King's College London

UNIVERSITY OF LONDON

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Candidate No: .... **Desk No:** ....

BSC AND MSCI EXAMINATION

6CCM332A INTRODUCTORY QUANTUM THEORY

SUMMER 2016

TIME ALLOWED: TWO HOURS

THIS PAPER CONSISTS OF TWO SECTIONS, SECTION A AND SECTION B.
SECTION A CONTRIBUTES HALF THE TOTAL MARKS FOR THE PAPER.
ANSWER ALL QUESTIONS IN SECTION A.
ALL QUESTIONS IN SECTION B CARRY EQUAL MARKS, BUT IF MORE THAN TWO QUESTIONS ARE ATTEMPTED, THEN ONLY THE BEST TWO WILL COUNT.

NO CALCULATORS ARE PERMITTED.

DO NOT REMOVE THIS PAPER FROM THE EXAMINATION ROOM

TURN OVER WHEN INSTRUCTED

SECTION A

A 1. (Schrödinger equation generalities) (16 percent)

Consider a particle of mass m moving in one dimension, with a real potential $V(t, x)$.

(i) Write down the Hamilton operator H and state the (full, time-dependent) Schrödinger equation for the wave function $\psi(t, x)$.

(ii) Assume that H does not depend on time t; derive the time-independent Schrödinger equation $H\,\phi(x) = E\,\phi(x)$ from the full Schrödinger equation.
Assuming that the eigenfunctions and eigenvalues of the Hamilton operator are $\phi_n(x)$ and E_n, respectively, write the most general solution $\psi(t, x)$ of the full Schrödinger equation expressed in terms of the E_n and $\phi_n(x)$.

(iii) Given a solution $\psi(t, x)$ of the Schrödinger equation, define the probability density $\rho(t, x)$.
What does it mean to say that the wave function $\psi(t, x)$ is normalised?
For a normalised $\psi(t, x)$, what is the interpretation of the integral $\int_I |\psi(t, x)|^2\,dx$, where I is some interval?
Derive the continuity equation $\partial_t \rho(t, x) + \partial_x j(t, x) = 0$, where $j(t, x)$ is the probability current; a formula for $j(t, x)$ should result from your derivation.

A 2. (Operators in Hilbert spaces, and QM) (14 percent)

(i) Let $\mathcal{H}$ be some Hilbert space with scalar product $\langle\cdot,\cdot\rangle$. State the definition of the adjoint of a linear operator on $\mathcal{H}$, the definition of a self-adjoint operator, and the definition of a unitary operator.

(ii) Give an example (other than the identity operator) of an operator that is both self-adjoint and unitary (no proof required).

(iii) Show that the eigenvalues of a self-adjoint operator are real numbers.

(iv) Let A be a self-adjoint operator and $\psi \in \mathcal{H}$ be a normalised state.
Give the definition of the expectation value $\langle A\rangle_\psi$ of A in the state ψ.
Briefly describe the physical meaning of $\langle A\rangle_\psi$ in quantum mechanics.

(v) Consider the momentum operator $\hat{p} = \frac{\hbar}{i}\frac{d}{dx}$ on the real line: give its eigenfunctions and show that they are not square-integrable.

A 3. (Uncertainty relation) (20 percent)

(i) Give the definition of the uncertainty $(\Delta A)_\psi$ of a self-adjoint operator A in a normalised state ψ.

(ii) Show that $(\Delta A)_\psi$ is zero if and only if ψ is an eigenstate of A.

(iii) What is the statement made by the "axiom of the collapse of the wave function"?
In the light of the fact shown in part (ii), why is it reasonable to impose this axiom? (A brief answer is sufficient.)

(iv) Now consider two self-adjoint operators A and B on $\mathcal{H}$, and let $\psi \in \mathcal{H}$ be a normalised state. You may assume that the expectation values $\langle A\rangle_\psi$ and $\langle B\rangle_\psi$ are zero.
Derive Heisenberg's uncertainty relation

$$(\Delta A)^2_\psi \, (\Delta B)^2_\psi \geq \tfrac{1}{4} \, | \, \langle \, [A,B] \, \rangle_\psi \, |^2 \ .$$

You may use the Cauchy-Schwarz inequality $\langle u,u\rangle\langle v,v\rangle \geq |\langle u,v\rangle|^2$ without proof.

SECTION B

B 4. (Harmonic oscillator) (25 percent)

Let $\hat{p}$ and $\hat{x}$ be the momentum resp. position operator in one dimension, and consider a Hamilton operator

$$H = c_1\hat{p}^2 + c_2\hat{x}^2 + c_3\hat{p}\hat{x} + c_4$$

where $c_1, \ldots, c_4$ are real constants, $c_1 > 0$ and $c_2 > 0$.
Also, define the operator $a := \alpha\hat{x} + i\,\beta\hat{p}$, where α and β are real constants.

(i) Find conditions on α, β and $c_1, \ldots, c_4$ such that one has $[a, a^\dagger] = 1$ and can write $H = \hbar\omega\; a^\dagger a$ with some positive constant ω (which in the end should be expressed in terms of the $c_1, \ldots, c_4$.)

(ii) From now on, assume that the conditions from part (i) are satisfied, and set $N := a^\dagger a$.
Let ψ_ν be a normalised eigenvector of N with eigenvalue $\nu \in \mathbb{R}$. One can show that $N\, a^k\psi_\nu = (\nu - k)\, a^k\psi_\nu$ and $N\,(a^\dagger)^k\psi_\nu = (\nu + k)\,(a^\dagger)^k\psi_\nu$ for any positive integer k.
Compute the norm square $\|\, a^k\psi_\nu \,\|^2$.
Use the result to show that ν must be a positive integer.

