1. Cohomology of Arithmetic Groups and Automorphic Forms

1.1. Symmetric Spaces. Let $G$ be a semisimple algebraic group defined over $\mathbb{Q}$: as an example, one could take a number field $F/\mathbb{Q}$ and then take $G = \text{Res}_{F/\mathbb{Q}} \text{SL}_n$. Then the symmetric space for $G$ is given by $X = G(\mathbb{R})/K_\infty$, where $K_\infty$ is a maximal connected compact subgroup of $G(\mathbb{R})$. If $G = \text{Res}_{F/\mathbb{Q}} (\text{SL}_2)$ as above and $F$ has signature $(r,s)$ (i.e. $F$ has $r$ real embeddings and $s$ pairs of complex embeddings), then $X = (\text{SL}_2(\mathbb{R})/\text{SO}_2(\mathbb{R}))^r \times (\text{SL}_2(\mathbb{C})/\text{SU}(2))^s$.

Note that $\text{SL}_2(\mathbb{R})/\text{SO}_2(\mathbb{R})$ is isomorphic to hyperbolic 2-space $\mathcal{H}^2$ (i.e. the complex upper half plane with the hyperbolic metric) and $\text{SL}_2(\mathbb{C})/\text{SU}(2)$ is similarly hyperbolic 3-space $\mathcal{H}^3$.

The symmetric space $X$ is a real manifold, but for dimension reasons may not have a complex structure: for example, if $F/\mathbb{Q}$ is an imaginary quadratic extension, then $X \cong \mathcal{H}^3$, which is a 3-dimensional real manifold.

1.2. Locally Symmetric Spaces. Let $K = \prod_p K_p \subset G(\mathbb{A}^\infty_{\mathbb{Q}})$, where $\mathbb{A}^\infty_{\mathbb{Q}}$ is the ring of finite adèles (of $\mathbb{Q}$) and each $K_p \subset G(\mathbb{Q}_p)$ is a compact open subgroup. Then the locally symmetric space attached to $K$ (and $G$) is $Y(K) := G(\mathbb{Q}) \backslash [X \times G(\mathbb{A}_F)/K]$. One can show that in fact, there exist finitely many arithmetic groups $\Gamma_i$ acting on $X$ for which $Y(K) = \bigsqcup_i \Gamma_i \backslash X$. When $\Gamma_i$ are small enough (neat) then each $\Gamma_i$ acts freely and properly discontinuously on $X$, and $Y(K)$ is naturally a smooth manifold (locally isomorphic to $X$, hence a locally symmetric space).

1.3. Cohomology. For us, the primary object of study will be the singular cohomology $H^*(Y(K), \mathbb{Z})$ which is alternatively computed as the group cohomology $\bigoplus_{i,n \geq 0} H^n(\Gamma_i, \mathbb{Z})$. There is a Hecke algebra $T$ acting on $H^*(Y(K), \mathbb{Z})$, and the complex vector space $H^*(Y(K), \mathbb{C})$, together with its $T$-module structure, can be described in terms of automorphic representations of $G$.

Example 1.3.1. If $G = \text{SL}_2(\mathbb{Q})$, then this is the relationship between modular forms and cohomology of modular curves via the Eichler-Shimura isomorphism. In general, this relationship is given by a theorem of Franke (or Matsushima’s formula if $Y(K)$ is compact, or we restrict attention to the contribution of cuspidal automorphic representations).

Date: January 9, 2019.
Remark 1.3.1. Only a special subset of automorphic representations contribute to $H^*(Y(K), \mathbb{C})$: for cuspidal automorphic representations $\pi$ we see a contribution to $H^*(Y(K), \mathbb{C})$ if $\pi_\infty$ has non-vanishing $(\mathfrak{g}, K_\infty)$-cohomology (c.f. Matsushima’s formula) and $(\pi_\infty)^K \neq 0$.

