Quantum Mechanics III Epiphany Term 2009
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Homework 3 – due 12 March 2009

1. For the isotropic three-dimensional simple harmonic oscillator (with $V(\mathbf{x}) = \frac{m\omega^2}{2}|\mathbf{x}|^2$), construct annihilation and creation operators for each of the x, y and z components and write down their commutation rules. From them find expressions for $\hat{H}$, $\hat{L}^2$, $\hat{L}_\pm$ and $\hat{L}_z$, and verify the commutation rules for these operators. In the two separate cases of energy $E = 3\hbar\omega/2$ and energy $E = 5\hbar\omega/2$, find the possible values of l (associated to the eigenvalues of $\hat{L}^2$ as usual), and express energy and angular momentum eigenstates $|Elm\rangle$ in terms of states $|n_x, n_y, n_z\rangle$, parametrised by the three harmonic-oscillator quantum numbers n_z, n_y, n_z associated with the three cartesian directions.

Answer

We just have to take the results of the one-dimenional harmonic oscillator, one for each coordinates, since the three-dimensional Hamiltonian is just a sum of independent one-dimensional Hamiltonians:

$$\hat{a}_j = \frac{\hat{P}_j + im\omega\hat{X}_j}{\sqrt{2m}}, \quad \hat{a}_j^\dagger = \frac{\hat{P}_j - im\omega\hat{X}_j}{\sqrt{2m}} \tag{1}$$

for $j = 1, 2, 3$ representing the three directions x, y, z. They satisfy the commutation relations

$$[\hat{a}_j, \hat{a}_k] = [\hat{a}_j^\dagger, \hat{a}_k^\dagger] = 0, \quad [\hat{a}_j^\dagger, \hat{a}_k] = \delta_{j,k}\hbar\omega. \tag{2}$$

Then, the full Hamiltonian is just

$$\hat{H} = \frac{3\hbar\omega}{2} + \sum_{j=1}^{3} \hat{a}_j\hat{a}_j^\dagger = \frac{3\hbar\omega}{2} + \sum_{j=1}^{3} \hat{n}_j \tag{3}$$

where we introduce the operators $\hat{n}_j = \hat{a}_j\hat{a}_j^\dagger$ for the components j, counting the energy difference with respect to the ground state energy of the associated Hamiltonian. For the angular momentum operator, we use

$$\hat{X}_j = \frac{\hat{a}_j - \hat{a}_j^\dagger}{i\omega\sqrt{2m}}, \quad \hat{P}_j = \sqrt{\frac{m}{2}}(\hat{a}_j + \hat{a}_j^\dagger) \tag{4}$$

and we have

$$\begin{aligned}
\hat{L}_z &= \hat{X}_1\hat{P}_2 - \hat{P}_1\hat{X}_2 = \frac{1}{i\omega}(\hat{c}_{12} - \hat{c}_{21}) \\
\hat{L}_+ &= \hat{L}_x + i\hat{L}_y = \frac{1}{i\omega}(\hat{c}_{23} - \hat{c}_{32} + i\hat{c}_{31} - i\hat{c}_{13}) \\
\hat{L}_- &= \hat{L}_+^\dagger = \frac{1}{i\omega}(\hat{c}_{23} - \hat{c}_{32} - i\hat{c}_{31} + i\hat{c}_{13}) \\
\hat{L}^2 &= \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2 = -\frac{1}{\omega^2}(\hat{c}_{12}^2 + \hat{c}_{21}^2 + \hat{c}_{23}^2 + \hat{c}_{32}^2 + \hat{c}_{31}^2 + \hat{c}_{13}^2 \\
&\quad -2(\hat{n}_1\hat{n}_2 + \hat{n}_2\hat{n}_3 + \hat{n}_3\hat{n}_1) - 2\hbar\omega(\hat{n}_1 + \hat{n}_2 + \hat{n}_3))
\end{aligned}$$

where we introduce the operators

$$\hat{c}_{ij} = \hat{a}_i \hat{a}_j^\dagger \tag{5}$$

(in particular, $\hat{c}_{ii} = \hat{n}_i$), and used the relations

$$\hat{c}_{12}\hat{c}_{21} = \hat{n}_1\hat{n}_2 + \hbar\omega\hat{n}_2, \quad \hat{c}_{21}\hat{c}_{12} = \hat{n}_1\hat{n}_2 + \hbar\omega\hat{n}_1 \tag{6}$$

and similar relations obtained by cyclic permuttations $1, 2, 3 \mapsto 2, 3, 1 \mapsto 3, 1, 2$. In order to verify the commutation relations, let us use the operators $\hat{n}_j$ and $\hat{c}_{ij}$. We have

$$[\hat{n}_i, \hat{n}_j] = 0 \tag{7}$$

(since the $\hat{n}_i$'s count an energy differences in different directions), and

$$[\hat{n}_i, \hat{c}_{jk}] = \hbar\omega(\delta_{ij} - \delta_{ik}). \tag{8}$$

Clearly, then, $[\sum_i \hat{n}_i, \hat{c}_{jk}] = 0$, so that $\hat{H}$ commutes with all angular momentum operators. For the other commutation relations, we use the commutation relations

$$[\hat{c}_{ij}, \hat{c}_{kl}] = \hbar\omega(\hat{c}_{il}\delta_{jk} - \hat{c}_{kj}\delta_{il}). \tag{9}$$

We have

$$[\hat{L}_z, \hat{L}_\pm] = -\frac{\hbar}{\omega}\left(\hat{c}_{13} - \hat{c}_{31} \mp i\hat{c}_{32} \pm i\hat{c}_{23})\right) = \pm\hbar\hat{L}_\pm \tag{10}$$

and since we know that $\hat{L}^2$ can be constructed out of $\hat{L}_\pm$ and $\hat{L}_z$, and its commutation relations follow from those of $\hat{L}_\pm$ and $\hat{L}_z$, this is sufficient.

