Fluctuation relations in non-equilibrium quantum steady states

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Mainly based on works with
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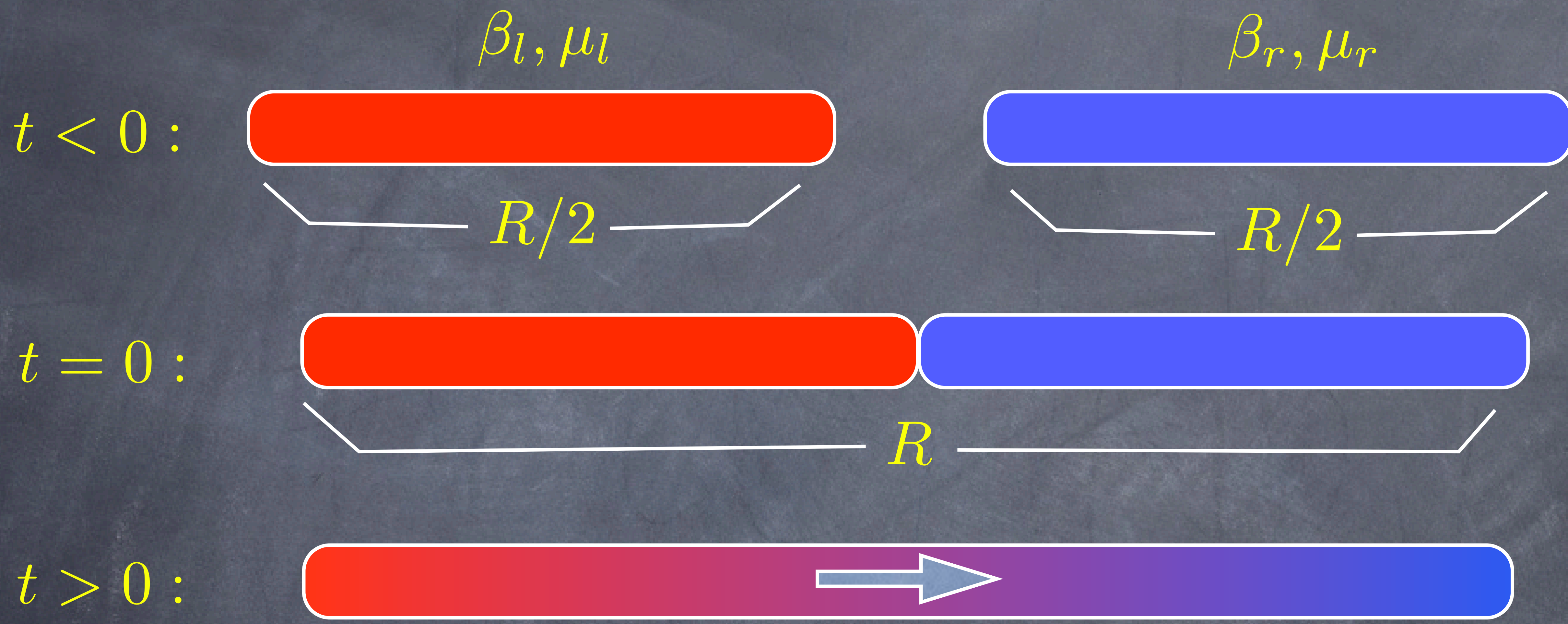
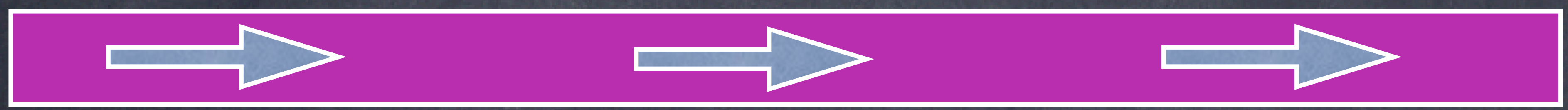
Constructing a homogeneous quantum steady state



$R \gg t v_F \gg$ observation lengths



Constructing a homogeneous quantum steady state (zero chemical potentials)

Steady state limit:

$$\langle \cdots \rangle_{\text{ness}} = \lim_{t\to\infty} \lim_{R\to\infty} \frac{\text{Tr}\left(e^{-iHt}\rho_0 e^{iHt}\cdots\right)}{\text{Tr}\left(\rho_0\right)}$$

