Quantum Mechanics III — Michaelmas Term 2009

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Homework 4 – due 15 December 2009

Angular momentum operators:

$$\hat{L}_x = \hat{y}\hat{p}_z - \hat{z}\hat{p}_y, \quad \hat{L}_y = \hat{z}\hat{p}_x - \hat{x}\hat{p}_z, \quad \hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x. \tag{1}$$

Abstract angular momentum algebra:

$$[\hat{J}_p, \hat{J}_q] = i\hbar\epsilon_{pqr}\hat{J}_r \tag{2}$$

(summation over repeated indices implied). Module construction: $\hat{J}_\pm = \hat{J}_x \pm i\hat{J}_y$, states $|jm\rangle$, $m \in \{-j, -j+1, \ldots, j\}$, $j = 0, 1/2, 1, 3/2, \ldots$ with

$$(\hat{J}_x^2 + \hat{J}_y^2 + \hat{J}_z^2)|jm\rangle = \hbar^2 j(j+1)|jm\rangle, \quad \hat{J}_z|jm\rangle = \hbar m|jm\rangle, \tag{3}$$
$$\hat{J}_+|jm\rangle = \hbar\sqrt{(j+m+1)(j-m)}|j, m+1\rangle, \quad \hat{J}_-|jm\rangle = \hbar\sqrt{(j-m+1)(j+m)}|j, m-1\rangle$$

1. Calculate the commutators $[\hat{L}_z, \hat{x}^2]$, $[\hat{L}_z, \hat{y}^2]$ and $[\hat{L}_z, \hat{z}^2]$, and deduce the commutator $[\hat{L}_z, \hat{x}^2 + \hat{y}^2 + \hat{z}^2]$. What is the geometric interpretation of the latter result?

2. (a) A representation is a linear map from abstract algebra elements to matrices, in such a way that the algebra relations are satisfied by the matrices under the usual matrix operations. Using completeness relations in fixed-j subspaces, show that the matrices M_q, $q = x, y, z$ whose matrix elements are given by $(M_q)_{mm'} = \langle jm|\hat{J}_q|jm'\rangle$ for a fixed j, form a representation of the abstract angular-momentum algebra – or $SU(2)$ algebra – defined by (2). These are called “spin-j representations”.

   (b) Construct the matrices in the spin-1/2 representation of $\hat{J}_x$, $\hat{J}_y$, $\hat{J}_z$.

3. An electron has a spin of 1/2, and the average of the z-component of its spin is $\hbar/2$. What normalised vector describes its state? If a measurement of the x-component of the spin is measured, what are the possible values that can be obtained? Calculate the probability of measuring a positive value.

Answers

1. First calculate

$$\begin{aligned}
[\hat{L}_z, \hat{x}] &= [\hat{x}\hat{p}_y - \hat{y}\hat{p}_x, \hat{x}] \\
&= -[\hat{y}\hat{p}_x, \hat{x}] \qquad \text{because both } \hat{x} \text{ and } \hat{p}_y \text{ commute with } \hat{x} \\
&= -\hat{y}[\hat{p}_x, \hat{x}] \qquad \text{because } \hat{y} \text{ commutes with } \hat{x} \\
&= -\hat{y}(-i\hbar) \\
&= i\hbar\hat{y}
\end{aligned} \tag{4}$$

then

$$\begin{aligned}
[\hat{L}_z, \hat{y}] &= [\hat{x}\hat{p}_y - \hat{y}\hat{p}_x, \hat{y}] \\
&= [\hat{x}\hat{p}_y, \hat{y}] \qquad \text{because both } \hat{y} \text{ and } \hat{p}_x \text{ commute with } \hat{y} \\
&= \hat{x}[\hat{p}_y, \hat{y}] \qquad \text{because } \hat{x} \text{ commutes with } \hat{y} \\
&= \hat{x}(-i\hbar) \\
&= -i\hbar\hat{x} \qquad (5)
\end{aligned}$$

and finally

$$\begin{aligned}
[\hat{L}_z, \hat{z}] &= [\hat{x}\hat{p}_y - \hat{y}\hat{p}_x, \hat{z}] \\
&= 0 \qquad \text{because } \hat{x},\ \hat{p}_y,\ \hat{y},\ \hat{p}_x \text{ all commute with } \hat{z} \qquad (6)
\end{aligned}$$

Then, we may evaluate the commutators we are looking for:

$$\begin{aligned}
[\hat{L}_z, \hat{x}^2] &= \hat{x}[\hat{L}_z, \hat{x}] + [\hat{L}_z, \hat{x}]\hat{x} \\
&= i\hbar(\hat{x}\hat{y} + \hat{y}\hat{x}) \\
&= 2i\hbar\hat{x}\hat{y} \qquad (7)
\end{aligned}$$

then

$$\begin{aligned}
[\hat{L}_z, \hat{y}^2] &= \hat{y}[\hat{L}_z, \hat{y}] + [\hat{L}_z, \hat{y}]\hat{y} \\
&= -i\hbar(\hat{y}\hat{x} + \hat{x}\hat{y}) \\
&= -2i\hbar\hat{x}\hat{y} \qquad (8)
\end{aligned}$$

and finally

$$\begin{aligned}
[\hat{L}_z, \hat{z}^2] &= \hat{z}[\hat{L}_z, \hat{z}] + [\hat{L}_z, \hat{z}]\hat{z} \\
&= 0 \qquad (9)
\end{aligned}$$

Hence, we find

$$[\hat{L}_z, \hat{x}^2 + \hat{y}^2 + \hat{z}^2] = 0 \qquad (10)$$

which simply means that the square of the length of the position vector is invariant under rotation with respect to the z axis, as it should.

