Simplicial commutative rings

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1 Classical setup

Let $k$ be a field and let $\text{Art}_k$ be the category of Artin local rings with residue field $k$ and let $\mathcal{F} : \text{Art}_k \to \text{Set}$ be a functor ("deformation problem"). We are interested in properties of these kinds of functors, for example (pro)-representability. Today we want to replace this classical setup with a derived setup. Replace sets with simplicial sets and $\text{Art}_k$ with $\text{sArt}_k$ and functors with simplicially enriched functors.

2 Simplicial commutative rings

Definition 1. The category of simplicial commutative rings $\text{sCR}$ is the category of simplicial objects in the category of commutative rings, i.e., the functor category $[\Delta^{\text{op}}, \text{CR}]$.

This is the same thing as ring objects in the category of simplicial sets (because limits are computed pointwise).

The free-forgetful adjunction

$$\text{Forget} : \text{CR} \leftrightarrow \text{Sets} : Z[-]$$

extends to an adjunction

$$\text{sCR} \leftrightarrow \text{sSets}$$

by applying the polynomial ring functor to the set of $n$-simplices. We can use this adjunction to transfer the model structure from $\text{sSets}$ to $\text{sCR}$, which has the following description: A map $f : R \to S$ is

- Weak equivalence if and only if the map of the underlying simplicial sets is a weak equivalence.
- Fibration if and only if the map of the underlying simplicial sets is a (Kan) fibration.
- Cofibration if and only if it satisfies the left lifting property (LLP) with respect to trivial fibrations.

Remark 1. Every simplicial commutative ring is in particular a simplicial (abelian) group, and so it is fibrant.
2.1 Enrichment

Recall that \textbf{sSets} is self-enriched, i.e., it has internal hom objects (this is true just because it is a presheaf category). These have the explicit description

\[
\text{Hom}(X,Y)_n := \text{hom}(\Delta^n, \text{Hom}(X,Y)) = \text{hom}(X \times \Delta^n, Y)
\]

and we use the notation \(Y^X := \text{Hom}(X,Y)\) when it is convenient.

**Fact 1.** If \(i : X \to Y\) is a cofibration and \(p : A \to B\) is a fibration (of simplicial sets), then the induced map

\[
A^Y \to A^X \times_{B^X} B^Y
\]

is a fibration and it is a trivial fibration if either \(i\) or \(p\) is trivial.

For simplicial commutative rings \(R,S\), we can form the equalizer

\[
\text{Hom}(R,S) \xrightarrow{0} S
\]

which is the subobject of "0-preserving maps". Similarly we can define the subobject of maps "Preserving 1", that are "additive", "multiplicative" and taking the intersection we get an object

\[
\text{sCR}(R,S).
\]

For \(n \geq 0\) the mapping complex \(S^\Delta^n\) has the structure of a simplicial ring and we can describe

\[
\text{sCR}(R,S)_n = \text{sCR}(R, S^\Delta^n).
\]

This gives the category \textbf{sCR} the structure of a category enriched over \textbf{sSets}.

3 Simplicial Artin local rings

Write \(I = \Delta^1\) and we define the boundary of the \(n\)-cube by

\[
\partial I^n = \bigcup_{1 \leq k \leq n} I^{k-1} \times \partial I \times I^{n-k}.
\]

The simplicial circle \(S^n\) is then defined to be the pushout (so it is naturally a pointed simplicial set)

\[
\partial I^n \longrightarrow I^n \quad \longrightarrow \quad S^n
\]

This is not the usual definition but it has the advantage that

\[
S^{n+m} := S^n \wedge S^m := (S^n \times S^m) / S^n \times \{\ast\} \cup \{\ast\} \times S^n
\]
holds on the nose, rather than up to homotopy. For a simplicial commutative ring $R$ we define
\[ \pi_n(R) := \hom((S^n, \{\ast\}), (R, 0))/ \sim = \pi_0(\Hom_*(S^n, R)). \]
Then we define the associated graded ring as
\[ \pi_*(R) := \bigoplus_{n \geq 0} \pi_n(R) \]
which is a graded ring because there are maps
\[ \Hom_*(S^n, R) \times \Hom_*(S^m, R) \to \Hom_*(S^n \times S^m, R \times R) \to \Hom_*(S^n \wedge S^m, R) \]
where the last map is induced by multiplication. If we now take connected components then we get maps
\[ \pi_n(R) \times \pi_m(R) \to \pi_{n+m}(R). \]

