1. Simplicial Galois Representations

1.1. Motivation. First we need to discuss how to define a derived version of a deformation of a Galois representation. Since the coefficients are now allowed to be simplicial Artin rings, we need a new definition.

So what do we mean by \( G_{K,S} \rightarrow G(A) \)?

Here \( G_{K,S} \) is the Galois group of the maximal algebraic extension of \( K \) unramified outside a finite set of places \( S \), \( G \) is a reductive group (later we may take \( G \) to be adjoint), and \( A \in \text{sArt}_k \) (the category of simplicial local Artin rings, as defined in Raffael’s talk).

The naive idea (which doesn’t work) is to define \( G(A) \) directly.

1. Let \( A \in \text{sArt}_k \). We could just say \( p \mapsto G(A_p) \), which will define a simplicial group. Unfortunately, this is not homotopy invariant: to see this note that

\[
G(A_p) = \text{Hom}_{\text{CR}}(\mathcal{O}_G, A_p) = \text{Hom}_{\text{sCR}}(\mathcal{O}_G, A^\Delta) = \text{Hom}_{\text{sCR}}(\mathcal{O}_G, A),
\]

where we view \( \mathcal{O}_G \) as a constant simplicial ring in the third and fourth term and the underline denotes simplicial enrichment. Therefore, our attempt is just

\[
A \mapsto \text{Hom}_{\text{sCR}}(\mathcal{O}_G, A).
\]

But \( \mathcal{O}_G \) (viewed as a discrete simplicial ring) will almost never be cofibrant, so there’s no reason to expect that we should get something homotopy invariant.

2. So instead we could define \( G(A) := \text{Hom}_{\text{sCR}}(c(\mathcal{O}_G), A) \). This is now homotopy invariant, but unfortunately it’s not a simplicial group, because cofibrant replacement won’t respect the Hopf algebra structure of \( \mathcal{O}_G \), so this isn’t quite what we want.

3. (Comment/speculation from the audience) maybe we could put a model structure on the category of simplicial Hopf algebras and then try to cofibrantly replace \( \mathcal{O}_G \), now viewed as a constant simplicial Hopf algebra? Unclear.

So instead of trying to define \( G(A) \) directly, we make the observation that actually \( G_{K,S} = \pi_1^{\text{et}}(\mathbb{Z}[[\frac{1}{S}]], \ast) \). But this profinite group can be viewed as the fundamental group of a pro-(pointed simplicial set) \( X \) (in fact there are two ways of doing this, which we will describe in a moment). Then we make the observation that in the discrete case, i.e when \( A \) is an ordinary ring,

\[
\{ \rho : G_{K,S} \rightarrow G(A) \} = \{ (G(A))\text{-torsors over } |X| \} = \text{Hom}_{\text{top}}(|X|, BG(A))/\sim
\]

where \( |X| \) denotes the geometric realization of the pro-(simplicial set) \( X \), \( BG(A) \) is the classifying space of the group \( G(A) \), and \( \sim \) means that we’re taking homotopy classes of morphisms.

So what are these spaces \( X \)? One candidate is the étale topological type defined in [?] following [?]. This is a pro-(simplicial set) \( (X_i)_i \) indexed by étale hypercoverings of the scheme \( \text{Spec} \mathbb{Z}[[\frac{1}{S}]] \), whose \( \pi_1(X_i, \ast) := \)

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lim_\alpha \pi_1(X_\alpha, \ast) recovers the étale fundamental group of Spec \mathbb{Z}[1/\mathfrak{p}]$. For our purposes we can do something simpler, which is to note that $G_{K,S} = \lim_\alpha G_\alpha$ is a profinite group, and then we can take $X$ to be the pro-system $(X_\alpha)_\alpha = (N(G_\alpha))_\alpha$, where $N$ denotes the nerve of a group, viewed as a one-object groupoid.