↘ Observables supported on a finite region

Initially: $\rho_0 = e^{-\beta_l H_l - \beta_r H_r}$

Evolution Hamiltonian: $H = H_l + H_r + H_{\text{contact}}$

* From now on, assume limit R -> infinity already taken

Large-time cumulant generating function

$$F(\lambda) = \lim_{t\to\infty} \frac{1}{t} \log\langle e^{i\lambda E(t)} e^{-i\lambda E}\rangle_{\text{ness}}$$

charge transfer: [Levitov and Lesovik, 1993; Klich, Schonhamer, D. Bernard and BD, and others]

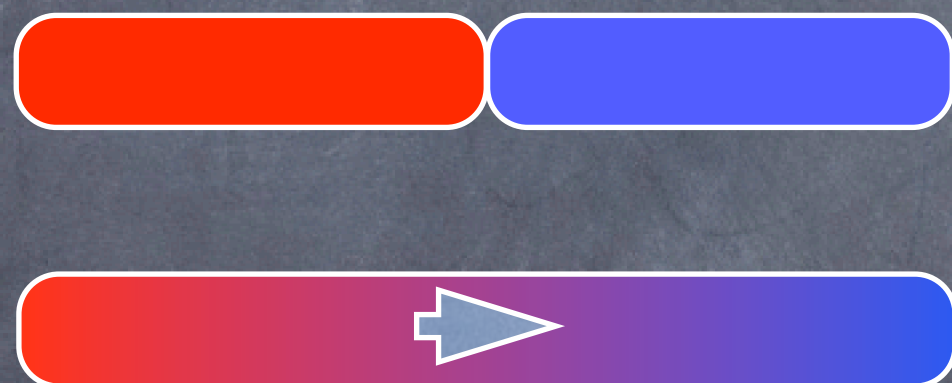
energy transfer in critical systems: [D. Bernard and BD, 2012 & 2013]

We measure energy transferred at large time:

$$E := \frac{1}{2}(H_l - H_r) \qquad E(t) = e^{iHt} E e^{-iHt}$$

$t = 0:$ quantum measure of E

$t > 0:$ quantum measure of E(t)



Deduce statistics of E(t)-E: cumulants diverge linearly in time, generating function:

$$F(\lambda) = i\lambda \underset{\uparrow\ \text{current}}{J} + \frac{(i\lambda)^2}{2} \underset{\nearrow\ \text{«noise»}}{N} + \ldots$$

Scattering formalism for the «non-equilibrium density matrix»

Non-equilibrium density matrix / scattering operator

$$S_+ := \lim_{t\to\infty} e^{-iHt} e^{i(H_l+H_r)t}$$

$$\langle \mathcal{O} \rangle_{\text{ness}} = \frac{\text{Tr}\left(\rho_{\text{ness}}\mathcal{O}\right)}{\text{Tr}\left(\rho_{\text{ness}}\right)} = \frac{\text{Tr}\left(\rho_0 S_+^{-1} \mathcal{O} S_+\right)}{\text{Tr}\left(\rho_0\right)}$$

Scattering operator intertwines between left-/right-Hamiltonians and their stationary correspondants

$$\rho_{\text{ness}} := S_+\, \rho_0\, S_+^{-1} = e^{-\beta_l H_l^+ - \beta_r H_r^+}$$

$$H_{l,r}^+ S_+ := S_+ H_{l,r}$$

Two possible conditions

- **Time-reversal symmetry:**

$$\tau H \tau^{-1} = H \qquad \tau H_{l,r} \tau^{-1} = H_{l,r} \qquad \tau \text{ anti-unitary}$$

Defining the opposite-time scattering operator

$$S_- := \lim_{t \to -\infty} e^{-iHt} e^{i(H_l + H_r)t} \qquad H^-_{l,r} S_- := S_- H_{l,r}$$

we immediately have

$$\tau S_\pm \tau^{-1} = S_\mp \qquad \tau H^\pm_{l,r} \tau^{-1} = H^\mp_{l,r}$$

- **Pure transmission:**

$$SE = -ES \qquad S := S_-^{-1} S_+$$

Main results

[D. Bernard and BD, 2013]

