Quantum Mechanics III — Epiphany Term 2008

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Homework 1 – due 1 February 2008

A quantum system has an observable $\hat{A}$ that can take values 1, 2 and 3. The corresponding orthonormal eigenstates are $|1\rangle$, $|2\rangle$ and $|3\rangle$. The system evolves in time with the hamiltonian $\hat{H}$ whose action on these states is given by

$$\begin{aligned}\hat{H}|1\rangle &= -|1\rangle + 2i|2\rangle \\ \hat{H}|2\rangle &= -2i|1\rangle + 2i|3\rangle \\ \hat{H}|3\rangle &= -2i|2\rangle + |3\rangle\end{aligned}$$

(a) Find the eigenvalues and eigenstates of the hamiltonian.

(b) Initially $\hat{A}$ is measured and found to be 2. The system then evolves for a time t. At that time, if a measure of $\hat{A}$ is performed, what values may be obtained and with what probabilities? What are the expectation value $\langle\hat{A}\rangle$ and standard deviation $\Delta\hat{A}$ of $\hat{A}$? If, instead, a measure of the energy is performed at the time t, what values may be obtained and with what probabilities?

Answer

(a) In matrix form:

$$|1\rangle = \begin{pmatrix}1\\0\\0\end{pmatrix}, \quad |2\rangle = \begin{pmatrix}0\\1\\0\end{pmatrix}, \quad |3\rangle = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

the hamiltonian is

$$\hat{H} = \begin{pmatrix}-1 & -2i & 0\\ 2i & 0 & -2i\\ 0 & 2i & 1\end{pmatrix} \tag{1}$$

The characteristc polynomial equation that gives the eigenvalues E is

$$0 = \det(H - E\mathbf{1}) = E(1-E^2) + 4(1+E) - 4(1-E) = E(1-E^2) + 8E$$

There is a factor E, so that $E = 0$ is one eigenvalue. If $E \neq 0$, then

$$0 = 1 - E^2 + 8 = 9 - E^2$$

so that $E = \pm 3$ are the two other eigenvalues. For the eigenstates, consider $|v\rangle = v_1|1\rangle + v_2|2\rangle + v_3|3\rangle$. The eigenvalue equation $\hat{H}|v\rangle = E|v\rangle$ gives:

$$\begin{aligned}-v_1 - 2iv_2 &= Ev_1 \\ 2iv_1 - 2iv_3 &= Ev_2 \\ 2iv_2 + v_3 &= Ev_3\end{aligned}$$

In the case $E = 0$, we have $v_1 = v_3$ from the second equation, and $v_1 = v_3 = -2iv_2$ consistently from the first and the last equation. In the case $E = 3$, we have $v_2 = 2iv_1$ from the first equation, $v_3 = iv_2 = -2v_1$ from the last equation, and this is consistent with the second equation. Finally, in the case $E = -3$, we have $v_2 = 2iv_3$ from the last equation, $v_1 = iv_2 = -2v_3$ from the first equation, and this is consistent with the second equation. Normalising to $\langle v|v\rangle = 1$, we find:

$$\begin{aligned} E = 0 \quad &: \quad |E = 0\rangle = \frac{1}{3}\left(-2i|1\rangle + |2\rangle - 2i|3\rangle\right) \\ E = 3 \quad &: \quad |E = 3\rangle = \frac{1}{3}\left(|1\rangle + 2i|2\rangle - 2|3\rangle\right) \\ E = -3 \quad &: \quad |E = -3\rangle = \frac{1}{3}\left(-2|1\rangle + 2i|2\rangle + |3\rangle\right) \end{aligned} \tag{2}$$

(b) When a measurement is performed, the state of the system just after the measurement is the state corresponding to the value observed. Hence, here we have $|\Psi(0)\rangle = |2\rangle$. We have to write this initial state in the basis of the hamiltonian eigenstates, in order to be able to evolve it in time. With the decomposition of the identity operator in the energy basis, $\mathbf{1} = |E = 0\rangle\langle E = 0| + |E = 3\rangle\langle E = 3| + |E = -3\rangle\langle E = -3|$, we have

$$|2\rangle = \langle E = 0|2\rangle\, |E = 0\rangle + \langle E = 3|2\rangle\, |E = 3\rangle + \langle E = -3|2\rangle\, |E = -3\rangle$$

so that

$$|\Psi(0)\rangle = \frac{1}{3}\left(|E = 0\rangle - 2i|E = 3\rangle - 2i|E = -3\rangle\right) \tag{3}$$

Evolving this state for a time t, we just have to multiply by $e^{-i\hat{H}t/\hbar}$, giving

$$|\Psi(t)\rangle = \frac{1}{3}\left(|E = 0\rangle - 2ie^{-3it/\hbar}|E = 3\rangle - 2ie^{3it/\hbar}|E = -3\rangle\right)$$

