Matrix Factorizations and Deformations of Topological Branes

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Abstract

This thesis describes a new computational approach to computing deformations of topological D-branes using the computer algebra package Singular.

D-branes are non-perturbative objects in string theory which have a world-sheet description as conformal boundary conditions. Topological twisting is a method to simplify the theory drastically while still retaining most of the information relevant for the effective low-energy physics encoded in a string model, and for the mathematical study of mirror symmetry. It was known for a long time that many string theory backgrounds have a description in terms of Landau-Ginzburg models, and more recently it was shown that supersymmetric boundary conditions can be implemented by matrix factorizations of the Landau-Ginzburg potential.

Given a matrix factorizations for a D-brane in the Landau-Ginzburg model, functions in the new Singular library file allow obstructions to the deformations of the branes to be calculated. These enable deformed matrix factorizations to be found and can allow the effective superpotential of the underlying string theory to be calculated. This computational approach is faster and more reliable than existing methods and outputs information directly relevant to theorists.

Using the Singular library file we find deformations of various matrix factorizations and introduce an idea of boosted matrix factorizations where the deformation parameters are taken to be matrices solving the zero locus of the obstruction polynomials.
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Introduction

String theory has now been with us for more than thirty years and in that time it has been lauded as a promising candidate for a “theory of everything” thanks in part to the graviton showing up in its particle spectrum. In this theory strings are fundamental 1-dimensional objects which can either be open or closed (i.e. strands or loops). As they move around they map out a 2-dimensional surface known as the string’s “world-sheet” on which live bosonic and fermionic fields mapping the world-sheet to our world (the “target space”). The necessitation of non-negative energy states requires the target space to be a 10 dimensional manifold which is a problem since we are only cognizant of 4 space-time dimensions. However, this is not as bad as it first seems - all that is necessary is that the extra 6 dimensions are so small that we are unable to notice them (i.e. of Planck-length scale). We say that these extra dimensions are “compactified”.

$N = 2$ superconformal field theories (SCFTs) with central charge 15 on the string world-sheet lead to a type IIA or IIB superstring theory. The $N = 2$ SCFTs which are the most interesting can be built from a tensor product of two smaller theories - an “external” one with central charge 6 and an “internal” one with central charge 9. The four degrees of freedom of Minkowski space-time can be thought of as four bosons and their fermionic superpartners whose charges total to 6. Therefore, we can think of the first of the two constituent theories as being Minkowski space-time. The best candidate for the $c = 9$ theory involves six compactified spacial dimensions which exist on such a small scale that it seems to us as though we exist only in the four Minkowski dimensions. It is necessary that the choice of $c = 9$ theory is made so that a consistent string theory can be formulated. This means that we must ensure conformal invariance and $N = 2$ supersymmetry which leads to conditions defining a Calabi-Yau three-fold. There are many Calabi-Yau three-folds and other than trial and error no way of determining which describes our universe. Many of these models
are related continuously by deformations of the SCFT described by local coordinates of the moduli space (coupling constants of the marginal operators of the SCFT) [57].

The world-sheet of an open string has a boundary (at the string’s endpoints). The boundary conditions which need to be introduced in a conformal field theory to account for this are known as D-branes and can be treated as separate, non-perturbative dynamical objects in string theory. One also finds bosons and fermions on D-branes (i.e. on world-sheet boundaries as opposed to the world-sheet “bulk”). The conformal field theory can be deformed by both bulk and boundary fields. One can impose either A- or B-type boundary conditions on the boundary states of the boundary conformal field theory (BCFT) which give A- respectively B-branes.

Landau-Ginzburg models flow to a conformal field theory at the infra-red fixed point of the renormalization group flow. From looking at the boundary Landau-Ginzburg action it can be seen to conserve supersymmetry if the LG-superpotential (a polynomial $W$) can be factorized into two polynomials. Recently it has been suggested by Kontsevich that this idea can be extended to matrices so that instead of factorizing the superpotential into two polynomials we find two matrices which multiply together to give the superpotential multiplied by an identity matrix

