COHOMOLOGY OF ARITHMETIC GROUPS

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Let $G$ be a semisimple real Lie group with $\Gamma$ a discrete cocompact subgroup that is torsion free. Let $K$ be a maximal compact subgroup of $G$ and let $V$ be a finite dimensional complex continuous representation of $G$ that we might assume irreducible. We are interested in studying the cohomology groups

$$H^*(\Gamma,V),$$

which, as we have seen in the previous lectures, are closely related to the theory of automorphic forms. We will show the following results:

- The cohomology groups can be computed as

$$H^n(\Gamma,V) = H^k(\mathfrak{g},K;C^\infty(\Gamma \setminus G) \otimes V).$$

- These $(\mathfrak{g},K)$-cohomology groups can be decomposed as

$$H^k(\mathfrak{g},K;C^\infty(\Gamma \setminus G) \otimes V) = \bigoplus_{\pi} \mu(\pi,\Gamma) H^n(\mathfrak{g},K;H_\pi \otimes V),$$

where $(\pi,H_\pi)$ runs over certain representations of $G$. This is known as Matsushima’s formula.

- We will finally study the groups

$$H^k(\mathfrak{g},K;H_\pi \otimes V)$$

in more detail. In particular, we will show that, for certain representations $\pi$, they vanish outside certain range and we will calculate their dimension.

1. Cohomology and differential forms

Reference: [BW00] §VII.2

Let $X := G/K$ denote the symmetric space associated to $G$, it is simply connected and contractible.

1.1. Differential forms. Let $\mathcal{A}^q = \mathcal{A}^q(X,V)$ denote the smooth $V$-valued differentials forms of degree $q$ on $X$ with the usual differentials $d : \mathcal{A}^q \to \mathcal{A}^{q+1}$ given by

$$d\omega(v_1,\ldots,v_q) = \sum_{i=1}^q (-1)^i v_i \cdot \omega(v_1,\ldots,\hat{v}_i,\ldots,v_q) + \sum_{i<j} (-1)^{i+j} \omega([v_i,v_j],v_1,\ldots,\hat{v}_i,\ldots,\hat{v}_j,\ldots,v_q),$$

where $v_i \cdot \omega(v_1,\ldots,\hat{v}_i,\ldots,v_q)$ denotes the differentiation of the function $\omega$ in the direction $v_i$, $[,]$ refers to the bracket of vector fields, and $\hat{\cdot}$ means omission of the corresponding argument.

\[1\] We do not say anything about the non-compact case due to lack of time, but that case is of particular interest. See [BW00] §XIV for the analogous statements in this context.
Proposition 1. There is an canonical isomorphism

\[ H^*(\Gamma, V) = H^*(\Gamma \backslash X, \tilde{V}) \]

where \( \tilde{V} \) is the local system on \( \Gamma \backslash X \) associated to \( V \).

Proof. This follows immediately from the fact that \( \Gamma \backslash X \) is a \( K(\Gamma,1) \)-space, i.e. \( \pi_1(\Gamma \backslash X) = \Gamma \) and all its other homotopy groups vanish. \( \square \)

The comparison between de Rham and singular cohomology now gives us the following Corollary:

Corollary 1. There are canonical isomorphisms

\[ H^*(\Gamma, V) \cong H^*(A^\bullet(\Gamma \backslash X, \tilde{V})) \cong H^*(A^\bullet(X, \tilde{V}))^\Gamma. \]

1.2. \((g,K)\) modules and \((g,K)\)-cohomology. Let \( g \) denote the Lie algebra of \( G \). Recall that a \((g,K)\)-module is a vector space \( W \) over \( \mathbb{R} \) which is a \( g \)-module and a \( K \)-module with the obvious compatibility condition. Namely we ask

- \( \pi(k) \cdot (\pi(X) \cdot v) = \pi(\text{Ad} k(X)) \cdot (\pi(k) \cdot v) \), for all \( k \in K, X \in g, v \in W \),
- If \( F \subseteq W \) is a \( K \)-stable finite dimensional subspace, then the representation of \( K \) is differentiable, and has \( \pi|_F \) as differential.

Example 1. If \( V \) is a representation of \( G \), then the subspace of smooth and \( K \)-finite vectors of \( V \) is a \((g,K)\)-module.

