We want to prove an $R = \mathbb{T}$ theorem, where $R$ is a Galois deformation ring and $\mathbb{T}$ is a Hecke-algebra. The ring $R$ will be a deformation ring which parametrizes lifts of $\overline{\rho}$ that should should arise from some space of automorphic forms $S$ (which essentially means the lifts are 'geometric'). The space of automorphic forms $S$ is acted on by a Hecke algebra $\mathbb{T}$. If we can prove an $R = \mathbb{T}$ theorem then such a lift $\rho : \text{Gal}(\overline{F}/F) \to \text{Gl}_n(\mathcal{O})$ gives rise to $\mathbb{T} \to \mathcal{O}$ which should correspond to an eigenform $f \in S$.

1 \ $R$

Let $F$ be a totally real field of degree $d = [F : \mathbb{Q}]$, let $p$ be an odd prime unramified in $F$ and let

$$G_{F, \Sigma_Q} = \text{Gal}(F_{\Sigma_Q}/F)$$

where $F_{\Sigma_Q} \subset \overline{F}$ is the maximal unramified outside of $\Sigma_Q$ extension of $F$ inside $\overline{F}$. Let $K/\mathbb{Q}_p$ be a field of coefficients with ring of integers $\mathcal{O}$ and residue field $k$. Consider

$$\overline{\rho} : G_{F, \Sigma_Q} \to \text{Gl}_2(k)$$

and assume that:

- The determinant of $\overline{\rho}$ is the inverse of the cyclotomic character $\overline{\tau}$.
- The representation $\overline{\rho}$ is unramified outside places $v \mid p$.
- The representation $\overline{\rho}|_{G_{F,v}}$ is absolutely irreducible.
- The representation $\overline{\rho}|_{G_{F,v}}$ is finite flat for all $v \mid p$ (here $G_{F,v}$ is the absolute Galois group of the completion $F_v$ of $F$ at $v$). Finite flat means that there is a finite flat group scheme $\mathcal{G}/\mathcal{O}_{F,v}$ such that

$$\tau \otimes \overline{\rho}|_{G_{F,v}} \sim \mathcal{G}(\overline{F_v}).$$

This condition is satisfied if for example $\overline{\rho}|_{G_{F,v}}$ is of the form

$$\begin{pmatrix} 1 & * \\ 0 & \overline{\tau}^{-1} \end{pmatrix}$$

split by $F_v(\mu_p, \sqrt[p]{u_1}, \cdots, \sqrt[p]{u_k})$ for some $u_i \in \mathcal{O}_{\overline{F_v}}^\times$. 
Define $R_{\mathfrak{p},Q}$ as in Misja’s talk parametrizing deformations $\rho : G_{F,\Sigma_Q} \to \text{Gl}_2(A)$ satisfying:

- The determinant $\det \rho = \epsilon^{-1}$.
- The restriction $\rho|_{G_{F,v}}$ is finite flat for all $v | p$, which means that $\rho \mod \mathfrak{m}_A^n$ is finite flat in the previous sense, for all $n$.
- The representation $\rho$ is unramified outside $\Sigma_Q$.

2  $T$

Let $G = \text{Res}_{F/Q} \text{Gl}_2/F$ (or more generally consider multiplicative group associated to a quaternion algebra $D$ over $F$ ramified precisely at $2m$ Archimedean primes and no finite primes, in which case replace $d$ by $d' := d - 2m$ below). Consider the associated symmetric space

$$Y(K_Q) = G(Q) \backslash X \times G(K_f)/K_Q = \coprod \Gamma_i \backslash X$$

where $X = (\mathbb{H}^\pm)^d$, $\mathbb{H}^\pm$ is the disjoint union of the upper and lower half plane and where $K_Q$ is given by

$$\prod_{v \notin Q} \text{Gl}_2(O_{F_v}) \times \prod_{v \in Q} K_v$$

where $K_v$ is to be described later. (Actually we need to fix some auxiliary level away from $Q$ to make sure $K_Q$ is neat or equivalently that $Y_{K_Q}$ is smooth and that certain coverings will be étale.) Then $H^i(Y(K_Q), \mathcal{O})$ has a natural action of Hecke operators $T_v$ for all places $v \notin Q$ and operators $U_v$ for $v \in Q$. Let $\mathbb{T}_Q$ be the $\mathcal{O}$-subalgebra of

