changes to the argument of Theorem 8.3.

If $\lambda \in S_2$ then for each $n \in \mathbf{N}$ there exist $x_n \in X$ and $h_n \in \mathcal{C}$ such that $\|h_n\| = 1$ and $\|(a(x_n) - \lambda I)^{-1} h_n\| > n$. Equivalently there exists $f_n \in \mathcal{C}$ such that $\|f_n\| = 1$ and $\|(a(x_n) - \lambda I) f_n\| < 1/n$. One then puts $g_n = |U_n|^{-1/2} \chi_{U_n} f_n$, where U_n is a small enough neighbourhood of x_n to ensure that $\|Ag_n - \lambda g_n\| < 1/n$; this is possible because $a(x)$ depends norm continuously on x. Letting $n \to \infty$, it follows that $\lambda \in \mathrm{Spec}(A)$. □

Problem 8.5 Following the assumptions and notation of Theorem 8.4, suppose in addition that $\mathcal{C}$ is a Hilbert space and that each operator $a(x)$ is normal. Prove that

$$\mathrm{Spec}(A) = \overline{S_1}.$$

□

8.6 Indefinite spectral problems

In this section we provide an introduction to the theory of indefinite spectral problems and explain the relevance of Krein spaces. This is a research field in its own right, and those who wish to pursue it could turn to one of the sources in the footnote.[1] We do not present the theory with maximum generality and often restrict attention to bounded operators for technical simplicity.

[1] References to some of further important papers and books may be found in H. Langer, Spectral functions of definitizable operators in Krein spaces, eds. D. Butkovic et al., pp. 1-46 in 'Functional analysis', Lect. Notes in Math. 948, Springer (1982); H. Langer, A. Markus, V. Matsaev, Locally definite operators in indefinite inner product spaces, Math. Ann. 308 (1997) 405-424; J. Behrndt, R. Möws and C. Trunk, Singular Indefinite Sturm-Liouville Operators with a Spectral Gap, J. Spectral Theory 1 (3) (2011) 327-347; A. Zettl, Sturm-Liouville Theory, Amer. Math. Soc., Providence, RI, 2005; J. Behrndt and C. Trunk, On the negative squares of indefinite Sturm-LiouvillE operators, J. Diff. Eqns. 238 (2007) 491-519; J. Behrndt and F. Philipp, Spectral analysis of singular ordinary differential operators with indefinite weights, J. Diff. Eqns. 248 (2010) 2015-2037.

Let H and B be self-adjoint operators on a Hilbert space $\mathcal{H}$, and assume for simplicity that B is bounded. One may wish to find the spectrum of the linear pencil $P(\lambda) = H - \lambda B$, that is the set of all $\lambda \in \mathbf{C}$ for which $P(\lambda)$ does not map $\mathrm{Dom}(H)$ one-one onto $\mathcal{H}$. More modestly one might seek non-zero solutions $f \in \mathrm{Dom}(H)$ and $\lambda \in \mathbf{C}$ of $Hf = \lambda Bf$.

In this paragraph we restrict attention to the important special case in which B is self-adjoint and bounded with a bounded inverse. If $\lambda \in \mathbf{C}$ and B is positive the spectrum of $P(\lambda)$ equals that of $K - \lambda I$, where $K = B^{-1}H$, which is self-adjoint with respect to the equivalent inner product $\langle f, g\rangle_B = \langle Bf, g\rangle$. Therefore the spectrum of the pencil is real. A similar reduction is possible if H is positive. If both operators are indefinite, we shall see that $\mathrm{Spec}(P(\cdot))$ need not be real.

There are two obvious ways of calculating $\mathrm{Spec}(P(\cdot))$. In the first, one puts $B = R^{-2}J$ where $R = |B|^{-1/2}$ and J is self-adjoint with $J^2 = I$. The spectral problem is equivalent to that for $R^2H - \lambda J$ and also to that for $RHR - \lambda J = \widetilde{H} - \lambda J$, and finally to that for $A - \lambda I$ where $A = J\widetilde{H}$. Therefore

$$\mathrm{Spec}(P(\cdot)) = \mathrm{Spec}(A).$$

In the second method, one finds all λ such that $0 \in \mathrm{Spec}(H - \lambda B)$ directly – perhaps, but not necessarily, by calculating the entire spectrum of $H - \lambda B$. This method does not require B to be bounded or to have a bounded inverse.

We next describe some of the basic ideas of the theory of Krein spaces. Let $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ be a Hilbert space and let $J : \mathcal{H} \to \mathcal{H}$ be a bounded linear operator satisfying $J = J^*$ and $J^2 = I$. The Krein space $(\mathcal{H}, [\![\cdot,\cdot]\!])$ is by definition the vector space $\mathcal{H}$ provided with the indefinite inner product $[\![f, g]\!] = \langle Jf, g\rangle$. It follows immediately that $[\![f, f]\!] \in \mathbf{R}$ for all $f \in \mathcal{H}$ and $|[\![f, g]\!]| \le \|f\|\,\|g\|$ for all $f,\, g \in \mathcal{H}$. Given $(\mathcal{H}, \langle\cdot,\cdot\rangle)$, J may be called the fundamental symmetry of the Krein space; however a Krein space has many different such symmetries associated with different choices of $[\![\cdot,\cdot]\!]$.

Every bounded linear functional $\phi : \mathcal{H} \to \mathbf{C}$ is of the form $\phi(f) = [\![f, g]\!]$ for some $g \in \mathcal{H}$, and $\|\phi\| = \|g\|$. The weak topology on $\mathcal{H}$ may therefore be defined by reference to all functionals $f \to [\![f, h]\!]$. Note that an operator on $\mathcal{H}$ is bounded if and only if it is weakly continuous, by the closed graph theorem, and that a linear subspace of $\mathcal{H}$ is norm closed if and only if it is weakly closed, by the Hahn-Banach theorem.

Given an unbounded operator $A = JH$, let $\mathcal{D}$ denote the set of all $f \in \mathcal{H}$ such that $[\![Ag, f]\!] = [\![g, h]\!]$ for all $g \in \mathrm{Dom}(A)$ and some (necessarily unique) $h \in \mathcal{H}$. We then define $A^\dagger f = h$, with $\mathrm{Dom}(A^\dagger) = \mathcal{D}$; the identity $A^\dagger = JA^*J$ is always valid. The identity $H = H^*$ implies $A = A^\dagger$ by a routine argument. If $A = A^\dagger$ and $Af = \lambda f$ then an elementary calculation implies that $\lambda \in \mathbf{R}$ unless $[\![f, f]\!] = 0$; however, complex eigenvalues may occur.

Theorem 8.6 *Let H be a self-adjoint operator acting in a Hilbert space $\mathcal{H}$, and suppose that there exists $c > 0$ such that $H \ge cI$. Suppose also that $A = JH$ where*

J is a bounded self-adjoint operator with spectrum $\{\pm 1\}$, so that $J^2 = I$. Then

$$\mathrm{Spec}(A) \subseteq (-\infty, -c\,] \cup [\,c, \infty).$$

Proof
Version 1 The hypotheses of the theorem imply that $\|J\| = 1$ and $\|H^{-1}\| \leq c^{-1}$. If $\lambda \in \mathbf{C}$ and $|\lambda| < c$ then $JH - \lambda I = JH(I - \lambda H^{-1}J)$ and an application of Theorem 1.2.11 of LOTS yields $\lambda \notin \mathrm{Spec}(JH)$ and

$$\|(JH - \lambda I)^{-1}\| \leq \frac{\|(JH)^{-1}\|}{1 - |\lambda|\,\|H^{-1}J\|} = \frac{\|H^{-1}\|}{1 - |\lambda|\,\|H^{-1}\|} \leq \frac{1}{c - |\lambda|}.$$

The identity $JH - \lambda I = \lambda J(\lambda^{-1}H - J)$ implies that $\lambda \notin \mathrm{Spec}(A)$ if and only if $\lambda^{-1}H - J$ is invertible. We use numerical range ideas to prove that $\lambda^{-1}H - J$ is invertible for all $\lambda \notin \mathbf{R}$. Suppose that $\lambda = r\mathrm{e}^{i\theta}$ where $r > 0$ and $0 < |\theta| < \pi$. If $f \in \mathrm{Dom}(H)$ then

$$\begin{aligned}
\|(\lambda^{-1}H - J)f\|\,\|f\| &\geq |\langle(\lambda^{-1}H - J)f, f\rangle| \\
&\geq |\mathrm{Im}\,\langle(\lambda^{-1}H - J)f, f\rangle| \\
&= r^{-1}|\sin(\theta)|\langle Hf, f\rangle \\
&\geq c\,r^{-1}|\sin(\theta)|\,\|f\|^2.
\end{aligned}$$

Therefore there exists $k > 0$ such that

$$\|(\lambda^{-1}H - J)f\| \geq k\|f\|$$

for all such f. This implies that $\mathrm{Ker}(\lambda^{-1}H - J) = \{0\}$ and that $\mathrm{Ran}(\lambda^{-1}H - J)$ is closed. If $f \perp \mathrm{Ran}(\lambda^{-1}H - J)$ then

$$f \in \mathrm{Ker}((\lambda^{-1}H - J)^*) = \mathrm{Ker}(\overline{\lambda^{-1}}H - J) = \{0\}$$

because $\overline{\lambda} \notin \mathbf{R}$. Therefore $\mathrm{Ran}(\lambda^{-1}H - J) = \mathcal{H}$ and $\lambda^{-1}H - J$ has a bounded inverse.

Version 2 This proof is easier and more natural, provided one puts a little time into understanding the theory of Krein spaces. The hypotheses of the theorem imply that $[\![Af, f]\!] \geq c\|f\|^2$ for all $f \in \mathrm{Dom}(A)$ and that A is invertible with $\|A^{-1}\| = \|H^{-1}\| \leq c^{-1}$. If $\lambda \in \mathbf{C}$ and $|\lambda| < c$ then $A - \lambda I = A(I - \lambda A^{-1})$ is invertible by Theorem 1.2.11 of LOTS and

$$\|(A - \lambda I)^{-1}\| \leq \frac{\|A^{-1}\|}{1 - |\lambda|\,\|A^{-1}\|} \leq \frac{1}{c - |\lambda|},$$

so such λ do not lie in $\mathrm{Spec}(A)$.

If $\lambda \notin \mathbf{R}$ then $\lambda = re^{i\theta}$ where $r > 0$ and $0 < |\theta| < \pi$. Then

$$\begin{aligned} \|(A-\lambda I)f\| \, \|f\| &\geq |\lambda| \, |[\![(\lambda^{-1}A - I)f, f]\!]| \\ &\geq |\lambda| \, |\mathrm{Im}\, [\![(\lambda^{-1}A - I)f, f]\!]| \\ &= |\sin(\theta)| \, [\![Af, f]\!] \\ &\geq k\|f\|^2 \end{aligned}$$

where $k = c|\sin(\theta)| > 0$. Therefore $A - \lambda I$ is one one with closed range.

We continue assuming that $\lambda \notin \mathbf{R}$. If $[\![(A-\lambda I)g, f]\!] = 0$ for all $g \in \mathrm{Dom}(A)$ then

$$f \in \mathrm{Ker}(A^\dagger - \overline{\lambda} I^\dagger) = \mathrm{Ker}(A - \overline{\lambda} I) = \{0\}.$$

because $\overline{\lambda} \notin \mathbf{R}$. Since $\mathrm{Ran}(A - \lambda I)$ is closed, we deduce that $A - \lambda I$ is invertible, and hence that $\lambda \notin \mathrm{Spec}(A)$.

Version 3 If H is bounded the following simpler proof is valid. One has $A = JH = H^{-1/2}BH^{1/2}$ where $B = H^{1/2}AH^{1/2} = B^*$. Therefore $\mathrm{Spec}(A) = \mathrm{Spec}(B) \subseteq \mathbf{R}$. The fact that A is similar to the self-adjoint operator B also shows that A has a full spectral calculus, including a bounded but non-self-adjoint spectral projection associate with every Borel subset of $\mathbf{R}$. ☐

One may classify simple eigenvalues of an operator $A \in \mathcal{L}(\mathcal{H})$ into positive, negative and neutral types depending on whether $[\![f, f]\!]$ is positive, negative or zero, where f is an eigenvector corresponding to the eigenvalue in question. The hypotheses of the next theorem may often be verified by using Theorem 1.5.6 of LOTS, or an extension of that theorem. We focus on the value $t = 0$ for simplicity, but the theorem may be applied to any other value by considering $B_t = A_{t+a}$.

Theorem 8.7 *Suppose that $A_t \in \mathcal{L}(\mathcal{H})$, $f_t \in \mathcal{H}\backslash\{0\}$ and $\lambda_t \in \mathbf{C}$ are defined for all sufficiently small real t and that $A_t f_t = \lambda_t f_t$ for all such t. Suppose also that each of the functions is norm differentiable at $t = 0$ and that $A_0 = A_0^\dagger$. Then $\lambda_0 \notin \mathbf{R}$ implies that λ_0 is of neutral type. If $[\![f_0, f_0]\!] \neq 0$, so that $\lambda_0 \in \mathbf{R}$, then*

$$\lambda_0' = \frac{[\![A_0' f_0, f_0]\!]}{[\![f_0, f_0]\!]}, \tag{8.3}$$

where the prime refers to the first derivative with respect to t.

Proof
Version 1 The assumptions imply that

$$[\![A_t f_t, f_t]\!] = \lambda_t [\![f_t, f_t]\!] \tag{8.4}$$

for all small enough $t \in \mathbf{R}$. The first statement of the theorem follows by putting $t = 0$ and using $A_0 = A_0^\dagger$ to prove that $[\![A_0 f_0, f_0]\!] \in \mathbf{R}$. Differentiating (8.4) at $t = 0$ yields

$$\begin{aligned} &[\![A_0' f_0, f_0]\!] + [\![A_0 f_0', f_0]\!] + [\![A_0 f_0, f_0']\!] \\ &\quad = \lambda_0' [\![f_0, f_0]\!] + \lambda_0 [\![f_0', f_0]\!] + \lambda_0 [\![f_0, f_0']\!]. \end{aligned}$$

Using $A_0 = A_0^\dagger$ again, this implies that

$$[\![A_0' f_0, f_0]\!] = \lambda_0' [\![f_0, f_0]\!],$$

which yields (8.3) immediately.

Version 2 The hypothesis can also be written in the form

$$H_t f_t = \lambda_t J f_t$$

for all sufficiently small $t \in \mathbf{R}$, where $H_t = J A_t$. From this point onwards we assume for simplicity that H_t and J are self-adjoint and bounded but do not need to assume that J is invertible. Differentiating both sides of

$$\langle H_t f_t, f_t \rangle = \lambda_t \langle J f_t, f_t \rangle$$

at $t = 0$ leads to the same conclusion as in Version 1, in the form

$$\lambda_0' = \frac{\langle H_0' f_0, f_0 \rangle}{\langle J f_0, f_0 \rangle}.$$

□

The following example and the program following it illustrate how the different types of eigenvalue move as the parameter c varies.

Example 8.8 Let $A = JH$ be the $N \times N$ matrix which is constructed by analogy with the case $N = 4$, for which

$$H = \begin{pmatrix} c & -1 & & \\ -1 & c & -1 & \\ & -1 & c & -1 \\ & & -1 & c \end{pmatrix}, \qquad J = \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}.$$

Theorem 8.6 implies that A has real spectrum if $c > 2$. However, simple numerical calculations indicate that A has some real and some complex eigenvalues if $0 < c < 2$. Figure 8.1 plots the eigenvalues of A for one particular case. □

Figure 8.1: Spectrum of A for $N = 40$ and $c = 1$

Example 8.9 The following MATLAB animation of the dependence of the spectrum on the parameter c is well worth running. The different types of eigenvalue are coloured red, green and black. The range of c considered depends on the choices of `cinit` and `cfinal`. The program may be downloaded from
http://www.mth.kcl.ac.uk/staff/eb_davies/colouranimate5.m

```
%  This calculates the eigenvalues of
%  an indefinite tridiagonal matrix
%  and plots it as a function of c

close('all')

% basic definitions
  M=25;
  N=2*M; % size of matrix
  J=diag([-ones(1,M)  ones(1,M)]);

 % set up plot data
  xmin = -5;
  xmax = 5;
  ymax = 0.2;
  ymin = -0.2;
  axis normal
  axis([xmin xmax ymin ymax]);
  hold on


  % initialize plot data off window
  xred=ones(1,N);
  yred=ones(1,N);
  xgreen=ones(1,N);
  ygreen=ones(1,N);
  xblack=ones(1,N);
  yblack=ones(1,N);

  % calculate eigenvalues and classify them into types
      fr=100; %number of frames to be shown
      cinit=0.9; % intial value of c
      cfinal=0.6;  % final value of c
  for r=1:fr
      c=cinit*(fr-r)/fr+cfinal*r/fr;
      H=diag(c*ones(1,N))-diag(ones(1,N-1),1)-diag(ones(1,N-1),-1);
      A=J*H;
      redvec=8*ones(1,N); % initialize plot data off window
```

```
    blackvec=8*ones(1,N); % initialize plot data off window
    greenvec=8*ones(1,N); % initialize plot data off window
    [V,D]=eig(A);
    for s=1:N
     ve=V(:,s);
     la=D(s,s);
        if ve'*J*ve>0.0001
          redvec(1,s)=la;
          elseif ve'*J*ve<-0.0001
          greenvec(1,s)=la;
          else blackvec(1,s)=la;
        end
    end
% clear previous plot
    plot(xred,yred,'w.','MarkerSize',10);
    plot(xgreen,ygreen,'w.','MarkerSize',10);
    plot(xblack,yblack,'wx','MarkerSize',4);
% set up new plot data
    xred=real(redvec);
    yred=imag(redvec); %+0.001*ones(1,N);
    xgreen=real(greenvec);
    ygreen=imag(greenvec);% -0.001*ones(1,N);
    xblack=real(blackvec);
    yblack=imag(blackvec);
% plot eigenvalues
    p1=plot(xred,yred,'r.','MarkerSize',10);
    p2=plot(xblack,yblack,'kx','MarkerSize',4);
    p3=plot(xgreen,ygreen,'g.','MarkerSize',10);
    drawnow
end;
```

□

We turn to the applications of the above theory to ordinary differential equations. The following sets the scene. A much more thorough account of such problems has recently been given in a series of papers by Behrndt, Möws and Trunk.

Problem 8.10 Let H be the self-adjoint operator acting in $L^2((\alpha,\beta),\mathrm{d}x)$ and given formally by

$$(Hf)(x) = -\frac{\mathrm{d}}{\mathrm{d}x}\left(p(x)\frac{\mathrm{d}f}{\mathrm{d}x}\right) + v(x)f(x)$$

subject to Dirichlet boundary conditions. We assume that p is positive, that v is real, that p, p^{-1} are bounded and that $v \in L^2$. The domain of H depends on p but H may also be defined as the self-adjoint operator associated with the closed quadratic form

$$Q(f) = \int_\alpha^\beta p(x)|f'(x)|^2 + v(x)|f(x)|^2\,\mathrm{d}x$$

defined on $W_0^{1,2}((\alpha,\beta),\mathrm{d}x)$.

Now let a be a real-valued function on (α,β) that is bounded and bounded away from zero, and define the operator J by $(Jf)(x) = \mathrm{sign}(a(x))f(x)$. Prove that if $|a|$ is sufficiently regular (specify) the eigenvalue problem $Hf = \lambda af$ is equivalent to the eigenvalue problem $\widetilde{H}f = \lambda Jf$, and hence also to the eigenvalue problem $J\widetilde{H}f = \lambda f$, where $\widetilde{H}$ is of the same form as H, but for functions $\widetilde{p}$ and $\widetilde{v}$ that you should determine. $\square$

The following example illustrates some general features of indefinite spectral problems.

