The Iwasawa Main Conjecture in Families

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Motivation by analogy - Weil Conjectures

Let $X/\mathbb{F}_\ell$ be a smooth projective curve of genus $g$ with Jacobian $J$.

**Theorem (Weil)**

$$Z(X, t) := \exp \left( \sum_{m \in \mathbb{N}} \frac{|X(\mathbb{F}_\ell^m)|}{m} t^m \right) = \frac{\prod_{i=1}^{2g} (1 - \alpha_i t)}{(1 - t)(1 - \ell t)} = Z(\mathbb{P}^1_{\mathbb{F}_\ell}, t) \times \det(1 - \text{Frob}_\ell t | T_p(J))$$

where $\alpha_i$ are the eigenvalues of $\ell$-Frobenius acting on the $p$-adic Tate module of $J$.

Can one prove analogues for a number field $F$?

<table>
<thead>
<tr>
<th>Number Fields</th>
<th>Function Fields</th>
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<tbody>
<tr>
<td>0</td>
<td>$\ell$</td>
</tr>
<tr>
<td>$\text{Spec } \mathcal{O}_F$</td>
<td>$X$</td>
</tr>
<tr>
<td>$F$</td>
<td>$K(X)$</td>
</tr>
<tr>
<td>$F(\mu_{p^\infty})/F$</td>
<td>$\overline{F}<em>\ell/\mathbb{F}</em>\ell$</td>
</tr>
<tr>
<td>$\text{Gal}(F(\mu_{p^\infty})/F) \sim \text{Cl}<em>p(F(\mu</em>{p^\infty}))$</td>
<td>$\text{Frob}_\ell \sim T_p(J)$</td>
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Here $\text{Cl}_p(F(\mu_{p^\infty})) := \varprojlim \text{Cl}_p(F(\mu_{p^m}))$, the structure of $\text{Cl}_p(F(\mu_{p^m}))$ should be related to the zeros of a $\zeta$ function.
Iwasawa’s Main Conjecture

Let’s focus on the case $\text{Spec } \mathbb{Z}$. The appropriate zeta function is $\zeta$, the Riemann $\zeta$-function. Note $\text{Gal}(\mathbb{Q}(\mu_{p\infty})/\mathbb{Q}) \cong \mathbb{Z}_p^\times = \Delta \times \mathbb{Z}_p$, and for technical reasons we replace $\mathbb{Q}(\mu_{p\infty})$ with the subfield $\mathbb{Q}_\infty = \mathbb{Q}(\mu_{p\infty})^+$, which is a $\mathbb{Z}_p^+ := \mathbb{Z}_p^\times / \langle c \rangle$-extension of $\mathbb{Q}$.

**Theorem (Kubota-Leopoldt)**

$\zeta$ extends to a function $\zeta_p \in \text{Frac}(\mathbb{Z}_p[[\mathbb{Z}_p^+]]) = \text{Frac}(\mathbb{Z}_p[\Delta/\langle c \rangle][[T]])$ which interpolates the negative odd integer values of $\zeta$.

**Remark**

One classifies f.g. torsion modules over $\mathbb{Z}_p[[\mathbb{Z}_p^+]]$ in terms of their characteristic ideals, which are generated by elements of $\mathbb{Z}_p[[\mathbb{Z}_p^+]].$

**Theorem (Iwasawa Main Conjecture for $\mathbb{Q}$, Mazur–Wiles, Coates–Sujatha)**

The characteristic ideal of the f.g. torsion $\mathbb{Z}_p[[\text{Gal}(\mathbb{Q}_\infty/\mathbb{Q})]]$-module $X_\infty = \text{Gal}(M_\infty/\mathbb{Q}_\infty)$ is equal to $I(\mathbb{Z}_p^+)\zeta_p$. Here $M_\infty$ is the maximal abelian extension of $\mathbb{Q}_\infty$ of $p$-power degree unramified away from the unique $p$ lying over $p$.

Goal of this talk: generalize this picture to modular forms, and then families of modular forms over the eigencurve.
Iwasawa main conjecture for modular forms

For this talk let $\Gamma = \Gamma_1(N) \cap \Gamma_0(p)$. Fix a normalized cuspidal eigenform $f \in S_k(\Gamma)$ of weight $k \geq 2$, nebentypus $\epsilon$ and of finite slope (i.e. the $U_p$-eigenvalue is nonzero).