(iii) Compute $a(t) := U(t)^{-1}\, a\, U(t)$, where $U(t) = \exp\{-\frac{i}{\hbar}H\,t\}$ is the time evolution operator.
(Hint: it is best to show $H^n a = a\,(H - \hbar\omega)^n$ first.)

(iv) Compute $\frac{d}{dt}\, a(t)$ and $[H, a(t)]$ separately and compare. (Alternatively, you may derive Heisenberg's equation of motion in general, if you prefer.)

B 5. (Two-state system) (25 percent)

On the space $\mathcal{H} = \mathbb{C}^2$ with standard scalar product, consider the self-adjoint operators

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

These satisfy the commutation relations $[\sigma_j, \sigma_k] = \sum_l 2i\, \epsilon_{jkl}\, \sigma_l$.

Assume that the Hamiltonian of the system is given by $H = E\,\sigma_3$ with some constant $E > 0$, and consider the observable $A = \sigma_1$.

Assume that three measurements are performed at times $t_0 < t_1 < t_2$, and that the system is left undisturbed in between.

(i) At time t_0, one measures H and finds the result E: what state is the system in immediately before the next measurement at t_1? And what axioms of QM are used to answer this?

(ii) At time t_1, one measures A and finds the positive eigenvalue. What state is the system in immediately after this measurement?

(iii) At time t_2, one measures A once more. What is the probability to find the negative eigenvalue?

(iv) Show that the only states $\psi = \begin{pmatrix} a \\ b \end{pmatrix}$ for which the uncertainty relation allows the product $(\Delta A)_\psi\, (\Delta H)_\psi$ to be zero are those where $a^* b$ is real.

B 6. (Symmetries, angular momentum) (25 percent)

(i) In three space dimensions, the self-adjoint angular momentum operators $L_1,\ L_2,\ L_3$ are defined by

$$L_j = \sum_{k,l=1}^{3} \epsilon_{jkl}\ \hat{x}_k\,\hat{p}_l$$

where ϵ_{jkl} is the totally antisymmetric tensor. (Explicitly, we have $L_1 = \hat{x}_2\hat{p}_3 - \hat{x}_3\hat{p}_2\,,\ L_2 = \hat{x}_3\hat{p}_1 - \hat{x}_1\hat{p}_3\,,\ L_3 = \hat{x}_1\hat{p}_2 - \hat{x}_2\hat{p}_1\,.$)
Using the commutation relations between position and momentum operators $[\,\hat{x}_j, \hat{p}_k\,] = i\hbar\,\delta_{j,k}$, compute the commutator $[\,L_1, L_2\,]$ and express it in terms of angular momentum components.
Furthermore, compute $[\,L_1, x_k\,]$ for $k = 1, 2, 3$.

(ii) Assume $\psi \in L^2(\mathbb{R}^3)$ is a normalised state in which the uncertainty of L_1 vanishes, i.e. such that $(\Delta L_1)_\psi = 0$. Show that ψ is orthogonal to $L_3\,\psi$.

(iii) Consider rotations R_α around the x_1-axis by an angle α, i.e. the coordinate transformation $\vec{x} \longmapsto R_\alpha\,\vec{x}$ with

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad \text{and} \quad R_\alpha = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix} .$$

Find a unitary operator U_α which implements such rotations on wave functions $\psi(\vec{x})$, i.e. which satisfies

$$\big(U_\alpha\,\psi\big)(\vec{x}) = \psi(R_\alpha^{-1}\,\vec{x}) \ .$$

Hint: consider rotations by infinitesimal α first, then use a suitable formula for the exponential to pass to finite angles. The result for U_α should be expressed in terms of L_1. You may of course assume that the function $\psi(\vec{x})$ has good analytic properties (in particular that it is differentiable).

B 7. (One-dimensional systems) (25 percent)

Consider a quantum mechanical particle of mass m confined to an interval $I := [0, l]$ of length l. Assume there is a constant potential inside the interval, i.e. $V(x) = V_0$ for all $x \in I$.

(i) First show, using arguments from the probabilistic interpretation, that "confined to the interval" means the wave function has to satisfy $\psi(t, x) = 0$ for all $x \notin I$.

(ii) Then find the most general normalisable solution $\phi(x)$ to the time-independent Schrödinger equation such that $\phi(x)$ is continuous for all $x \in \mathbb{R}$.
In particular, give the energy values E_n of the Hamilton operator.

(iii) Briefly compare this energy spectrum to what one finds in the analogous classical system.

(iv) In what way are the energy values found in (ii) special from the point of view of de Broglie's wave interpretation of particles?
(Reminders and hints: De Broglie suggested that to a particle of classical momentum p_c one can assign a wave length $\lambda_{\text{dB}} = \frac{2\pi\hbar}{p_c}$. Take the E_n from above, extract the classical momentum from writing E_n as the sum of the kinetic energy $\frac{p_c^2}{2m}$ and the potential energy. Compute λ_{dB} for each E_n and compare to the intrinsic length scale of the system.)

(v) Next, consider a particle confined to an interval as above, but with a more general potential $V(x)$ inside the interval: assume only that $V(x)$ is (continuous and) bounded from below by some constant $V_{\min}$.
Show that the expectation value of the Hamiltonian H in any normalised wave function ψ satisfies $\langle H \rangle_\psi \geq V_{\min}$.
(Hint: To compute the expectation value, use the scalar product $\langle \psi_1, \psi_2 \rangle = \int \psi_1^* \psi_2 \, dx$ as usual. And you may assume that ψ is continuous and that $\partial_x \psi$ does not diverge at the endpoints of the interval.)