1.2. Defining $BG$. So we now need to define some notion of $BG(A)$ for $A \in \text{sArt}_k$. For ordinary commutative rings $A$, note $BG(A)$ is the geometric realization of the nerve of $G(A)$: i.e. if $N_p(G(A))$ denotes the $p$-simplices of the nerve, then the functor of points

$$A \mapsto N_p(G(A))$$

is represented by $G \times^p$. Why is this true? To construct $BG$ for a discrete group, we construct $EG$ a contractible space and has a free action of $G$, and then we take $BG = EG/G$.

To do this, let $C$ be the category whose objects are indexed by elements of $G$, and whose morphisms are $g \to gh$. Let $D$ have one object, with morphisms labelled by $G$ and composition is multiplication. Then there’s a map $C \to D$.

In general, if $C$ is a small category, then $NC$ is a simplicial set where the 0-simplices are objects of $C$, and for $k > 0$, the $k$-simplices are $k$-tuples of composable morphisms.

So essentially, the nerve of $C$ (above) is contractible, and if we quotient by $G$, then we get the nerve of $D$.

With this in mind, we can now define $BG$ for a simplicial ring.

**Definition 1.1.** Consider the bisimplicial set $[p] \mapsto \text{Hom}_{\text{sCR}}(c(\mathcal{G}N,G),A)$. Then $BG(A)$ is $\text{Ex}^\infty$ (fibrant replacement) of the geometric realization of $\text{Hom}_{\text{sCR}}(c(\mathcal{G}N,G),A)$: note the geometric realization can be computed either by taking the ”total simplicial set” of the bisimplicial set, or by taking the diagonal: in fact these are homotopy equivalent (this is not easy: see [? ]).

Concretely, if $A$ is discrete, then $BG(A)$ is weakly equivalent to $NG(\pi_0(A))$. In the definition, we need the cofibrant replacements and the $\text{Ex}^\infty$ fibrant replacement in order for this thing to behave well, at least homotopy theoretically.

1.3. Galois Deformations. Now we can talk about Galois deformations. So let $(X_\alpha)_\alpha$ be either the étale topological type for Spec $\mathbb{Z}[1/\mathfrak{p}]$, or the pro-simplicial set $NG_\alpha$ where $\alpha$ varies over the finite Galois groups.

**Definition 1.2.** Now fix a map $\vec{p} : X \to BG(k)$ in pro $\text{sSet}$. Then define the **unframed deformation functor**

$$F_{\mathbb{Z}[1/\mathfrak{p}],\vec{p}} = \text{Hom}_{\text{pro} - \text{sSet}}(X, BG(A)) \times^h_{\text{Hom}_{\text{pro} - \text{sSet}}(X, BG(k))} \vec{p}$$

where $\text{Hom}_{\text{pro} - \text{sSet}}(X, BG(A)) = \colim_\alpha \text{Hom}_{\text{sSet}}(X_\alpha, BG(A))$ and $\vec{p}$ is really $\Delta^0$ with the map to $\text{Hom}_{\text{pro} - \text{sSet}}(X, BG(k))$ given by $\vec{p}$. There is also a framed version, where one replaces pro $\text{sSet}$ with pro $\text{sSet}_*$, the pro-category of pointed simplicial sets (and choosing basepoints for $X$ and $BG$). Keeping track of this basepoint can be roughly thought of as keeping track of a basis, which explains why this is the framed thing.

2. Pro-Representability

Recall the derived Schlessinger criterion from last week. This says that if $F : \text{sArt}_k \to \text{sSet}$ is formally cohesive, then it is pro-representable if and only if $\pi_i(tF) = 0$ for $i > 0$, where $tF$ is the tangent complex of $F$ as defined by Dougal last week.

