Quantum Mechanics III | Michaelmas Term 2008

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Homework 2 – due 17 November 2008

1. Calculate the commutator
$$-i[\hat{X}^2, \hat{P}^2] \tag{1}$$
What observable does this correspond to? Explain using the quantisation conditions.

2. Given an observable $\hat{A}$, we can take the set of its independent eigenvectors as a basis of the Hilbert space. We may then define the operator $f(\hat{A})$, for any function $f$ from the spectrum of $\hat{A}$ to $\mathbb{C}$, by its action on all eigenvectors of $\hat{A}$:
$$f(\hat{A})|a\rangle = f(a)|a\rangle \tag{2}$$
for any eigenvector $|a\rangle$ with $\hat{A}$-eigenvalue $a$.

   (a) Show that with this definition, we have for $n$ a non-negative integer,
   $$f(\hat{A}) = \hat{A}\hat{A}...\hat{A} \ (n \text{ times}) \quad \text{if the function is} \quad f(a) = a^n. \tag{3}$$

   (b) Consider a unitary operator $U$. Show by directly taking the $n^{\text{th}}$ power of the observables that
   $$(U\hat{A}U^\dagger)^n = U\hat{A}^n U^\dagger \tag{4}$$

   (c) Show from the general definition that
   $$f(U\hat{A}U^\dagger) = Uf(\hat{A})U^\dagger \tag{5}$$

   (d) If $\hat{P}_a^{\hat{A}}$ is the projection operator on the $\hat{A}$-eigenspace of eigenvalue $a$, deduce that
   $$U\hat{P}_a^{\hat{A}}U^\dagger = \hat{P}_a^{U\hat{A}U^\dagger} \tag{6}$$

3. (a) Evaluate
   $$e^{-i\hat{P}x/\hbar}\,\hat{X}\,e^{i\hat{P}x/\hbar} \tag{7}$$
   where $x$ is some real number.

   (b) Consider a normalised state-vector $|\psi\rangle$. Is the vector $e^{i\hat{P}x/\hbar}|\psi\rangle$ still normalised? If $|\psi\rangle$ has average position $x_0$, what is the average position of $e^{i\hat{P}x/\hbar}|\psi\rangle$?

**Answers**

1. We must use repeatedly the general formulas $[\hat{A}\hat{B}, \hat{C}] = [\hat{A}, \hat{C}]\hat{B} + \hat{A}[\hat{B}, \hat{C}]$ and $[\hat{A}, \hat{B}\hat{C}] = [\hat{A}, \hat{B}]\hat{C} + \hat{B}[\hat{A}, \hat{C}]$, and we must use $[\hat{X}, \hat{P}] = i\hbar$. We have

$$\begin{aligned}
-i[\hat{X}^2, \hat{P}^2] &= -i([\hat{X}, \hat{P}^2]\hat{X} + \hat{X}[\hat{X}, \hat{P}^2]) \\
&= -i([\hat{X}, \hat{P}]\hat{P}\hat{X} + \hat{P}[\hat{X}, \hat{P}]\hat{X} + \hat{X}[\hat{X}, \hat{P}]\hat{P} + \hat{X}\hat{P}[\hat{X}, \hat{P}]) \\
&= -i(i\hbar\hat{P}\hat{X} + \hat{P}i\hbar\hat{X} + \hat{X}i\hbar\hat{P} + \hat{X}\hat{P}i\hbar) \\
&= 2\hbar(\hat{P}\hat{X} + \hat{X}\hat{P}) \qquad (8)
\end{aligned}$$

In order to identify the observable, we reason as follows: First, we know that the observable $\widehat{X^2}$, corresponding to the classical phase-space variable $X^2$, is just $\hat{X}^2$. Similarly, $\widehat{P^2} = \hat{P}^2$. But by the Dirac's quantisation conditions, we have that

$$[\widehat{X^2}, \widehat{P^2}] = i\hbar\widehat{\{X^2, P^2\}} \qquad (9)$$

and evaluating the Poisson brackets gives

$$\{X^2, P^2\} = \frac{\partial(X^2)}{\partial X}\frac{\partial(P^2)}{\partial P} - \frac{\partial(X^2)}{\partial P}\frac{\partial(P^2)}{\partial X} = 4XP \qquad (10)$$

