Quantum Mechanics III | Epiphany Term 2008

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Homework 3 – due 29 February 2008

In one dimension a stream of particles of mass m and energy E is incident on a potential "hole"

$$V(x) = \begin{cases} 0 & x < 0 \\ -V_0 & 0 < x < a \\ 0 & x > a \end{cases}$$

with $V_0 > 0$. Find the transmission and reflection coefficients. Show that when

$$2m(E + V_0) = \frac{n^2\pi^2\hbar^2}{a^2}, \quad n = 1, 2, 3, \ldots$$

there is no reflection.

ANSWER

This is a scattering problem with potential having the same finite value at both $x = \pm\infty$, so particles transmitted have the same energy as that of incident and reflected particles. Let us assume that particles are incident from the left. Then, we have

$$\psi(x) = \begin{cases} \psi_<(x) \equiv e^{ikx} + Ae^{-ikx} & x \to -\infty \\ \psi_>(x) \equiv Be^{ikx} & x \to \infty \end{cases}$$

with $k = \sqrt{2mE/\hbar^2}$ (since the potential is 0 at $x = \pm\infty$) where $E > 0$ is the energy of the incident, reflected and transmitted particles. The reflection R and transmission T coefficients are simply

$$R = |A|^2 , \quad T = |B|^2$$

and we know that $R+T = 1$ by conservation of the current. The potential is constant for $x < 0$ and for $x > a$ so that in fact $\psi(x) = \psi_<(x)$ for $x < 0$ and $\psi(x) = \psi_>(x)$ for $x > a$. In the region $0 < x < a$, the potential is also constant but with a different value $-V_0$, so that in that region

$$\psi(x) = \psi_0(x) \equiv Ce^{iqx} + De^{-iqx} \quad 0 < x < a$$

with $q = \sqrt{2m(E + V_0)/\hbar^2}$. Note that since the energy has to be positive, then we necessarily have oscillating exponentials in $0 < x < a$, where the potential is negative.

We now need to implement continuity of the wave function and its derivatives at $x = 0$ and $x = a$. We have 4 equations:

$$\begin{aligned} \psi_<(0) = \psi_0(0) \quad &\Rightarrow \quad 1 + A = C + D \\ \psi'_<(0) = \psi'_0(0) \quad &\Rightarrow \quad k(1 - A) = q(C - D) \\ \psi_>(a) = \psi_0(a) \quad &\Rightarrow \quad Be^{ika} = Ce^{iqa} + De^{-iqa} \\ \psi'_>(a) = \psi'_0(a) \quad &\Rightarrow \quad kBe^{ika} = q(Ce^{iqa} - De^{-iqa}) \end{aligned}$$

Isolating A in the first equation and replacing it into the second leads to

$$k(2 - C - D) = q(C - D)$$

On the other hand, isolating B in the third equation and replacing it into the fourth leads to

$$k(Ce^{iqa} + De^{-iqa}) = q(Ce^{iqa} - De^{-iqa})$$

Hence we have two equations for two variables, C and D. One way to solve them is by writing them using matrices:

$$\begin{pmatrix} k+q & k-q \\ (k-q)e^{iqa} & (k+q)e^{-iqa} \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} 2k \\ 0 \end{pmatrix}$$

Inverting the 2 by 2 matrix gives

$$\begin{pmatrix} C \\ D \end{pmatrix} = \frac{1}{(k+q)^2 e^{-iqa} - (k-q)^2 e^{iqa}} \begin{pmatrix} (k+q)e^{-iqa} & q-k \\ (q-k)e^{iqa} & k+q \end{pmatrix} \begin{pmatrix} 2k \\ 0 \end{pmatrix}$$

That is,

$$\begin{aligned} C &= \frac{2k(k+q)e^{-iqa}}{(k+q)^2 e^{-iqa} - (k-q)^2 e^{iqa}} \\ D &= \frac{2k(q-k)e^{iqa}}{(k+q)^2 e^{-iqa} - (k-q)^2 e^{iqa}} \end{aligned}$$

Hence, we can find B from its relation to C and D above:

$$Be^{ika} = \frac{2k(k+q) + 2k(q-k)}{(k+q)^2 e^{-iqa} - (k-q)^2 e^{iqa}} = \frac{4kq}{(k+q)^2 e^{-iqa} - (k-q)^2 e^{iqa}}$$

This gives for the transmission coefficient $T = |B|^2$:

$$T = \frac{16k^2q^2}{(k+q)^4 + (k-q)^4 - 2(k+q)^2(k-q)^2\cos(2qa)}$$

We see immediately that when $2m(E + V_0) = n^2\pi^2\hbar^2/a^2$ with $n = 1, 2, 3, \ldots$, then $q = n\pi/a$ and $\cos(2qa) = 1$ so that

$$T = \frac{16k^2q^2}{(k+q)^4 + (k-q)^4 - 2(k+q)^2(k-q)^2} = \frac{16k^2q^2}{((k+q)^2 - (k-q)^2)^2} = \frac{16k^2q^2}{(4kq)^2} = 1$$

so that there is total transmission at these energies. Hence, since $R = 1 - T$, the reflection is zero at these energies. In general, the reflection coefficient is

$$R = 1 - \frac{16k^2q^2}{(k+q)^4 + (k-q)^4 - 2(k+q)^2(k-q)^2\cos(2qa)} = \frac{4(k+q)^2(k-q)^2\sin^2(qa)}{(k+q)^4 + (k-q)^4 - 2(k+q)^2(k-q)^2\cos(2qa)}$$