But in our situation, $BG(k)$ will not be contractible, so $BG$ won’t be formally cohesive. So instead of having a tangent complex, we get a local system $tBG$ on $BG(k)$, i.e. a functor $L : \text{Simp}(BG(k)) \to \text{Ch}(k)$ sending all morphisms to quasi-isomorphisms (this was defined by Dougal last week). Recall the following result:
Proposition 2.1. If $F : \mathsf{Art}_k \to \mathsf{sSet}$ is now any homotopy invariant functor which preserves pullbacks (in the sense of Dougal’s talk), and given $\rho : X \to \mathcal{F}(k)$, consider the new functor

$$\text{F}_X,\rho(A) := \text{hofib}(\text{Hom}_{\mathsf{sSet}}(X, F(A)) \to \text{Hom}_{\mathsf{sSet}}(X, F(k))).$$

This is formally cohesive, and the tangent complex is

$$\text{tF}_X,\rho \cong C^*(X, \rho^*tF)$$

where $C^*$ is the cochains construction introduced last week.

So all we need to know is that $BG$ is homotopy invariant and preserves homotopy pullbacks, and then we can hope to apply the Derived Schlessinger Criterion by computing the homotopy groups of $\text{tF}_X,\rho$.

Note $BG$ is homotopy invariant because of the fibrant replacement we took in the definition. There’s a criterion to check that $BG$ preserves homotopy pullbacks.

Proposition 2.2. If $F : \mathsf{Art}_k \to \mathsf{sSet}$ is homotopy invariant, $F(A)$ is path-connected for all $A$, $A \to \Omega F(A)$ (loop space) preserves homotopy pullbacks, and $\pi_0\Omega F(A) \to \pi_0\Omega F(B)$ is surjective whenever $\pi_0 A \to \pi_0 B$ is surjective, then $F$ preserves homotopy pullbacks.

To apply this, we use that $G(A) := \text{Hom}_{s\mathsf{CR}}(\mathcal{C}(\mathcal{G}), A) \to \Omega BG(A)$ is a weak equivalence, which should heuristically be true by looking at the homotopy groups.

Lemma 2.3. The tangent complex of $A \to BG(A)$ is a local system on $BG(k)$ whose homology is $\mathfrak{g}$, the Lie algebra of $G(k)$, concentrated in degree 1 with a $G(k)$-action (via the adjoint action, conjugation) at a basepoint.

Note the $G(k)$-action arises because for any $Z$ a simplicial set and $L : \text{Simp}(Z) \to \text{Ch}(k)$ a local system, one can check directly that $\pi_1(Z, z)$ naturally acts on $H_*(L_z)$, where $z : \Delta^0 \to Z$ is some basepoint.

Proposition 2.4. The tangent complex $\text{tF}_{Z[1/\mathfrak{s}]},\rho$ is quasi-isomorphic to $C^{*+1}(X, \rho^*\mathfrak{g})$, and

$$\pi_{-i}(\text{tF}_{Z[1/\mathfrak{s}]},\rho) \cong H^{i+1}(X, \rho^*\mathfrak{g}) = H^{i+1}(Z[1/\mathfrak{s}]), \text{ad } \rho)$$

for $i \geq -1$ (for $i > 1$ we have $\pi_i(\text{tF}_{Z[1/\mathfrak{s}]},\rho) = 0$).

The last identification with étale cohomology (i.e. continuous group cohomology in this case) can be seen by identifying the cochains construction with étale cochains.

So if $G$ is an adjoint group (i.e. has trivial centralizer) and $\rho$ is Schur (i.e. the centralizer of $\rho$ is the center of the group), then this is telling us that $H^0(Z[1/\mathfrak{s}], \text{ad } \rho) = 0$, so we’re pro-representable by derived Schlessinger’s criterion. In general, one can modify this construction to take into account groups whose center is non-trivial (like $\text{GL}_n$): for the purposes of this study group, we’ll ignore this, but the details are worked out in Section 5.4 of [?].

Lemma 2.5. The functor $\pi_0\text{F}_{Z[1/\mathfrak{s}]},\rho : \mathsf{Art}_k \to \mathsf{Set}$ is isomorphic to the usual deformation functor if $\rho$ is Schur, i.e. the centralizer of $\rho$ is $Z(G)$.