$$W.1 = E \cdot J.$$ 

We then have a matrix

$$Q = \begin{pmatrix} 0 & E \\ J & 0 \end{pmatrix}$$

which squares to the superpotential multiplied by the identity. Only B-type boundary conditions lead to matrix factorizations. Each matrix factorization describes a B-brane and the particle spectrum of the B-brane is given by the cohomology of the operator $D_Q$ acting on boundary fields $\Phi$ as

$$D_Q\Phi = Q\Phi - \Phi^\sigma Q.$$ 

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Deformations of D-branes can be found by deforming matrix factorizations. In addition to deforming matrix factorizations by fermionic elements of the cohomology we can deform by objects related to the fermions which are in the kernel of $D_Q$. These can be found using Massey products and the algorithmic tools of deformation theory [45–47, 51, 52, 70]. This process allows us to obtain a deformed matrix factorization modulo the ideal generated by a set of “obstruction” polynomials. In addition, these obstructions should allow computation of the effective superpotential of the underlying string theory.

B-type branes can be described by both bound states in RCFT and by matrix factorizations of superpotentials in Landau-Ginzburg models. Evidence for the consistency of the boundary states of the two approaches has been gathered in [7, 8, 10, 22, 41].

There exists a geometrical description of matrix factorizations [35, 39, 40, 42, 43, 54, 61]: On Calabi-Yau manifolds A- or B-branes are given an $A_\infty$ structure by the tree-level topological correlators. The corresponding derived categories contain the gauge invariant information of the branes. The category of A-branes is equivalent to the Fukaya category (which may need to be enlarged for the Homological Mirror Symmetry conjecture to be true in some cases), and the category of B-branes is equivalent to the derived category of coherent sheaves. The Homological Mirror Symmetry conjecture says that two Calabi-Yaus $X$ and $Y$ are mirrors of each other if the derived category of coherent sheaves on $X$ is equivalent to the Fukaya category of $Y$ [49, 61].

In the papers [45–47] the algorithm used to compute deformations of matrix factorizations using Massey products has been completed by hand or by “black-box” computations using the “mod_versal” procedure in the computer algebra package Singular [29]. The main problem with doing the calculations by hand is that for all but the simplest examples the process is extremely time consuming and prone to error. Computations using “mod_versal” return the correct obstructions but no additional information like the open string spectrum meaning that one cannot make any link between the variables showing up in the obstructions and the bosons. This is necessary in order to determine R-charges of the deformation parameters and compute the
effective superpotential. Because of this it is also unable to produce deformed matrix factorizations.

This thesis outlines a new library file written for Singular which automates the process for a given matrix factorization and returns all information relevant to physicists (including deformed matrix factorizations) [19].

**Overview of thesis** In chapter 2 I review some ideas from conformal field theory and introduce $N = 1$ and $N = 2$ superconformal field theories. Then we look at boundary conformal field theory and D-branes (symmetry preserving boundary conditions) are introduced. The chapter ends by looking at bulk and boundary topological field theories.

Chapter 3 looks at Landau-Ginzburg models and how the preservation of supersymmetry leads naturally to the factorization condition of the Landau-Ginzburg superpotential. From this we get to the idea of matrix factorizations of the superpotential which represent D-branes.

Chapter 4 gives a short introduction to the minimal model branes and their different matrix factorizations. We also look at tensor products of minimal models and their treatment in terms of tensor product matrix factorizations and of linear matrix factorizations.

The 5th chapter explains the Massey product algorithm used to construct deformed matrix factorizations. First a general description is given then it is applied more concretely to our purpose. A simple example is explicitly calculated to further illustrate the algorithm. The chapter ends with a couple of general results for A-type minimal models.

In chapter 6 the implementation of the algorithm is explained in pseudo-code before descriptions of the main functions in the new library file for Singular are given.

Chapter 7 contains some results obtained by using the program on several matrix factorizations. For some examples the effective superpotential is calculated using a method explained in the next chapter. Further examples can be found in the appendices.
In chapter 8 it is explained how one can compute the effective superpotential using R-charges and the obstruction polynomials output by the program. The idea of replacing deformation parameters by matrices which still solve the obstruction zero-locus is introduced. Then we discuss how setting deformation parameters to be variables can change the deformed matrix factorization into a matrix factorization for a different superpotential.

Chapter 9 contains conclusions are made and open questions stated. A list of desirable improvements to the library file is also given.