Theorem 8.11 *Let $A = JH$ act in $L^2(\mathbf{R})$ with domain $\mathcal{D} = W^{2,2}(\mathbf{R})$, where $(Jf)(x) = \mathrm{sign}(x)f(x)$, $Hf = -f'' + Vf$ and the real potential V is taken to be bounded with finite limits $c_{\pm\infty}$ at $\pm\infty$. Then the essential spectrum of A is given by*

$$\mathrm{Ess}(A) = [c_\infty,\infty) \cup (-\infty,-c_{-\infty}].$$

Proof Let $A_0 = JH_0$ where H_0 is the same operator as H but with domain $\mathcal{D}_0 = \mathcal{D}_+ \oplus \mathcal{D}_-$, where $\mathcal{D}_+ = W_0^{2,2}(0,\infty)$ and $\mathcal{D}_- = W_0^{2,2}(-\infty,0)$. A and A_0 are both closed because J is invertible and H, H_0 are closed, indeed self-adjoint. Neither domain contains the other, but Lemma 11.3.2 of LOTS implies that A and A_0 have the same essential spectrum. Now A_0 is the direct sum of H_+ and H_- where $H_+f = -f'' + V$ acts in $L^2(0,\infty)$, while $H_-f = f'' - V$ acts in $L^2(-\infty,0)$, subject to Dirichlet boundary conditions at 0 in both cases. Therefore

$$\mathrm{Ess}(A) = \mathrm{Ess}(A_0) = \mathrm{Ess}(H_+) \cup \mathrm{Ess}(H_-).$$

One may write $H_+ = B + C$ where $Bf = -f'' + c_\infty f$ subject to Dirichlet boundary conditions at 0, while $Cf = Vf - c_\infty f$. Now C is relatively compact perturbation of B by a slight modification of Theorem 5.7.1 of LOTS. Therefore

$$\mathrm{Ess}(H_+) = \mathrm{Ess}(B) = [c_\infty, \infty).$$

The proof that $\mathrm{Ess}(H_-) = (-\infty, -c_{-\infty}]$ is similar. □

The operator A of Theorem 8.11 is not elliptic and it may have complex eigenvalues; because the operator is real, these must occur in complex conjugate pairs. This issue may be investigated at a general level, but we shall give a complete analysis of an exactly soluble example, which involves a 'delta function potential' concentrated at 0, because it illustrates some of the possibilities.

Example 8.12 Given γ, $\delta > 0$, we define a self-adjoint operator $H_{\gamma,\delta}$ acting in $L^2(\mathbf{R})$ as follows. Formally $(H_{\gamma,\delta}f)(x) = -f''(x) + \gamma f(x)$ on the domain $\mathcal{D}$ consisting of all functions $f = f_- \oplus f_+$ where $f_- \in W^{2,2}(-\infty, 0)$ and $f_+ \in W^{2,2}(0, \infty)$ satisfy $f_-(0) = f_+(0)$ and $f_+'(0) - f_-'(0) = -\delta f_+(0)$. Alternatively, $H_{\gamma,\delta}$ is the self-adjoint operator associated with the closed quadratic form

$$Q(f) = \int_{\mathbf{R}} \left(|f'(x)|^2 + \gamma |f(x)|^2\right) \mathrm{d}x - \delta |f(0)|^2,$$

defined on the domain $W^{1,2}(\mathbf{R})$. We are interested in finding the spectrum of $A_{\gamma,\delta} = JH_{\gamma,\delta}$, where $(Jf)(x) = \mathrm{sign}(x) f(x)$, the domain of $A_{\gamma,\delta}$ being the same as that of $H_{\gamma,\delta}$. □

Theorem 8.13 *Let $A_{\gamma,\delta}$ be the operator defined in Example 8.12. Then*

$$\mathrm{Spec}(A_{\gamma,\delta}) = S_1 \cup S_2 \tag{8.5}$$

where

$$S_1 = \mathrm{Ess}(A_{\gamma,\delta}) = (-\infty, -\gamma] \cup [\gamma, \infty) \tag{8.6}$$

and $S_2 = \mathrm{Eig}(A_{\gamma,\delta})$ depends on δ, γ as follows.

1. *If $0 < \delta^2 \le 2\gamma$ then $S_2 = \emptyset$.*
2. *If $2\gamma < \delta^2 \le 4\gamma$ then $S_2 = \{\pm\delta\sqrt{\gamma - \delta^2/4}\}$.*
3. *If $4\gamma < \delta^2 < \infty$ then $S_2 = \{\pm i\delta\sqrt{\delta^2/4 - \gamma}\}$.*

Moreover

$$\mathrm{Ind}(A_{\gamma,\delta} - \lambda I) = 0 \text{ for all } \lambda \notin \mathrm{Ess}(A_{\gamma,\delta}). \tag{8.7}$$

Proof We abbreviate $A_{\gamma,\delta}$ to A throughout. The proof of (8.6) follows that of Theorem 8.11 closely. In the following calculations, $\sqrt{z}$ always stands for the square root whose argument lies in $(-\pi, \pi]$.

If λ is an eigenvalue of A then the corresponding eigenfunction must be

$$f(x) = \begin{cases} \mathrm{e}^{-\sqrt{\gamma-\lambda}\,x} & \text{if } x > 0, \\ \mathrm{e}^{\sqrt{\gamma+\lambda}\,x} & \text{if } x < 0. \end{cases}$$

The conditions for λ to be an eigenvalue are

$$\begin{aligned} \mathrm{Re}\,\sqrt{\gamma-\lambda} &> 0, & (8.8) \\ \mathrm{Re}\,\sqrt{\gamma+\lambda} &> 0, & (8.9) \\ \sqrt{\gamma-\lambda}+\sqrt{\gamma+\lambda} &= \delta. & (8.10) \end{aligned}$$

Putting $\mu = \lambda/\gamma \in \mathbf{C}$ and $\tau = \delta/\sqrt{\gamma} > 0$, the conditions become

$$\begin{aligned} \mathrm{Re}\,\sqrt{1-\mu} &> 0, & (8.11) \\ \mathrm{Re}\,\sqrt{1+\mu} &> 0, & (8.12) \\ \sqrt{1-\mu}+\sqrt{1+\mu} &= \tau. & (8.13) \end{aligned}$$

Squaring both sides of (8.13) yields

$$2 + 2\sqrt{1-\mu^2} = \tau^2$$

and then

$$\mu = \pm\tau\sqrt{1-\tau^2/4}. \tag{8.14}$$

However, the same 'solutions' are obtained from all four of the equations

$$\pm\sqrt{1-\mu} \pm \sqrt{1+\mu} = \tau,$$

so some caution is necessary. Given $\tau > 0$ we put $\mu = \tau\sqrt{1-\tau^2/4}$ and have to determine whether μ satisfies (8.11-8.13). We consider a series of cases, applicable for different values of τ.

Case τ1. If $0 < \tau < \sqrt{2}$ then $0 < \mu < 1$ because the function $g(\tau) = \tau\sqrt{1-\tau^2/4}$ is strictly monotone increasing on $[0, \sqrt{2}]$ with $g(0) = 0$ and $g(\sqrt{2}) = 1$. The function $f(t) = \sqrt{1-t} + \sqrt{1+t}$ is strictly concave on $(-1, 1)$ with $f(\pm 1) = \sqrt{2}$. Therefore $\sqrt{2} < f(\mu) = \tau$. The contradiction implies that (8.13) has no solution.

Case τ2. If $\tau = \sqrt{2}$ then $\mu = 1$ and $f(\mu) = \sqrt{2}$. The condition (8.11) fails so there is no solution of (8.11-8.13).

Case τ3. If $\sqrt{2} < \tau < 2$ then $0 < \mu < 1$ because the function g is strictly monotone decreasing on $[\sqrt{2}, 2]$ with $g(\sqrt{2}) = 1$ and $g(2) = 0$. Therefore (8.11) and (8.12) are valid. If one puts $\nu = \sqrt{1+\mu} + \sqrt{1-\mu}$ then the properties of the function f ensure that $\sqrt{2} < \nu < 2$. Moreover $\mu = \nu\sqrt{1-\nu^2/4}$ by the arguments used to prove (8.14). Therefore $g(\tau) = g(\nu)$. The strict motonicity of g implies that $\tau = \nu$, so (8.13) also holds. We conclude that there are two solutions of (8.11-8.13), both real, namely $\mu = \pm\tau\sqrt{1-\tau^2/4}$.

Case $\tau 4$**.** If $\tau = 2$ then $\mu = 0$ and $f(\mu) = 2$. Also $\sqrt{1+\mu} = \sqrt{1-\mu} = 1$ so the conditions (8.11-8.13) have a single solution, namely $\mu = 0$.

Case $\tau 5$**.** If $\tau > 2$ then $\tau^2/4 - 1 > 0$ so $\mu = i\tau\sqrt{\tau^2/4-1}$ is purely imaginary. Therefore $|\arg(1 \pm \mu)| < \pi/2$ and $|\arg\sqrt{1 \pm \mu}\,| < \pi/4$. This implies (8.11) and (8.12).

Also

$$1 - \mu^2 = 1 + \tau^2(\tau^2/4 - 1) = (\tau^2/2 - 1)^2$$

so

$$\sqrt{1-\mu^2} = \tau^2/2 - 1 > 0.$$

This is equivalent to

$$\left(\sqrt{1+\mu} + \sqrt{1-\mu}\right)^2 = \tau^2,$$

and implies (8.13) by an application of (8.11) and (8.12). We conclude that there are two solutions of (8.11-8.13), namely $\mu = \pm i\tau\sqrt{\tau^2/4-1}$.

We can now prove the assertions of the theorem. Item 1 is a direct consequence of Cases $\tau 1$ and $\tau 2$, once one translates from τ and μ back to λ, γ and δ. Item 2 follows from Cases $\tau 3$ and $\tau 4$, while Item 3 follows from Case $\tau 5$.

We next prove (8.7). The identities

$$(A - \lambda I)^* = (J(H - \lambda J))^* = (H - \overline{\lambda}J)J = J(A - \overline{\lambda}I)J$$

imply that

$$\dim(\mathrm{Ker}((A - \lambda I)^*) = \dim(\mathrm{Ker}(A - \overline{\lambda}I)).$$

Items 1 to 3 establish that

$$\dim(\mathrm{Ker}(A - \overline{\lambda}I)) = \dim(\mathrm{Ker}(A - \lambda I)),$$

the value being 0 or 1. Therefore

$$\mathrm{Ind}(A - \lambda I) = \dim(\mathrm{Ker}(A - \lambda I)) - \dim(\mathrm{Ker}(A - \lambda I)^*) = 0$$

for all $\lambda \notin \mathrm{Ess}(A)$. Finally (8.7) implies that

$$\mathrm{Spec}(A)\backslash\mathrm{Ess}(A) \subseteq \mathrm{Eig}(A),$$

which yields (8.5) immediately. $\square$

Chapter 9

Quantitative bounds on operators

9.1 Pseudospectra

page 251. More on the Airy operator

The fact that $\|T_t\| \leq 1$ for all $t \geq 0$ implies that the numerical range of A lies in $\{z : \mathrm{Re}\,(z) \leq 0\}$ and that the resolvent operators $(zI - A)^{-1}$ satisfy

$$\|(zI - A)^{-1}\| \leq \mathrm{Re}\,(z)^{-1}$$

if $\mathrm{Re}\,(z) > 0$. Bounds on $\|(zI - A)^{-1}\|$ for $\mathrm{Re}\,(z) < 0$ have been obtained by several people. The sharpest current result, below, is taken from 'W Bordeaux-Montrieux, Estimation de résolvante et construction de quasimode près du bord du pseudospectre, preprint 2010'.

Theorem 9.1 *The quantity $\|((x + iy)I - A)^{-1}\|$ is independent of y and satisfies*

$$\|((x + iy)I - A)^{-1}\| \sim \sqrt{\frac{\pi}{2}}|x|^{-1/4}\exp(4|x|^{2/3}/3)$$

as $x \to -\infty$.

9.2 Generalized spectra and pseudospectra

page 252. Before Example 9.2.3 insert: We discuss the concept of numerical range for operator pencils in a new Section 9.8.

page 257,6. Replace 'local minimum' by 'strict local minimum'.

page 261,-2. The dot after $\mathcal{C}$ is intended to mean 'such that'. p273,3 Interchange minimum and maximum on this line.

9.2.1 Local constancy of the resolvent norm

Theorem 9.2.8 of LOTS should state that the resolvent norm of an operator on a Banach space is continuous and cannot have a *strict* local maximum. This does not, however, prevent it from being locally constant.

Shargorodsky[1] has given a detailed account of the state of the art in this field, as well as being responsible for most of the recent results, and we summarize some of the known results without proof; see footnote 1 for references to the original papers.

Proposition 9.2 *Let $\mathcal{H}$ be an infinite-dimensional Hilbert space.*

1. ***(Daniluk)*** *If A is a bounded operator on $\mathcal{H}$, then $\rho(z) = \|(A - zI)^{-1}\|$ cannot be constant on any open subset of $U = \mathbf{C}\backslash\mathrm{Spec}(A)$.*

2. ***(Globevnik)*** *If A is a bounded operator on a Banach space $\mathcal{B}$, and $U = \mathbf{C}\backslash\mathrm{Spec}(A)$ is connected then $\rho(z) = \|(A-zI)^{-1}\|$ cannot be constant on any open subset of U.*

3. ***(Globevnik, Shargorodsky and Shkarin)*** *There exists a bounded operator A on a separable, strictly convex, reflexive Banach space $\mathcal{B}$ such that $\rho(\cdot)$ is constant in a neighbourhood of 0. However, this is not possible if $\mathcal{B}$ is complex uniformly convex.*

4. ***(Shargorodsky)*** *There exists a closed densely defined operator A on a separable Hilbert space $\mathcal{H}$ and a non-empty open subset of $U = \mathbf{C}\backslash\mathrm{Spec}(A)$ on which $\rho(\cdot)$ is constant.*

5. ***(Shargorodsky)*** *Let A be the generator of a strongly continuous one-parameter semigroup acting on $\mathcal{H}$. If V is an open connected subset of $U = \mathbf{C}\backslash\mathrm{Spec}(A)$ and $\rho(z) \leq M$ for all $z \in V$ then $\rho(z) < M$ for all $z \in V$.*

The following result is unpublished and is included with the permission of the author.

Theorem 9.3 (Shargorodsky) *Let X be a complex strictly convex Banach space (X is not required to be complex uniformly convex) and let $B : X \to X$ be a closed densely defined operator with a compact resolvent $R(\lambda) := (B - \lambda I)^{-1}$. Let Ω be*

[1] E. Shargorodsky, On the level sets of the resolvent norm of a linear operator, Bull. London Math. Soc. 40 (2008) 493-504; E. Shargorodsky, Pseudospectra of semigroup generators, Bull. London Math. Soc. 42 (2010) 1031-1024.

a connected open subset of the resolvent set of B. If $\|R(\lambda)\| \le M$ for all $\lambda \in \Omega$, then $\|R(\lambda)\| < M$ for all $\lambda \in \Omega$.

Proof The proof is similar to that of Theorem 2.6 in Shargorodsky.[2]

Suppose the contrary: there exists $\lambda_0 \in \Omega$ such that $\|R(\lambda_0)\| = M$. Then Theorem 2.1 in footnote 2 or the maximum principle[3] imply that $\|R(\lambda)\| = M$, $\forall \lambda \in \Omega$. Shifting the independent variable if necessary, we can assume that $0 \in \Omega$.

According to Lemma 1.1 of Globevnik and Vidav,[4] there exists $r > 0$ such that $|\lambda| \le r$ implies

$$\left\|R(0) + \lambda R^2(0)\right\| = \|R(0) + \lambda R'(0)\| \le M.$$

Since $\|R(0)\| = M$, there exist $u_n \in X$, $n \in \mathbb{N}$ such that $\|u_n\| = \frac{1}{M}$ and $\|R(0)u_n\| \to 1$ as $n \to \infty$. Since $R(0)$ is compact, one can assume, after going to a subsequence, that $R(0)u_n$ converges to a vector $x \in X$ and $\|x\| = 1$. Then $y := rR(0)x \ne 0$ and $|\zeta| \le 1$ implies

$$\begin{aligned} \|x + \zeta y\| &= \lim_{n\to\infty} \left\|R(0)u_n + \zeta r R^2(0)u_n\right\| \\ &\le \left\|R(0) + \zeta r R^2(0)\right\| \, \|u_n\| \\ &\le M/M = 1. \end{aligned}$$

This contradicts the complex strict convexity of X. □

9.3 The numerical range

page 268,2. In this example the convex hull and the closed convex hull coincide.

page 269,-9. Replace $\mathrm{Num}(A)$ by $\mathrm{Num}(J_n)$.

page 270. Lemma 9.3.14

In the final equation of the statement of the lemma, either side is allowed to be infinite. The proof of the lemma is too brief and may be expanded as follows.
Let $a, b \notin \overline{\mathrm{Num}}(A)$ where $a \notin \mathrm{Spec}(A)$. By the connectedness hypothesis there exists a continuous curve $\gamma : [0,1] \to \mathbf{C}\backslash\overline{\mathrm{Num}}(A)$ such that $\gamma(0) = a$ and $\gamma(1) = b$.

[2] E. Shargorodsky 'On the level sets of the resolvent norm of a linear operator', *Bull. Lond. Math. Soc.* 40 (2008) 493–504.

[3] see, e.g., Theorem 3.13.1 in E. Hille and R.S. Phillips, *Functional analysis and semigroups* (American Mathematical Society, Providence, R. I., 1957) or Ch. III, Sect. 14 in N. Dunford and J.T. Schwartz, *Linear Operators. I. General theory* (Interscience Publishers, New York and London, 1958)

[4] J. Globevnik and I. Vidav, 'On operator-valued analytic functions with constant norm', *J. Funct. Anal.* 15 (1974) 394–403.

We claim that there exists $\delta > 0$ such that if $\gamma(\sigma) \notin \mathrm{Spec}(A)$ for some $\sigma \in [0,1]$ then $\gamma(s) \notin \mathrm{Spec}(A)$ whenever $|s-\sigma| < \delta$. This is enough to prove that $\gamma(1) \notin \mathrm{Spec}(A)$ and hence that $\mathrm{Spec}(A) \subseteq \overline{\mathrm{Num}}(A)$.
The proof of the claim follows the given proof of the lemma. The fact that $\delta > 0$ may be chosen to be independent of σ depends on the lower bound

$$\min\{\mathrm{dist}(\gamma(s), \overline{\mathrm{Num}}(A)) : s \in [0,1]\} > 0.$$

page 274,13. Replace k_β by A_β.

page 274,-2. Replace ‘except’ by ‘expect’.

page 275,-2. Insert space before orthonormal.

page 276,13. Replace imaginary by real.

9.4 Higher order hulls and ranges

page 278. Lemma 9.4.4

This lemma requires the definition of $\mathrm{Num}_2(A)$ to be extended to unbounded self-adjoint operators. The calculations may be justified by using the spectral theorem for self-adjoint operators in Section 5.4, starting on page 143.

page 279,-10. Replace $p_1(z) \notin \overline{\mathrm{Num}}(A)$ by $p_1(z) \notin \overline{\mathrm{Num}}(p_1(A))$.

page 282,-5. Replace $\mathrm{Num}(A^2)$ by $\overline{\mathrm{Num}}(A^2)$.

page 283-8. I should have written $\lambda := \beta^2/\alpha^2$ and $\mu := 2\beta^2$.

9.5 Von Neumann's theorem

9.6 Peripheral point spectrum

page 289,5 and 6. Replace 1 by I three times.

page 291,-1. Extra (at start of subscript.

page 292,14. In the definition of $H_{i,j}$ replace n by $n+1$ twice.

9.7 2×2 block operator matrices

The following new section combines a few results already in the book with new material. A much more complete account may be found in 'C Tretter, Spectral Theory of Block Operator Matrices and Applications, Imperial College Press, London, 2008', referred to below as Tretter.

Inverting block matrices Some special results are available for operators that act in a Hilbert space that is decomposed in a natural way as a direct sum of two orthogonal subspaces, $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$. This is particularly relevant to the study of self-adjoint operators that are not bounded above or below and have a gap in the spectrum containing 0. Here are a few representative theorems on the subject. We assume that the operators in question are given in block form with respect to the direct sum decomposition of $\mathcal{H}$, and that the blocks are all bounded; this condition can often be weakened.