$$\text{End}_\mathcal{O}(H^d(Y(K_Q), \mathcal{O}))$$

generated by the operators $T_v, U_v$. There is a Hecke-equivariant map

$$S_{(2,\ldots,2)}(K_Q) \to H^d_{\text{cusp}}(Y_{K_Q}, \mathbb{C}) \subset H^d(Y_{K_Q}, \mathbb{C})$$

where $S_{(2,\ldots,2)}(K_Q)$ is the space of parallel weight 2 Hilbert modular cuspforms of level $K_Q$. So if $f \in S_Q$ is an eigenform for the Hecke operators with eigenvalues in $\mathcal{O}_f \subset \mathbb{C}$ (and assume $\mathcal{O}_f \subset \mathcal{O}$ which can be achieved by enlarging $\mathcal{O}$), then we get a map

$$\mathbb{T}_Q \to \mathcal{O}$$

$$T_v \mapsto a_v$$

where $a_v$ is the $T_v$ eigenvalue of $f$.

3  $R_Q \to \mathbb{T}_Q$

We start with a result due to Carayol and Taylor:
Theorem 1. For \( f \) as above there is an associated Galois representation 
\[
\rho_f : G_{F, \Sigma_Q} \to \text{GL}_2(O)
\]
such that for all \( v \notin \Sigma_Q \) that characteristic polynomial of \( \rho_f(\text{Frob}_v) \) is 
\[
X^2 - a_v X + \text{Nm}_{F/Q} v.
\]
Moreover the representation \( \rho_f|_{G_{F_v}} \) is finite flat for all \( v \mid p \) and the \( \text{GL}_2(K) \) valued representation is irreducible.

Assume that \( \rho = \overline{\rho_f} \) for some \( f \) as above (in fact, then there is such an \( f \) with \( Q = \emptyset \) by level lowering results of Jarvis, Rajaei, Fujiwara, Gee). Then for any such \( f \) we get a diagram
\[
\begin{array}{ccc}
R_{\rho,Q} & \longrightarrow & \mathcal{O} \\
\downarrow & & \downarrow \\
\mathbb{T}_Q & \longrightarrow & k
\end{array}
\]
such that \( T_v \mapsto a_v \mapsto \text{Tr}(\rho(\text{Frob}_v)) \). This map does not depend on \( f \) and gives us a kernel \( m_Q \subset \mathbb{T}_Q \). Using the fact that \( \overline{\rho} \) is irreducible be know by Dimitrov (but it is easy if \( d - 2m = 0,1 \)) that
\[
H^i(Y(K_Q), \mathcal{O})_{m_Q} = \begin{cases} 0 & \text{if } i \neq q \\ H^d_{\text{cusp}}(Y(K_Q), \mathcal{O})_{m_Q} & \text{if } i = d \end{cases}
\]
and moreover that this is torsionfree. But the Hecke algebra \( \mathbb{T}_{Q,m_Q} \) acts faithfully on \( H^d(Y(K_Q), \mathcal{O})_{m_Q} \), so in fact every map
\[
\mathbb{T}_{Q,m_Q} \rightarrow \mathcal{O}
\]
comes from \( f \) as above so we get a commutative diagram
\[
\begin{array}{ccc}
R_{\rho,Q} & \longrightarrow & \Pi_f \mathcal{O}, \\
\downarrow & & \downarrow \\
\mathbb{T}_{Q,m_Q} & \longrightarrow & \mathcal{O}
\end{array}
\]
(using that \( \mathbb{T}_{Q,m_Q} \rightarrow \Pi_f \mathcal{O} \) is injective and \( R_{\rho,Q} \) is generated by traces of images under the universal deformation).