We get a similar result for the framed deformations, without assuming the Schur condition.

Proof. This is basically unwinding definitions. We’re asking about components of $\text{Hom}_{\mathsf{sSet}}(X, BG(A))$, which correspond to isomorphism classes of $G(A)$-torsors over $X$, which in turn correspond to conjugacy classes of Galois representations $\rho : Z[1/S] \to G(A)$.
To dig a bit into why this should be true, consider the following equalities in the framed case. Suppose $A \in \text{Art}_k$ is an ordinary (underived) Artin ring with residue field $k$. Then if $G_{K,S} = \lim \alpha G_{\alpha}$

$$\pi_0 \text{Hom}_{\text{pro-}(s\text{Set})}((N(G_{\alpha}),*,(BG(A),*)) = \pi_0 \text{colim} \alpha \text{Hom}_{s\text{Set}}((N(G_{\alpha}),*,(BG(A),*))$$

$$= \text{colim} \alpha \pi_0 \text{Hom}_{s\text{Set}}((N(G_{\alpha}),*,(BG(A),*))$$

$$= \text{colim} \alpha \pi_0 \text{Hom}_{\text{Set}}((N(G_{\alpha}),*,(N(G(A)),*))$$

$$= \text{colim} \alpha \text{Hom}_{\text{Grp}}(G_{\alpha},G(A))$$

$$= \text{Hom}_{\text{pro-}(\text{Grp})}((G_{\alpha}),G(A))$$

$$= \text{Hom}_{\text{cont}}(\lim_{\alpha} G_{\alpha},G(A)).$$

The first equality is the definition of Hom sets in the pro-category, the second is the fact that $\pi_0$ commutes with filtered colimits, the third is the equivalence of $BG(A)$ with $N(G(A))$ when $A$ is discrete, the fourth is the adjunction between $\pi_1$ and the classifying space (in the homotopy category), the fifth is the definition of Hom in a pro-category again, and the sixth is the fact that pro-(finite groups) are the same as profinite groups with the profinite topology. \qed

3. Local Conditions

Let $\overline{\rho} : \pi_1 \text{Spec } Z[\frac{1}{\mathfrak{p}}] \to G(k)$ be a fixed Galois representation. If $v \in S$ is some finite place, let $F_{Q_v,\overline{\rho}}$ denote the deformation functor for $\overline{\rho}$ pulled back to $\pi_1 \text{Spec } Q_v$. We then get a natural transformation

$$F_{Z[\frac{1}{\mathfrak{p}}],\overline{\rho}} \to F_{Q_v,\overline{\rho}}$$

**Definition 3.1.** A local condition is a simplicially enriched functor $D_v : s\text{Art}_k \to s\text{Set}$ equipped with a natural transformation

$$D_v \to F_{Q_v,\overline{\rho}}.$$  

The corresponding global deformation functor with local conditions is defined to be

$$F_{Z[\frac{1}{\mathfrak{p}}],\overline{\rho}}^D := F_{Z[\frac{1}{\mathfrak{p}}],\overline{\rho}} \times_{F_{Q_v,\overline{\rho}}} D_v$$

**Remark 3.2.** We don’t necessarily need a map $D_v \to F_{Q_v,\overline{\rho}}$: we can take a zig-zag instead, where the maps going the “wrong way” are weak equivalences, and still make the theory work. See the remark after (9.1) in [?].