A list of mathematical definitions is given in appendix A which is followed by an index of those definitions on page 148.

Appendix B contains some additional calculations which are useful for and referenced in parts of the text.

Appendix C contains progress on a very slow running quintic example using Singular.
Conformal Field Theory

In this chapter, we give a very short sketch of some basic properties of conformal field theories (CFTs) to provide some of the background and context of (topological) branes. We first collect some material on two-dimensional CFTs in general, then highlight some special features of $N = 2$ supersymmetric CFTs, which are particularly important for string theory, and which allow for so-called topological twisting, a procedure which is reviewed briefly in section 2.4. Before that, we outline boundary CFT in section 2.3; the conformal boundary states introduced there correspond to D-branes in string theory, and their topological “skeletons” are what can be described by matrix factorizations in Landau-Ginzburg models. Far more detailed overviews of conformal field theories are given e.g. in [24,26,56,64,66].

2.1 Some Elements of Conformal Field Theory in 2 dimensions

A CFT is a quantum field theory (QFT) where the conformal group (or equivalently the Lie algebra of conformal transformations) acts covariantly on the space of states and fields. Conformal transformations $x \rightarrow x'$ on $\mathbb{R}^d$ with flat metric $g_{\mu\nu}$ preserve angles

$$g_{\mu\nu}(x) \rightarrow g'_{\mu\nu}(x') = \Omega(x) g_{\mu\nu}(x)$$

(2.1)

where $\Omega(x)$ is some (positive) function. If $d > 2$ the conformal group is finite-dimensional and is generated by translations, rotations, scale transformations and special conformal transformations (inversion plus translation).

In two dimensions, however, the conformal group is infinite-dimensional, therefore covariance of a QFT under conformal transformations is a much more restrictive requirement.
To see that the conformal group is infinite-dimensional, let us take the 2-dimensional manifold (called the world-sheet) to be a cylinder with coordinates \((t,x) \in \mathbb{R} \times S^1\), i.e. with compact space direction (and supplied with standard Minkowski metric). After a Wick rotation in time, passage to light-cone coordinates, and exponentiation, one arrives at the complex coordinates

\[ z := e^{t+ix}, \quad \bar{z} := e^{t-ix}. \]

(Note that the distant past \((t = -\infty)\) corresponds to the origin \(z = 0\) in the complex plane.) It is easy to see that, in these coordinates, (2.1) is satisfied by arbitrary analytic coordinate transformations

\[ z \mapsto f(z), \quad \bar{z} \mapsto \bar{f}(\bar{z}) \quad (2.2) \]

and these form an infinite-dimensional space.

In quantum theory, one is interested in the Lie algebra of transformation groups. To study this, we use a Laurent expansion (about zero) of the coordinate transformations (2.2),

\[ f(z) = \sum_{n \in \mathbb{Z}} a_n z^n. \quad (2.3) \]

So there are infinitesimal transformations \(z \mapsto z' = z - \sum_{n \in \mathbb{Z}} \epsilon_n z^{n+1}\) (where \(\epsilon_n\) is a small parameter). \(f(z)\) has Taylor expansion

\[ f(z + \delta z) = f(z) + \delta z \partial_z f(z) + \cdots \quad (2.4) \]

similarly for \(\bar{z}\). Setting \(\delta z = -\epsilon_n z^{n+1}\) in (2.4) gives

\[ f(z') = f(z) - \epsilon_n z^{n+1} \partial_z f(z) + \cdots \quad (2.5) \]

showing that the generators of infinitesimal transformations are

\[ l_n = -z^{n+1} \partial_z, \quad \bar{l}_n = -\bar{z}^{n+1} \partial_{\bar{z}}. \]

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l_n, \bar{l}_n generate local conformal mappings on the space of functions. These satisfy the so-called Witt algebra relations (two commuting copies)

\[
[l_m, l_n] = (m - n)l_{m+n}, \\
[l_m, \bar{l}_n] = (m - n)\bar{l}_{m+n}, \\
[l_m, \bar{l}_n] = 0.
\]

(2.6)

As the “left-moving” generators l_n commute with the “right-moving” ones \bar{l}_n, one typically treats the coordinate \z as completely independent of \bar{z}, i.e. works on \mathbb{C}^2 rather than the complex plane. One needs to remember to identify \z with the complex conjugate of \z at the end of computations, e.g. if one wants to obtain correlation functions.