I. If there is time-reversal symmetry, then standard Fluctuation Relations hold:

$$F(\lambda) = F(i(\beta_l - \beta_r) - \lambda)$$

see [C. Jarzynski and D. K. Wojcik, 2004] and review [Esposito, Harbola, Mukamel 2009]
classical: [Evans et al., Gallavoti and Cohen, Kurchan, Lebowitz and Spohn, ... 1993-]

As usual, this means an exponential decay of ratio of probabilities:

$$\frac{P_t(\Delta E)}{P_t(-\Delta E)} \sim e^{(\beta_l - \beta_r)\Delta E}$$

II. If there is pure transmission, then «Extended Fluctuation Relations» hold:

$$-i\frac{dF(\lambda)}{d\lambda} = J(\beta_l + i\lambda, \beta_r - i\lambda)$$

Equilibrium fluctuations => non-equilibrium fluctuations

Non-equilibrium current => non-equilibrium fluctuations

Non-equilibrium conductance = noise

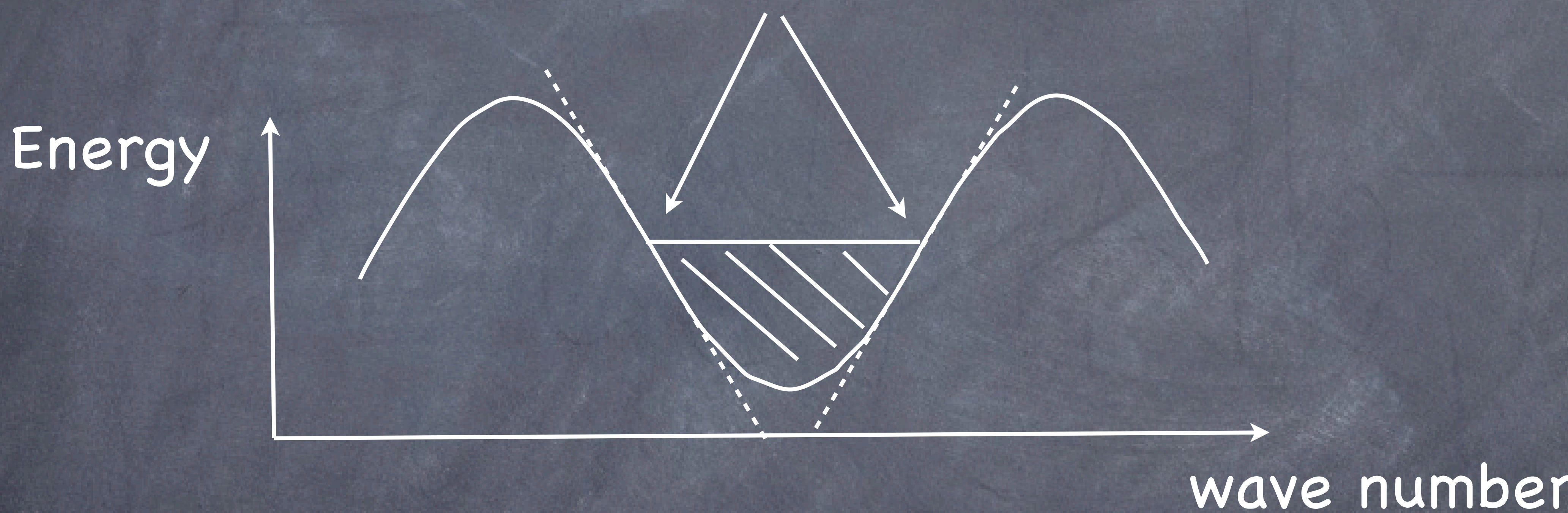
Examples

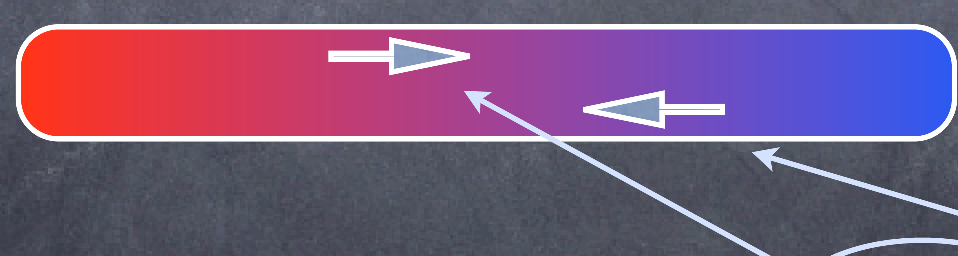
- Most physical systems of interest have time-reversal symmetry, including the well-known non-equilibrium Kondo and Anderson models describing quantum dots.