4 Taylor-Wiles primes

Suppose that \( v \in Q \) satisfy \( \text{Nm}(v) = 1 \mod p \) and that \( \overline{\rho}(\text{Frob}_v) \) has distinct eigenvalues in \( k \). Then
\[
\rho_{Q}^{\text{univ}} : G_{F, \Sigma_Q} \to \text{GL}_2(R_{\rho,Q})
\]
has the property that
\[
\rho_{Q}^{\text{univ}}|_{G_{F_v}} \sim \begin{pmatrix} \chi & 0 \\ 0 & \epsilon^{-1} \chi^{-1} \end{pmatrix}
\]
where (here $I_{F,v} \subset G_{F_v}$ is the inertia group)

$$\chi|_{I_{F_v}}$$

factors as a character (the first map is local class field theory).

$$I_{F,v} \to O_{F_v}^\times \to k_v^\times \to \Delta_v.$$  

Where $\Delta_v$ is the maximal pro-$p$ quotient of $k_v^\times$, which is nontrivial because $\# k_v^\times$ is divisible by $p$. All in all $\chi$ will determine a character $\Delta_v \to R[\Delta]_Q$ leading to a ring homomorphism

$$O[\Delta_Q] \to R[\Delta]_Q$$

where

$$\Delta_Q = \prod_{v \in Q} \Delta_v.$$  

Recall that we haven’t defined the groups $K_v$ yet (for $v \in Q$). We define

$$K_{v,0} = \left( \begin{array}{cc} * & * \\ 0 & * \end{array} \right) \mod v$$

containing

$$K_v = \left\{ \left( \begin{array}{cc} a & * \\ 0 & d \end{array} \right) \mod v \left| ad^{-1} \mapsto 1 \in \Delta_v \right. \right\}.$$  

Note that we trivially get $K_v/K_{v,0} = \Delta_v$ for all $v \in Q$. So now we get a $\Delta_v$ torsor

$$\begin{array}{c}
Y(K_Q) \\
\downarrow \\
Y(K_{Q,0})
\end{array}$$

inducing a map on cohomology

$$M_Q := H^d(Y(K_Q), O)_{m_Q} \to H^d(Y(K_{0,Q}, O)_{m_Q}.$$  

This gives two actions of $\Delta_Q$ on the space of modular forms $M_Q$. One defined geometrically as above and one via

$$O[\Delta_Q] \to R[\Delta]_Q \to T_{Q,m_Q}.$$  

The fact that the two actions coincide is local-global compatibility of Langlands correspondences at $v \in Q$. Furthermore $M_Q$ is free over $O[\Delta_Q]$ (this is a general fact about Galois covers and proved using the Hochschild-Serre spectral sequence).
5 Galois cohomology

Recall that $R^{\text{univ}}_{\overline{\rho}, Q}$ is generated by

$$\dim_k H^1(G_{F, \Sigma}, \text{Ad} \overline{\rho})$$

generators as an $O$-algebra. Once we start adding conditions we will need less generators:

- Fixing determinant $\det \rho = \epsilon^{-1}$ means we replace $\text{Ad} \overline{\rho}$ with $\text{Ad}^0 \overline{\rho}$ (space of trace zero endomorphisms).
- The finite flat (think crystalline) condition tells us we get classes with image in a certain subspace

$$L_v \subset H^1(G_{F,v}, \text{Ad}^0 \overline{\rho})$$

of the local Galois cohomology groups.

Let us denote this subspace by $H^1_Q$ and its dimension by $r_Q$. Local Tate duality tells us that

$$H^i(G_{F,v}, \text{Ad}^0 \overline{\rho}) \cong H^{2-i}(G_{F,v}, \text{Ad}^0 \overline{\rho}(1))^\vee.$$ 

For $i = 1$ the former space contains $L_v$ and we let $L^\perp_v$ be its orthogonal complement, so we get Selmer groups (global classes that map to $L_v$ or $L^\perp_v$ for all $v$)

$$H^1_Q, H^1_Q \perp.$$

For $v \in Q$ we have that $L_v = H^1$ and $L^\perp_v = 0$, and in this case $\dim H^0 = 1$ and $\dim H^1 = 2$ by the local Euler characteristic formula

$$\dim H^1 - (\dim H^0 + \dim H^2) = \begin{cases} 0 & \text{if } v \nmid p \\ j \cdot 3 & \text{if } v \mid p \end{cases}$$

where $j = [F_v : \mathbb{Q}_p]$ and $3 = \dim \text{Ad}^0$. For $v \mid p$, we have $\dim L_v = [F_v : \mathbb{Q}_p] + \dim H^0$. (This is harder and uses that $p$ is unramified in $F$.) Now global duality and Euler characteristic computations gives us the formula of Wiles

$$\# H^1_Q / \# H^1_Q \perp = \frac{\#H^0(G_F, \text{Ad}^0 \overline{\rho})}{\#H^0(G_f, \text{Ad}^0 \overline{\rho}(1))} \cdot \prod_v \frac{\#L_v}{\#H^0(G_{F,v}, \text{Ad}^0 \overline{\rho})}.$$ 