**Theorem (Eichler-Shimura, Deligne, Serre, ...)**

There exists a unique semisimple cts representation $\rho_f : G_{\mathbb{Q}, Np} \to \text{GL}_2(\overline{\mathbb{Q}}_p)$ such that,

$$\text{tr} \rho_f(\text{Frob}_\ell) = a_\ell \text{ and } \det \rho_f(\text{Frob}_\ell) = \epsilon(\ell)\ell^{k-1}$$

for all primes $\ell \nmid Np$, where $a_\ell$ is the $\ell$th Fourier coefficient of $f$.

Let $\mathcal{D} = \text{Spf}(\mathbb{Z}_p[[\mathbb{Z}_p^\times]])^{\text{rig}}$, union of rigid analytic open unit disks.

<table>
<thead>
<tr>
<th>Spec $\mathbb{Z}$</th>
<th>$f$</th>
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<tbody>
<tr>
<td>$\zeta_p \in \mathbb{Z}_p[[\mathbb{Z}_p^\times / \langle c \rangle]]$</td>
<td>$L_p(f) \in \mathcal{O}(\mathcal{D}) \supset \mathbb{Z}_p[[\mathbb{Z}_p^\times / \langle c \rangle]]$</td>
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<tr>
<td>$X_\infty$ characteristic ideal in $\mathbb{Z}_p[[\mathbb{Z}_p]]$</td>
<td>the Selmer group $\text{Sel}(f)$, a torsion coherent sheaf on $\mathcal{D}$ characteristic ideal sheaf in $\mathcal{O}_\mathcal{D}$</td>
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**Conjecture (Iwasawa Main Conjecture)**

*The characteristic ideal sheaf of $\text{Sel}(f)$ is the vanishing ideal sheaf of $L_p(f)$.*
**Definition**

If \( f = \sum_{n \geq 1} a_n q^n \in S_{k+2}(\Gamma, \mathbb{C}) \) is a cusp form and \( \chi : (\mathbb{Z}/D)^\times \rightarrow \mathbb{C}^\times \) is a Dirichlet character of conductor \( D \), its \( \chi \)-twisted \( L \)-function is

\[
L(f, \chi, s) = \sum_{n \geq 1} \frac{\chi(n)a_n}{n^s}.
\]

**Proposition**

We have analytic continuation to entire functions on \( \mathbb{C} \) via

\[
L(f, s) = \frac{(-2\pi i)^s}{\Gamma(s)} \int_0^{i\infty} f(z)z^{s-1} dz
\]

and

\[
L(f, \chi, s) = \frac{(-2\pi i)^s}{\Gamma(s)G(\chi^{-1})} \sum_{a \in (\mathbb{Z}/D)^\times} \chi^{-1}(a) \int_{a/D}^{\infty} f(z) \left(z - \frac{a}{D}\right)^{s-1} dz
\]
If \( f \in S_{k+2}(\Gamma, \mathbb{C}) \) is a newform with \( U_p f = \alpha f \) and \( v_p(\alpha) < k + 1 \) (\( f \) has non-critical slope), then there exists a \( p \)-adic \( L \)-function \( L_p(f) \in \mathcal{O}(\mathcal{D}_L) \) for some finite extension \( L/\mathbb{Q}_p \) such that if \( \chi \) has conductor \( p^n \) and \( 0 \leq j < k + 1 \),

\[
L_p(f)(\chi^{-1}(x)x^j) = -\alpha^{-n} \left( 1 - \chi^{-1}(p) \frac{p^j}{\alpha} \right) G(\chi^{-1}) \cdot j! \cdot p^{nj} \left[ \frac{L(f, \chi, j + 1)}{(2\pi i)^{j+1} \Omega_f^\pm} \right]
\]

Roughly, one algebraizes the notion of period integral, and then \( p \)-adically interpolates it.
Selmer Groups

Recall $(\rho_f, V)$ is the attached Galois representation. $\mathcal{D}$ acquires a Galois action via

$$\chi_{\text{cyc}} : G_{\mathbb{Q}, Np} \to \text{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q}) \cong \mathbb{Z}_p^\times$$

Definition

The **cyclotomic deformation** is

$$V_{\text{cyc}} := V \otimes \mathcal{O}_\mathcal{D}$$

which is a coherent sheaf over $\mathcal{D}$ with $G_{\mathbb{Q}, Np}$-action given diagonally by the natural action on $V$ and $\chi_{\text{cyc}}^{-1}$ on $\mathcal{O}_\mathcal{D}$. 
Selmer Groups

**Definition**

The Selmer group $\text{Sel}(f)$ is a subspace

$$\text{Sel}(f) := H^2_{\text{Sel}}(G_{\mathbb{Q}, Np}, V_{\text{cyc}}) \subseteq H^2(G_{\mathbb{Q}, Np}, V_{\text{cyc}})$$

with

- unramified condition at $\ell \mid N$
- trianguline condition at $p$.