The following theorem was proved in a matrix context by Schur. See Grushin for applications to differential operators. We assume that $\mathcal{H}$ denotes a Hilbert space, although in some theorems it could be a Banach space.

Lemma 9.4 *Let $L : \mathcal{H} \to \mathcal{H}$ be defined by*

$$L := \begin{pmatrix} A & 0 \\ C & D \end{pmatrix},$$

where $\mathcal{H} := \mathcal{H}_1 \oplus \mathcal{H}_2$ and A, C, D are bounded operators acting between the appropriate subspaces. If A is invertible in $\mathcal{H}_1$ then L is invertible in $\mathcal{H}$ if and only if D is invertible in $\mathcal{H}_2$. Moreover

$$L^{-1} = \begin{pmatrix} A^{-1} & \\ -D^{-1}CA^{-1} & D^{-1} \end{pmatrix}.$$

Proof Put

$$M = \begin{pmatrix} E & F \\ G & H \end{pmatrix}$$

and expand $LM = ML = I$ into its constituent equations. The claimed result is then routine algebra. □

Theorem 9.5 (Schur) *Let $L : \mathcal{H} \to \mathcal{H}$ be defined by*

$$L := \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where $\mathcal{H} := \mathcal{H}_1 \oplus \mathcal{H}_2$ and A, B, C, D are bounded operators acting between the appropriate subspaces. If A is invertible in $\mathcal{H}_1$ then L is invertible in $\mathcal{H}$ if and only if $S := D - CA^{-1}B$ is invertible in $\mathcal{H}_2$.

Proof Combine the lemma above with the elementary formula

$$\begin{pmatrix} A^{-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} I & A^{-1}B \\ 0 & I \end{pmatrix} = \begin{pmatrix} I & 0 \\ C & D - CA^{-1}B \end{pmatrix}$$

□

Corollary 9.6 *In the above theorem if $\mathcal{H}$ is finite-dimensional and A is invertible then*

$$\det(L) = \det(A)\det(D - CA^{-1}B).$$

Example 9.2.3 is a typical application of the theorem.

Block matrices and numerical range The following is taken from 'H. Langer et al., A new concept for block operator matrices: the quadratic numerical range, Lin. Alg. and Appl. 330 (2001) 89-112'. This paper further develops the notion of quadratic numerical range, introduced in 'H. Langer, C. Tretter, Spectral decomposition of some non-self-adjoint block operator matrices, Operator Theory, 39 (1998) 339-359'. The extension of the theory to unbounded operators may be found in 'C. Tretter, Spectral inclusion for unbounded block operator matrices, J. Funct. Anal. 256 (2009) 3806-3829'. See also C. Tretter, Spectral Theory of Block Operator Matrices and Applications, Imperial College Press, 2008 .

Theorem 9.7 *Let $X = \left(\begin{smallmatrix} A & B \\ B^* & -C \end{smallmatrix}\right)$ where $A + A^* \geq 2\alpha I$, $C + C^* \geq 2\alpha I$ and $\alpha > 0$. Then the spectrum of X is disjoint from the set $\{x + iy : |x| < \alpha\}$.*

Proof If we put $\widetilde{A} = A - \alpha I$ and $\widetilde{C} = C - \alpha I$ and $\widetilde{X} = \left(\begin{smallmatrix} \widetilde{A} & B \\ B^* & -\widetilde{C} \end{smallmatrix}\right)$ then

$$\begin{aligned} X^*X &= \widetilde{X}^*\widetilde{X} + \alpha \begin{pmatrix} \widetilde{A} + \widetilde{A}^* & 0 \\ 0 & \widetilde{C} + \widetilde{C}^* \end{pmatrix} + \alpha^2 I \\ &\geq \alpha^2 I. \end{aligned}$$

Combining this with a similar inequality for XX^*, one deduces that X is invertible and $\|X^{-1}\| \leq 1/\alpha$. The theorem follows by applying this result to $X + (u + is)I$ for all $s \in \mathbf{R}$ and suitable $u \in \mathbf{R}$. □

If $X = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ then one defines

$$W^2(X) = \left\{ \lambda \in \mathbf{C} : \det(X_{f,g} - \lambda I_2) = 0 \text{ for some } \begin{pmatrix} f \\ g \end{pmatrix} \in \Sigma \right\}$$

where

$$\begin{aligned}
\Sigma &= \left\{ \begin{pmatrix} f \\ g \end{pmatrix} : f \in \mathcal{H}_1, g \in \mathcal{H}_2, \|f\| = \|g\| = 1 \right\}, \\
X_{f,g} &= \begin{pmatrix} \langle Af, f\rangle & \langle Bg, f\rangle \\ \langle Cf, g\rangle & \langle Dg, g\rangle \end{pmatrix}.
\end{aligned}$$

Theorem 9.8 *One has*

$$\mathrm{Spec}(X) \subseteq \overline{W^2(X)} \subseteq \overline{\mathrm{Num}(X)} \tag{9.1}$$

for all operators X on $\mathcal{H}$. If $X = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$, $\dim(\mathcal{H}_1) > 1$ *and* $\dim(\mathcal{H}_2) > 1$ *then*

$$\mathrm{Num}(A) \cup \mathrm{Num}(D) \subseteq W^2(X).$$

Proof See 'V Kostrykin, K A Makarov, A K Motovilov, Perturbation of spectra and spectral subspaces, Trans. Amer. Math. Soc. 359 (1) (2007) 77-89' and Tretter. The inclusions

$$\mathrm{Eig}(X) \subseteq W^2(X) \subseteq \mathrm{Num}(X)$$

depend on calculations with 2×2 matrices that are similar to those below; they suffice to prove (9.1) if $\mathcal{H}$ is finite-dimensional. We will prove the general case of the first inclusion in (9.1).

If $\lambda \in \mathrm{Spec}(X)$ then either there exists a sequence of unit vectors $v_n = (f_n, g_n)'$ such that $\|Xv_n - \lambda v_n\| \to 0$ as $n \to \infty$ or there exists a sequence of unit vectors $v_n = (f_n, g_n)'$ such that $\|X^* v_n - \overline{\lambda} v_n\| \to 0$ as $n \to \infty$. We treat only the first case, the other one being similar.

The assumptions imply that

$$\begin{aligned}
\|(A - \lambda I)f_n + Bg_n\| &\to 0 \\
\|Cf_n + (D - \lambda I)g_n\| &\to 0
\end{aligned}$$

as $n \to \infty$, and hence

$$\begin{aligned}
|\langle (A - \lambda I)f_n + Bg_n, (f_n/\|f_n\|)\rangle| &\to 0 \\
|\langle Cf_n + (D - \lambda I)g_n, (g_n/\|g_n\|)\rangle| &\to 0.
\end{aligned}$$

These equations may be written in the form

$$\begin{aligned}
|(\alpha_n - \lambda)u_n + \beta_n v_n| &\to 0 \\
|\gamma_n u_n + (\delta_n - \lambda)v_n| &\to 0
\end{aligned}$$

where $\alpha_n = \langle Af_n, f_n\rangle/\|f_n\|^2$, $\beta_n = \langle Bg_n, f_n\rangle/(\|f_n\|\,\|g_n\|)$, $\gamma_n = \langle Cf_n, g_n\rangle/(\|f_n\|\,\|g_n\|)$, $\delta_n = \langle Dg_n, g_n\rangle/\|g_n\|^2$, $u_n = \|f_n\|$ and $v_n = \|g_n\|$. We next observe that α_n, β_n, γ_n, δ_n are uniformly bounded and that $|u_n|^2 + |v_n|^2 = 1$ for all n. Therefore

$$\lim_{n\to\infty} \det(M_n - \lambda I) = 0$$

where $M_n = \left(\begin{smallmatrix} \alpha_n & \beta_n \\ \gamma_n & \delta_n \end{smallmatrix}\right)$, and λ is a limit of the eigenvalues of the matrices M_n as $n \to \infty$. This implies that $\lambda \in \overline{W^2(X)}$. $\square$

Theorem 9.9 *If X is self-adjoint with $a = \min(\mathrm{Spec}(X))$ and $b = \max(\mathrm{Spec}(X))$ then either $W^2(X) = [a,b]$ or there exist c, d such that*

$$W^2(X) = [a,c] \cup [d,b].$$

Block matrices and pseudospectra

The following ideas are due to 'R. Byers, A bisection method for measuring the distance of a stable matrix to the unstable matrices, SIAM J. Sci. Statist. Comput., 9 (1988) 875-881'. They were developed further in 'M. A. Freitag, A. Spence, The calculation of the distance to instability by the computation of a Jordan block, preprint 2010', where the resulting algorithms were applied to a range of matrices associated with certain differential operators. We assume that A is an $n \times n$ matrix; if this matrix arises by applying the finite element method to a differential operator, one should also have appropriate bounds on the numerical range or resolvent norm of the original operator for $z \in \mathbf{C}$ with large imaginary and small real parts to ensure that instability problems do not arise for such z. We assume that the spectrum of A is contained in $\mathbf{C}_- = \{z : \mathrm{Re}\ (z) < 0\}$. We define the degree of instability ε of A to be the norm of the smallest perturbation E such that the spectrum of $A + E$ does not lie in $\mathbf{C}_-$.

The theory of pseudospectra implies that

$$\varepsilon^{-1} = \sup\{\|(zI - A)^{-1}\| : z \in i\mathbf{R}\}.$$

Equivalently

$$\varepsilon^2 = \inf_{\omega \in \mathbf{R}} \sigma\left((A - i\omega)^*(A - i\omega)\right),$$

where $\sigma(B)$ is defined to be the smallest eigenvalue of $B = B^*$, which is positive in our case.

Given $\omega \in \mathbf{R}$, let σ be the smallest eigenvalue of $(A - i\omega)^*(A - i\omega)$ and x the corresponding normalized eigenvector. If we put $y = \varepsilon^{-1}(A - i\omega)x$ then

$$\begin{aligned} (A - i\omega)x &= \varepsilon y \\ (A^* + i\omega)y &= \varepsilon x. \end{aligned}$$

Equivalently, introducing the matrix B, we have

$$B\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} A & -\varepsilon \\ \varepsilon & -A^* \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = i\omega\begin{pmatrix} x \\ y \end{pmatrix}.$$

The matrix B is Hamiltonian in the sense that $B^* = -JBJ^{-1}$ where $J = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. Therefore the set of eigenvalues of B is invariant under reflection in the imaginary axis. If $\varepsilon = 0$ then B has no purely imaginary eigenvalues. These facts imply that one can follow its eigenvalues as ε increases until two of them meet on the imaginary axis. This critical value of ε is the measure of instability of A. The paper of Freitag and Spence provides efficient algorithms for implementing the procedure just described.

9.8 The numerical range for operator pencils

Let $A(\lambda) = \sum_{r=0}^{n} \lambda^r A_r$, where A_r are bounded operators on a Hilbert space $\mathcal{H}$ and $A_n = I$. In Section 9.3 we defined the spectrum of $A(\cdot)$ to be the set of $\lambda \in \mathbf{C}$ such that $A(\lambda)$ is not invertible. We define the numerical range $\mathrm{Num}(A(\cdot))$ by

$$\mathrm{Num}(A(\cdot)) = \bigcup_{0 \neq x \in \mathcal{H}} \{\lambda : \langle A(\lambda)x, x\rangle = 0\}$$

Note that for each non-zero $x \in \mathcal{H}$, $p_x(\lambda) = \langle A(\lambda)x, x\rangle$ is a polynomial with degree n, and so has at most n zeros. It is easy to prove that if $A(\lambda) = B - \lambda I$, then $\mathrm{Num}(A(\cdot))$ equals the numerical range of B as defined in Section 9.3. The results in this section are well-known in the operator pencil community; they are sometimes called variational bounds.[5]

Theorem 9.10 *One has*

$$\mathrm{Spec}(A(\cdot)) \subseteq \overline{\mathrm{Num}(A(\cdot))}$$

Proof The inclusion $\mathrm{Eig}(A(\cdot)) \subseteq \mathrm{Num}(A(\cdot))$ is elementary, and this suffices if $\mathcal{H}$ is finite-dimensional.

In the general case if $\lambda \in \mathrm{Spec}(A(\cdot))$ then there exists a sequence $x_r \in \mathcal{H}$ such that $\|x_r\| = 1$ for all r and either $\lim_{r\to\infty} \|A(\lambda)x_r\| = 0$ or $\lim_{r\to\infty} \|A(\lambda)^* x_r\| = 0$. In both cases we deduce that

$$\lim_{r\to\infty} \langle A(\lambda)x_r, x_r\rangle = 0.$$

If

$$\langle A(z)x_r, x_r\rangle = \prod_{s=1}^{n} (z - \gamma_{r,s})$$

for all $z \in \mathbf{C}$, then we may deduce that

$$\lim_{r\to\infty} \min_{1\leq s\leq n} |\lambda - \gamma_{r,s}| = 0.$$

[5]For further developments and references see D Eschwé and M Langer, Variational principles for eigenvalues of self-adjoint operator functions, Int. Eqns. Oper. Theory 49 (2004) 287-321 and M Langer and C Tretter, Variational principles for eigenvalues of the Klein-Gordon equation, J. Math. Phys. 47 (2006) 103506, 18 pp.

Since $\gamma_{r,s} \in \mathrm{Num}(A(\cdot))$ for all r, s, it follows that $\lambda \in \overline{\mathrm{Num}(A(\cdot))}$. $\square$

From this point onwards we restrict attention to the special case $A(\lambda) = \lambda^2 I + \lambda B + C$, where B and C are bounded self-adjoint operators. Extensions to unbounded operators are possible under suitable conditions. There is a large literature on such problems, which may have non-real eigenvalues. We are interested in conditions which imply that the spectrum of such a pencil is real. It should be mentioned that generalizations of the results below have been applied to the Klein-Gordon equation.[6]

The pencil $A(\cdot)$ is said to be hyperbolic if it satisfies the condition

$$\langle Bx, x\rangle^2 > a\|x\|^2\langle Cx, x\rangle$$

for all non-zero $x \in \mathcal{H}$. This is equivalent to assuming the positivity of certain discriminants and to the conditions of the following theorem.[7]

Theorem 9.11 *The following are equivalent.*

1. *There exists $\alpha \in \mathbf{R}$ such that $A(\alpha) < 0$ in the sense that* $\mathrm{Spec}(A(\alpha)) \subseteq (-\infty, c]$ *for some $c < 0$.*
2. *The set $Num(A(\cdot))$ is the union of two disjoint, separated, real intervals I, J.*

Proof Assume (1). If $A(\alpha) < 0$ then $A(s) < 0$ for some $\varepsilon > 0$ and all $s \in (\alpha-\varepsilon, \alpha+\varepsilon)$. If $0 \neq x \in \mathcal{H}$ then p_x is a quadratic polynomial with real coefficients. Since $p_x(s) \to +\infty$ as $s \to \pm\infty$ through real values and $p_x(s) < 0$ for all $s \in (\alpha-\varepsilon, \alpha+\varepsilon)$, the roots of $p_x(\lambda) = 0$ are real and distinct. If we denote these roots by β_x and γ_x, where $\beta_x < \gamma_x$, then each depends continuously on x so

$$I = \{\beta_x : \|x\| = 1\}, \qquad J = \{\gamma_x : \|x\| = 1\}$$

are non-empty intervals. Moreover $I \subseteq (-\infty, \alpha-\varepsilon]$ and $J \subseteq [\alpha+\varepsilon, \infty)$.

Conversely, given (2), each polynomial p_x has real roots, which we denote by β_x and γ_x, satisfying $\beta_x \leq \gamma_x$. The intervals I, J are assumed to be disjoint so there exists α and $\varepsilon > 0$ such that $I \subseteq (-\infty, \alpha-\varepsilon]$ and $J \subseteq [\alpha+\varepsilon, \infty)$, Since $p_x(\alpha) < 0$ for all non-zero $x \in \mathcal{H}$ we deduce that $A_\alpha < 0$. $\square$

Corollary 9.12 *If there exists $\alpha \in \mathbf{R}$ such that $A(\alpha) < 0$ then* $\mathrm{Spec}(A(\cdot)) \subseteq \mathbf{R}$.

[6]See 'Matthias Langer and Christiane Tretter, Variational principles for eigenvalues of the Klein-Gordon equation, J. Math. Phys. 47 (2006) 103506, 18 pp.'

[7]See R J Duffin, A minimax theory for overdamped networks, J. Rational Mech. Anal. 4, (1955) 221-233, M G Krein and G K (Heinz) Langer, The spectral function of a selfadjoint operator in a space with indefinite metric, Dokl. Akad. Nauk SSSR 152 (1963) 39-42, Section 31 of A S Markus : Introduction to the spectral theory of polynomial operator pencils, vol.71, Translation of mathematical monographs, American Mathematical Society 1988, and F Tisseur, K Meerbergen, The quadratic eigenvalue problem, SIAM Review 43, No. 2 (2001) 235-286.

In principle the computation of $\mathrm{Spec}(A(\cdot))$ is easy in this situation, at least for matrices. One simply computes $\mathrm{Eig}(A(s))$ for each $s \in \mathbf{R}$, to obtain a family of real curves depending continuously on s. One then determines the set of s for which one of the eigenvalues vanishes.

9.9 Jacobi matrices

The theory of Jacobi matrices and their associated orthogonal polynomials is vast, and we can do no more than provide an introduction. One says that a tridiagonal $N \times N$ matrix A is a Jacobi matrix if it is real and symmetric and $A_{r,s} > 0$ whenever $r - s = \pm 1$. One could weaken the last condition to $A_{r,s} \geq 0$ whenever $r - s = \pm 1$, but if any such $A_{r,s}$ vanishes, the matrix can be decomposed into two or more independent blocks. We will say that A is a infinite bounded Jacobi matrix if one replaces $\{1, 2, \ldots, N\}$ by $\mathbf{Z}^+ = \mathbf{N} \cup \{0\}$ and there is a uniform bound on the coefficients. It follows directly that if A is an infinite bounded Jacobi matrix then it determines a bounded self-adjoint operator on $\ell^2(\mathbf{Z}^+)$; moreover the vector $e_0 \in \ell^2(\mathbf{Z}^+)$ defined by $e_{0,0} = 1$ and $e_{0,n} = 0$ for all other n is cyclic in the sense that the linear span of $\{A^n e_0 : n \in \mathbf{Z}^+\}$ is norm dense in $\ell^2(\mathbf{Z}^+)$. Further results on the subject of this section may be found in Sections 4.4 and 9.3 of LOTS. Cyclic vectors also play a role in Example 1.5.7 and Lemma 11.2.9 of LOTS, although this is not mentioned there.