The first fraction on the right hand side is zero by our irreducibility assumption on $\overline{\rho}$. Putting everything together we get

$$\dim H^1_Q - \dim H^1_Q \perp = \sum_{v \mid \infty} (-1) + \sum_v [F_v : \mathbb{Q}_p] + \#Q,$$

The 'numerical coincidence' occurs since we are working with $\text{Gl}_2$ and since $F$ is totally real we see that

$$\sum_{v \mid \infty} (-1) = \sum_{v \mid p} [F_v : \mathbb{Q}_p]$$

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since they are both just equal to the the degree of $F$ over $Q$. Let $r = \dim H^1_{\emptyset}$ and use the Chebotarev density theorem to choose for each $n$ a set of Taylor-Wiles primes $Q_n = \{v_1, \cdots, v_r\}$ such that $Nm(v) \equiv 1 \mod p^n$ and

$$H^1_{\emptyset}(G_F, \Ad^0 \overline{\rho}) \to \bigoplus_{v \in Q} H^1(G_{F_v}, \Ad^0 \overline{\rho}(1))$$

is an isomorphism. The idea for the proof is to realize this in terms of conditions on $\text{Frob}_v \in \text{Gal}(L_n/F)$ where $L_n = L(\zeta_{p^n})$ and $L$ is the splitting field of $\Ad^0 \overline{\rho}$. Then

$$H^1_{Q_n^+} = 0$$

and so

$$\#H^1_{Q_n} = \#H^1_{\emptyset}$$

and so

$$R_{\overline{\rho}, Q_n}$$

is generated by $r = \#Q_n$ elements.

6 Patching

Now for each $n \geq 1$ we have a map

$$\mathcal{O}[\Delta_{Q_n}] \to R_{\overline{\rho}, Q_n} \to T_{Q_n} \to \text{End}(M_{Q_n})$$

which fits into a diagram as follows (taking compatible presentations of both $R_{\overline{\rho}, Q_n}$ and $\mathcal{O}[\Delta_{Q_n}]$):

$$\mathcal{O}[\Delta_{Q_n}] \to R_{\overline{\rho}, Q_n} \to T_{Q_n} \to \text{End}(M_{Q_n})$$

with $M_{Q_n}/(S_1, \cdots, S_r) = M_{\emptyset}$.

Now we essentially want to take the limit over $n$. Since there are only finitely many such data mod $m^n$, there is a compatible subsequence and we can take limits to obtain

$$\mathcal{O}[S_1, \cdots, S_r] \to \mathcal{O}[T_1, \cdots, T_r] \to R_\infty \to T_\infty \to \text{End}(M_\infty)$$

with $M_\infty/(S_1, \cdots, S_r) = M_{\emptyset}$. Moreover we know that $M_\infty$ is free over the first ring (lets call it $A$ and lets call the second ring $B$), we want to show that $M_\infty$ is also free over $B$: The Auslander-Buchsbaum formula tells us that

$$\text{depth}_B M_\infty + \text{proj dim}_B M_\infty = r + 1$$
and since $S_1, \cdots, S_r, \omega_\mathcal{O}$ is a regular sequence we know that

$$\text{depth}_B M_\infty = r + 1.$$ 

Hence the projective dimension is zero, i.e., $M_\infty$ is free over $B$. This means that $M_\infty$ is a faithful $B$-module and so the map from $B$ to $\text{End}_{M_\infty}$ is injective. This tells us immediately that

$$B \cong R_\infty \cong T_\infty$$

over which $M_\infty$ is free. Going back down this gives $R = T$.

Remark 1. You don’t actually need to patch. See the paper by Brochard (Compositio, 2017) which proves a commutative result that implies you only need $Q_1$. 