Definition 1. For \( V \) a \((g,K)\)-module we define

\[ C^q(g, K; V) = \text{Hom}_K(\wedge^q(g/K, V) = (\wedge^q \mathfrak{p}^* \otimes V)^K/K^q \]

where \( g = \mathfrak{t} \oplus \mathfrak{p} \) is the Cartan decomposition. There are differentials defined in the same way as was done in [1], which gives us a complex \( C^\bullet(g, K; V) \) and we define the \((g,K)\)-cohomology of \( V \) as the cohomology groups of this complex:

\[ H^*(g, K; V) := H^*(C^\bullet(g, K; V)). \]

The left translation by elements \( g \in G \) provides an isomorphism between the tangent space at \( g \) and the tangent space at the identity element, and hence an identification

\[ A^q(\Gamma \backslash X, V) = \text{Hom}_K(\wedge^q(g/K, C^\infty(\Gamma \backslash G) V) = C^q(g, K; C^\infty(\Gamma \backslash G) \otimes V). \]

An explicit computation of the differentials now gives:

Proposition 2. There is a canonical isomorphism

\[ H^*(\Gamma \backslash X, \tilde{V}) = H^*(g, K; C^\infty(\Gamma \backslash G) \otimes V). \]
2. Matsushima’s Formula

Reference: [BW00, VII.3-6]

Let $L^2(\Gamma \backslash G, V)$ be the space of square-integrable $V$-valued functions of $\Gamma \backslash G$. This is acted upon by $G$ and decomposes as

$$L^2(\Gamma \backslash G) = \bigoplus \pi m(\pi, \Gamma) \cdot H_\pi,$$

a direct sum of irreducible representations with finite multiplicities. Moreover one has

$$C^\infty(\Gamma \backslash G) = (L^2(\Gamma \backslash G))^\infty = \left(\bigoplus \pi m(\pi, \Gamma) \cdot H_\pi\right)^\infty,$$

where $(-)^\infty$ means taking smooth vectors.

**Proposition 3** (Matsushima’s formula, [BW00, VII.3.2 Theorem]). We have

$$H^* (\g, K; C^\infty(\Gamma \backslash G) \otimes V) = \bigoplus \pi m(\pi, \Gamma) \cdot H^* (\g, K; H_\pi \otimes V)$$

where the direct sum is now finite.

**Proof.** The previously stated facts give us

$$H^* (\g, K; C^\infty(\Gamma \backslash G) \otimes V) = H^* (\g, K; \left(\bigoplus \pi m(\pi, \Gamma) \cdot H_\pi\right)^\infty \otimes V).$$

We want to show that the right hand side term equals

$$\bigoplus \pi m(\pi, \Gamma) H^* (\g, K; H_\pi \otimes V).$$

Let now $S \subseteq \hat{G}$ be a finite set of representations. Then we can decompose

$$H^* (\pi, \Gamma) = \bigoplus_{\pi \in S} m(\pi, \Gamma) H^* (\g, K; H_\pi \otimes V) \oplus H^* (\g, K; \left(\bigoplus_{\pi \notin S} \pi m(\pi, \Gamma) \cdot H_\pi\right)^\infty \otimes V).$$

The compactness assumption on $\Gamma$ tells us that the cohomology $H^* (\Gamma, V)$ of the arithmetic group is finite dimensional. We deduce, for dimension reasons, that, for a large enough $S$, we have

$$H^* (\g, K; H_\pi \otimes V) = 0 \quad \forall \pi \notin S.$$

We have hence reduced then to proving that, if each $(\g, K)$-cohomology of a countable collection of irreducible unitary representations of $G$ vanishes, then the cohomology of its closed direct sum vanishes as well. This is not very hard and follows from a topological argument (cf. [BW00, VII.3.3 Lemma] for the details).

Summarising, we have reduced our computation of $H^* (\Gamma, V)$ to the study the $(\g, K)$ cohomology groups of certain representations of the form $H_\pi \otimes V$, where $H_\pi$ is a unitary $(\g, K)$-module and $V$ is a finite dimensional (irreducible) complex continuous representation of $G$.

**Remark 1.** We will give later a necessary condition for $\pi$ to appear in the the above sum. A precise characterisation of the representations $\pi$ that contribute to $H^* (\Gamma, V)$ have been described by Vogan-Zuckerman.
3. Calculation of the \((g, K)\)-cohomology

Reference: \[\text{BW00} \text{ \S II}\].