**Example 3.3** (Sanity Check). Suppose $\overline{\rho}$ is actually unramified at $v$, and let $S' = S \setminus \{v\}$. Then we have a natural transformation

$$F_{Z[\frac{1}{\mathfrak{p}}],\overline{\rho}} \to F_{Q_v,\overline{\rho}}.$$  

If we take $D_v = F_{Z[\frac{1}{\mathfrak{p}}],\overline{\rho}}$, then the global deformation functor

$$F_{Z[\frac{1}{\mathfrak{p}}],\overline{\rho}}^D = F_{Z[\frac{1}{\mathfrak{p}}],\overline{\rho}} \times_{F_{Q_v,\overline{\rho}}} D_v$$

is weakly equivalent to $F_{Z[\frac{1}{\mathfrak{p}}],\overline{\rho}}$: in [?] they prove this by noting that each functor is formally cohesive, so it suffices to check that the induced fiber sequence of tangent complexes is an isomorphism: see Section 8 of their paper for the details.

In practice, Galatius and Venkatesh want to turn underived local conditions into derived local conditions. Assume $F_{Q_v,\overline{\rho}}$ is pro-representable (this is the only case they will care about later) with simplicial pro-ring $\mathcal{R}_v$. Then we have maps

$$\mathcal{R}_v \to \pi_0 R_v =: R_v \to R_v^D,$$

where $R_v^D$ is the underived local condition. Now let

$$D_v := \text{Hom}(c(R_v^D),-).$$
We then get a zig-zag
\[ R_v \xrightarrow{\sim} c(R_v) \to c(R_v) \to c(R_v^D), \]
and by taking Hom we get
\[ \text{Hom}_{sSet}(c(R_v^D), -) \to \text{Hom}_{sSet}(c(R_v), -) \to \text{Hom}_{\text{pro-sSet}}(c(R_v), -) \xrightarrow{\sim} \text{Hom}_{\text{pro-sSet}}(R_v, -). \]
Now use Remark 3.2 to obtain a local condition.

**Theorem 3.4.** Suppose $R_v^D$ is formally smooth. Then
\[ t^i F_{D[1/\mathcal{S}], \mathfrak{p}} \cong H^{i+1}(Z[1/\mathcal{S}], \text{ad} \rho). \]

**Proof Sketch.** We have a map $tD_v \to tF_{Q_v, \mathfrak{p}}$ and a quasi-isomorphism $\tau_{\geq 0}(tD_v) \to tD_v$. Therefore,
\[ tF_{Z[1/\mathcal{S}], \mathfrak{p}} \xrightarrow{\sim} \text{hofib}(tF_{Z[1/\mathcal{S}], \mathfrak{p}} \oplus \tau_{\geq 0}(tD_v) \to tF_{Q_v, \mathfrak{p}}). \]
is a natural weak equivalence.

But we have a factorization $\tau_{\geq 0}(D_v) \to \tau_{\geq 0}(tF_{Q_v, \mathfrak{p}}) \to tF_{Q_v, \mathfrak{p}}$. The source and target of the first map have homotopy only in degree 0, so the first map induces a quasi-isomorphism onto the subcomplex $\tau_{\geq 0}(tF_{Q_v, \mathfrak{p}})$ whose cohomology is $H^1_{\text{dR}}(Q_v, \text{ad} \rho)$. Note the fact that this is true is not obvious.

Going forward, we have some extra assumptions on $\mathfrak{p}$:

1. $H^0(Q_p, \text{ad} \mathfrak{p}) = H^2(Q_p, \text{ad} \mathfrak{p}) = 0$: this means that at $p$, the universal deformation problem is pro-representable and formally smooth.

2. For $v \in S \setminus \{p\}$, $H^j(Q_v, \text{ad} \mathfrak{p}) = 0$ for $j = 0, 1, 2$: this means that we have trivial deformation theory away from $p$ in $S$.

3. (big image) The image of $\mathfrak{p}|_{Q(\zeta_p)}$ contains the image of $G^{sc}(k)$ (simply connected cover) in $G(k)$.

4. At $p$, $\mathfrak{p}$ is torsion crystalline, and there is an unobstructed subfunctor $\text{Def}^{\text{cris}} \subset \text{Def}_p$ such that the tangent space is $H^1_{\text{dR}}(Q_v, \text{ad} \mathfrak{p})$. 