**Definition 2.** Let $k$ be a field considered as a discrete simplicial set, then we define the category $\sArt_k$ of simplicial Artin local rings as the full subcategory of $\sCR/k$ (simplicial commutative rings with a fixed map to $k$) on the objects $R$ satisfying:
- The discrete ring $\pi_0(R)$ is an Artian local ring with residue field $k$
- The associated graded ring $\pi_*(R)$ is a finitely generated $\pi_0(R)$ module.

## 4 Deformation problems

We will study functors $F : \sArt_k \to \sSets$.

**Definition 3.** We call $F$ homotopy invariant if it preserves weak equivalences. A simplicial enrichment of $F$ is a choice of morphisms
\[ \sArt_k(R, S) \to \Hom(F(R), F(S)) \]
for each $R, S \in \sArt_k$ which is compatible with compositions and extending the usual functoriality of $F$ on zero simplices.

**Lemma 1.** Important example: If $R \in \sArt_k$ is cofibrant, then
\[ \sArt_k(R, -) \]
is simplicially enriched and homotopy invariant.

**Proof.** Simplicial enrichment: For $S, T \in \sArt_k$ we want to define a map
\[ \sArt_k(S, T) \to \Hom(\sArt_k(R, S), \sArt_k(R, T)). \]
By adjunction this would correspond to a map (by the exponential law)
\[ \sArt_k(S, T) \times \sArt_k(R, S) \to \sArt_k(R, T) \]
which we can take to be the composition morphism, which clearly extends the usual functoriality on zero simplices. For homotopy invariance we note the following: Since every simplicial commutative ring is fibrant, every weak equivalence is a weak equivalence between fibrant objects. By Ken Brown’s Lemma, it suffices to show the functor $\mathbf{sArt}_k(R, -)$ preserves trivial fibrations. So let $f : S \to T$ be such a trivial fibration and $X \to Y$ a cofibration between simplicial sets. Then we want to show that any diagram

$$
\begin{array}{ccc}
X & \xrightarrow{g} & \mathbf{sArt}_k(R, S) \\
\downarrow & & \downarrow f \\
Y & \xrightarrow{h} & \mathbf{sArt}_k(R, T)
\end{array}
$$

has a lifting, proving that $f$ is a trivial fibration. We claim that the lifting in the diagram is equivalent to a lift in the following diagram (using the exponential law)

$$
\begin{array}{ccc}
S^Y & \xrightarrow{\epsilon} & TY \\
\downarrow & & \downarrow \\
R & \xrightarrow{\delta} & S^X \times_T TY.
\end{array}
$$

But since $X \to Y$ is cofibrant and $S \to T$ is a trivial fibration we find that the vertical map is a trivial fibration (by the important fact stated in the beginning). We conclude that a lift exists since $R$ is cofibrant.

**Definition 4.** A natural weak equivalence $\eta : \mathcal{F} \to \mathcal{G}$ between functors $\mathcal{F}, \mathcal{G} : \mathbf{sArt}_k \to \mathbf{sSets}$ is a natural transformation such that all components

$$
\eta_R : \mathcal{F}(R) \to \mathcal{G}(R)
$$

are weak equivalences. The functors $\mathcal{F}, \mathcal{G}$ are called naturally weakly equivalent if there is a zig-zag of natural weak equivalences.

**Lemma 2** (Technical Lemma). If $\mathcal{F}$ is homotopy invariant, then there exists an $\mathcal{F}'$, which is simplicially enriched and has values in Kan complexes, and a natural weak equivalence

$$
\mathcal{F} \to \mathcal{F}'.
$$

Moreover we can make $\mathcal{F} \to \mathcal{F}'$ functorial in $\mathcal{F}$.

The importance of this functoriality is that we can replace a zig-zag of homotopy invariant functors by a a weakly equivalent zig-zag such that all functors (except possibly the endpoints) are simplicially enriched and Kan valued.