Theorem 9.13 (M. H. Stone) *Let H be a bounded self-adjoint operator on the infinite-dimensional Hilbert space $\mathcal{H}$, and suppose that there is a unit vector $e_0 \in \mathcal{H}$ that is cyclic in the sense that the linear span of $\{H^n e_0 : n \in \mathbf{Z}^+\}$ is norm dense in $\mathcal{H}$. Then there exists a complete orthonormal sequence $\{e_n\}_{n \in \mathbf{Z}^+}$ in $\mathcal{H}$ such that the matrix $A_{m,n} = \langle He_n, e_m \rangle$ is an infinite Jacobi matrix whose entries are bounded.*

Proof If one applies the Gram-Schmidt orthogonalization procedure to the sequence $f_n = H^n e_0$, $n \in \mathbf{Z}^+$, one obtains an orthonormal sequence $\{e_n\}_{n \in \mathbf{Z}^+}$ such that $\mathrm{lin}\{e_n : n \in \mathbf{Z}^+\} = \mathrm{lin}\{f_n : n \in \mathbf{Z}^+\}$. The latter is dense by cyclicity, so $\{e_n\}_{n \in \mathbf{Z}^+}$ is a complete orthonormal sequence. The construction yields

$$He_n = A_{n+1,n} e_{n+1} + A_{n,n} e_n + \ldots A_{0,n} e_0 \tag{9.2}$$

where $A_{n+1,n} > 0$ for all n and $A_{m,n} = 0$ if $m > n+1$. This yields $\langle He_n, e_m \rangle = A_{m,n}$ for all m, $n \geq 0$. The self-adjointness of H implies that $A_{n,m} = A_{m,n} = 0$ if $m > n + 1$, so A is tridiagonal. The definition of $A_{m,n}$ implies that $|A_{m,n}| \leq \|H\|$ for all $m, n \in \mathbf{Z}_+$. $\square$

Theorem 9.14 *Let H be a bounded self-adjoint operator acting in $\mathcal{H}$ with cyclic vector e. Then there exists a probability measure μ on $\mathbf{R}$ with support $S = \mathrm{Spec}(A)$ and a unitary map $U : \mathcal{H} \to L^2(S, \mu)$ such that $Ue = 1$ and $(UHU^{-1} f)(x) = x f(x)$ for all $f \in L^2(S, \mu)$ and almost all $x \in S$ with respect to μ .*

Proof This is a direct statement of the spectral theorem for some versions of that theorem. In others it follows from the spectral theorem. See, for example, Theorem 2.5.2 of E B Davies, Spectral Theory and Differential Operators, Camb. Univ. Press, 1995. $\square$

These theorems are related to the theory of orthogonal polynomials. This subject goes back to the nineteenth century, and the classical families of orthogonal polynomials are associated with unbounded self-adjoint differential operators acting in $L^2(a,b)$ for some interval (a,b), possibly of infinite length. There is also a well developed theory of orthogonal polynomials on the unit circle, associated in a similar way to a unitary operator with a cyclic vector.[8]

Theorem 9.15 *Let μ be a probability measure on* $\mathbf{R}$ *with infinite compact support X. Then the operator $H : L^2(X,\mu) \to L^2(X,\mu)$ defined by $(Hf)(x) = xf(x)$ is bounded and self-adjoint with $\|H\| = \max\{|x| : x \in \mathrm{supp}(\mu)\}$. The unit vector $p_0(x) = 1$ is cyclic with respect to H. If $\{p_n\}_{n\in\mathbf{Z}^+}$ is the orthonormal basis constructed as in Theorem 9.13 then p_n is a polynomial of degree n with positive leading coefficient. The polynomials satisfy the second order recurrence relation*

$$A_{n+1,n}p_{n+1}(x) + (A_{n,n} - x)p_n(x) + A_{n-1,n}p_{n-1}(x) = 0. \tag{9.3}$$

subject to $p_0 = 0$ and $p_1 = 1$.

Proof The cyclicity of p_0 with respect to H follows by the Stone-Weierstrass theorem: the set of all polynomials on X is uniformly dense in $C(X)$ and hence norm dense in $L^2(X,\mu)$. The space $L^2(X,\mu)$ is infinite-dimensional because we are assuming that the support of μ is infinite. If $F_n = H^n p_0$ then f_n is a polynomial of degree n, and the nature of the Gram-Schmidt construction implies that p_n is of degree n with positive leading coefficient for all n. The recurrence relation (9.3) is equivalent to the more abstract identity

$$Hp_n = A_{n+1,n}p_{n+1} + A_{n,n}p_n + A_{n-1,n}p_{n-1},$$

which is a special case of (9.2), subject to the fact that $A_{m,n} = 0$ if $|m-n| > 1$; see the proof of Theorem 9.13. $\square$

[8] A recent account of this was given in B. Simon, Orthogonal polynomials on the unit circle, Parts 1 and 2, Amer. Math. Soc., Providence, RI, 2005.

Chapter 10

Quantitative bounds on semigroups

10.1 Long term growth bounds

page 296,(ii). This should assume that Z is closed or refer to Problem 6.1.2 page 165, where it is proved that closedness follows from the other assumptions.

10.2 Short term growth bounds

page 300,-13. Replace n by $n-1$ except on its third occurrence in this equation.

page 303. This page assumes a degree of familiarity with the Legendre transform and associated ideas from convexity theory. There are ample resources on the web about this.

page 306. More on Example 10.2.9 .

In semiclassical analysis one replaces a pure differential expression D^α by $h^{|\alpha|}D^\alpha$ and studies the asymptotic behaviour of the resulting operator as $h \to 0$. In Example 10.2.9 this leads to the study of the paradigmatic Hamiltonian operator

$$(L_h f)(x) = hf'(x) + v(x)f(x),$$

acting in $L^2(\mathbf{R})$ for small $h > 0$. The corresponding classical Hamiltonian is $\ell(x,\xi) = v(x) + i\xi$. The semiclassical spectrum of L_h is, by definition, the closure of $\{\ell(x,\xi) : x,\xi \in \mathbf{R}\}$. This equals $\{(x,\xi) : a \leq x \leq b\}$ where $a = \inf \mathrm{Re}\, \{v(x) :$

$x \in \mathbf{R}\}$ and $b = \sup \operatorname{Re}\{v(x) : x \in \mathbf{R}\}$. This happens to coincide with $\overline{\operatorname{Num}}(L_h)$ for all $h > 0$.

J Sjöstrand, M Zworski, N Dencker, M Hager and others have studied the connection between the pseudospectra of much more general operators and the semiclassical spectrum, following an initial insight in 'E B Davies, Semi-classical states for non-self-adjoint Schrödinger operators, Comm. Math. Phys. 200(1) (1999) 35-41'. The semiclassical limit of the spectrum itself is harder to analyze.

Lemma 10.1 *If the function*

$$a(x) = \int_0^x v(s)\,\mathrm{d}s$$

is bounded, then the spectrum of L_h equals $i\mathbf{R}$ for all $h > 0$.

Proof One may use the bounded functions $\exp(\pm h^{-1}a(\cdot))$ to prove that L_h is similar to the operator $h\frac{\mathrm{d}}{\mathrm{d}x}$, whose spectrum equals $i\mathbf{R}$. Note, however, that the condition number of the similarity transformation increases exponentially as $h \to 0$. $\square$

The above example is not typical – it does not satisfy a natural extension of the Weyl law to non-self-adjoint operators. In her PhD thesis Hager proved that if one considers very small random perturbations of a periodic potential that satisfies the conditions of the lemma, then the spectrum of the perturbed L_h becomes dense in the semiclassical spectrum as $h \to 0$. She has also elucidated the asymptotic behaviour of the spectrum near the boundaries $\{(x, \xi) : x = a, b\}$. This work was generalized in many ways in higher dimensions in 'M Hager, J Sjöstrand, Eigenvalue asymptotics for randomly perturbed non-self-adjoint operators, Math. Ann, 342(1) (2008) 177-243' and 'W Bordeaux Montrieux, Johannes Sjöstrand, Almost sure Weyl asymptotics for non-self-adjoint elliptic operators on compact manifolds, preprint 2009'.

page 307,3. This should be

$$f(x) := \exp\left\{zx - cx^{1-\gamma}\right\}.$$

10.3 Contractions and dilations

10.4 The Cayley transform

page 310. At various places I am using Problem 6.1.2 on page 165 to deduce the closedness of Z without mentioning that fact.

page 311,12. This multi-line equation could be expanded to

$$\begin{aligned}
\|C(\delta H+f)\|^2 &= \|\delta Ch+f\|^2\\
&= |\delta|^2\,\|Ch\|^2+2\varepsilon|\langle Ch,f\rangle|^2+\|f\|^2\\
&\geq 2\varepsilon|\langle Ch,f\rangle|^2+\|f\|^2\\
&> \varepsilon^2|\langle Ch,f\rangle|^2\|h\|^2+\|f\|^2\\
&= |\delta|^2\,\|h\|^2+\|f\|^2\\
&= \|\delta h+f\|^2
\end{aligned}$$

for all small enough $\varepsilon > 0$.

page 311,17. Replace 'terms' by 'a term'.

page 312,-14 and -11. Replace Lemma 5.4.4 by Theorem 5.4.5.

page 313, Theorem 10.4.4. Replace the final H by iH.

10.5 One-parameter groups

page 316,11. The first A on the RHS of this displayed equation should be A^r.

page 318,2. Replace $a \in S$ by $a \in X$.

page 318,12 and 14. Replace c^2 by c^4.

10.6 Resolvent bounds in Hilbert space

page 321,6. It would be better to replace $\mathcal{H}$ by $\mathcal{B}$.

page 324, Theorem 10.6.5. Replace λ by z.

No analogue of the Eisner-Zwart theorem exists in a Banach space context, even if it is assumed to be reflexive. If one does not assume that e^{At} is a one-parameter semigroup this is contained in Theorem 8.3.10. If one does make such an assumption then one may consider $(Af)(x) = (i+\varepsilon)f''(x)$ acting in $L^p(\mathbf{R})$ where $1 \leq p < 2$, and use Theorem 8.1.3 to prove that a suitable bound that is uniform with respect to $\varepsilon > 0$ cannot exist.

page 324. Integral conditions for exponential decay

The following theorem has something in common with the contents of the new Section 10.8. The question addressed is that of obtaining an upper bound on

$$\omega_0 := \inf_{0<t<\infty} \{t^{-1} \log(\|\mathrm{e}^{At}\|)\} = \lim_{0<t<\infty} \{t^{-1} \log(\|\mathrm{e}^{At}\|)\} \tag{10.1}$$

from weak decay conditions involving certain integrals. See Theorem 10.1.6 of LOTS for the proof of (10.1). By considering the case in which $\dim(\mathcal{B}) = 1$, one sees that the first part of the following lemma is optimal of its type.

Lemma 10.2 *If* e^{At} *is a one-parameter semigroup on the Banach space* $\mathcal{B}$ *and* $0 < p < \infty$ *then*

$$k := \int_0^\infty \|\mathrm{e}^{At}\|^p \,\mathrm{d}t < \infty$$

implies that

$$\omega_0 \leq -\frac{1}{kp}.$$

Either $\|\mathrm{e}^{At}\| = 0$ *for all* $t > k$ *or there exists* $t < k$ *such that* $\|\mathrm{e}^{At}\| < 1$.

Proof We start by observing that (10.1) implies that $\mathrm{e}^{\omega_0 t} \leq \|\mathrm{e}^{At}\|$ for all $t \geq 0$. Therefore

$$\int_0^\infty \mathrm{e}^{\omega_0 tp} \,\mathrm{d}t \leq \int_0^\infty \|\mathrm{e}^{At}\|^p \,\mathrm{d}t = k < \infty.$$

This implies that $\omega_0 < 0$ and $k \geq (|\omega_0|p)^{-1}$. The stated bound follows immediately.

If we put

$$E = \{t \geq 0 : \|\mathrm{e}^{At}\| \geq 1\}$$

then $k \geq |E|$. If $E \supseteq [0, k)$ then E equals $[0, k)$ or $0, k]$ and $\|\mathrm{e}^{At}\| = 0$ for almost all $t > k$. The subadditivity of the norm then implies that $\|\mathrm{e}^{At}\| = 0$ for all $t > k$. If $E \supseteq [0, k)$ is false then there exists $t < k$ for which $\|\mathrm{e}^{At}\| < 1$. □

Surprisingly one can obtain more detailed conclusions from weaker hypotheses in the Hilbert space context. Theorems 10.3, 10.4 and 10.10 are of this type. No analogue of the Eisner-Zwart theorem exists in a Banach space context.

Theorem 10.3 *Let* e^{At} *be a one-parameter semigroup acting on the Banach space* $\mathcal{B}$ *and let* $0 < k < \infty$. *Each of the following statements implies the next.*

1.

$$\int_0^\infty \|\mathrm{e}^{At} v\| \,\mathrm{d}t \leq k\|v\| \text{ for all } v \in \mathcal{B}.$$

2. *If*

$$M = \sup_{0 \le t \le k} \|e^{At}\| \tag{10.2}$$

then $\|e^{At}\| \le M$ *for all* $t > k$*. Moreover* $\mathrm{Spec}(A) \subseteq \{z : \mathrm{Re}\,(z) < 0\}$ *and*

$$\|(iyI - A)^{-1}\| \le k$$

for all $y \in \mathbf{R}$*.*

3. $\mathrm{Spec}(A) \subseteq \{z : \mathrm{Re}\,(z) \le -1/k\}$ *and*

$$\|(zI - A)^{-1}\| \le \frac{2M}{\mathrm{Re}\,(z) + 1/k} \tag{10.3}$$

for all z *such that* $\mathrm{Re}\,(z) > -1/k$*.*

4. *Assume in addition that* $\mathcal{B}$ *is a Hilbert space. Then there exists* $K < \infty$ *such that*

$$\|e^{At}\| \le K(1+t)e^{-t/k}$$

for all $t \ge 0$*.*

Proof

1⇒2. If $t > k$ then

$$\|e^{At}v\| \le \|e^{As}\|\,\|e^{A(t-s)}v\| \le M\|e^{A(t-s)}v\|$$

for all $s \in [0,k]$ and $v \in \mathcal{H}$. Therefore

$$\|e^{At}v\| \le \frac{M}{k}\int_0^k \|e^{A(t-s)}v\|\,\mathrm{d}s \le M\|v\|$$

for all $v \in \mathcal{H}$. This proves the first assertion of item 2. The proof of the second assertion involves a small modification of the proof of Theorem 8.2.1 of LOTS.

2⇒3. This follows Lemma 3.11.7 of 'O Staffan, Well-posed Linear Systems, Encyclopedia of Mathematics and its Applications, no. 103, Camb. Univ. Press, 2009'. We break the proof into three cases, and use the inequality $M \ge 1$. If $z = -u - iv$ where $0 \le u < 1/k$ and $v \in \mathbf{R}$, we use the resolvent perturbation expansion (8.3) and Corollary 8.1.4 of LOTS. We have

$$\begin{aligned}
\|(zI - A)^{-1}\| &= \|\{uI + (A + ivI)\}^{-1}\| \\
&\le \frac{1}{\|(A + ivI)^{-1}\|^{-1} - u} \\
&\le \frac{1}{1/k - u} \\
&= \frac{1}{1/k + \mathrm{Re}\,(z)}.
\end{aligned}$$

If $0 \le \text{Re }(z) \le 1/k$ then

$$\|(zI - A)^{-1}\| \le k \le \frac{2}{1/k + \text{Re }(z)}.$$

Finally if $1/k \le \text{Re }(z) < \infty$ then (10.2) implies that

$$\|(zI - A)^{-1}\| \le \frac{M}{\text{Re }(z)} \le \frac{2M}{1/k + \text{Re }(z)}.$$

3⇒4. If one puts $Z = A + k^{-1}I$ and $w = z + 1/k$ then (10.3) becomes $\|(wI - Z)^{-1}\| \le 2M/\text{Re }(w)$ for all w such that $\text{Re }(w) > 0$, so the Eisner-Zwart theorem (Theorem 10.6.5 of LOTS) is directly applicable. □

A slight modification to the proof yields a more general result below. A sharper result for the case $p = 2$ is presented in Theorem 10.10.

Theorem 10.4 *Let* e^{At} *be a one-parameter semigroup acting on the Banach space* $\mathcal{B}$. *Let* $0 < k < \infty$ *and* $1 < p < \infty$. *Each of the following statements implies the next.*

1.
$$\int_0^\infty \|\mathrm{e}^{At}v\|^p \,\mathrm{d}t \le k\|v\|^p \text{ for all } v \in \mathcal{B}.$$

2. *If*
$$M = \sup_{0 \le t \le k} \|\mathrm{e}^{At}\|$$
then $\|\mathrm{e}^{At}\| \le M$ *for all* $t > k$. *Moreover* $\text{Spec}(A) \subseteq \{z : \text{Re }(z) \le 0\}$ *and*
$$\|(zI - A)^{-1}\| \le \frac{k^{1/p}}{(qx)^{1/q}}$$
for all z *such that* $Re(z) > 0$, *where* $\frac{1}{p} + \frac{1}{q} = 1$.

3. $\text{Spec}(A) \subseteq \{z : \text{Re }(z) \le -1/(pk)\}$ *and*
$$\|(zI - A)^{-1}\| \le \frac{2M}{\text{Re }(z) + 1/(pk)} \tag{10.4}$$
for all z *such that* $\text{Re }(z) > -1/(pk)$.

4. *Assume in addition that* $\mathcal{B}$ *is a Hilbert space. Then there exists* $K < \infty$ *such that*
$$\|\mathrm{e}^{At}\| \le K(1 + t)\mathrm{e}^{-t/(pk)}$$
for all $t \ge 0$.

Proof

1⇒2. The proof of the first assertion of item 2 is a small adaptation of that in Theorem 10.3. If $z = x + iy$ where $x > 0$ and $y \in \mathbf{R}$ then

$$\begin{aligned}
\|(zI - A)^{-1}\| &\leq \int_0^\infty \|\mathrm{e}^{At}\| \mathrm{e}^{-xt}\, \mathrm{d}t \\
&\leq \left\{ \int_0^\infty \|\mathrm{e}^{At}\|^p \, \mathrm{d}t \right\}^{1/p} \left\{ \int_0^\infty \mathrm{e}^{-xtq} \, \mathrm{d}t \right\}^{1/q} \\
&\leq \frac{k^{1/p}}{(qx)^{1/q}}.
\end{aligned}$$

2⇒3. If $x > 0$ is small enough, we deduce from the resolvent perturbation expansion that

$$\begin{aligned}
\|(iyI - A)^{-1}\| &\leq \| \{xI - (zI - A)\}^{-1} \| \\
&\leq \frac{\|(zI - A)^{-1}\|}{1 - x\|(zI - A)^{-1}\|} \\
&= \left\{ \|(zI - A)^{-1}\|^{-1} - x \right\}^{-1} \\
&\leq \left\{ \frac{(qx)^{1/q}}{k^{1/p}} - x \right\}^{-1} \\
&= \left\{ x \left(\frac{q}{(qxk)^{1/p}} - 1 \right) \right\}^{-1}.
\end{aligned}$$

On putting $x = \frac{1}{qk}$ we obtain

$$\|(iyI - A)^{-1}\| \leq pk$$

for all $y \in \mathbf{R}$. The remainder of the proof of the theorem uses $2 \Rightarrow 3$ and then $3 \Rightarrow 4$ of Theorem 10.3, but with k replaced by pk. □

10.7 Growth bounds using the Schur decomposition

In this section we restrict attention to $n \times n$ matrices. The ideas in this section are due to C F van Loan.

The Schur decomposition theorem states that for every square matrix A, there exists a unitary matrix U such that $UAU^* = D + T$ where D is diagonal and T is strictly upper triangular. If the eigenvalues of A are all distinct there are $n!$ such decompositions, up to trivial phases, but if A has any degenerate eigenvalues there may be infinitely many decompositions. It is not clear how to find the decomposition that minimizes the norm of T. If one uses the Frobenius (Hilbert-Schmidt) norm this problem does not arise because T then has the same norm for all decompositions, by

$$\|A\|_F^2 = \|D\|_F^2 + \|T\|_F^2 = \sum_{i=1}^{n} |\lambda_i(A)|^2 + \|T\|_F^2.$$

Since $\|B\| \leq \|B\|_F \leq \sqrt{n}\|B\|$ for all $n \times n$ matrices B, the issue is not a huge one. $\|T\|_F$ is sometimes used as a measure of non-normality.

Forgetting the unitary transformation, if $A = D + T$ where D is diagonal with entries $\lambda_r = -\mu_r + i\nu_r$ where $\mu_r > 0$ and $\nu_r \in \mathbf{R}$ and T is strictly upper triangular then e^{At} decreases exponentially for large $t > 0$ with approximate decay rate $\mathrm{e}^{-\mu_1 t}$, assuming that $\mu_1 \leq \mu_r$ for all $r > 1$.