The values of $\hat{A}$ that can be obtained are just those stated in the question: 1, 2 and 3. In order to find their respective probabilities, we must re-write the evolved state in the basis of the $\hat{A}$-eigenstates, $|1\rangle$, $|2\rangle$ and $|3\rangle$, using the explicit expressions we found for $|E = 0\rangle$, $|E = 3\rangle$ and $|E = -3\rangle$. This gives:

$$|\Psi(t)\rangle = \frac{1}{9}\left((-2i - 2ie^{-\frac{3it}{\hbar}} + 4ie^{\frac{3it}{\hbar}})|1\rangle + (1 + 4e^{-\frac{3it}{\hbar}} + 4e^{\frac{3it}{\hbar}})|2\rangle + (-2i + 4ie^{-\frac{3it}{\hbar}} - 2ie^{\frac{3it}{\hbar}})|3\rangle\right)$$

The probability for measuring a value n is just $P(n) = |\langle n|\Psi(t)\rangle|^2$, for $n = 1, 2, 3$. Hence, the probabilities are as follows:

$$\begin{aligned} P(1) &= \frac{1}{81}\left(24 - 4e^{-\frac{3it}{\hbar}} - 4e^{\frac{3it}{\hbar}} - 8e^{-\frac{6it}{\hbar}} - 8e^{\frac{6it}{\hbar}}\right) = \frac{1}{81}\left(24 - 8\cos\frac{3t}{\hbar} - 16\cos\frac{6t}{\hbar}\right) \\ P(2) &= \frac{1}{81}\left(33 + 8e^{-\frac{3it}{\hbar}} + 8e^{\frac{3it}{\hbar}} + 16e^{-\frac{6it}{\hbar}} + 16e^{\frac{6it}{\hbar}}\right) = \frac{1}{81}\left(33 + 16\cos\frac{3t}{\hbar} + 32\cos\frac{6t}{\hbar}\right) \\ P(3) &= P(1) = \frac{1}{81}\left(24 - 8\cos\frac{3t}{\hbar} - 16\cos\frac{6t}{\hbar}\right) \end{aligned}$$

Note that they add up to 1 for any t, as it should. Also, at $t = 0$, we have $P(1) = P(3) = 0$ and $P(2) = 1$, as it should also, since the initial state was just after measuring the value 2. The expectation value of $\hat{A}$ is just $P(1) + 2P(2) + 3P(3)$, that is

$$\langle \hat{A} \rangle = 2$$

Interestingly, it is time-independent. This is not generic, it is particular to this system and to the initial condition we chose. It is due to a very special transformation that keeps the hamiltonian almost unchanged. Let us consider the hamiltonian in the basis $|1\rangle, |2\rangle, |3\rangle$, as in (1). Then we see that if we do a *complex conjugation* of the matrix $iHt/\hbar$, as well as the *change of basis* $|1\rangle \to |3\rangle$, $|3\rangle \to |1\rangle$ *and* $|2\rangle \to -|2\rangle$, it stays exactly the same (in other words, the hamiltonian H itself changes its sign under this transformation). This is true for any power of the matrix $iHt/\hbar$, because complex conjugation just applies to each factor independently, and the rest is a change of basis. Hence it is also true for $e^{-iHt/\hbar}$. Then we have that

$$(\langle 1|e^{-iHt/\hbar}|2\rangle)^* = -\langle 3|e^{-iHt/\hbar}|2\rangle \ , \quad (\langle 3|e^{-iHt/\hbar}|2\rangle)^* = -\langle 1|e^{-iHt/\hbar}|2\rangle$$

Since the probabilities are $P(n) = |\langle n|e^{-iHt/\hbar}|2\rangle|^2$, we find $P(1) = P(3)$ for all times: the probability is always distributed equally on the values 1 and 3 of the operator $\hat{A}$. Then, $P(1) + 2P(2) + 3P(3) = 4P(1) + 2P(2)$, and the condition $1 = P(1) + P(2) + P(3) = 2P(1) + P(2)$ implies $4P(1) = 2 - 2P(2)$ so that the average is always 2.

There is a different transformation that is similar in spirit and more common: complex conjugation *in the basis of the hamiltonian eigenstates*, which is called *time-inversion symmetry*. It keeps the hamiltonian invariant because in this basis, it is a diagonal matrix with real elements, so it is a true *symmetry*. This also holds for any power of the hamiltonian. But under this symmetry, the evolution operator $e^{-iHt/\hbar}$ transforms into $e^{iHt/\hbar}$: this is a time inversion.

As for the variance, we have

$$\Delta\hat{A} = \langle \hat{A}^2 \rangle - \langle \hat{A} \rangle^2 = P(1) + 4P(2) + 9P(3) - 4 = \frac{16}{27} - \frac{16}{81}\cos\frac{3t}{\hbar} - \frac{32}{81}\cos\frac{6t}{\hbar}$$

Finally, if we measure the energy instead, the values we can obtain are 0, 3 or -3, and the probabilities are just given by the modulus squared of each the coefficients in the decomposition (3) (time evolution doesn't affect these probabilities, because $\hat{H}$ is a conserved quantity):

$$\begin{aligned} P(E=0) &= \frac{1}{9} \\ P(E=3) &= \frac{4}{9} \\ P(E=-3) &= \frac{4}{9} \end{aligned}$$

which add up to 1 as it should.