Hence, we find that

$$-i[\hat{X}^2, \hat{P}^2] = -i[\widehat{X^2}, \widehat{P^2}] = \hbar\widehat{\{X^2, P^2\}} = 4\hbar\widehat{XP} \qquad (11)$$

Hence, $-i[\hat{X}^2, \hat{P}^2]$ is the observable corresponding to $4\hbar XP$. With (8), we thus find the observable corresponding to the product of classical phase-space variables $XP$ in a nice symmetric form:

$$\widehat{XP} = \frac{\hat{P}\hat{X} + \hat{X}\hat{P}}{2} \qquad (12)$$

2. (a) From the definition, if $f(a) = a^n$, then

$$f(\hat{A})|a\rangle = f(a)|a\rangle = a^n|a\rangle \qquad (13)$$

But the right-hand side of (3) is, by repeatedly using the eigenvalue equation $\hat{A}|a\rangle = a|a\rangle$,

$$\hat{A}\hat{A}...\hat{A}|a\rangle = \hat{A}\hat{A}...a|a\rangle = \cdots = \hat{A}a...a|a\rangle = aa\ldots a|a\rangle = a^n|a\rangle \qquad (14)$$

Hence both left- and right-hand sides are the same when they act on any eigenstate of $\hat{A}$. Since these states form a basis, this means that both sides are the same as operators on the Hilbert space.

(b) All we have to use is that if some given operator $U$ is unitary, then this means that $U^\dagger U = 1$. In order to directly take the $n^{\text{th}}$ power, perhaps the best proof is by induction. Clearly formula (4) is true for $n = 1$. Let us assume it is true for $n = m - 1$. Then

$$(U\hat{A}U^\dagger)^m = U\hat{A}U^\dagger(U\hat{A}U^\dagger)^{m-1} = U\hat{A}U^\dagger U\hat{A}^{m-1}U^\dagger = U\hat{A}\hat{A}^{m-1}U^\dagger = U\hat{A}^mU^\dagger \qquad (15)$$

hence formula (4) also holds for $n = m$. This completes the induction.

(c) In order to use the general definition of a function of an observable, we must act on some vector $|v\rangle$ with both the left-hand side and the right-hand side, and verify that we get the same result for any vector $|v\rangle$ in a given basis. First, recall that if $|a\rangle$ is eigenstate of $\hat{A}$ with eigenvalue $a$, then $U|a\rangle$ is eigenstate of $U\hat{A}U^\dagger$ with the same eigenvalue $a$; that is becasue $U\hat{A}U^\dagger U|a\rangle = U\hat{A}|a\rangle = Ua|a\rangle = aU|a\rangle$. So, for the vector $|v\rangle$ we may take $U|a\rangle$, and from the general definition, we have for the left-hand side of (5),

$$f(U\hat{A}U^\dagger)U|a\rangle = f(a)U|a\rangle \tag{16}$$

On the other hand, the right-hand side, acting on the same vector $U|a\rangle$, gives

$$Uf(\hat{A})U^\dagger U|a\rangle = Uf(\hat{A})|a\rangle = Uf(a)|a\rangle = f(a)U|a\rangle \tag{17}$$

Hence both are the same. Now the vectors $U|a\rangle$ themselves form a basis. Indeed, as we said they are eigenvectors of the observable $U\hat{A}U^\dagger$, but also they are *all* of its eigenvectors. That's because if $|a\rangle'$ is an eigenvector of $U\hat{A}U^\dagger$ with eigenvalue $a$, then $U^\dagger|a\rangle'$ is an eigenvector of $\hat{A}$ with the same eigenvalue (just reverse the process: $\hat{A}U^\dagger|a\rangle' = U^\dagger U\hat{A}U^\dagger|a\rangle' = U^\dagger a|a\rangle' = aU^\dagger|a\rangle'$). So for any $U\hat{A}U^\dagger$-eigenvector $|a\rangle'$, there is a $\hat{A}$-eigenvetor $|a\rangle$ such that $U^\dagger|a\rangle' = |a\rangle$, that is, $|a\rangle' = U|a\rangle$.