**Remark**

- When $f$ is $U_p$-ordinary, then $\rho_{f,p}$ is reducible (so has an ordinary subspace), and the local condition at $p$ is defined using the ordinary subspace.
- If not, $\rho_{f,p}$ could be irreducible, but its $(\varphi, \Gamma)$-module will be reducible, with a **trianguline** subspace, hence the condition above.
Characteristic Ideals

**Definition**

A **Krull domain** is an integral domain $A$ such that $A_p$ is a DVR for all height one primes, $A = \bigcap_p A_p$ (only height 1 primes) and any nonzero element of $A$ is contained in only a finite number of height 1 primes (dimension 1 Krull domain $=$ Dedekind domain).

Krull domains have a well-behaved notion of characteristic ideal.

**Definition**

If $M$ is a finitely generated torsion $A$-module, there exists a pseudo-isomorphism

$$M \to \bigoplus_{i=1}^n A/p_i^{m_i}$$

where each $p_i$ is a height 1 prime. So we let

$$\text{char}_A(M) := \prod_{i=1}^n p_i^{m_i}$$
We can make the following definition over $\mathcal{D}$ (which is not affinoid, but the increasing union of affinoid Krull domains):

**Definition (Pottharst)**

If $\mathcal{M}$ is a torsion coherent $\mathcal{O}_\mathcal{D}$-module, its characteristic ideal sheaf is

\[
\text{“char}_\mathcal{D}(\mathcal{M}) := \prod_{\mathcal{P} \text{ closed point}} \mathcal{P}^{\ell_{\mathcal{P}}(\mathcal{M})}
\]

(This doesn’t literally make sense: define on affinoid covers and take an inverse limit to make this precise)
By taking the vanishing locus of the global section $L_p(f)$ of $\mathcal{D}$, we obtain a coherent ideal sheaf $\mathcal{I}_f \subseteq \mathcal{O}_D$.

**Conjecture (Iwasawa main conjecture for $f$)**

$H^2_{\text{Sel}}(G_{\mathbb{Q}, Np}, V_{\text{cyc}})$ is a torsion coherent $\mathcal{O}_D$-module and

$$\text{char}_D(H^2_{\text{Sel}}(G_{\mathbb{Q}, Np}, V_{\text{cyc}})) = \mathcal{I}_f$$

Known in a lot of, but not all, cases by work of Kato, Skinner-Urban and Wan.
The eigencurve

The eigencurve is the space of finite slope overconvergent modular forms, or more precisely, their Hecke eigensystems. Actually, we’ll use overconvergent modular symbols.

To construct the eigencurve, one varies the weight $p$-adically: the classical weights are given by maps $w_k : \mathbb{Z}_p^\times \to \mathbb{C}_p^\times$ taking $x \mapsto x^k$. More generally, you can take more general weights $\mathbb{Z}_p^\times \to \mathbb{R}_p^\times$ in “pseudoaffinoid rings” and consider a mixed characteristic adic space of weights

$$\mathcal{W} = \text{Spa}(\mathbb{Z}_p[[\mathbb{Z}_p^\times]], \mathbb{Z}_p[[\mathbb{Z}_p^\times]])^{an} \cong \text{Spa}(\mathbb{Z}_p((\mathbb{Z}/p)^\times)[[T]], \mathbb{Z}_p((\mathbb{Z}/p)^\times)[[T]])^{an}$$

Every point except for the boundary point has residue field defined over $\mathbb{Q}_p$, and the boundary point is defined over $\mathbb{F}_p(\!(T)\!)$. 

\[ k \in \mathbb{Z}_p \ (x \mapsto x^k) \]
The eigencurve

The (extended) eigencurve of modular symbols of level \( N \) is an adic curve \( w : \mathcal{E}(N) \to \mathcal{W} \) such that

the fiber \( \mathcal{E}(N)_{\kappa} \) over any \( L \)-point \( \kappa \in \mathcal{W}(L) \) (for \( L/\mathbb{Q}_p \) finite or \( L = \mathbb{F}_p((T)) \)) is the space of Hecke eigensystems in \( \text{Symb}_\Gamma(D_{\kappa}(L)) \) with nonzero \( U_p \)-eigenvalue \( (\text{finite slope}) \).