Fix a cuspidal tempered system of Hecke eigenvalues $\chi : \mathbb{T} \to \mathbb{Z}$. Here we say that a system of Hecke eigenvalues is cuspidal and tempered if all of the automorphic representations $\pi$ with this system of Hecke eigenvalues are cuspidal and tempered (at Archimedean places).

Proposition 1.3.1 (Borel–Wallach + Matsushima’s Formula). The generalized $\chi$-eigenspaces $H^i(Y(K), \mathbb{Q})_\chi$ satisfy:

1. $H^i(Y(K), \mathbb{Q})_\chi = 0$ if $i \notin [q_0(G), q_0(G) + \ell_0(G)]$
2. $\dim H^i(Y(K), \mathbb{Q})_\chi = \dim H^{0i(G)}(Y(K), \mathbb{Q})_\chi \times (\ell_0(G))_{i-q_0(G)}$

The integers $\ell_0(G)$ and $q_0(G)$ (especially $\ell_0(G)$) will be extremely important for the rest of the study group: we define

$$\ell_0(G) = \text{rank } G(\mathbb{R}) - \text{rank } K_\infty,$$

and we define

$$q_0(G) = \frac{1}{2} (\dim Y(K) - \ell_0(G)).$$

Example 1.3.2. Say $F/\mathbb{Q}$ has signature $(r, s)$, as before. Let $G = \text{Res}_{F/\mathbb{Q}} \text{SL}_2, F$. Then $\ell_0(G) = s$ and $q_0(G) = r + s$. In this case

$$X = (H^2)^r \times (H^3)^s,$$

and $\dim Y(K) = 2r + 3s$. Therefore, $H^i(Y(K), \mathbb{Q})_\chi$ is nonzero for $i = r + s, \ldots, r + 2s$.

Note that $(\ell_0(G))_{i-q_0(G)}$ is the dimension of $\bigwedge^{i-q_0(G)} V$ where $V$ is a vector space of dimension $\ell_0(G)$. Thus, we are tempted by 1.3.1(2) to guess that $H^*(Y(K), \mathbb{Q})_\chi$ is generated by $H^{0i(G)}(Y(K), \mathbb{Q})_\chi$ by the action of some exterior algebra acting on cohomology. In fact, this is exactly what Venkatesh hopes to be true:

Conjecture 1.3.1 (Venkatesh, see for example the introduction to [7]). $H^*(Y(K), \mathbb{Q})_\chi$ is generated by $H^{0i(G)}(Y(K), \mathbb{Q})_\chi$ by the action of some exterior algebra acting on cohomology. In particular, this exterior algebra should come from a motivic cohomology group.

1.3.1. Addendum: The case of $\text{Res}_{F/\mathbb{Q}} \text{GL}_1$. It’s instructive to consider the example $G = \text{Res}_{F/\mathbb{Q}} \text{GL}_1, F$ with signature $(r, s)$. Of course this isn’t semisimple, but we can set up everything discussed above for reductive groups as well. The analogue of the symmetric space is:

$$X := \text{GL}_1(F \otimes_{\mathbb{Q}} \mathbb{R})/\mathbb{R}^*_0 K_\infty$$

where $K_\infty = \text{SU}(1)^s$ is the maximal connected compact subgroup of $G(\mathbb{R})$.

In this case, for $K \subset \text{GL}_1(A_F^\infty)$ compact open, we have a surjective map

$$Y(K) := F^\times \backslash [X \times \text{GL}_1(A_F^\infty)/K] \to \text{Cl}(K) := F^\times \backslash [\text{GL}_1(A_F^\infty)/((\mathbb{R}^*_0)^r \times (\mathbb{C}^\times)^s) K]$$

to a (finite) adelic generalized class group (if $K = \hat{\mathbb{O}}_F$ then $\text{Cl}(K)$ is the narrow class group of $F$). Assuming that $F^\times \cap K$ is sufficiently small (more precisely, that $F^\times \cap K$ is a torsion-free finite index subgroup of the totally positive global units $\mathcal{O}_F^\times^{+\infty}$), Dirichlet’s unit theorem implies that the fibres of this map can be identified with a real torus of dimension $r + s - 1$:

$$T(K) = (F^\times \cap K) \backslash ((\mathbb{R}^*_0)^r \times (\mathbb{C}^\times/\text{SU}(1)^s))/\mathbb{R}^*_0.$$ 

In particular, the cohomology of $Y(K)$ is a direct sum of $|\text{Cl}(K)|$ copies of an exterior algebras on the free rank $r + s - 1$ Abelian group $F^\times \cap K$. 