(d) Here the easiest way is to use the definition of the projection operator as follows[1]:

$$\hat{P}_a^{\hat{A}} = f_a(\hat{A}) \quad \text{where} \quad f_a(x) = \begin{cases} 1 & x = a \\ 0 & x \neq a \end{cases} \tag{18}$$

This indeed gives the fundamental property of the projection operator:

$$\hat{P}_a^{\hat{A}}|\tilde{a}\rangle = \begin{cases} |\tilde{a}\rangle & \tilde{a} = a \\ 0 & \tilde{a} \neq a \end{cases} \tag{19}$$

Hence we have

$$U\hat{P}_a^{\hat{A}}U^\dagger = Uf_a(\hat{A})U^\dagger = f_a(U\hat{A}U^\dagger) = \hat{P}_a^{U\hat{A}U^\dagger} \tag{20}$$

3. (a) Here, we may use the general formula

$$e^{\hat{A}}\hat{B}e^{-\hat{A}} = e^{[\hat{A},\cdot]}\hat{B} \equiv \hat{B} + [\hat{A},\hat{B}] + \frac{1}{2}[\hat{A},[\hat{A},\hat{B}]] + \frac{1}{3!}[\hat{A},[\hat{A},[\hat{A},\hat{B}]]] + \ldots \tag{21}$$

Happily, in the case at hand, only the first and second terms are non-zero, since $[\hat{P},\hat{X}] = -i\hbar$ which commutes with everything. Hence, we find

$$e^{-i\hat{P}x/\hbar}\,\hat{X}\,e^{i\hat{P}x/\hbar} = \hat{X} - x \tag{22}$$

[1] We take discrete eigenvalues – otherwise we would have needed the projection operator on an *interval* $[a,a']$ of eigenvalues, $\hat{P}_{[a,a']}^{\hat{A}}$.

(b) If $|\psi\rangle$ is normalised, this means that $\langle\psi|\psi\rangle = 1$. Then, with $|\phi\rangle = e^{i\hat{P}x/\hbar}|\psi\rangle$, we have

$$\langle\phi|\phi\rangle = \langle\psi|e^{-i\hat{P}x/\hbar}e^{i\hat{P}x/\hbar}|\psi\rangle = \langle\psi|e^{-i\hat{P}x/\hbar+i\hat{P}x/\hbar}|\psi\rangle = \langle\psi|\psi\rangle = 1 \tag{23}$$

which is still normalised. Here, we used the general result

$$e^{\hat{A}}e^{\hat{B}} = e^{\hat{A}+\hat{B}} \quad \text{if} \quad [\hat{A},\hat{B}] = 0 \tag{24}$$

(or equivalently, we used (21) with $\hat{B} = 1$). Supposing that $\langle\psi|\hat{X}|\psi\rangle = x_0$, we can evaluate the average position corresponding to the vector $|\phi\rangle$:

$$\begin{aligned}
\langle\phi|\hat{X}|\phi\rangle &= \langle\psi|e^{-i\hat{P}x/\hbar}\hat{X}e^{i\hat{P}x/\hbar}|\psi\rangle \\
&= \langle\psi|(\hat{X}-x)|\psi\rangle \\
&= \langle\psi|\hat{X}|\psi\rangle - x\langle\psi|\psi\rangle \\
&= x_0 - x
\end{aligned} \tag{25}$$

That is, it is just shifted by the value $-x$.