Here \( D_{\kappa}(L) \) is an algebra of locally analytic distributions with a weight \( \kappa \)-action, which is a huge \( p \)-adic coefficient system. But more importantly, the eigencurve gives \( p \)-adic variation:

the fiber \( \mathcal{E}(N)_U \) over any (Tate) open affinoid \( U = \text{Spa}(R) \subseteq \mathcal{W} \) is the space of systems of Hecke eigenvalues in \( \text{Symb}_\Gamma(D_R) \) with invertible \( U_p \)-eigenvalue \( (\text{finite slope at all specializations}) \).

Again, \( D_R \) is an algebra of locally analytic distributions with a weight \( \kappa_U \)-action (family of \( p \)-adic coefficient systems).

This construction is originally due to Coleman and Mazur in [CM98] in characteristic 0, and was extended to characteristic \( p \) by Andreatta-Iovita-Pilloni in [AIP18]. The version I use is due to Johansson-Newton and Gulotta in [JN19] and [Gul19].

In particular, there are spaces of “\( T \)-adic” modular forms over the boundary weights. These have perfectoid properties (Heuer).
Main Theorem

Now let $\mathcal{D} = \text{Spa}(\mathbb{Z}_p[[\mathbb{Z}_p^\times]], \mathbb{Z}_p[[\mathbb{Z}_p^\times]])^{an}$. Following ideas of Joël Bellaiche and David Hansen, we prove the following.

Theorem (I.)

There exists a non-trivial line bundle $\mathcal{L}$ over $\tilde{E}(N)_{new} \times \mathcal{D}$ and a global section $L \in \mathcal{O}(\mathcal{L})$ which specializes to the usual $p$-adic $L$-function at all classical points on $E(N)_{new}$ up to a unit.

Non-triviality of the line bundle allows for ambiguity up to a unit. Trivialization amounts to picking a unit.

Idea of proof.

Instead of working with classical modular symbols, work with modular symbols valued in huge $p$-adic coefficient systems (mixed-characteristic locally analytic distribution modules), and construct it in the same way.

Take an $\mathbb{F}_q((T))$-point on the boundary. Then it has a $p$-adic $L$-function! What is the arithmetic significance of this?
Sheaf of Selmer groups

We adapt the previous formalism.

- Over $\widetilde{\mathcal{E}}(N)_{\text{new}}^{\text{irr}}$ there is a canonical sheaf of 2-dimensional $G_{\mathbb{Q}, Np}$ representations $\mathcal{V}$, and we can cyclotomically deform it to get a coherent sheaf $\mathcal{V}_{\text{cyc}}$ over $\widetilde{\mathcal{E}}(N)_{\text{new}}^{\text{irr}} \times \mathcal{D}$.

- We can define the Selmer group $H^2_{\text{Sel}}(G_{\mathbb{Q}, Np}, \mathcal{V}_{\text{cyc}})$ in a similar way. The local condition at $p$ is defined by a triangulation (i.e. a rank 1 subsheaf of $\mathfrak{D}_{\text{rig}}^{\dagger}(\mathcal{V}_{\text{cyc}})$) constructed in forthcoming work of Bellovin.

- (In progress) One defines characteristic ideal sheaves of torsion coherent sheaves over $\widetilde{\mathcal{E}}(N)_{\text{new}}^{\text{irr}} \times \mathcal{D}$ similarly, by doing it affinoid-locally.
Two-variable main conjecture

Conjecture

\[ H^2_{\text{Sel}}(G_{\mathbb{Q}}, N_p, V_{\text{cyc}}) \text{ is a torsion coherent } \mathcal{O}_{\tilde{E}(N)_{\text{new}}} \times \mathcal{D} \text{-module and} \]

\[ \text{char} \tilde{E}(N)_{\text{new}} \times \mathcal{D}^* H^2_{\text{Sel}}(G_{\mathbb{Q}}, N_p, V_{\text{cyc}}) = \mathcal{I}_f \]

This conjecture implies the Iwasawa main conjecture (as stated earlier) at a classical point on the eigencurve.

Future work

- In work in progress, I am trying to prove one inclusion using existing work of David Hansen on a two-variable Euler system over \( \tilde{E}(N)_{\text{new}} \).
- One can specialize this conjecture over a characteristic \( p \) point in \( \tilde{E}(N)_{\text{new}}^{\text{irr}} \) to obtain a main conjecture in characteristic \( p \). What is the arithmetic significance of this?
References