The Witt algebra contains a finite sub-algebra consisting of the l_n with n \in \{-1, 0, 1\} which generate SL(2, \mathbb{R}). Transformations from these generate two copies of the Möbius group

\[
z \mapsto \frac{az + b}{cz + d}, \quad \bar{z} \mapsto \frac{a'\bar{z} + b'}{c'\bar{z} + d'}
\]

with

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in SL(2, \mathbb{R}).
\]

For general reasons (Wigner’s theorem), in quantum theory one needs to admit projective representations of a Lie algebra in quantum theory, which motivates to consider central extensions of the Lie algebra in question. The central extension of the Witt algebra (2.6) is the Virasoro algebra, Vir, which has generators L_n and \bar{L}_n.

\[
[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12}m(m^2 - 1)\delta_{m+n, 0}, \\
[\bar{L}_m, \bar{L}_n] = (m - n)\bar{L}_{m+n} + \frac{c}{12}m(m^2 - 1)\delta_{m+n, 0}, \\
[L_m, \bar{L}_n] = 0.
\]

(2.7)

c is the central charge (also called the conformal anomaly).

We can now rephrase our definition of a conformal field theory in two dimensions: it is
a quantum field theory in two dimensions such that the space of states and the space of fields carry a representation of (two commuting copies of) the Virasoro algebra. The full set of symmetries of a CFT may have additional generators \( W_{(l)}^{(l)} \) apart from the Virasoro modes \( L_n, \overline{L}_n \). We use the notation \( \mathcal{W} \times \overline{\mathcal{W}} \) for the full symmetry algebra (also referred to as \( W \)-algebra in this context). The additional generators have fixed commutation relations with the \( L_n \),

\[
[L_n, W_{m}^{(l)}] = ((h_l - 1)n - m)W_{n+m}, \quad [L_n, \overline{W}_{m}^{(l)}] = 0 \quad (2.8)
\]

(analogously with the right-moving \( \overline{L}_n \)) – which mean that the \( W_{n}^{(l)} \) are the modes of a left-moving primary field of conformal dimension \( h_l \) (here: integer or half-integer), see below. The possible commutation relations of the \( W_{m}^{(l)} \) among each other depend heavily on the set of \( h_l \), to find consistent \( W \)-algebras is a complicated task as soon as some of the \( h_l \geq 2 \). \( W \)-algebras with additional generators of conformal dimension \( h_l = 1 \) are affine Lie algebras, examples with generators of dimension \( h_l = \frac{3}{2} \) will be given below in the section on superconformal field theory.

In a CFT, the symmetry algebra \( \mathcal{W} \times \overline{\mathcal{W}} \) acts on the space of states \( \mathcal{H} \). We assume (as is true for all unitary CFTs) that \( \mathcal{H} \) decomposes into a direct sum of irreducible representations

\[
\mathcal{H} = \bigoplus_{(\xi, \overline{\xi}) \in I} \mathcal{H}_\xi \otimes \mathcal{H}_{\overline{\xi}} \quad (2.9)
\]

such that each \( \mathcal{H}_\xi \) is a highest weight representation of \( \mathcal{W} \), and analogously for \( \mathcal{H}_{\overline{\xi}} \) and \( \overline{\mathcal{W}} \). (\( I \) is some index set satisfying certain conditions, see below.) For \( \mathcal{W} = Vir \), this means that there is a highest weight state \( |h_\xi\rangle \) in \( \mathcal{H}_\xi \) which satisfies

\[
L_0 |h_\xi\rangle = h_\xi |h_\xi\rangle, \quad L_n |h_\xi\rangle = 0 \text{ for } n > 0 \quad (2.10)
\]

where \( h_\xi \) is called the weight or conformal dimension of the state \( |h_\xi\rangle \). If \( \mathcal{W} \) contains further symmetry generators, there are similar conditions for the \( W_n \). We will write \( |h_\xi, \overline{h}_{\overline{\xi}}\rangle := |h_\xi\rangle \otimes |\overline{h}_{\overline{\xi}}\rangle \).