2. $\ell_0(G)$ and Galois Cohomology

Let $G = \text{Res}_{F/Q} \text{SL}_n,F$, and let $\chi : T \to Z$ be a cuspidal tempered system of Hecke eigenvalues. Assume that $H^0(G)(Y(K), Q)_\chi \neq 0$.

We assume the following (vaguely stated) conjecture about the existence of Galois representations:

**Conjecture 2.0.1.** For each prime $p$ there exists a geometric Galois representation $\rho_\chi : G_F = \text{Gal}(\bar{F}/F) \to \text{PGL}_n(\overline{Q}_p)$ such that $\rho_\chi(\text{Frob}_v)$ is described in terms of $\chi$ for almost all places $v$ of $F$.

When $F$ is CM and $p$ is sufficiently large (depending on $n$), this conjecture is known (modulo the difference between $\text{SL}_n$ and $\text{GL}_n$, which is not so serious) [5, 6, 1].

We can define Galois cohomology groups for the adjoint representation $\text{Ad} \rho_\chi$ (which is an $n^2-1$ dimensional representation of $G_F$) and a Bloch-Kato Selmer group

$$H^1_f(G_F, \text{Ad} \rho_\chi) \subset H^1(G_F, \text{Ad} \rho_\chi).$$

There is also a dual Selmer group

$$H^1_f(G_F, (\text{Ad} \rho_\chi)^*(1)) \subset H^1(G_F, (\text{Ad} \rho_\chi)^*(1)).$$

where (1) denotes a Tate twist by the cyclotomic character.

**Fact 2.0.1** (Greenberg-Wiles). Assuming $\rho_\chi$ is irreducible and odd (odd says something about the image of complex conjugation under $\rho_\chi$)

$$\ell_0(G) = \dim H^1_f(G_F, (\text{Ad} \rho_\chi)^*(1)) - \dim H^1_f(G_F, \text{Ad} \rho_\chi).$$

This fact follows from a computation using Tate global duality, which is also a key computation in the Taylor-Wiles method (and it’s extension by Calegari and Geraghty to situations with $l$ such $p$ is not so serious) [5, 6, 1].

We should also note that we expect $\dim H^1_f(G_F, \text{Ad} \rho_\chi) = 0$ — this would be a consequence of the Bloch-Kato conjecture. See [3].

Thus we see the constant $\ell_0(G)$ defined before on the “automorphic side” appearing in the completely different “Galois” side.

3. Patching and $H_*(Y(K), Z_p)$

Here we give a Galois theoretic explanation of Venkatesh’s conjecture and the exterior algebra structures appearing in Proposition 1.3.1, via the obstructed Taylor-Wiles method.

Let $\chi : T \to Z$ be as before, and fix a prime $p$. Then we may look at the reduced system of eigenvalues $\chi : T \to \overline{F}_p$. This determines a maximal ideal $m \subset T$, and then $T_m$ is a local $Z_p$-algebra which can be shown to act on $H_*(Y(K), Z_p)_m$ (we have switched from cohomology to homology here for convenience, although they encode the same information).