- There are also very important examples of models with pure transmission of energy!

(1) Low-energy universal regime of any **quantum critical chain** with unit dynamical exponent => 1+1 D Conformal Field Theory (CFT)

Linear dispersion relation,
separation between right- and left- movers





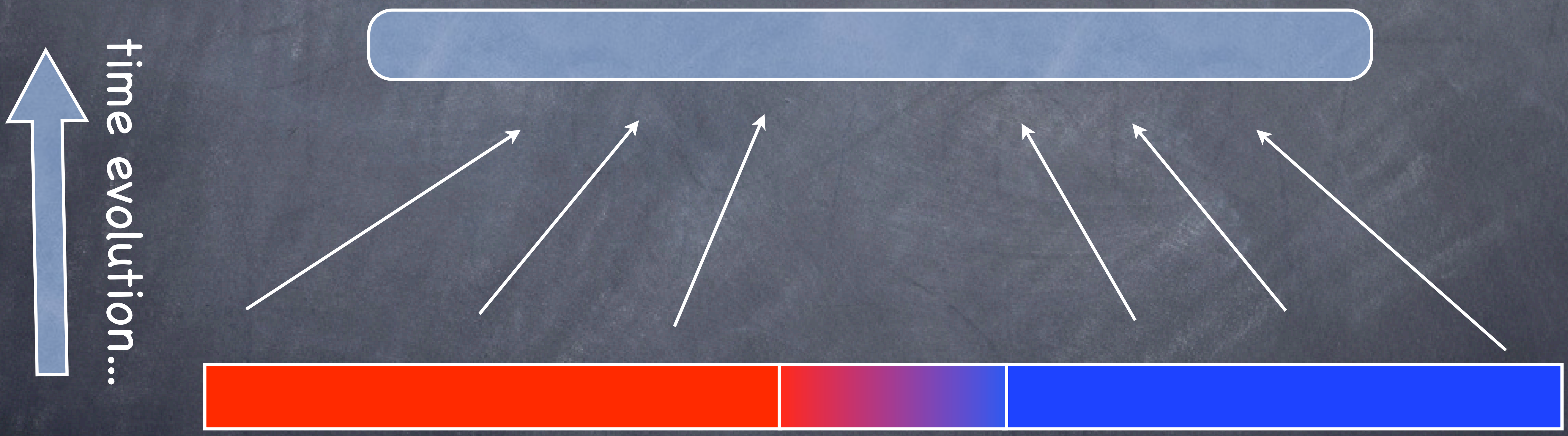
energy density: $h(x) = h_{\rightarrow}(x) + h_{\leftarrow}(x)$

momentum density: $p(x) = h_{\rightarrow}(x) - h_{\leftarrow}(x)$

[D. Bernard and BD, 2012]

Non-equilibrium density matrix: right-movers feel left temperature, and vice versa:

$$H^+_{l,r} = \text{energy of right-, left- movers} = \int_{-\infty}^{\infty} dx\, h_{\rightarrow,\leftarrow}(x)$$



$$H_{l,r} = \int_{x \lessgtr 0} dx\, h(x)$$

By dimensional analysis, left-right-mover separation, and a standard energy-density calculation, energy current is

$$J(\beta_l, \beta_r) = \frac{c\pi}{12\hbar}\left(\frac{1}{\beta_r^2} - \frac{1}{\beta_l^2}\right)$$

CFT central charge

[D. Bernard and BD, 2012]

numerics found agreement with this [C Karrasch, R. Ilan & J. E. Moore, 2012]

Left-right-mover separation immediately implies pure transmission (no reflection!), so that, by our result II,

$$F(\lambda) = \frac{c\pi}{12\hbar}\left(\frac{i\lambda}{\beta_r(\beta_r - i\lambda)} - \frac{i\lambda}{\beta_l(\beta_l + i\lambda)}\right)$$

First calculated from assumption of fluctuation relations [D. Bernard and BD, 2012]

Calculated using a local field construction of CFT [D. Bernard and BD, 2013]

Calculated using Virasoro algebra and extended to star-graph circuits [BD, M. Hoogeveen and D. Bernard, 2013]