Let \((\rho, E)\) be a finite dimensional irreducible complex representations of \(G\) and \((\sigma, H)\) be a unitary \((g, K)\)-module. Let \(V = H \otimes E\) and \(\tau = \rho \otimes \sigma\). With an eye on Matsushima’s formula, we want to study the cohomology groups \(H^*(g, K; V)\) in this particular case.

3.1. The Casimir element. Let \((y_i)\) be a basis of \(g\) and \((y'_i)\) be its dual basis with respect to the Killing form. Then

\[
C = \sum_i y_i \cdot y'_i
\]

is an element of the center of the universal enveloping algebra \(U(g)\) of \(g\), independent of the choice of basis, and is called the Casimir element. By Schur’s lemma, \(C\) must act as a scalar on any representation.

**Proposition 4** (\[\text{BW00} \text{ \S II.3.1 Proposition}\]). Assume that \(\rho(C) = s \cdot \text{Id}\) and \(\sigma(C) = r \cdot \text{Id}\). Then

- If \(r \neq s\) then \(H^*(g, K; V) = 0\).
- If \(r = s\) then \(H^*(g, K; V) = \text{Hom}_K(\wedge^q p, V)\).

**Proof.** The proof follows these steps:

1. One defines an inner product on

\[
\mathcal{D}^q(V) := \text{Hom}_{\mathbb{R}}(\wedge^q p, V) = (\wedge^q p) \otimes H \otimes E
\]

(observe that we are just taking \(\mathbb{R}\)-linear homomorphisms and hence \(\mathcal{D}^q(V)\) is bigger than \(C^q(V)\)) by taking the tensor products of the inner products on each term, which we call \((-,-)_V\). We can then define an adjoint \(\partial : \mathcal{D}^q \rightarrow \mathcal{D}^{q-1}\) of \(d\) for the inner product \((-,-)_V\) and shows the that

\[
\Delta := d\partial + \partial d
\]

acts on \(C^q(g, K; V)\) as \((\rho(C) - \sigma(C)) \cdot \text{Id}\) and that if \(\Delta = 0\) then \(d = \partial = 0\) (the first assertion follows from a direct calculation and the last one follows from the non-degeneracy of the bilinear pairing).

2. If \(r \neq s\) then for \(\eta \in C^q(g, K; V)\) a \(q\)-cocycle, we have

\[
\Delta \eta = d\partial \eta + \partial d\eta = d\partial \eta
\]

and so \(\eta = (r - s)^{-1} d\partial \eta\) is a coboundary.

3. If \(r = s\) then \(\Delta = 0\) and so \(d = 0\) and hence every chain is closed, which gives

\[
H^q(g, K; V) = \text{Hom}_K(\wedge^q p, V).
\]

\[\square\]

**Corollary 2.** For trivial coefficients, we can identify

\[
(\wedge^q p)^K
\]

with the \(G\)-invariant differential forms on \(G/K\). The result says that all such forms are harmonic, recovering an old result of Cartan.

\[\text{\footnote{We switch notation and denote the representation \(V\) from the previous sections by \(E\).}}\]
Corollary 3. The representations \( \pi \) contributing to the some in Proposition 3 are such that \( \chi_\pi = \chi_\rho^* \) and \( \omega_\pi = \omega_\rho^* \), where \( \rho^* \) denotes the contragredient representation of \( \rho \) and \( \chi_\pi \) (resp. \( \chi_\rho^* \)) and \( \omega_\pi \) (resp. \( \omega_\rho^* \)) denote the infinitesimal and central characters of \( \pi \) (resp. \( \rho^* \)).

4. Cohomology of tempered representations

Reference: [BW00, §III].

In this section, we calculate the dimension of the \((g,K)\)-cohomology groups \( H^*(g,K;V) \) for certain representations \( V = H \otimes E \). In particular, we will see that they vanish outside a certain range which is given in purely in terms of \( G \) and \( K \).

4.1. Parabolic induction. Let’s start with a definition.

Definition 2. A parabolic pair is a pair \((P,A)\) where \( P \) is a parabolic subgroup and \( A \) is a split component of a maximal torus in the Levi of \( P \). Say \((P,A) < (P',A')\) if \( P \subset P' \) and \( A \supset A' \) and we fix a minimal parabolic pair \((P_0,A_0)\). We say that a parabolic \((P,A)\) is standard (w.r.t. the chosen minimal parabolic pair) if it is greater than the minimal one.

Let \((P,A)\) be a standard parabolic pair. Recall the Levi decomposition \( P = MN = A_0MN \).