Theorem 10.5 *Continuing with the notation above we have*

$$\mathrm{e}^{-\mu_1 t} \leq \|\mathrm{e}^{At}\| \leq \mathrm{e}^{-\mu_1 t}\mathrm{e}_n(\|T\|t)$$

for all $t \geq 0$, *where*

$$\mathrm{e}_n(s) = \sum_{r=0}^{n-1} \frac{s^r}{r!}.$$

Proof The lower bound is an immediate consequence of the fact that every diagonal entry of D is an eigenvalue of A. To obtain the upper bound we estimate the terms in the finite perturbation expansion

$$\mathrm{e}^{At} = \mathrm{e}^{Dt} + \sum_{r=1}^{n-1} J_r(t)$$

where

$$J_r(t) = \int_{A(r,t)} \mathrm{e}^{D(t-s_r)}T\mathrm{e}^{D(s_r-s_{r-1})}T \ldots \mathrm{e}^{D(s_2-s_1)}T\mathrm{e}^{Ds_1}\,\mathrm{d}^r s$$

and $A(r,t) = \{s \in \mathbf{R}^r : 0 \leq s_1 \leq \ldots \leq s_r \leq t\}$. The remaining terms in the infinite perturbation expansion (11.10) of LOTS vanish because T is strictly upper triangular and e^{Ds} is diagonal for all $s \in \mathbf{R}^r$. Now

$$\begin{aligned}\|J_r(t)\| &\leq \int_{A(r,t)} \|\mathrm{e}^{D(t-s_r)}T\mathrm{e}^{D(s_r-s_{r-1})}T \ldots \mathrm{e}^{D(s_2-s_1)}T\mathrm{e}^{Ds_1}\|\,\mathrm{d}^r s \\ &\leq \mathrm{e}^{-\mu_1 t}\|T\|^r t^r/r!\end{aligned}$$

because $\|\mathrm{e}^{Ds}\| = \mathrm{e}^{-\mu_1 s}$ for all $s \geq 0$ and $|A(r,t)| = t^r/r!$ The theorem follows immediately. $\square$

Example 10.6 The leading term in the long time asymptotics can be determined exactly if $D = 0$. One then has

$$\mathrm{e}^{Tt} = \sum_{r=0}^{n-1} T^r t^r / r!$$

so

$$\|\mathrm{e}^{Tt}\| = \|T^{n-1}\| t^{n-1}/(n-1)! + O(t^{n-2})$$

as $t \to \infty$. Moreover $T^{n-1}_{r,s} = 0$ unless $r = 1$ and $s = n$, so

$$\|T^{n-1}\| = |(T^{n-1})_{1,n}| = |T_{1,2}T_{2,3} \ldots T_{n-1,n}|.$$

□

10.8 Growth bounds using a Liapounov operator

K Veselić and Yu M Nechepurenko have introduced another measure of non-normality which leads to different bounds on $\|\mathrm{e}^{At}\|$. The results below are taken from 'K Veselić, Bounds for exponentially stable semigroups, Lin. Alg. Appl. 358 (2003) 309-333'. Similar bounds for matrices were obtained in 'Yu M Nechepurenko, A bound for the matrix exponential, J. Comp. Math. Phys. 42 (2002) 131-141'. The methods used are very closely related to those described in Section 10.6 of LOTS, but Lemma 10.8 provides a method of computing the relevant index $\|X\|$ in applications.

We consider an $n \times n$ matrix A with eigenvalues $\lambda_r = -\mu_r + i\nu_r$ where $\nu_r \in \mathbf{R}$ for all r and $0 < \mu_1 \leq \mu_2 \leq \ldots$. We then define

$$X = \int_0^\infty \mathrm{e}^{A^*t}\mathrm{e}^{At}\,\mathrm{d}t. \tag{10.5}$$

This integral converges and defines a non-negative self-adjoint matrix X. The next lemma shows that the constant $c = 2\mu_1\|X\|$ could be used as a measure of non-normality of A.

Lemma 10.7 *One has $c \geq 1$ for all A of the above form. If A is normal then $c = 1$.*

Proof If $Ax = \lambda_1 x$ and $\|x\| = 1$ then

$$\|X\| \geq \langle Xx, x\rangle = \int_0^\infty \|\mathrm{e}^{At}x\|^2\,\mathrm{d}t = \int_0^\infty |\mathrm{e}^{\lambda_1 t}|^2\,\mathrm{d}t = \frac{1}{2\mu_1}.$$

If A is normal and $UAU^* = D$ for some unitary U and diagonal D then

$$\|X\| = \|UXU^*\| = \|\int_0^\infty \mathrm{e}^{D^*t}\mathrm{e}^{Dt}\,\mathrm{d}t\| = \frac{1}{2\mu}$$

assuming that $\max \operatorname{Re} \operatorname{Spec}(D) = -\mu < 0$. $\square$

The matrix X may be computed numerically by using standard routines for the continuous Liapounov problem (these involve reducing A and A^* to triangular form independently) once the following is established.

Lemma 10.8 *The matrix X is the unique solution of*

$$A^*X + XA = -I. \tag{10.6}$$

Proof The uniqueness of the solution of (10.6) follows from the fact that $\lambda + \overline{\lambda}$ is non-zero for every eigenvalue λ of A. If we define the 'superoperator' L on the space of $n \times n$ matrices by $L(B) = A^*B + BA$ then

$$\mathrm{e}^{Lt}(B) = \mathrm{e}^{A^*t}B\mathrm{e}^{At}$$

and $\mathrm{e}^{Lt} \to 0$ as $t \to +\infty$. Moreover

$$\begin{aligned} -I &= \int_0^\infty L\mathrm{e}^{Lt}(I)\,\mathrm{d}t \\ &= L\int_0^\infty \mathrm{e}^{A^*t}\mathrm{e}^{At}\,\mathrm{d}t \\ &= L(X) \\ &= A^*X + XA. \end{aligned}$$

$\square$

Veselić contains the following results, attributing the second one to Godunov, Kiriljuk and Kostin in 1990. He assumes that e^{At} is a strongly continuous one-parameter semigroup acting on the Hilbert space $\mathcal{H}$ and that the weakly convergent integral (10.5) converges to define a non-negative bounded self-adjoint operator X. Typically X^{-1} is unbounded.

Theorem 10.9

$$\|X^{1/2}\mathrm{e}^{At}\| \le \|X^{1/2}\|\mathrm{e}^{-t/(2\|X\|)}$$

for all $t \ge 0$ and hence

$$\|\mathrm{e}^{At}\|^2 \le \|X\|\,\|X^{-1}\|\mathrm{e}^{-t/\|X\|}$$

The second bound may not be useful if X^{-1} has a very large norm or if it is unbounded, but Veselić and Nechepurenko have other bounds in that case. The following is the simplest.

Theorem 10.10 (Veselić, Theorem 2) *Let e^{At} be a one-parameter semigroup on the Hilbert space $\mathcal{H}$ and suppose that the non-negative self-adjoint operator X defined by*

$$\langle Xu, v\rangle = \int_0^\infty \langle \mathrm{e}^{At}u, \mathrm{e}^{At}v\rangle\,\mathrm{d}t$$

is bounded. Then

$$\|\mathrm{e}^{At}\| \le \left(\sup_{0\le\tau\le\|X\|} \|\mathrm{e}^{A\tau}\| \right) \exp\left(-\frac{t-\|X\|}{2\|X\|} \right)$$

for all $t \ge 0$.

Proof Let $\psi \in \mathcal{H}$ and put $c = \|X\|$. Then the differential inequality

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\langle X\mathrm{e}^{At}\psi, \mathrm{e}^{At}\psi\rangle &= \frac{\mathrm{d}}{\mathrm{d}t}\int_t^\infty \langle \mathrm{e}^{As}\psi, \mathrm{e}^{As}\psi\rangle \,\mathrm{d}s \\ &= -\langle \mathrm{e}^{At}\psi, \mathrm{e}^{At}\psi\rangle \\ &\le -c^{-1}\langle X\mathrm{e}^{At}\psi, \mathrm{e}^{At}\psi\rangle \end{aligned}$$

implies that

$$\langle X\mathrm{e}^{At}\psi, \mathrm{e}^{At}\psi\rangle \le \mathrm{e}^{-t/c}\langle X\psi, \psi\rangle$$

for all $t \ge 0$. If $a > 0$ and $0 \le t-a \le s \le t$ then

$$\|\mathrm{e}^{At}\psi\|^2 \le \|\mathrm{e}^{A(t-s)}\|^2\|\mathrm{e}^{As}\psi\|^2 \le C\|\mathrm{e}^{As}\psi\|^2$$

where $C = \sup_{0\le u\le a}\|\mathrm{e}^{Au}\|^2$. If $t \ge a$ we deduce that

$$\begin{aligned} \|\mathrm{e}^{At}\psi\|^2 &= \frac{1}{a}\int_{t-a}^t \|\mathrm{e}^{At}\psi\|^2 \,\mathrm{d}s \\ &\le \frac{C}{a}\int_{t-a}^t \|\mathrm{e}^{As}\psi\|^2 \,\mathrm{d}s \\ &\le \frac{C}{a}\int_{t-a}^\infty \|\mathrm{e}^{As}\psi\|^2 \,\mathrm{d}s \\ &= \frac{C}{a}\,\langle X\mathrm{e}^{A(t-a)}\psi, \mathrm{e}^{A(t-a)}\psi\rangle \\ &\le \frac{C}{a}\,\mathrm{e}^{-(t-a)/c}\langle X\psi, \psi\rangle \\ &\le \frac{Cc}{a}\,\mathrm{e}^{-(t-a)/c}\|\psi\|^2. \end{aligned}$$

The proof is completed by putting $a = c$, and noting that the case $0 \le t \le \|X\|$ is trivial. □

Note that if one has the further inequality

$$\mathrm{Re}\,\langle A\psi, \psi\rangle \le a\|\psi\|^2$$

for all $\psi \in \mathrm{Dom}(A)$ then $\|\mathrm{e}^{At}\| \le \mathrm{e}^{at}$ for all $t \ge 0$, so the theorem implies that

$$\|\mathrm{e}^{At}\| \le \exp\left(a\|X\| - \frac{t-\|X\|}{2\|X\|} \right)$$

for all $t \geq \|X\|$.

The following example shows that Theorem 10.10 yields very poor long term decay bounds in some cases. Theorem 10.3 is no better. However, the long time asymptotic decay rate of a semigroup whose generator is a moderately sized Jordan matrix is very unstable with respect to small perturbations, so one cannot expect good bounds in such cases.

Example 10.11 If

$$A = \begin{pmatrix} -1 & c \\ 0 & -1 \end{pmatrix}$$

where $c \geq 0$, then

$$\mathrm{e}^{At} = \mathrm{e}^{-t} \begin{pmatrix} 1 & ct \\ 0 & 1 \end{pmatrix}$$

for all $t \geq 0$, so e^{At} is a one-parameter contraction semigroup if $0 \leq c \leq 1$, but not if $c \geq 3$. A direct calculation shows that

$$X = \begin{pmatrix} 1/2 & c/4 \\ c/4 & 1/2 + c^2/4 \end{pmatrix}.$$

If $c = 1$ then $\|X\| \sim 1.184$ so $1/(2\|X\|) < 1/2$, although the correct exponent for this case is 1. For very large c one obtains

$$\frac{1}{2\|X\|} = \frac{2}{c^2} + O(c^{-1})$$

but the correct exponent remains 1 for all $c > 0$. $\square$

Chapter 11

Perturbation Theory

11.1 Perturbations of unbounded operators

page 329,-11. Replace the final f by f_θ.

The following continuation of the section obtains a more detailed analysis of perturbations of the spectrum than that in Rellich's Theorem 11.1.6 when the spectral projection is of finite rank. We also assume that the operator A is bounded. This does not result in an essential loss of generality, because the spectrum of an unbounded operator A is directly related to that of any of its resolvent operators by Lemma 8.1.9. Perturbation theory is well understood when all the operators concerned are self-adjoint, and we concentrate on issues that arise when non-trivial Jordan forms are involved.

For the remainder of the section we assume that A is a bounded operator on the Banach space $\mathcal{B}$, that $a \in \mathbf{C}$ is an isolated point of $\mathrm{Spec}(A)$, and that the spectral projection P_0 of A associated with a by Theorem 1.5.4 has finite rank N. The number N is called the algebraic multiplicity of a while the dimension $M \leq N$ of the space of all eigenvectors of A associated with the eigenvalue a is called its geometric multiplicity. In the definition of P_0 we take the closed contour to be a circle γ with centre a and radius $\delta > 0$, where δ is small enough so that a is the only point of $\mathrm{Spec}(A)$ inside γ.

We next assume that A_z is an analytic family of bounded operators in the sense of Section 1.4, defined for all $z \in \mathbf{C}$ such that $|z| < \varepsilon$ for some $\varepsilon > 0$. We also assume that $A_0 = A$.

Theorem 11.1 *If $\varepsilon > 0$ is small enough then $\mathrm{Spec}(A_z) \cap \gamma = \emptyset$ for all z such that $|z| < \varepsilon$. The spectral projection P_z associated with the part of the spectrum of A_z*

inside γ depends analytically on z, and it has rank N for all such z.

Proof This is an obvious adaptation of the proof of Theorem 11.1.6. One also needs to note that $F(\lambda, z) = (\lambda I - A_z)^{-1}$ exists and is a jointly analytic function of (λ, z) provided λ is in a small enough neighbourhood of γ and $|z|$ is sufficiently small; this uses (1.5) in Theorem 1.2.9. $\square$

Theorem 11.2 *If $N = 1$ then A_z has a single eigenvalue λ_z inside γ for all small enough z. Moreover λ_z is an analytic function of z and it satisfies*

$$\lambda_z = a + z\langle A'_0 x_0, \phi_0\rangle + O(z^2)$$

where $Ax_0 = ax_0$, $A^*\phi_0 = a\phi_0$ and $\langle x_0, \phi_0\rangle = 1$.

Proof The spectral projection P_0 of A corresponding to the eigenvalue a is of rank 1 and therefore may be written in the form $P_0 f = \langle f, \phi_0\rangle x_0$ where $Ax_0 = ax_0$ and $\langle x_0, \phi_0\rangle = 1$. Since P_0^* is the corresponding projection of A^*, it also follows that $A^*\phi_0 = a\phi_0$.

If $\widetilde{x}_z = P_z x_0$ then $\widetilde{x}_z$ is an analytic function of z, $A_z\widetilde{x}_z = \lambda_z\widetilde{x}_z$ and $\langle \widetilde{x}_z, \phi_0\rangle \neq 0$ provided z is small enough. Putting

$$x_z = \widetilde{x}_z / \langle \widetilde{x}_z, \phi_0\rangle$$

we deduce that x_z is an analytic function of z, $A_z x_z = \lambda_z x_z$ and $\langle x_z, \phi_0\rangle = 1$ for all small enough z. Hence $\langle x'_0, \phi_0\rangle = 0$. A similar construction yields an analytic function $\phi_z \in \mathcal{B}^*$ such that $A_z^*\phi_z = \lambda_z\phi_z$, $\langle x_0, \phi_z\rangle = 1$ for all small enough z and hence $\langle x_0, \phi'_0\rangle = 0$.

We now expand both sides of

$$\langle A_z, x_z, \phi_z\rangle = \lambda_z\langle x_z, \phi_z\rangle$$

as power series in z and identify the coefficients of 1 and z. This yields $\lambda_0 = a$ and $\lambda'_0 = \langle A'_0 x_0, \phi_0\rangle$ after simplification. $\square$

The next lemma will allow us to replace A_z by a similar operator for which the corresponding spectral projection does not depend on z.

Lemma 11.3 (Kato) [1] *Let P, Q satisfy $P^2 = P$ and $Q^2 = Q$ and let $S = I - (P - Q)^2$ and $T = I - P - Q$. Then $T^2 = S$ and $TQ = PT$. If $\|P - Q\| < 1$ then T is invertible and $TQT^{-1} = P$.*

Proof The identities in the first sentence are proved by direct algebraic calculations. Under the final condition, S is invertible by Problem 1.2.8, so T is invertible and $T^{-1} = S^{-1}T = TS^{-1}$. The final identity follows. $\square$

[1]See section 1.4.6 of T. Kato, Perturbation Theory of Linear Operators, 1st edition, Springer, 1966. Note that $T = (I - P)(I - Q) - PQ$, whereas Kato considers $T' = (I - P)(I - Q) + PQ$.

Lemma 11.4 *There exists $\sigma > 0$ and an analytic family of bounded invertible operators T_z defined for all z such that $|z| < \sigma$ and satisfying $T_0 = I$ and*

$$T_z P_z T_z^{-1} = P_0.$$

Proof This follows directly from Lemma 11.3 by putting $P = P_0$ and $Q = P_z$. $\Box$

If one puts $B_z = T_z A_z T_z^{-1}$ then B_z is an analytic family of operators whose spectra coincide with those of A_z for every z. However, the spectral projection of B_z associated with the region inside γ equals P_0 by Lemma 11.4. Therefore we can determine the spectrum of A_z inside γ by considering the restrictions $M_z = B_z|_{\mathcal{D}}$ to the fixed finite-dimensional subspace $\mathcal{L} = P_0\mathcal{B}$.

If $N > 1$ then the eigenvalues of A_z need not be analytic functions of z. However, the invariance of the trace under similarity transformations implies that

$$\mathrm{tr}[A_z P_z] = \mathrm{tr}[B_z P_0] = \mathrm{tr}[M_z]$$

for all small enough z. Therefore the trace is an analytic function of z. Formulae for calculating the coefficients of its power series expansion have been given by Kato.[2]

For the remainder of this section we consider the linear family $A + zB$ of $N \times N$ matrices, assuming that A takes its Jordan canonical form for the standard basis of $\mathbf{C}^N$ and that 0 is the only eigenvalue of A. More explicitly A has a diagonal block matrix in which each diagonal entry is an elementary $M \times M$ Jordan matrix J_M of the form

$$J_{M,r,s} = \begin{cases} 1 & \text{if } s = r+1 \text{ and } 1 \le r \le M-1, \\ 0 & \text{otherwise.} \end{cases}$$

The eigenvalues of $A + zB$ coincide with the zeros λ of

$$F(\lambda, z) = \lambda^N + \lambda^{N-1} f_{N-1}(z) + \ldots + \lambda f_1(z) + f_0(z)$$

where f_r are polynomials. The asymptotic form of the zeros as $|z| \to 0$ can be analyzed using Rouche's theorem in complex analysis. The range of possibilities is very complicated and we content ourselves with a few examples and some general comments.

Problem 11.5 Write down the formula for the eigenvalues of $A + zB$ where

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

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[2]See section 2.2 of T. Kato, Perturbation Theory of Linear Operators, 1st edition, Springer, 1966.

and prove that the eigenvalues have leading asymptotics

$$\lambda = \pm c^{1/2} z^{1/2} + O(|z|)$$

as $|z| \to 0$. This example suggests, correctly, that certain coefficients of the perturbation might be more important than others in determining the leading asymptotics for other values of N. $\square$

Problem 11.6 We shall need the following result. Let B be the $N \times N$ matrix with entries

$$B_{r,s} = \begin{cases} b_r & \text{if } s = r+1 \text{ and } 1 \le r \le N-1, \\ b_N & \text{if } r = N \text{ and } s = 1. \\ 0 & \text{otherwise.} \end{cases}$$

Prove that B is similar to the matrix bU where $b = (b_1 b_2 \ldots b_N)^{1/N}$ and U is the unitary matrix that corresponds to the choice $b_r = 1$ for all r above. Deduce that

$$\mathrm{Spec}(B) = \{b\omega^r : 0 \le r \le N-1\}$$

where $\omega = \mathrm{e}^{2\pi i/N}$. $\square$

Problem 11.7 Let $A = J_N$ and let B be the $N \times N$ matrix with entries

$$B_{r,s} = \begin{cases} 1 & \text{if } r = N \text{ and } s = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Prove that

$$\mathrm{Spec}(A + zB) = \{z^{1/N}\omega^r : 0 \le r \le N-1\}$$

where $\omega = \mathrm{e}^{2\pi i/N}$. $\square$

Problem 11.8 Let $A = \begin{pmatrix} J_M & 0 \\ 0 & J_{N-M} \end{pmatrix}$ for some $M < N$, and let B be the $N \times N$ matrix with entries

$$B_{r,s} = \begin{cases} 1 & \text{if } r = M \text{ and } s = M+1, \\ 1 & \text{if } r = N \text{ and } s = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Prove that

$$\mathrm{Spec}(A + zB) = \{z^{2/N}\omega^r : 0 \le r \le N-1\}$$

where $\omega = \mathrm{e}^{2\pi i/N}$. $\square$

Problem 11.9 Let $A = J_N$ and $B = J_N^*$. Prove that $A + zB$ is similar to $z^{1/2}(J_N + J_N^*)$. This implies that every eigenvalue of $A + zB$ is of order $z^{1/2}$ as $z \to 0$. Many other fractional powers of z appear in other examples. $\square$

Typically the eigenvalues of $A+zB$ are uniformly distributed around a circle whose radius is a fractional power of $|z|$, to leading order asymptotically as $|z| \to 0$. Rigorous studies of this problem started with Lidskii, but have continued up to the present time.[3]

There is another approach to this problem, which assumes that the coefficients of B are chosen randomly and independently according to some probability law. The asymptotic behaviour of the spectrum of $A + zB$ may then be determined with probability 1 in a suitable sense.[4]

11.2 Relatively compact perturbations

page 332,1. Replace 'by Theorem 4.2.4' by 'see Section 4.2'.

page 333,-9. Replace the displayed equation by

$$R(a, H) = R(a, H_0)(I - VR(a, H_0))^{-1}$$

11.3 Constant coefficient differential operators on the half-line

page 337,-5. In order to apply Lemma 11.2.1 one has to prove that $g_n := f_n/|||f_n|||$ converges weakly to zero in $\mathcal{D}_L$. Since $|||g_n||| = 1$ for all n, a density argument implies that it is sufficient to prove that $\langle g_n, h\rangle \to 0$ for all $h \in C_c[0, \infty)$. This follows from the fact that $\text{supp}(g_n) \subseteq [n, 4n]$.

page 338,1. Replace $a_{2n} = 0$ by $a_{2n} = 1$.

page 338,10. Replace γ by σ.

page 338. A formulation of Theorem 11.3.4 in terms of the Fredholm index and subject to general boundary conditions at 0 is given in Theorem XVIII.6.2 of I.