Deduced from Extended Fluctuation Relations as done here [D. Bernard and BD, 2013]

(2) Any **integrable model**: scattering states represent «particles» that are subject to purely elastic scattering (Bethe ansatz), hence there is pure transmission

- The Ising quantum chain is a simple example:

$$H = -\frac{1}{2}\sum_j \left(\sigma_j^x \sigma_{j+1}^x + h\sigma_j^z\right)$$

Its «Bethe particles» are free fermions, with energies and momenta

$$\epsilon(\theta) = \sqrt{h^2 + 1 - 2h\cos\theta} \qquad p(\theta) = h\sin\theta \quad (\theta \in [-\pi, \pi])$$

Non-equilibrium density matrix: again, right-movers feel left temperature, and vice versa: [W. H. Aschbacher & C.-A. Pillet, 2003]

$$H^+_{l,r}|\theta\rangle = W_{l,r}(\theta)|\theta\rangle \qquad W_{l,r}(\theta) = \begin{cases} \epsilon(\theta) & \theta \gtrless 0 \\ 0 & \theta \lessgtr 0 \end{cases}$$

The current is then easy to calculate (free fermions!)

$$J(\beta_l,\beta_r) = \frac{h}{2\pi}\int d\theta\, \frac{h\sin\theta}{1+e^{-\beta_l W_l(\theta)-\beta_r W_r(\theta)}}$$

from which we immediately find the large-time cumulants,

$$F(\lambda) = \frac{h}{2\pi}\int d\theta\, \frac{h\sin\theta}{W_l(\theta)-W_r(\theta)}\log\left(1+e^{(\beta_l+i\lambda)W_l(\theta)+(\beta_r-i\lambda)W_r(\theta)}\right)$$

[A. De Luca, J. Viti, D. Bernard and BD 2013; D. Bernard and BD, 2013]
Also agreement in harmonic oscillator case with results of
[K. Saito and A. Dhar, 2007]

- Non-equilibrium density matrix for energy transfer known in any homogeneous Quantum Field Theory (QFT) [BD, 2012]

One can then deduce the cumulant generating function in any homogeneous Integrable QFT by calculating the non-equilibrium current using Thermodynamic Bethe Ansatz:

$$J(\beta_l, \beta_r) = \int d\theta \, p(\theta) \mu_{\beta_l, \beta_r}(\theta)$$

Thermodynamic Bethe Ansatz

[O. A. Castro Alvaredo, Y. Chen, BD and M. Hoogeveen, work in progress]

Main steps of proofs

Recall:

$$F(\lambda) = \lim_{t\to\infty} \frac{1}{t} \log\langle e^{i\lambda E(t)} e^{-i\lambda E}\rangle_{\text{ness}}$$

Differentiate:

«normalized»

$$-i\frac{dF(\lambda)}{d\lambda} = \lim_{t\to\infty} \frac{1}{t}\text{Tr}\left(\mathfrak{n}\left[e^{-i\lambda E}\,\rho_{\text{ness}}\,e^{i\lambda E(t)}\right]\int_0^t \hat{\jmath}(s)\,ds\right)$$

$$= \lim_{t\to\infty} \text{Tr}\left(\mathfrak{n}\left[e^{-i\lambda E(-t/2)}\,\rho_{\text{ness}}\,e^{i\lambda E(t/2)}\right]\hat{\jmath}(0)\right)$$

$$= \text{Tr}\left(\mathfrak{n}\left[e^{-i\lambda E^+}\,\rho_{\text{ness}}\,e^{i\lambda E^-}\right]\hat{\jmath}(0)\right)$$

then use time-reversal invariance or pure transmission, and

$$\rho_{\text{ness}} = e^{-\beta_l H_l^+ - \beta_r H_r^+} \qquad E^\pm = \frac{1}{2}(H_l^\pm - H_r^\pm)$$

Conclusion

- We have shown fluctuation relations in quantum non-equilibrium steady states built from a «bipartite» initial condition.
- Extended fluctuation relations are a powerful tool for exact calculations in CFT and in integrable models.
- Similar results hold for charge transfer, with exact universal results in CFT.
- Generalization to presence of impurities in integrable case (not homogeneous)?
- Generalization of such fluctuation relations to quantum circuits (ex. star-graph configuration)?