In any CFT, there must be a vacuum state \( |0, 0\rangle \) of left- and right-moving weights zero,
which is in fact invariant under global conformal transformations, i.e. annihilated by $L_0, L_1, L_{-1}$ (as well as by their right-moving counterparts).

The irreducible representations $\mathcal{H}_i$ are spanned by their highest weight state together with descendant states, found by acting with $L_{-m}$, $m > 0$ (or in general further modes $W_n$) on the highest weight states,

$$L_{-n_k} \cdots L_{-n_1} | h \rangle ;$$

this descendant has weight $n_k + \ldots + n_1 + h$. A representation of a Virasoro algebra where all $L_{-n} | h \rangle$ are independent is called a **Verma module** $\mathcal{V}_h$. In general, there are linear dependencies among states of the same weight, leading to so-called singular states (or null-states) within the representation. Singular states are descendant states which satisfy equation (2.10) but with a weight strictly greater than that of their highest weight state. In such a case, singular states together with their descendants have to be divided out in order to obtain an irreducible $\mathcal{W}$-representation.

Similar to representations of finite-dimensional Lie algebras, one can consider tensor products of $\mathcal{W}$-representations, although the precise definition (see e.g. [59]) is more intricate due to the central term in the Virasoro algebra. Decomposition of tensor products of representations into irreducibles leads to the fusion rules of the symmetry algebra $\mathcal{W}$, written symbolically as

$$[i] \times [j] = \sum_{k \in I_W} N^{k}_{ij} [k] \quad (2.11)$$

where $[i]$ denotes an (equivalence class of an) irreducible representation and the $N^{k}_{ij} \in \mathbb{Z}_+$ are multiplicities. $I_W$ denotes the set of all possible irreducible highest weight representations of $\mathcal{W}$.

An irreducible representation $\mathcal{H}_i$ can be partially characterised by its **conformal character**

$$\chi_i(q) = \text{Tr}_{\mathcal{H}_i} q^{L_0 - \frac{c}{24}} \quad (2.12)$$

which counts the degeneracies of $L_0$-eigenvalues. Setting $q = e^{2\pi i \tau}$ with $\Im(\tau) > 0$ en-
sures that the power series $\chi_i(q)$ actually converge for highest weight representations. The remarks on the geometric meaning of the Virasoro generators made before allow to relate $L_0 + \bar{L}_0$ to the Hamilton operator on the cylinder, so that the quantity

$$Z(q, \bar{q}) = \text{Tr}_H q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{c}{24}} = \sum_{(i, \bar{i}) \in I} \chi_i(q) \chi_{\bar{i}}(\bar{q})$$

is the \textbf{partition function} of the CFT. At least formally, it can be written as a path integral of the CFT defined on a torus with modular parameter $\tau$, and using this point of view one can argue that $Z(q, \bar{q})$ should be invariant under modular transformations $\tau \mapsto \tau + 1$ and $\tau \mapsto -1/\tau$. In particular the latter transformation, also called modular $S$-transformation, leads to strong constraints on the admissible index sets $I$ occurring in (2.9,2.13) [12]. Choices that are always admissible are the so-called diagonal modular invariant ($\bar{i} = i$) and the charge conjugate partition function ($\bar{i} = i^+$, where $i^+$ denotes the conjugate representation to $i$), where $i$ runs over the set $I_W$ of all possible irreducible highest weight representations of $\mathcal{W}$ in both cases. 

A \textbf{rational conformal field theory} is one for which $I$ is a finite set. It can be shown in some generality that the characters themselves have a rather simple behaviour under modular transformations, namely there exists a matrix $S$ such that

$$\chi_i(\bar{q}) = \sum_{j \in I_W} S_{ij} \chi_j(q)$$

$$S^* = S^{-1}, \quad S = S^t, \quad S^2 = C$$

where $C_{ij} = \delta_{j,i^+}$. There is a surprising relation between $S$-matrix elements and the fusion rules, the so-called Verlinde formula [73],

$$N_{ij}^k = \sum_{m \in I_W} \frac{S_{im} S_{jm} S_{km}^*}{S_{0m}}$$

which will in particular be needed below to show that Cardy boundary states satisfy Cardy’s conditions.
In a CFT, the symmetry algebra also acts on the space of field operators. In fact, there is a one-to-one correspondence between fields and states, thus the field space decomposes into the same irreducibles as the state space. Given a field \( \phi(z, \bar{z}) \), one obtains a state by applying the field at \( z = \bar{z} = 0 \) to the vacuum state \( |0,0\rangle \):

\[
|\phi\rangle := \phi(0,0) |0,0\rangle .
\]