Assuming Conjecture 2.0.1, we then have a Galois representation $\rho_\chi$ attached to $\chi$, and we can look at its reduction mod $p$, which we will denote

$$\overline{\rho}_m = \overline{\rho}_\chi : G_F \to \text{PGL}_n(\overline{F}_p).$$

In optimal circumstances, the Calegari-Geraghty method of [2] allows us to describe $H_*(Y(K), Z)_m$ in a rather elaborate way, using the following auxiliary objects:

- A map of power series algebras over $Z_p$, $S_\infty \to R_\infty$ such that $\dim R_\infty = \dim S_\infty - \ell_0(G)$. (This numerology arises from the same Galois cohomology calculation as Fact 2.0.1)

---

1: These properties are expected to always hold for $\rho_\chi$
• A free $R_\infty$-module $M_\infty$, and an isomorphism
  \[ R_\infty \otimes_{S_\infty} \mathbb{Z}_p \cong R_{\overline{\chi}}, \]
  where the map $S_\infty \to \mathbb{Z}_p$ is given by sending all of the power series variables to 0, and where $R_{\overline{\chi}}$ is a certain geometric deformation ring of $\overline{\chi}$.

In nice enough cases (and assuming enough conjectures), Calegari–Geraghty show that $R_{\overline{\chi}} \cong T_m$ and that
  \[ H_{q_0(G)+i}(Y(K), \mathbb{Z}_p)_m = \text{Tor}^S_{i}(M_\infty, \mathbb{Z}_p). \]

Note that since $R_\infty$ acts on $M_\infty$, we get a graded action of $\text{Tor}^S_{\infty}(R_\infty, \mathbb{Z}_p)$ on $H_*(Y(K), \mathbb{Z}_p)$.

**Example 3.0.1.** To see how this relates to Ventakesh’s conjecture, suppose $T_m = \mathbb{Z}_p$ (so we have a Galois representation $\rho_m$ with $\mathbb{Z}_p$ coefficients lifting $\overline{\chi}_m$). In this case we can take $R_\infty = \mathbb{Z}_p$ as well, and $S_\infty = \mathbb{Z}_p[x_1, \ldots, x_{\ell_0(G)}]$. Then
  \[ \text{Tor}^S_{\infty}(R_\infty, \mathbb{Z}_p) = \text{Tor}^S_{\infty}(\mathbb{Z}_p[x_1, \ldots, x_{\ell_0(G)}], \mathbb{Z}_p, \mathbb{Z}_p), \]
  which is the exterior algebra of a free rank $\ell_0(G)$ $\mathbb{Z}_p$-module. See, example, Corollary 4.5.5 and the subsequent exercises in [8].

Thus, we get the conjectured graded action, which should be motivic in origin — indeed the free rank $\ell_0(G)$ $\mathbb{Z}_p$-module which appears can be identified with the Selmer group $H^1_F(G_F, (\text{Ad} \rho_m)^*1))$.

In [4], Galatius and Venkatesh describe $\text{Tor}^S_{\infty}(R_\infty, \mathbb{Z}_p)$ as the homotopy groups of a simplicial ring, which is the derived deformation ring of the Galois representation $\overline{\chi}$. This recovers the Tor-algebra in a canonical way. In [7] (assuming various hypotheses and conjectures), Venkatesh shows that the action of the Tor-algebra on homology is also canonical, using the derived Hecke algebra which we briefly introduce next.

### 4. Derived Hecke Algebra

In addition to the Galois-theoretic explanation of the exterior algebra action, Venkatesh also gives a Hecke-theoretic explanation. One of the goals of [7] is to upgrade the action of $T$ on $H^*(Y(K), \mathbb{Z}_p)$ to an action of a graded algebra $\hat{T}$, whose degree zero part is $T$. This is the “derived Hecke algebra”. In particular, the action is graded, and we want a surjection (perhaps only after inverting $p$)
  \[ \hat{T} \otimes T H^{q_0(G)}(Y(K), \mathbb{Z}_p)_m \twoheadrightarrow H^*(Y(K), \mathbb{Z}_p)_m. \]

When $T_m = \mathbb{Z}_p$, Venkatesh proves that
  \[ T_m = \land^* H^1_F(G_F, (\text{Ad} \rho_m)^*1)) * \]
  and compares the action of the derived Hecke algebra with the action of $\text{Tor}^S_{\infty}(R_\infty, \mathbb{Z}_p) = \land^* H^1_F(G_F, (\text{Ad} \rho_m)^*1))$.

### References