[3] J. Moro, J. V. Burke and M. L. Overton, On The Lidskii-Vishik-Lyusternik perturbation theory for eigenvalues of matrices with arbitrary Jordan structure, Siam J. Matrix Anal. Appl. 18, no. 4 (1997) 793-817; A. C. M. Rana, M. Wojtylak, Eigenvalues of rank one perturbations of unstructured matrices, Linear Alg. Applic. 437 (2012) 589600.

[4] M. Hager and J. Sjöstrand, Eigenvalue asymptotics for randomly perturbed non-selfadjoint operators, Math. Annalen 342 (1) (2008) 177-243; E. B. Davies and M. Hager: Perturbations of Jordan matrices. J. Approx. Theory 156 (2009) 82-94.

Gohberg, A. Goldberg and M. A. Kaashoek, Classes of Linear Operators, vol. 1, Birkhäuser, Basel, 1990.

page 339,6,8,12. Replace real by imaginary and ξ_r by $i\xi_r$ in several places.

page 340,1. Delete the two brackets.

page 340,13. The term $\|f\|$ is missing.

page 350,17. Replace Section 5.1 by Section 11.1.

page 351,-5. a core

page 353,-3 and -2 and -1. Replace $(b+|c+\omega|)$ by $(b+\varepsilon|c+\omega|)$ on each line.

page 354,1 and 2 and 3. Replace b by bN on each line.

Chapter 12

Markov chains and graphs

page 357,7. Delete commas around word Markov.

page 357,-16. Replace subscript 2 by subscript 1.

page 358,-6. Delete final) .

page 360,4. If S is any subset of X

page 360,-9. End equation with ... $= c\|f\|$.

page 370, eq (12.11). Replace t on second line of equation by n.

page 374,-2. See Theorem 2.4.4 on p. 65.

page 375, Lemma 12.6.1. State that J is the incidence matrix of a k-tree.

Chapter 13

Positive semigroups

13.6 Positive semigroups on $C(X)$

page 405. The following new material should be included just before Theorem 13.6.12.

Theorem 13.6.12 extends one result of the Perron-Frobenius theory, which applies to all non-negative $n \times n$ matrices. We start with one of the original results of the P-F theory.

Lemma 13.1 *Let M be an $n \times n$ Markov matrix and let $S \subseteq \{1, \ldots, n\}$ be invariant in the sense that $i \in S$ and $M_{i,j} > 0$ implies $j \in S$. Then the restriction A of M to $i, j \in S$ is also a Markov matrix and* $\mathrm{Spec}(A) \subseteq \mathrm{Spec}(M)$.

Proof If one permutes the indices so that $S = \{1, \ldots, m\}$ then one may write

$$M = \begin{pmatrix} A & 0 \\ C & D \end{pmatrix}$$

from which all of the assertions follow by inspection. □

Lemma 13.2 *Let M be an $n \times n$ Markov matrix and let $Mf = zf$ where $|z| = 1$ and $\|f\|_\infty = 1$. Then $S = \{i : |f_i| = 1\}$ is an invariant set. Moreover $f_j = zf_i$ if $i, j \in S$ and $M_{i,j} > 0$.*

Proof If $i \in S$ then

$$1 = \sum_{j=1}^{n} M_{i,j} \frac{f_j}{zf_i} = \sum_{j=1}^{n} M_{i,j} \mathrm{Re}\left(\frac{f_j}{zf_i}\right) \leq \sum_{j=1}^{n} M_{i,j} \left|\frac{f_j}{zf_i}\right| \leq 1.$$

Therefore $|f_j| = |zf_i| = 1$ and $f_j = zf_i$ whenever $M_{i,j} > 0$. □

Theorem 13.3 (Frobenius) *If M is an $n \times n$ Markov matrix and z is an eigenvalue of M satisfying $|z| = 1$ then z^m is an eigenvalue of M for all $m \in \mathbf{Z}$. Moreover $z^s = 1$ for some $s \in \{1, \ldots, n\}$.*

Proof The two lemmas allow us to reduce to the case in which $|f_i| = 1$ for all $i \in \{1, \ldots, n\}$. Since $M\overline{f} = \overline{z}\overline{f}$, it is sufficient to prove the theorem for positive integers m, and we do this by induction. If $Mg = z^m g$ then

$$\begin{aligned}(M(fg))_i &= \sum_{j=1}^{n} M_{i,j} f_j g_j \\ &= \sum_{j=1}^{n} M_{i,j} z f_i g_j \\ &= (z f_i)(z^m g_i) \\ &= z^{m+1}(fg)_i,\end{aligned}$$

so z^{m+1} is an eigenvalue with eigenvector fg. $\square$

See Schaefer, Banach Lattices and Positive Operators, Theorem 1.2.7, p.8.

Chapter 14

NSA Schrödinger operators

14.3 Bounds in one space dimension

The comment immediately after the proof of Theorem 14.3.1 is incorrect. One may use the reflection principle to obtain a similar, but worse, bound on the half-line subject to Dirichlet boundary conditions at 0. However, this does not yield a sharp result, which, surprisingly, is different from that for the whole line. The theorem below refers to a function $g : \mathbf{R} \to [1,2)$ defined by

$$g(a) = \sup_{y \geq 0} |\mathrm{e}^{iay} - \mathrm{e}^{-y}|.$$

One easily sees that $g(0) = 1$ and $\lim_{a \to \pm\infty} g(a) = 2$. We do not attempt to specify the precise domain of the operator H involved.

Theorem 14.1 (Frank, Laptev, Seiringer[1]) *Let $H = H_0 + V$ act in $L^2(0,\infty)$, where $H_0 f = -f''$ subject to Dirichlet boundary conditions at 0 and $V \in L^1(0,\infty)$, so that H has essential spectrum $[0,\infty)$ together with a possible sequence of discrete eigenvalues, each with finite algebraic multiplicity. If $\lambda = |\lambda|\mathrm{e}^{i\theta}$ is an eigenvalue of H with $0 < \theta < 2\pi$, then*

$$|\lambda| \leq \frac{[g(\cot(\theta/2))]^2}{4} \|V\|_1^2.$$

This bound is sharp.

The proof of Theorem 14.1 is similar to, but more difficult than, that of Theorem 14.3.1, relying heavily on bounds derived from the formula for the Green function $G(z,x,y)$ of $(H_0 + z^2)^{-1}$, namely

$$G(z,x,y) = \frac{\mathrm{e}^{-z|x-y|}}{2z} - \frac{\mathrm{e}^{-z(x+y)}}{2z}.$$

[1] R. L. Frank, A. Laptev, R. Seiringer, A sharp bound on eigenvalues of Schrödinger operators on the half-line with complex-valued potentials, Oper. Theory: Adv. and Applic., 214 (2010) 39-44.

Since LOTS was written the multi-dimensional analogue of this theory has developed a lot. New results are due to Laptev, Safronov and Frank. The following is one of several such theorems.

Theorem 14.2 (Frank [2]) *Let $d \geq 2$ and $0 < \gamma \leq 1/2$. Then any eigenvalue $\lambda \in \mathbf{C}\backslash[0,\infty)$ of the Schrödinger operator $-\Delta + V$ with complex potential V acting in $L^2(\mathbf{R}^d)$ satisfies*

$$|\lambda|^\gamma \leq D_{\gamma,d} \int_{\mathbf{R}^d} |V(x)|^{\gamma+d/2}\,\mathrm{d}x.$$

The proof of Theorem 14.2 is not applicable to the results in this section dealing with long range potentials, i.e. potentials that decay slowly as $|x| \to \infty$.

14.5 The NSA harmonic oscillator

page 425. Theorem 14.5.1 establishes that the biorthogonal pair of sequences ϕ_n and ϕ_n^* is wild in the sense defined on page 83.

[2] R. L. Frank, Eigenvalue bounds for Schrödinger operators with complex potentials, Bull. London Math. Soc. 43(4) (2011) 745-750.

Chapter 15

Finite range matrices

15.1 Introduction

The material in this new chapter can be presented at many different levels. The underlying space may be $\mathbf{R}^N$, a Riemannian manifold, a metric space, $\mathbf{Z}^N$ or $\mathbf{N}$, the set of natural numbers, for example. The operator A of interest may be a differential operator or a bounded operator presented by means of its matrix elements. Finite range operators may be investigated directly or by using C*-algebra theory, which enables one to classify the essential spectra of such operators into parts geometrically.[1]

In this chapter we restrict attention to a discrete underlying space X. We assume that $(X, \mathcal{E})$ is a countable discrete graph in which every edge in $\mathcal{E}$ is an unordered pair (x, y) of vertices in X; we will write $x \sim y$ instead of $(x, y) \in \mathcal{E}$. We assume that there is a uniform upper bound k on the degrees of the vertices and that X is connected. Let $d(x, y)$ denote the graph distance between x and y, where each edge is taken to have length 1. If $X = \mathbf{Z}^N$ or $X = \mathbf{N}$ then we assume that the associated graph structure is invariant under translations.

A bounded operator A on $\ell^2(X)$ is said to be have finite range (or band width) ρ if $A_{m,n} = 0$ whenever $d(m, n) > \rho$. A finite range operator A acting in $\ell^2(\mathbf{Z})$ or $\ell^2(\mathbf{Z}_+)$ is said to have a band matrix, and is said to be tridiagonal if it has range 1, i.e. $A_{r,s} = 0$ whenever $|r - s| > 1$.

The inverse of a band matrix is almost never a band matrix, but there are some important cases in which this happens.[2] This is relevant to wavelet transforms,

[1] See E. B. Davies, Decomposing the essential spectrum, J. Funct. Anal. 257 (2009) 506-536; E. B. Davies and V. Georgescu, C*-algebras associated with some second order differential operators, preprint 2011; S. N. Chandler-Wilde and M. Lindner, Limit operators, Collective Compactness, and the Spectral Theory of Infinite Matrices, Mem. Amer. Math. Soc. No. 989, 2011, and many further references there.

[2] G. Strang, Fast transforms: Banded matrices with banded inverses, PNAS July 13, 2010 vol. 107 no. 28, 12413-12416.

and allows very rapid numerical computations of the transform and its inverse. Note that the set of all matrices that are banded and inverse banded is a group.

Operators that are periodic with respect to some discrete group arise in many different contexts and are the subject of Section 15.2. The material here is classical, but it is usually written down for differential operators.[3] Section 15.5 describes the spectra of infinite triangular matrices that have different periodic structures on the positive half-line and the negative half-line. Some of the results were motivated by the study of certain classes of infinite random tridiagonal matrices,[4] but the treatment here is more systematic and involves the use of what we call the stable spectrum of an operator. The fact that infinite triadiagonal matrices can be investigated by using transfer matrices as in Section 15.4 is classical and many of the formulae in this chapter can be extended to block tridiagonal matrices.[5]

Let X be a graph with the above properties and let $\mathcal{K}$ be a finite-dimensional Hilbert space and let $\ell^2(X,\mathcal{K})$ denote the space of square-summable $\mathcal{K}$-valued functions on X. The theory developed in this chapter is much more limited than that in other recent literature, where $X = \mathbf{Z}^N$, ℓ^2 is replaced by ℓ^p and $\mathcal{K}$ is allowed to be an infinite-dimensional Banach space.[6] On the other hand, the final theorem of the chapter, Theorem 15.20, can be extended to the graph context.[7] Unfortunately the standard theory of limit operators cannot be applied as it stands to graphs because there is no translation group acting on X.

If $A_{m,n} \in \mathcal{L}(\mathcal{K})$ for every m, $n \in X$, and $A_{m,n} = 0$ if $d(M,n) > \rho$, then the formula

$$(Af)_m = \sum_{n\in X} A_{m,n} f_n \tag{15.1}$$

may be used to evaluate Af for any $f : X \to \mathcal{K}$ because the finite range condition implies that the sums involved are all finite. Specifically, given $m \in X$

$$\#\{n \in X : d(m,n) \leq \rho\} \leq k^{\rho+1}.$$

[3]See, for example, M. S. P. Eastham, Spectral Theory of Periodic Differential Equations, Scottish Acad. Press, London, 1973, P. Kuchment, Floquet theory for partial differential equations, Birkhäuser, Basel, 1993, and M. Reed and B. Simon, Methods of Modern Mathematical Physics, IV, Academic Press, New York, 1975.

[4]See E. B. Davies, Spectral properties of random non-self-adjoint matrices and operators, Proc. Roy. Soc. London A 457 (2001) 191-206 and E. B. Davies, Spectral Theory of Pseudo-ergodic Operators, Commun. Math. Phys. 216 (2001) 687-704.

[5]See L G Molinari, Determinants of block tridiagonal matrices, Linear Alg. Appl. 429 (2008) 2221-2226, and other sources cited there.

[6]See V. S. Rabinovich, S. Roch, and B. Silbermann, Limit Operators and Their Applications in Operator Theory, Birkhäuser, 2004, and S. N. Chandler-Wilde and M. Lindner, Limit Operators, Collective Compactness, and the Spectral Theory of Infinite Matrices, Mem. Amer. Math. Soc. no. 989, Amer. Math. Soc., Providence, RI, 2011.

[7]See E. B. Davies, Stable Spectrum of an Operator on an Infinite Discrete Graph, preprint, 2011.

Lemma 15.1 *If A is an infinite $\mathcal{L}(\mathcal{K})$-valued matrix with finite range ρ then A is associated with a bounded operator on $\ell^2(X,\mathcal{K})$ if and only if the constant*

$$c = \sup\{\|A_{m,n}\| : m,\, n \in X\}$$

is finite. In this case

$$c \leq \|A\| \leq ck^{\rho+1}.$$

Proof The lower bound is elementary and the upper bound is a discrete version of Corollary 2.2.15 of LOTS. □

Operators with finite range also act on certain weighted spaces $\ell^2(X,w)$, in which the norm is given by

$$\|f\|_w^2 = \sum_{x\in X} |f(x)|^2 w(x).$$

Lemma 15.2 *Let A satisfy the conditions of Lemma 15.1. Let $c > 0$ and let $w : X \to (0,\infty)$ satisfy $c^{-1} \leq w(x)/w(y) \leq c$ for all x, y satisfying $x \sim y$. Then the formula (15.1) defines a bounded operator A_w on $\ell^2(X,w)$.*

Proof Let $U : \ell^2(X) \to \ell^2(X,w)$ be the unitary operator $(Uf)(x) = w(x)^{-1/2} f(x)$ and let $B = U^{-1}A_wU$, so that A_w is a bounded operator on $\ell^2(X,w)$ if and only if B is a bounded operator on $\ell^2(X)$. We have $(Bf)(x) = \sum_{y\in X} B_{x,y} f(y)$ where

$$B_{x,y} = w(x)^{1/2} w(y)^{-1/2} A_{x,y}$$

for all $x, y \in X$. Since $|A_{x,y}| \leq \|A\|$ and all x, $y \in X$ and $A_{x,y} = 0$ unless $d(x,y) \leq \rho$, we deduce that $|B_{x,y}| \leq c^{\rho/2}\|A\|$ for all x, $y \in X$ and $B_{x,y} = 0$ unless $d(x,y) \leq \rho$. Therefore B is a bounded operator on $\ell^2(X)$ by Lemma 15.1. □

The following theorem is one of a range of related results, many of which relate to differential operators. We say that f is subexponential at infinity if $f_\varepsilon \in \ell^2(\mathbf{Z}^N,\mathcal{K})$ for all $\varepsilon > 0$, where $f_{\varepsilon,n} = \mathrm{e}^{-\varepsilon|n|} f_n$ for all $n \in \mathbf{Z}^N$.

Theorem 15.3 (Sch'nol) *Let A be a bounded operator on $\ell^2(\mathbf{Z}^N,\mathcal{K})$ with finite range ρ. If $f : \mathbf{Z}^N \to \mathcal{K}$ is subexponential at infinity and it is not identically zero and $Af = \lambda f$, then λ lies in the ℓ^2 spectrum of A.*

Proof Given f as above and $\varepsilon > 0$, let $g_\varepsilon = Af_\varepsilon - \lambda f_\varepsilon$. Then

$$\begin{aligned}
\|g_{\varepsilon,n}\| &= \|\sum_{|s|\leq\rho} A_{n,n+s}\mathrm{e}^{-\varepsilon|n+s|} f_{n+s} - \lambda \mathrm{e}^{-\varepsilon|n|} f_n\| \\
&= \|\sum_{|s|\leq\rho} A_{n,n+rs}\mathrm{e}^{-\varepsilon|n+s|} f_{n+s} - \lambda \mathrm{e}^{-\varepsilon|n|} f_n - \mathrm{e}^{-\varepsilon|n|}(\sum_{|s|\leq\rho} A_{n,n+s} f_{n+s} - \lambda f_n)\| \\
&= \|\sum_{|s|\leq\rho} A_{n,n+s}\left(\mathrm{e}^{-\varepsilon|n+s|} - \mathrm{e}^{-\varepsilon|n|}\right) f_{n+s}\|.
\end{aligned}$$

We now use the bound

$$|\mathrm{e}^{-\varepsilon|n+s|} - \mathrm{e}^{-\varepsilon|n|}| \leq 2\varepsilon\rho\mathrm{e}^{-\varepsilon|n+s|}$$

provided $|s| \leq \rho$ and $0 < \varepsilon\rho < 1$. This yields

$$\begin{aligned}
\|g_{\varepsilon,n}\|^2 &\leq 4\varepsilon^2\rho^2 \Big(\sum_{|s|\leq\rho} \|A_{n,n+s}\| \, \|f_{\varepsilon,n+s}\| \Big)^2 \\
&\leq 4\varepsilon^2\rho^2 k^2 \Big(\sum_{|s|\leq\rho} \|f_{\varepsilon,n+s}\| \Big)^2 \\
&\leq 4\varepsilon^2\rho^2 k^2 (2\rho+1)^N \sum_{|s|\leq\rho} \|f_{\varepsilon,n+s}\|^2
\end{aligned}$$

where k is the constant in Lemma 15.1. Summing over n, we obtain

$$\|(A-\lambda I)f_\varepsilon\|_2^2 \leq 4\varepsilon^2\rho^2 k^2 (2\rho+1)^{2N} \|f_\varepsilon\|_2^2.$$

Since $\varepsilon > 0$ may be arbitrarily small, it follows that $A-\lambda I$ cannot have a bounded inverse, so $\lambda \in \mathrm{Spec}(A)$. □

15.2 Periodic matrices

The study of periodic differential operators has obvious importance in the quantum theory of electron transport in crystal lattices. This is also true in two dimensions when modelling surface waves. However, there is now another application, to the propagation of EM waves in periodic microstructures, which can be manufactured to have a wide range of forms. Application to optical, acoustic and water wave cloaking are now being investigated using similar ideas. This section provides an introduction to the underlying mathematics. A substantial part of this section may also be found in Section 4.4 of LOTS, which also contains further results.