Now, consider the case $E = 3\hbar\omega/2$. There, we must have $n_1 = n_2 = n_3 = 0$, so that the only state is $|000\rangle$. Since all operators $\hat{c}_{ij}$ and $\hat{n}_i$ annihilate $|000\rangle$, because the $\hat{a}_i^\dagger$ are always placed on the right, it follows that both $\hat{L}^2$ and $\hat{L}_z$ are zero on $|000\rangle$. Hence, this is directly an eigenstate of $\hat{L}^2$ and $\hat{L}_z$ with $l = 0$ and $m = 0$: $|E = 3\hbar\omega/2,\, l = 0,\, m = 0\rangle = |000\rangle$.

Finally, consider the case $E = 5\hbar\omega/2$. There are three states that have this energy:

$$|100\rangle, \quad |010\rangle, \quad |001\rangle$$

which correspond to the three solutions to the equation $n_1 + n_2 + n_3 = 1$. The action of angular momentum operators on these states can be obtained by realising that $\hat{c}_{ij}$, on these states, essentially moves the value of n_j to the ith position if n_j is nonzero, and puts a factor $\hbar\omega$ in front; otherwise it gives 0:

$$\hat{c}_{12}|100\rangle = 0, \quad \hat{c}_{12}|010\rangle = \hbar\omega|100\rangle, \quad \hat{c}_{12}|001\rangle = 0$$

and similar equations for $\hat{c}_{23}$ and $\hat{c}_{31}$, etc. In general, $\hat{c}_{ij}$ decreases the number on the jth place, and increase the number on the ith place. On the other hand, the operators $\hat{n}_j$ act like

$$\hat{n}_1|100\rangle = \hbar\omega|100\rangle, \quad \hat{n}_1|010\rangle = \hat{n}_1|001\rangle = 0$$

and similarly for $\hat{n}_2$ and $\hat{n}_3$. (These formulas are derived from the action of $\hat{a}_j$ and $\hat{a}_j^\dagger$, see homework 3). Then, we find

$$\hat{L}^2|100\rangle = 2\hbar^2|100\rangle, \quad \hat{L}^2|010\rangle = 2\hbar^2|010\rangle, \quad \hat{L}^2|001\rangle = 2\hbar^2|001\rangle$$

where only the last term $-2\hbar\omega(\hat{n}_1 + \hat{n}_2 + \hat{n}_3)$ in the parenthesis in the expression of $\hat{L}^2$ actually contributes. Hence, all these states have $l = 1$ (since then $l(l+1)\hbar^2 = 2\hbar^2$). These states then form a subspace of fixed l value, and in this subspace, we need to diagonalise $\hat{L}_z$ in order to find eigenvectors with fixed values of $\hat{m}$. It is clear that $\hat{L}_z|001\rangle = 0$, so that we already have an eigenvector with $m = 0$. That is, we already found that $|E = 5\hbar\omega/2,\, l = 1,\, m = 0\rangle = |001\rangle$. For the other two possible values of m, which are ± 1 since we have $l = 1$, we need to write down the eigenvalue equation on some arbitrary linear combination of $|100\rangle$ and $|010\rangle$. So, we say, for $m = 1$:

$$\begin{aligned} \hat{L}_z\left(A|100\rangle + B|010\rangle\right) &= \hbar\left(A|100\rangle + B|010\rangle\right) \\ \Rightarrow \frac{1}{i\omega}\left(-A\hbar\omega|010\rangle + B\hbar\omega|100\rangle\right) &= \hbar\left(A|100\rangle + B|010\rangle\right) \end{aligned}$$

which gives two equations, when we look at the coefficients of $|000\rangle$ and $|010\rangle$. These two equations are consistent, and give

$$iA = B$$

so that, with proper normalisation, we have

$$|E = 5\hbar\omega/2,\, l = 1,\, m = 1\rangle = \frac{1}{\sqrt{2}}(|100\rangle + i|010\rangle).$$

Similarly, for $m = -1$, we have

$$\begin{aligned} \hat{L}_z\left(A|100\rangle + B|010\rangle\right) &= -\hbar\left(A|100\rangle + B|010\rangle\right) \\ \Rightarrow \frac{1}{i\omega}\left(-A\hbar\omega|010\rangle + B\hbar\omega|100\rangle\right) &= -\hbar\left(A|100\rangle + B|010\rangle\right) \end{aligned}$$

so that

$$-iA = B$$

and, with proper normalisation, we have

$$|E = 5\hbar\omega/2,\, l = 1,\, m = -1\rangle = \frac{1}{\sqrt{2}}(|100\rangle - i|010\rangle).$$