**Definition 5.** We call a functor $\mathcal{F} : \mathbf{sArt}_k \to \mathbf{sSets}$ representable if it is naturally weakly equivalent to $\mathbf{sArt}_k(R, -)$ for some cofibrant $R \in \mathbf{sArt}_k$.

We remark that any representable functor is homotopy invariant, since $\mathbf{sArt}_k(R, -)$ is and homotopy invariance is preserved by natural weak equivalence.

If $\mathcal{F}, \mathcal{G}$ are simplicially enriched, then there is a simplicial set $\text{Nat}(\mathcal{F}, \mathcal{G})$ whose simplices are described by

$$
\text{Nat}(\mathcal{F}, \mathcal{G})_n := \{\text{Natural transformations } \Delta^n \times \mathcal{F} \to \mathcal{G}\}$$
where $\Delta^n$ denotes the constant functor with value $\Delta^n$. We get an enriched Yoneda lemma:

$$\text{Nat}(\text{sArt}_k(R,-), F) \cong F(R).$$

**Proposition 1.** If $F$ is simplicially enriched then $F$ is representable if and only if there exists a cofibrant $R$ and a vertex $v \in F(R)_0$ such that the corresponding map (coming from the enriched Yoneda Lemma)

$$\text{sArt}_k(R,-) \to F$$

is a natural weak equivalence.

**Proof.** If there is such a vertex, then $F$ is representable by definition.

Now first suppose that there is a natural weak equivalence $\eta : F \to \text{sCR}(R,-)$. Choose $v \in F(R)_0$ such that $\eta(v)$ is in the same connected component as the identity in $\text{sCR}(R,R)_0$, let

$$\nu : \text{sArt}_k(R,-) \to F$$

be the corresponding map (under enriched Yoneda).

Then $\eta \circ \nu : \text{sArt}_k(R,-) \to \text{sArt}_k(R,-)$ corresponds to $\eta(v)$. Since $\text{sArt}_k(R,R)$ is Kan there is an actual homotopy $H$ between $\eta(v)$ and the identity map. By simplicial Yoneda this correspond to a homotopy

$$\Delta^1 \times \text{sArt}_k(R,-) \to \text{sArt}_k(R,-)$$

between the identity natural transformation and $\eta \circ \nu$. This implies that $\nu$ is a natural weak equivalence. This basically means that we can now get rid of hats.

![Diagram](attachment:image.png)

Now suppose we have

$$\text{F} \leftarrow \text{G} \leftrightarrow \text{sArt}_k(R,-)$$

with $\mathcal{G}$ Kan valued and simplicially enriched. Then we choose $x \in F(R)_0$ such that $\Phi(x) \sim w$. Since $\mathcal{G}(R)$ is Kan we can find a homotopy $H : \Delta^1 \to \mathcal{G}(R)$ between $\phi(x)$ and $w$ which shows that $x$ is a natural weak equivalence.

For the general case, we have a zig-zag of natural weak equivalences

$$\text{F} \leftarrow \mathcal{G}_1 \leftrightarrow \mathcal{G}_2 \leftarrow \ldots \leftarrow \mathcal{G}_{n-1} \leftarrow \text{sArt}_k(R,-)$$

All the functors in this zig-zag are homotopy invariant (because representable) and hence by the Technical Lemma we may assume that $\mathcal{G}_1, ..., \mathcal{G}_n$ are simplicially enriched and Kan valued. We may then argue by induction on $n$ using the two cases above. □
Definition 6. We say that a functor $F : \mathbf{sArt}_k \to \mathbf{sSets}$ is pro-representable if there is a filtered category $J$ and a pro-object (with $R_j$ cofibrant)

$$D : J \to \mathbf{sArt}_k$$

$$j \mapsto R_j$$

such that $F$ is naturally weakly equivalent to the functor

$$\text{colim}_{J^{op}} \mathbf{sArt}_k(R_j, -).$$

The functor $F$ is sequentially pro-representable if we can choose $J = (\mathbb{N}, <)$ by which we mean the category

$$\{ \cdots \to * \to * \}.$$