An operator A on $\ell^2(\mathbf{Z}^N)$ is said to be G-periodic if G is a group of translations on $\mathbf{Z}^N$ and $AU_g = U_gA$ for all $g \in G$, where $(U_gf)_n = f_{n+g}$ for all $f \in \ell^2(\mathbf{Z}^N)$. If $N = 1$ and $G = p\mathbf{Z}$, we say that A has period p. Our first result about periodic operators holds at a greater level of generality.

Theorem 15.4 *Let A be a bounded operator on the Hilbert space $\mathcal{H}$ and let $AU = UA$ where U is a unitary operator such that* $\lim_{n\to\infty} U^nf = 0$ *weakly for all $f \in \mathcal{H}$. Then*

$$\mathrm{Spec}(A) = \mathrm{Ess}(A).$$

where $\mathrm{Ess}(A)$ *denotes the essential spectrum of A. Moreover every eigenvalue of A has infinite multiplicity.*

Proof Let $\lambda \in \mathrm{Spec}(A)$. Lemma 1.2.13 of LOTS implies that one of the two following cases must occur.

Case 1. There exists a sequence $f_n \in \mathcal{H}$ such that $\|f_n\| = 1$ for all n and $\lim_{n\to\infty} \|Af_n - \lambda f_n\| = 0$. Let $\{e_r\}_{r=1}^{\infty}$ be a complete orthonormal sequence in $\mathcal{H}$. For each n put $g_n = U^{m(n)} f_n$ where $m(n)$ is large enough that $|\langle g_n, e_r\rangle| < 1/n$ for all $1 \leq r \leq n$; this is possible by the weak convergence assumption on the powers of U. Then $\|g_n\| = 1$, $\lim_{n\to\infty} g_n = 0$ weakly and

$$\begin{aligned}\|Ag_n - \lambda g_n\| &= \|AU^{m(n)} f_n - \lambda U^{m(n)} f_n\| \\ &= \|U^{m(n)}(Af_n - \lambda f_n)\| \\ &= \|Af_n - \lambda f_n\| \\ &\to 0\end{aligned}$$

as $n \to \infty$. Therefore $\lambda \in \mathrm{Ess}(A)$ by Lemma 4.3.15 of LOTS.

Case 2. There exists a sequence $f_n \in \mathcal{H}$ such that $\|f_n\| = 1$ for all n and $\lim_{n\to\infty} \|A^* f_n - \overline{\lambda} f_n\| = 0$. Since $(U^*)^n f$ converges weakly to 0 as $n \to \infty$ for every $f \in \mathcal{H}$, by applying Case 1 to A^* one sees that $\overline{\lambda} \in \mathrm{Ess}(A^*)$. By applying Theorem 4.3.9 of LOTS and using the fact that $\mathcal{H}$ is reflexive, one may deduce that $\lambda \in \mathrm{Ess}(A)$; a more explicit proof is given in Theorem 4.11.

Finally, suppose that λ is an eigenvalue of A and that $\mathcal{L}$ is the corresponding eigenspace. The fact that $AU = UA$ implies that $U(\mathcal{L}) = \mathcal{L}$. If $0 \neq f \in \mathcal{L}$ then $f_n = U^n f \in \mathcal{L}$ is a sequence of vectors with $\|f_n\| = \|f\| \neq 0$ for all n, and f_n converges weakly to 0. This can only happen if $\mathcal{L}$ is infinite-dimensional. $\square$

The spectrum of a periodic operator can often be determined by using the Bloch decomposition, which applies Fourier analysis methods to the abelian group of translations which commute with the operator.

For the remainder of this section we restrict attention to periodic tridiagonal matrices. The restriction to one space dimension is justified for two reasons. The first is that it gives an insight into the higher dimensional theory while avoiding the notational complexity of the latter. The second is that there is much recent interest in quantum wires and the more general quantum graphs, which are well approximated by one-dimensional systems; even more recently optical communication devices and the nascent field of optical computers involve understanding the passage of light along narrow channels.

The statement and proof of Theorem 15.5 below can be extended to periodic operators with finite range acting on $\ell^2(\mathbf{Z}^N, \mathcal{K})$; we avoid writing down the more complicated formulae that this involves. The proof of the theorem uses a general technique for reducing the range of an operator at the cost of increasing the dimension of the auxiliary space $\mathcal{K}$.

Theorem 15.5 *Let A be the bounded operator that acts on $\ell^2(\mathbf{Z}, \mathcal{K})$ according to the formula*

$$(Af)_n = a_n f_{n-1} + b_n f_n + c_n f_{n+1}$$

where a_n, b_n, $c_n \in \mathcal{L}(\mathcal{K})$. *If* a_n, b_n, c_n *are all periodic with period* p *then*

$$\mathrm{Spec}(A) = \mathrm{Ess}(A) = \bigcup_{\theta\in[-\pi,\pi]} \mathrm{Spec}\,(M_\theta) \tag{15.2}$$

where M_θ *is the* $p \times p$ $\mathcal{L}(\mathcal{K})$*-valued matrix*

$$(M_\theta)_{r,s} = \mathrm{e}^{i\theta} K_{-1} + K_0 + \mathrm{e}^{-i\theta} K_1$$

and

$$\begin{aligned}
(K_0)_{r,s} &= \begin{cases} a_r & \text{if } r = s+1, \\ b_r & \text{if } r = s, \\ c_r & \text{if } r = s-1, \end{cases} \\
(K_1)_{r,s} &= \begin{cases} a_1 & \text{if } r = 1 \text{ and } s = p, \\ 0 & \text{otherwise}, \end{cases} \\
(K_{-1})_{r,s} &= \begin{cases} c_p & \text{if } r = p \text{ and } s = 1, \\ 0 & \text{otherwise}, \end{cases}
\end{aligned}$$

where $1 \leq r \leq p$ *and* $1 \leq s \leq p$ *throughout.*

Proof We define the unitary operator $U : \ell^2(\mathbf{Z}, \mathcal{K}) \to \ell^2(\mathbf{Z}, \mathcal{K}^p)$ by

$$(Uf)_{n,r} = f_{np+r}$$

for all $n \in \mathbf{Z}$ and $1 \leq r \leq p$. Then A has the same spectrum as $B = UAU^{-1}$, where B is a translation invariant operator on $\ell^2(\mathbf{Z}, \mathcal{K}^p)$. Indeed $B_{r,s} = K_{r-s}$ for all r, $s \in \mathbf{Z}$ where K_t are the $p \times p$ matrices defined above for $t = 0, \pm 1$ and we put $K_t = 0$ if $|t| \geq 2$. The proof is completed by using Fourier series methods to represent B as a matrix-valued multiplication operator on $L^2([-\pi, \pi], \mathcal{K}^p, \mathrm{d}\theta)$ as in Theorem 2.3 and then using Lemma 8.3. □

Theorem 15.6 *Let* A *be as in Theorem 15.5. Then* $\lambda \in \mathrm{Spec}(A)$ *if and only if there is a bounded function* $f : \mathbf{Z} \to \mathcal{K}$ *such that* $f_{n+p} = \mathrm{e}^{i\theta} f_n$ *for some* $\theta \in \mathbf{R}$ *and all* $n \in \mathbf{Z}$ *and* $Af = \lambda f$ *pointwise.*

Proof If a function f with the stated properties exists than $\lambda \in \mathrm{Spec}(A)$ by Theorem 15.3.

Conversely if $\lambda \in \mathrm{Spec}(A)$ then $\lambda \in \mathrm{Spec}(M_\theta)$ for some θ by Theorem 15.5. Let $\phi \in \mathcal{K}^p$ be a corresponding eigenvector. If $f : \mathbf{Z} \to \mathcal{K}$ is the unique function such that $f_{n+p} = \mathrm{e}^{i\theta} f_n$ for all $n \in \mathbf{Z}$ and $f_r = \phi_r$ for $1 \leq r \leq p$, then a direct calculation shows that $Af = \lambda f$. □

Theorem 15.7 *Let* A *be as in Theorem 15.5. Then* $\lambda \in \mathrm{Spec}(A)$ *if and only if there is a sequence* $h_n \in \ell^2(\mathbf{Z})$*, each term of which is a function of finite support and norm* 1*, and*

$$\lim_{n\to\infty} \|Ah_n - \lambda h_n\| = 0. \tag{15.3}$$

One can also require that $\mathrm{supp}(h_n) \subseteq [n, \infty)$ *or* $\mathrm{supp}(h_n) \subseteq (-\infty, n]$ *for every* n.

Proof If $\lambda \in \mathrm{Spec}(A)$ then $\lambda \in \mathrm{Spec}(M_\theta)$ for some $\theta \in [-\pi, \pi]$ by (15.2). The proof of Lemma 8.3 yields a sequence $f_n \in \ell^2(\mathbf{Z}, \mathcal{K})$ such that $\|f_n\| = 1$ and $\lim_{n\to\infty} \|Af_n - \lambda f_n\| = 0$. By truncating each f_n far enough away from 0, one obtains a sequence $g_n \in \ell^2(\mathbf{Z}, \mathcal{K})$, each term of which has finite support, and such that $\|g_n\| = 1$ and $\lim_{n\to\infty} \|Ag_n - \lambda g_n\| = 0$. Finally one may translate g_n by a distance that is a multiple of p to obtain a sequence $h_n \in \ell^2(\mathbf{Z}, \mathcal{K})$ such that $\mathrm{supp}(h_n) \subseteq [n, \infty)$ or $\mathrm{supp}(h_n) \subseteq [-\infty, n)$ for every n. Since A is periodic with period p, $\|Ah_n - \lambda h_n\| = \|Ag_n - \lambda g_n\|$ for every n and (15.3) follows.

The converse statement of the theorem follows directly from Lemma 4.3.15 of LOTS. $\square$

Theorem 15.8 *Let A be as in Theorem 15.5, but with $\mathcal{K} = \mathbf{C}$. Then $\lambda \in \mathrm{Spec}(A)$ if and only if there exists $\theta \in [-\pi, \pi]$ such that*

$$\gamma e^{i\theta} - \beta_p(\lambda) + \alpha e^{-i\theta} = 0 \tag{15.4}$$

where $\alpha = a_1 a_2 \ldots a_p$, $\gamma = c_1 c_2 \ldots c_p$ and

$$\beta_p(\lambda) = \det(\lambda I - M_0) + \alpha + \gamma \tag{15.5}$$

is a monic polynomial of degree p in λ.

Equivalently $\lambda \in \mathrm{Spec}(A)$ if and only if one of the roots $z \in \mathbf{C}$ of the fundamental polynomial

$$q(z) = \gamma z^2 - \beta_p(\lambda) z + \alpha \tag{15.6}$$

satisfies $|z| = 1$.

Proof An examination of $-\det(\lambda I - M_\theta)$ shows that its dependence on θ is of the form written in (15.4); this also allows one to verify that α and γ are as claimed. The value of $\beta_p(\lambda)$ is determined by putting $\theta = 0$ in the formula for $-\det(\lambda I - M_\theta)$. The equation (15.6) follows by replacing $e^{i\theta}$ by z. $\square$

Lemma 15.9 *If*

$$q(z) = \gamma z^2 - \beta_p(\lambda) + \alpha \tag{15.7}$$

is the fundamental polynomial associated with $A - \lambda I$ as in Theorem 15.8, then the fundamental polynomial associated with $A^* - \overline{\lambda} I$ is

$$\widetilde{q}(z) = \overline{\alpha} z^2 - \overline{\beta_p(\lambda)} + \overline{\gamma}. \tag{15.8}$$

Therefore

$$\overline{\widetilde{q}(\overline{z})} = z^2 q(z^{-1}) \tag{15.9}$$

for all z, $\lambda \in \mathbf{C}$.

Proof

Let $\widetilde{M_\theta}$ be the matrices associated with A^*. An inspection of their coefficients shows that $\widetilde{M_\theta} = (M_\theta)^*$. This identity also follows immediately from the representation of A as a matrix-valued multiplication operator on $\ell^2([-\pi,\pi],\mathbf{C}^p,\mathrm{d}\theta)$; see the proof of Theorem 15.5. Therefore

$$\begin{aligned}\det(\overline{\lambda}I - \widetilde{M_\theta}) &= \det((\lambda I - M_\theta)^*)\\ &= \overline{\det(\lambda I - M_\theta)}\\ &= \overline{\gamma \mathrm{e}^{i\theta} - \beta_p(\lambda) + \alpha \mathrm{e}^{-i\theta}}\\ &= \overline{\alpha}\mathrm{e}^{i\theta} - \overline{\beta_p(\lambda)} + \overline{\gamma}\mathrm{e}^{-i\theta}.\end{aligned}$$

This proves (15.8), and (15.9) follows immediately. ☐

In the context of Theorem 15.5, (15.2) suggests that the spectrum of A might be the union of p closed curves, but the situation is more complicated than this because of possible crossings and degeneracies.

Problem 15.10 Let $A : \ell^2(\mathbf{Z}) \to \ell^2(\mathbf{Z})$ be the bounded operator defined by

$$(Af)_n = a_n f_{n-1} + c_n f_{n+1}$$

where $a_n = a_{n+2}$ and $c_n = c_{n+2}$ for all $n \in \mathbf{Z}$. Determine the spectrum of A and work out how many components it has. ☐

15.3 The index of a Toeplitz operator

Theorem 4.4.2 of LOTS presented a simple version of theorem about the index of a Toeplitz operator acting on $\ell^2(\mathbf{N})$, and we need to obtain similar results for matrix-valued Toeplitz operators. In the next two sections we carry out the necessary calculations from first principles for the operator of interest; see Theorem 15.16 below. In this section we put the problem in a more general context and state the relevant theorem without proof.[8] Let $\mathcal{K}$ be a finite-dimensional Hilbert space and let P be the orthogonal projection of $L^2([-\pi,\pi],\mathcal{K})$ onto the Hardy subspace H^2 consisting of all functions $f \in L^2([-\pi,\pi],\mathcal{K})$ whose Fourier coefficients f_n vanish for all $n < 0$. Also let $M : [-\pi,\pi] \to \mathcal{L}(\mathcal{K})$ be a continuous periodic function and let B be the bounded multiplication operator on $L^2([-\pi,\pi],\mathcal{K})$, defined by $(Bf)(\theta) = M_\theta f(\theta)$. According to Theorem 8.3

$$\mathrm{Spec}(B) = \mathrm{Ess}(B) = \bigcup_{-\pi\le\theta\le\pi} \mathrm{Spec}(M_\theta).$$

[8] Comprehensive accounts of the theory of Toeplitz operators may be found in A. Böttcher and S. M. Grudsky, Spectral Properties of Banded Toeplitz Matrices, SIAM, 2005, A. Böttcher and B. Silbermann, Introduction to Large Truncated Toeplitz Matrices. Springer, New York, 1999, A. Böttcher and B. Silbermann, Analysis of Toeplitz Operators, Springer Monographs in Mathematics, second edition, 2010, and I. Gohberg and I. A. Feldman, Convolution Equations and Projection Methods for Their Solution. Amer. Math. Soc. Providence, RI, 1974.

The Toeplitz operator associated with B is defined by $T_B f = PBf$, where f and $T_B f$ lie in H^2.

Theorem 15.11[9] *The essential spectrum of* T_B *equals that of* B. *If* $\lambda \notin \mathrm{Ess}(T_B)$ *then* $\mathrm{Ind}(T_B - \lambda I)$ *equals minus the winding number around the origin of the function* $\delta : [-\pi, \pi] \to \mathbf{C}$ *defined by* $\delta(\theta) = \det(M_\theta - \lambda I)$.

In order to translate this into our terminology, let $\mathcal{F} : L^2([-\pi, \pi], \mathcal{K}) \to \ell^2(\mathbf{Z}, \mathcal{K})$ be the unitary operator associated with the Fourier series expansion. Then $\mathcal{F}(H^2) = \ell^2(\mathbf{N} \cup \{0\}, \mathcal{K})$ and $A_+ = \mathcal{F} T_B \mathcal{F}^{-1}$ is the truncation of the operator $A = \mathcal{F} B \mathcal{F}^{-1}$ to $\ell^2(\mathbf{N} \cup \{0\}, \mathcal{K})$. The operator A always commutes with translations in $\ell^2(\mathbf{Z}, \mathcal{K})$, but it need not be convolution by a function in ℓ^1, let alone by a function of finite support.

15.4 Transfer matrices

In this section we provide a new derivation of the formula (15.6) for the fundamental polynomial by means of the theory of transfer matrices. Although the assumptions are more stringent, the new method also allows one to determine the asymptotic forms at $\pm\infty$ of the solutions of $Af = \lambda f$ for any $\lambda \in \mathbf{C}$. The theory here may be extended to block tridiagonal matrices.[10]

We assume that $(Af)_n = a_n f_{n-1} + b_n f_n + c_n f_{n+1}$ as before. If a_n and c_n are non-zero for all $n \in \mathbf{Z}$, then the solution space of the equation

$$a_n f_{n-1} + b_n f_n + c_n f_{n+1} = \lambda f_n \tag{15.10}$$

is two-dimensional for every $\lambda \in \mathbf{C}$. The spectral character of λ is determined by the asymptotic behaviour of these solutions.

The recurrence relation (15.10) can be rewritten in the form

$$\begin{aligned} \begin{pmatrix} f_n \\ f_{n+1} \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ -a_n/c_n & (\lambda - b_n)/c_n \end{pmatrix} \begin{pmatrix} f_{n-1} \\ f_n \end{pmatrix} \\ &= X_n \begin{pmatrix} f_{n-1} \\ f_n \end{pmatrix} \\ &= T_n \begin{pmatrix} f_0 \\ f_1 \end{pmatrix} \end{aligned}$$

where $T_n = X_n X_{n-1} \dots X_1$.

[9] See Theorem 6.5 in Böttcher and Silbermann, 1999, or Theorem 1.9 of Böttcher and Grudsky, 2005.

[10] See D K Salkuyeh, Comments on "A note on a three-term recurrence for a tridiagonal matrix", Appl. Math. Comp. 176 (2006) 442-444; T Sogabe, On a two-term recurrence for the determinant of a general matrix, Appl. Math. Comp. 187 (2007) 785-788; L G Molinari, Determinants of block tridiagonal matrices, Linear Alg. Appl. 429 (2008) 2221-2226.

The following theorem should be compared with Theorem 15.8.

Theorem 15.12 *Let A be a bounded operator on $\ell^2(\mathbf{Z})$ defined by*

$$(Af)_n = a_n f_{n-1} + b_n f_n + c_n f_{n+1}$$

where a_n and c_n are non-zero for all $n \in \mathbf{Z}$. If A is periodic with period p then the equation $Af = \lambda f$ has a solution $f \in \ell^\infty(\mathbf{Z})$ if and only if one of the solutions z of

$$\gamma z^2 - \tau_p(\lambda) z + \alpha = 0 \tag{15.11}$$

satisfies $|z| = 1$, where $\alpha = a_1 a_2 \ldots a_p$, $\gamma = c_1 c_2 \ldots c_p$ and $\tau_p(\lambda) = \gamma\, \mathrm{tr}(T_p)$ is a monic polynomial of degree p in λ. The following cases arise.