2. (a) Let us evaluate $M_qM_r - M_rM_q$, where we have the matrix products for both orders of the matrices. We should find $i\hbar\epsilon_{qrs}M_s$ (summation over s implied), the matrix representation of $i\hbar\epsilon_{qrs}\hat{J}_s$. Let us look at the matrix element labelled by m, m', for both terms separately. We have, explicitly writing the matrix product,

$$\begin{aligned}
(M_qM_r)_{mm'} &= \sum_{m''=-j}^{j} (M_q)_{mm''}(M_r)_{m''m'} \\
&= \sum_{m''=-j}^{j} \langle jm|\hat{J}_q|jm''\rangle\langle jm''|\hat{J}_r|jm'\rangle \\
&= \langle jm|\hat{J}_q\hat{J}_r|jm'\rangle
\end{aligned}$$

where in the last step we used completeness on the j subspace,

$$\sum_{m''=-j}^{j} |jm''\rangle\langle jm''| = \mathbf{1}_j \tag{11}$$

(that is, this is 1 when acting on any vector in the j subspace – more precisely, it is a projector on the subspace with $\hat{J}^2$ eigenvalue j). Similarly, for the other term we have

$$(M_r M_q)_{mm'} = \langle jm|\hat{J}_r \hat{J}_q|jm'\rangle \tag{12}$$

Hence

$$\begin{aligned}(M_q M_r - M_r M_q)_{mm'} &= \langle jm|(\hat{J}_q\hat{J}_r - \hat{J}_r\hat{J}_q)|jm'\rangle \\ &= i\hbar\epsilon_{qrs}\langle jm|\hat{J}_s|jm'\rangle \\ &= i\hbar\epsilon_{qrs}(M_s)_{mm'}\end{aligned} \tag{13}$$

which shows that this is a representation of the angular-momentum algebra.

(b) We just need to evaluate explicitly the matrix elements using the known action of $\hat{J}_x$, $\hat{J}_y$, $\hat{J}_z$ on the vectors with $j = 1/2$, that is, the vectors $|1/2, -1/2\rangle$ and $|1/2, 1/2\rangle$. For simplicity, we will denote these vectors by $|+\rangle = |1/2, 1/2\rangle$ and $|-\rangle = |1/2, -1/2\rangle$; the first has a spin "up" in the z direction, and the second has a spin "down". We have

$$\begin{aligned}\hat{J}_x|-\rangle &= \frac{\hat{J}_+ + \hat{J}_-}{2}|-\rangle = \frac{1}{2}\hat{J}_+|-\rangle = \frac{\hbar}{2}|+\rangle \\ \hat{J}_x|+\rangle &= \frac{\hat{J}_+ + \hat{J}_-}{2}|-\rangle = \frac{1}{2}\hat{J}_-|+\rangle = \frac{\hbar}{2}|-\rangle \\ \hat{J}_y|-\rangle &= \frac{\hat{J}_+ - \hat{J}_-}{2i}|-\rangle = \frac{1}{2i}\hat{J}_+|-\rangle = -\frac{i\hbar}{2}|+\rangle \\ \hat{J}_y|+\rangle &= \frac{\hat{J}_+ - \hat{J}_-}{2i}|-\rangle = -\frac{1}{2i}\hat{J}_-|+\rangle = \frac{i\hbar}{2}|-\rangle \\ \hat{J}_z|-\rangle &= -\frac{\hbar}{2}|-\rangle \\ \hat{J}_z|+\rangle &= \frac{\hbar}{2}|+\rangle\end{aligned}$$

Hence, if we take the "standard" basis of column vectors:

$$|+\rangle \mapsto \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |-\rangle \mapsto \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{14}$$

the matrices in the spin-1/2 representation are

$$M_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad M_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad M_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{15}$$

Note that M_z is diagonal: our basis elements $|-\rangle$ and $|+\rangle$ are eigenvectors of $\hat{J}_z$.

3. Since the electron has a spin 1/2, this means that $j = 1/2$, so the maximum value the z component of its spin can have is $\hbar/2$. Since the average gives exactly this value, it must be a state with this value for sure, corresponding to the vector $|1/2, 1/2\rangle$. Hence the normalised vector representing its state is

$$|\psi\rangle = |+\rangle \tag{16}$$

(using the notation of the previous question), or, in matrix notation,

$$\psi = v^z_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{17}$$

For a measurement of the x component, the possibilities are the same as the measurements of the z component (or a measurement in any direction), that is $\hbar/2$ and $-\hbar/2$. This can be seen quite explicitly, by diagonalising the matrix M_x in order to find its eigenvectors and eigenvalues. It is simple to diagonalise: the eigenvectors $v^x_\pm$ and eigenvalues $\lambda_\pm$ are

$$v^x_+ = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \lambda_+ = \frac{\hbar}{2}; \quad v^x_- = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \lambda_- = -\frac{\hbar}{2} \tag{18}$$

The two eigenvalues are indeeed $\pm\hbar/2$. The probability of measuring a positive value is the probability of measuring $\hbar/2$. Hence, we need to take the absolute value squared of the overlap (the matrix product) between the dual of the eigenvector v^x_+ of M_x, with the state vector $\psi = v^z_+$. The dual vector of v^x_+ is just $(v^x_+)^\dagger$, that is, the transpose-and-complex-conjugate. Hence, we have

$$P(J_x = \hbar/2) = |(v^x_+)^\dagger v^z_+|^2 = \frac{1}{2}\left|\begin{pmatrix} 1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right|^2 = \frac{1}{2} \tag{19}$$

This means that if we know for sure that the z component is $\hbar/2$, then we have no idea what the x component may be!