Let \((\sigma,H_\sigma)\) be an admissible representations of \( \sigma \) with infinitesimal character \( \chi_\sigma \) and let \( \nu \in a^*_C \). Given this data, we define the parabolically induced representation as follows:

\[
I_{P,\sigma,\nu} = \text{Ind}_P^G(H_\sigma \otimes C_{\rho_\nu + \nu})
= \{ f \in C^\infty(G,H_\sigma) \ f(man \cdot g) = a^{\rho_\nu + \nu}(n)f(g) \},
\]

where \( \rho_\nu \in a^*_C \) is a usual normalisation factor defined as \( \rho_\nu(a) = \det(\text{Ad}a|_{\mathfrak{a}_P})^{1/2} \). This has an action of \( G \) by right translation which makes it into an admissible representations of \( G \), which is unitary if \( \sigma \otimes \nu \) is unitary, with infinitesimal character \( \chi_{\rho_\nu + \nu} \).

Definition 3. We call \((P,A)\) cuspidal if \( ^0M \) has a compact Cartan subgroup.

4.2. Cohomology of induced representations.

Proposition 5 ([BW00 III.5.1 Theorem]). Let \((P,A)\) be a standard cuspidal parabolic pair of \( G \). Let \((\sigma,H_\sigma)\) be a discrete series representations of \( ^0M \), \( \nu \in a^*_C \) purely imaginary and \( I = I_{P,\sigma,\nu} \). Finally let \( E \) be an irreducible and finite dimensional complex representation of \( G \). Then

1. \( H^q(g,K;I \otimes E) = 0 \) if \( q \not\in [q_0,q_0 + \ell_0] \)
2. If \( H^q(g,K;I \otimes E) \neq 0 \) then it has dimension

\[
\frac{\ell_0}{q - q_0}.
\]

Recall that the invariants \( q_0,\ell_0 \) are defined as \( \ell_0 = \ell_0(G) = \text{rk}(G) - \text{rk}(K) \) and that

\[
q_0 = q_0(G) = \frac{\dim(G/K) - \ell_0}{2}
\]
Sketch of proof. This is a very deep result with a very involved proof. We content ourselves with sketching the main steps of its proof, unfortunately omitting way too many details.

(1) First one proves (cf. [BW00, §III.2.5]) a version Shapiro’s Lemma and obtains
$$H^q(g, K; I \otimes E) = H^q(p, K_p; H_{\sigma,\nu} \otimes E),$$
where $H_{\sigma,\nu} = H_{\sigma} \otimes \mathbb{C}_{p^\nu}$.

(2) There is a Hochschild-Serre spectral sequence (cf. [BW00, §I.6.5]) which reads
$$E^{p,q}_2 := H^p(m, K_P; H^q(n, K_N; E) \otimes H_{\sigma,\nu}) \implies H^{p+q}(p, K_p; H_{\sigma,\nu} \otimes E).$$

(3) One needs now to understand the groups $H^q(n, K_N; E)$ as $m$-representations. This is a theorem of Kostant ([BW00, III.3.1 Theorem]):
$$H^q(n, E) = \bigoplus_{s \in W_P, \ell(s) = q} L_s$$
where $W_P$ is a system of representatives of $W_M \backslash W_G$ (the quotient of the Weyl groups), and the $L_s$ are certain representations which depend on $s$ and on the maximal weight $\lambda$ of $E$. Using this one shows that
$$H^{q+l(s)}(g, K; I \otimes E) = (H^*(m, K_P; H_{\sigma} \otimes L_s) \otimes \wedge^* a_C^*)^q.$$  

The first factor of the RHS is concentrated in degree $q^0(M) := (\dim^0 M - \dim K \cap ^0 M)/2$ and has dimension 1 ([BW00, §II.5.4 and II.6.7]). Observe also that $a_C^*$ has dimension $\ell_0$. This proves that
$$H^{q+l(s)}(g, K, I \otimes E) = \wedge^j a_C^*$$
where $j = q - q^0(M)$. This already shows that the dimensions of the cohomology groups are given by some combinatorial numbers and one needs to check that the non-vanishing range is the one claimed in the statement of the Proposition. Finally one shoes that in fact $l(s) = \frac{\dim N}{2}$ and that moreover
$$q_0(G) = q^0(M) + \frac{\dim N}{2},$$
and the result follows.

□

References