1. *The equation (15.11) has a root with modulus 1. This happens if and only if $\lambda \in \mathrm{Spec}(A) = \mathrm{Ess}(A)$.*
2. *We write $\lambda \in W_2(A)$ if both solutions of (15.11) satisfy $|z| < 1$. For such λ all non-zero solutions f of $Af = \lambda f$ decay exponentially as $n \to +\infty$ and grow exponentially as $n \to -\infty$.*
3. *We write $\lambda \in W_0(A)$ if both solutions of (15.11) satisfy $|z| > 1$. For such λ all non-zero solutions f of $Af = \lambda f$ grow exponentially as $n \to +\infty$ and decay exponentially as $n \to -\infty$.*
4. *We write $\lambda \in W_1(A)$ if one solution of (15.11) satisfies $|z| < 1$ and the other satisfies $|z| > 1$. For such λ one solution f of $Af = \lambda f$ decays exponentially as $n \to +\infty$ and grows exponentially as $n \to -\infty$, another grows exponentially as $n \to +\infty$ and decays exponentially as $n \to -\infty$, and all other non-zero solutions grow exponentially as $n \to \pm\infty$.*

The sets $W_0(A)$, $W_1(A)$ and $W_2(A)$ are disjoint and open and their union is $\mathbf{C}\backslash\mathrm{Spec}(A)$.

Proof The asymptotic behaviour as $n \to \pm\infty$ of the solutions of $Af = \lambda f$ are determined by the behaviour of $(T_p)^m$ as $m \to \pm\infty$. This is turn is determined by the eigenvalues of T_p, which are the solutions z of (15.11) because

$$\det(T_p) = \prod_{r=1}^{p} \det(X_r) = \frac{\alpha}{\gamma}$$

and $\tau_p(\lambda) = \gamma\, \mathrm{tr}(T_p)$. It is easy to verify that $\tau_p(\lambda)$ is of the stated form.

Cases 1 follows directly from Theorems 15.5 and 15.6.

Cases 2 to 4 are easy to prove if T_p is diagonalizable. The idea is the same in all cases. Each root z of (15.11) is an eigenvalue of T_p and is associated with an eigenvector of T_p. If we denote this by (f_0, f_1) then the recurrence relation may be solved for this initial condition, and the resulting function $f : \mathbf{Z} \to \mathbf{C}$ satisfies $Af = \lambda f$ and $f_{n+p} = zf_n$ for all $n \in \mathbf{Z}$. Its asymptotic form is therefore determined by the value of z. If T_p is not diagonalizable one obtains similar conclusions by using its Jordan canonical form.

The sets $W_r(A)$ are obviously disjoint with the stated union, but we have to prove that they are open. This follows from the fact that the roots of any polynomial depend continuously on its coefficients; this is a special case of a theorem about the zeros of analytic functions. In this case the coefficients are polynomials in λ. $\square$

Corollary 15.13 *Let A be as in Theorem 15.12 and let U be a connected component of $\mathbf{C}\backslash\mathrm{Spec}(A)$. Then there exist $r \in \{0, 1, 2\}$ such that $U \subseteq W_r(A)$. The value of r can be determined by solving (15.11) for any single point in U.*

Proof Because U is connected exactly one of the intersections on the right hand side of the identity

$$U = (U \cap W_0(A)) \cup (U \cap W_1(A)) \cup (U \cap W_2(A))$$

must be non-empty. If $U \cap W_r(A) \neq \emptyset$ then $U = U \cap W_r(A)$, so $U \subseteq W_r(A)$. The final statement follows directly. $\square$

An analogue of the following theorem for block triadiagonal matrices has been proved by Molinari,[11] who calls the identity of two closely related polynomials a duality relation.

Theorem 15.14 *The function $\beta_p(\lambda)$ defined in Theorem 15.8 coincides with the function $\tau_p(\lambda)$ defined in Theorem 15.12.*

Proof If $p = 2$ direct computations of both polynomials establish that

$$\beta_2(\lambda) = \tau_2(\lambda) = (\lambda - b_1)(\lambda - b_2) - a_1 c_2 - a_2 c_1$$

but for general p we adopt a more indirect approach.

It follows from the two stated theorems that the two polynomials

$$\begin{aligned} q(\lambda) &= \beta_p(\lambda) - \alpha - \gamma = \det(\lambda I - M_0), \\ \widetilde{q}(\lambda) &= \tau_p(\lambda) - \alpha - \gamma = \gamma \, \mathrm{tr}(T_p) - \alpha - \gamma, \end{aligned}$$

vanish if and only if $Af = \lambda f$ has a solution satisfying $f_{n+p} = f_n$ for all $n \in \mathbf{Z}$. Since q and $\widetilde{q}$ are both monic polynomials of degree p, it follows that they must coincide provided the roots of q are all distinct.

We deal with the case in which q has repeated roots by a perturbation argument. We fix a_r, c_r for all $r \in \{1, \ldots, p\}$ and make explicit the dependence of $q(\lambda)$ on

[11] L G Molinari, Determinants of block tridiagonal matrices, Linear Alg. Appl. 429 (2008) 2221-2226, (arXiv:0712.0681v3).

$b = (b_1, \ldots, b_p)$. If we put $\widehat{b}_r = rN$ for $1 \le r \le p$ then for all large enough N the roots of $q_{\widehat{b}}$ are close to rN and are therefore all distinct. The set of $t \in \mathbf{R}$ for which the roots of $q_{(1-t)b+t\widehat{b}}$ are all distinct is a dense open subset of $\mathbf{R}$ by Theorem 1.9. Since $q = \widetilde{q}$ for all such t a limiting argument implies that they are also equal for $t = 0$. □

Theorem 15.15 *Let A be a periodic finite range operator acting on $\ell^2(\mathbf{Z}) = \ell^2(\mathbf{M}) \oplus \ell^2(\mathbf{N})$ where $\mathbf{N}$ is the set of natural numbers and $\mathbf{M} = \mathbf{Z}\backslash\mathbf{N}$. Let A have the matrix form $= \begin{pmatrix} A_- & C \\ D & A_+ \end{pmatrix}$ where $A_- : \ell^2(\mathbf{M}) \to \ell^2(\mathbf{M})$, $A_+ : \ell^2(\mathbf{N}) \to \ell^2(\mathbf{N})$ and C, D are finite rank matrices. Then*

$$\mathrm{Ess}(A) = \mathrm{Ess}(A_+) = \mathrm{Ess}(A_-).$$

Proof The method of Corollary 4.3.8 of LOTS yields

$$\mathrm{Ess}(A) = \mathrm{Ess}\begin{pmatrix} A_+ & 0 \\ 0 & A_- \end{pmatrix} = \mathrm{Ess}(A_+) \cup \mathrm{Ess}(A_-).$$

We next prove that $\mathrm{Ess}(A) \subseteq \mathrm{Ess}(A_+)$. If $\lambda \in \mathrm{Ess}(A)$ then Theorem 15.7 yields a sequence $h_n \in \ell^2(\mathbf{Z})$ satisfying $\mathrm{supp}(h_n) \subset [n, \infty)$ and $\|h_n\| = 1$ and $\lim_{n\to\infty} \|Ah_n - \lambda h_n\| = 0$. This implies that $h_n \in \ell^2(\mathbf{N})$ and that h_n converges weakly to 0 as $n \to \infty$ and $\lim_{n\to\infty} \|A_+ h_n - \lambda h_n\| = 0$. This implies that $\lambda \in \mathrm{Ess}(A_+)$ by Theorem 4.3.15 of LOTS.

The proof that $\mathrm{Ess}(A) \subseteq \mathrm{Ess}(A_-)$ is similar. □

The following theorem is a closely related to a similar result for Toeplitz operators and has an analogue for constant coefficient differential operators on the half-line.[12] The case in which some of the a_n and c_n vanish is deduced from the following 'regular' version of the theorem.

Theorem 15.16 *Let A be a bounded operator on $\ell^2(\mathbf{Z})$ defined by*

$$(Af)_n = a_n f_{n-1} + b_n f_n + c_n f_{n+1}$$

where a_n and c_n are non-zero for all $n \in \mathbf{Z}$. Suppose that A is periodic with period p and that $A_\pm$ are defined as in Theorem 15.15. Given $\lambda \in \mathbf{C}$ let z_1, z_2 be the two roots of the fundamental polynomial (15.11). The following cases cover all possible values of λ.

1. *$\lambda \in \mathrm{Ess}(A_\pm)$ if and only if one of the roots z_r has absolute value* 1.
2. *If $|z_1| < 1$ and $|z_2| < 1$ then $A_\pm - \lambda I$ are Fredholm operators with* $\mathrm{Ind}(A_+ - \lambda I) = 1$ *and* $\mathrm{Ind}(A_- - \lambda I) = -1$.
3. *If $|z_1| > 1$ and $|z_2| > 1$ then $A_\pm - \lambda I$ are Fredholm operators with* $\mathrm{Ind}(A_+ - \lambda I) = -1$ *and* $\mathrm{Ind}(A_- - \lambda I) = 1$.

[12] See Corollary 7.4(iii) of D. E. Edmunds and W. D. Evans, Spectral Theory and Differential Operators, OUP, 1987.

4. If $|z_1| < 1$ and $|z_2| > 1$ then $A_\pm - \lambda I$ are Fredholm operators with $\mathrm{Ind}(A_+ - \lambda I) = \mathrm{Ind}(A_- - \lambda I) = 0.$

Proof For most of the proof we only consider the operator A_+. The results for A_- follow by a simple trick explained at the end. The index of $A_+ - \lambda I$ is calculated by using the formula

$$\begin{aligned}\mathrm{Ind}(A_+ - \lambda I) &= \dim(\mathrm{Ker}(A_+ - \lambda I)) - \dim(\mathrm{Coker}(A_+ - \lambda I)) \\ &= \dim(\mathrm{Ker}(A_+ - \lambda I)) - \dim(\mathrm{Ker}(A_+^* - \overline{\lambda} I)).\end{aligned}$$

This reduces the problem to finding the dimensions of certain eigenspaces.

Case 1. Theorem 15.12 case 1 implies that the existence of a root z such that $|z| = 1$ is equivalent to $\lambda \in \mathrm{Ess}(A)$. Theorem 15.15 then establishes the equivalence to $\lambda \in \mathrm{Ess}(A_\pm)$.

Case 2. The space of all solutions of $Af = \lambda f$ is two-dimensional, so there must exist a solution with $f_0 = 0$; this extra condition is equivalent to $A_+ f = \lambda f$. Any solution of $A_+ f = \lambda f$ is uniquely determined by the value of f_1 and it decays exponentially as $n \to +\infty$ by the hypothesis that $|z_1| < 1$ and $|z_2| < 1$. Therefore $\ker(A_+ - \lambda I)$ is one-dimensional.

Lemma 15.9 implies that the roots $\widetilde{z}_1$ and $\widetilde{z}_2$ of the fundamental polynomial associated with $A^* - \overline{\lambda} I$ satisfy $|\widetilde{z}_1| > 1$ and $|\widetilde{z}_2| > 1$. Therefore every solution of $A^* f = \overline{\lambda} f$ grows exponentially as $n \to +\infty$. Therefore $\ker(A_+^* - \overline{\lambda} I) = \{0\}$ and $\dim(\mathrm{Coker}(A_+ - \lambda I)) = 0$. Hence $\mathrm{Ind}(A_+ - \lambda I) = 1$.

Case 3. This is similar to Case 2, but with the roles of A and A^* interchanged.

Case 4. Lemma 15.9 implies that one of the roots $\widetilde{z}_1$ of the fundamental polynomial associated with $A^* - \overline{\lambda} I$ satisfies $|\widetilde{z}_1| < 1$ while the other satisfies $|\widetilde{z}_2| > 1$.

The assumptions imply that, up to multiplicative constants, there is only one solution of $Af = \lambda f$. The restriction of f to $\mathbf{N}$ is a solution of $A_+ f = \lambda f$ if and only if $f_0 = 0$. Therefore $\dim(\mathrm{Ker}(A_+ - \lambda I))$ equals 1 if $f_0 = 0$ and equals 0 otherwise. A similar argument applies to $A_+^* - \overline{\lambda} I$, the corresponding function being denoted $\widetilde{f}$. We have to deal with several cases.

If $f_0 \neq 0$ and $\widetilde{f}_0 \neq 0$ then $\dim(\mathrm{Ker}(A_+ - \lambda I)) = 0$ and $\dim(\mathrm{Ker}(A_+^* - \overline{\lambda} I)) = 0$, so $\mathrm{Ind}(A_+ - \lambda I) = 0$.

If $f_0 = 0$ then $\dim(\mathrm{Ker}(A_+ - \lambda I)) = 1$ so $\mathrm{Ind}(A_+ - \lambda I)$ equals 1 or 0. We now define A_{++} to be the restriction of A to $\ell^2(N_+)$ where $N_+ = \{n \in \mathbf{Z} : n \geq 2\}$. The assumption $f_0 = 0$ implies that $f_1 \neq 0$, because the recurrence relation is second order. Therefore λ is not an eigenvalue of A_{++} and $\mathrm{Ind}(A_{++})$ equals 0 or -1. Also $\mathrm{Ind}(A_{++}) = \mathrm{Ind}(A_+)$ by Theorem 4.1. Combining these facts yields $\mathrm{Ind}(A_+ - \lambda I) = 0$.

If $\widetilde{f}_0 \neq 0$ then a similar argument implies that $\mathrm{Ind}(A_+ - \lambda I) = 0$.

We finally explain how to obtain the claimed results for A_-. Since

$$A = A_+ + A_- + K$$

where K is finite rank and $A_\pm$ act independently on orthogonal subspaces, Corollary 4.3.8 of LOTS enables one to deduce that

$$\mathrm{Ind}(A - \lambda I) = \mathrm{Ind}(A_+ - \lambda I) + \mathrm{Ind}(A_- - \lambda I).$$

But $\mathrm{Ind}(A - \lambda I) = 0$ for all $\lambda \notin \mathrm{Ess}(A)$ by Theorem 15.4, so

$$\mathrm{Ind}(A_- - \lambda I) = -\mathrm{Ind}(A_+ - \lambda I)$$

for all such λ. □

If one or more of the coefficients a_n and c_n vanish one can obtain a result similar to that in Theorem 15.16 by two methods. The first involves modifying the method to take account of the new situation and the second is to calculate the required indexes by a limiting argument that uses Theorem 15.16. We give examples of both methods.

Theorem 15.17 *Suppose that $A : \ell^2(\mathbf{Z}) \to \ell^2(\mathbf{Z})$ is as in Theorem 15.16, except that $a_n = 0$ for at least one $n \in \mathbf{Z}$ and $c_m = 0$ for at least one $m \in \mathbf{Z}$. Then* $\mathrm{Spec}(A)$ *consists of a finite set S of eigenvalues, each of infinite multiplicity. Moreover,* $\mathrm{Ess}(A_\pm) = S$ *and* $\mathrm{Ind}(A_\pm - \lambda I) = 0$ *for all $\lambda \notin S$.*

Proof The new assumptions imply that $\alpha = \gamma = 0$, so the spectrum of M_θ, obtained from (15.4), does not depend on θ. If S is the finite set of eigenvalues of M_0 then $\mathrm{Spec}(A) = S$ by (15.2), and each $\lambda \in S$ is an eigenvalue of infinite multiplicity. The identity $\mathrm{Ess}(A_\pm) = S$ is a consequence of Theorem 15.15 and the vanishing of the index for all $\lambda \notin S$ follows from Theorem 4.3.18 of LOTS. □

The following theorem is of value because $\mathrm{Ind}(A_s - \lambda I)$ can be calculated for all $s \neq 0$ by using Theorem 15.16.

Theorem 15.18 *Suppose that $A : \ell^2(\mathbf{Z}) \to \ell^2(\mathbf{Z})$ is as in Theorem 15.16, but omitting the assumption that a_n and c_n are all non-zero. Given $s \in \mathbf{R}$ let A_s be the operator*

$$(A_s f)_n = a_{n,s} f_{n-1} + b_{n,s} f_n + c_{n,s} f_{n+1}$$

where $a_{n,s} = a_n$ unless $a_n = 0$, in which case $a_{n,s} = s$, $b_{n,s} = b_n$ for all $n \in \mathbf{Z}$, and $c_{n,s} = c_n$ unless $c_n = 0$, in which case $c_{n,s} = s$. Then A_s depends norm continuously on s as does $\mathrm{Spec}(A_s)$*, if one uses the Hausdorff metric for compact sets. If $\lambda \notin$* $\mathrm{Spec}(A)$ *then*

$$\mathrm{Ind}(A_{s,\pm} - \lambda I) = \mathrm{Ind}(A_\pm - \lambda I)$$

for all sufficiently small $s \in \mathbf{R}$.

Proof The norm continuity of A_s as a function of s follows directly from Lemma 15.1, but this is not sufficient to prove the continuous dependence of the spectrum on s; see Problem 15.19 below. Theorem 15.5 yields

$$\mathrm{Spec}(A_s) = \bigcup_{\theta \in [-\pi,\pi]} \mathrm{Spec}(M_{s,\theta})$$

where $M_{s,\theta}$ are $p \times p$ matrices that depend jointly continuously on s, θ. This implies the continuous dependence of $\mathrm{Spec}(A_s)$ on s.

If $\lambda \notin \mathrm{Spec}(A)$ then $\lambda \notin \mathrm{Ess}(A_{s,\pm})$ for all small enough s. Therefore $A_\pm - \lambda I$ and $A_{s,\pm} - \lambda I$ are Fredholm operators. They have equal indexes for small enough s by Theorem 4.3.11 of LOTS and the fact that $\lim_{s\to 0} \|A_{s,\pm} - A_\pm\| = 0$. □

Problem 15.19 Let $A_s : \ell^2(\mathbf{Z}) \to \ell^2(\mathbf{Z})$ be defined by $(A_s f)_n = c_{s,n} f_{n+1}$ for all $n \in \mathbf{Z}$ where $c_{s,n} = 1$ if $n \neq 0$ and $c_{s,0} = s$. Prove that $\mathrm{Spec}(A_s) = \{z : |z| = 1\}$ for all $s \neq 0$ and calculate $\mathrm{Spec}(A_0)$. □

15.5 Doubly periodic tridiagonal matrices

In this section we describe the spectra of infinite tridiagonal matrices that have different periodic structures on the right and left half-lines. Our main result, Theorem 15.20, can be extended to suitable operators on an infinite discrete graph X, under the assumption that the operator has a periodic structure on each of the infinite leads of X.[13] The key to the proof of our main result is the use of the stable spectrum, as defined in Section 4.5.

Theorem 15.20 *Let B be a tridiagonal operator acting on $\ell^2(\mathbf{Z})$ and satisfying*

$$B_{r,s} = \begin{cases} B_{1,r,s} & \text{if } r \geq a, \\ B_{2,r,s} & \text{if } r \leq -a, \end{cases}$$

for some $a > 0$, where B_1 and B_2 are periodic tridiagonal matrices. Suppose also that $B_{r,s} \neq 0$ for all r, s such that $|r - s| = 1$. Then

$$\mathrm{Stab}(B) = \mathrm{Ess}(B_1) \cup \mathrm{Ess}(B_2) \cup \bigcup_{m+n \neq 0} \{U_m(B_{1,+}) \cap U_n(B_{2,-})\}$$

where

$$U_n(X) = \{\lambda \in \mathbf{C} : X - \lambda I \text{ is Fredholm and } \mathrm{Ind}(X - \lambda I) = n\}$$

and $B_{1,+}$ denotes the restriction of B_1 to $\ell^2(\mathbf{Z} \cap [a, \infty))$ and $B_{2,-}$ denotes the restrictions of B_2 to $\ell^2(\mathbf{Z} \cap (-\infty, -a]))$.

[13] See E. B. Davies, Stable Spectrum of an Operator on an Infinite Discrete Graph, preprint, 2011.

Proof Since $B - B_{1,+} - B_{2,-}$ has finite rank, it follows that

$$\mathrm{Ess}(B) = \mathrm{Ess}(B_{1,+}) \cup \mathrm{Ess}(B_{2,-}) = \mathrm{Ess}(B_1) \cup \mathrm{Ess}(B_2). \tag{15.12}$$

The second equality in (15.12) is proved by applying Theorem 15.15 with A replaced successively by B_1 and B_2. The proof is then completed by applying Theorems 4.8, 15.16 and 15.18. $\square$