Those fields that produce highest weights states \( |h_\phi, \bar{h}_\phi\rangle \) are called primary fields; they can also be characterised by their commutation relations with the Virasoro modes:

\[
[L_n, \phi(z, \bar{z})] = z^{n+1} \partial_z \phi(z, \bar{z}) + (n+1)h_\phi z^n \phi(z, \bar{z})
\]

(2.17) together with analogous relations involving \( \bar{L}_n \), \( \bar{z} \) and the right-moving conformal dimension \( \bar{h}_\phi \) of \( \phi(z, \bar{z}) \). Yet another way to characterise primary fields is through their behaviour under arbitrary conformal transformations \( z \mapsto f(z) \) and \( \bar{z} \mapsto \bar{f}(\bar{z}) \):

\[
\phi\left(f(z), \bar{f}(\bar{z})\right) = \left(\frac{df}{dz}\right)^{-h} \left(\frac{d\bar{f}}{d\bar{z}}\right)^{-\bar{h}} \phi(z, \bar{z}) .
\]

(2.18) A quasi-primary field is a field which satisfies (2.17) for \( n \in \{-1, 0, 1\} \) or, equivalently, (2.18) for any global conformal transformation. Note that, for a quasi-primary field \( \phi \) of conformal dimension \( h_\phi \), the state \( \phi(0,0) |0,0\rangle \) has conformal weight (i.e. \( L_0 \)-eigenvalue) \( h_\phi \), as well.

Arbitrary descendants (or “members of the conformal family of”) a primary field \( \phi(z, \bar{z}) \) are obtained by the action of Virasoro (or more generally \( W \)) modes, in particular one has

\[
L_{-1} \phi(z, \bar{z}) = \partial_z \phi(z, \bar{z}) ;
\]

(2.19) explicit expressions for other descendants \( L_n \phi(z, \bar{z}) \) can be derived from the operator product expansion of \( T(z) \) with \( \phi(z, \bar{z}) \), see below. It can be shown that any field in a CFT can be written as a linear combination of quasi-primary fields and their derivatives.

One can construct chiral fields (depending only on \( z \) or on \( \bar{z} \)) from the modes of
the symmetry algebra. In particular, the left- and right-moving components of the **energy-momentum tensor** are given by

\[ T(z) = \sum_{n \in \mathbb{Z}} z^{-n-2} L_n, \quad \overline{T}(\overline{z}) = \sum_{n \in \mathbb{Z}} \overline{z}^{n-2} \overline{L}_n \]

\((T(z)\) and \(\overline{T}(\overline{z})\) can also be thought of as arising from the familiar energy-momentum tensor \(T_{\mu\nu}\), which is conserved, symmetric and, in a conformal theory, traceless, upon passing to light-cone coordinates.) One can recover the Virasoro modes from \(T(z)\) upon contour integration,

\[ L_n = \frac{1}{2\pi i} \oint_{C_u} T(z) z^{n+1} dz. \quad (2.20) \]

\(T(z)\) is an example for a field that is quasi-primary, but not primary, which can be checked using the Virasoro commutation relations (the central term does not show up in commutators of \(L_0\) and \(L_{\pm1}\)).

In the same way, one can associate (holomorphic) fields \(W^{(l)}(z)\) to any further (left-moving) \(\mathcal{W}\)-generators. Using the commutation relations (2.8), one can show that the field \(W^{(l)}(z)\) is a primary field of conformal dimension \(h_l\).

The commutation relations used so far can all be encoded in **operator product expansions** (OPEs). The intuitive idea behind an OPE is that, seen from “far away”, two fields inserted close to each other can be replaced by a linear combination of single field insertions; the expansion coefficients should be functions of the two original insertion points, in general with singularities as the two points approach each other. In a conformal QFT, scale invariance leads to additional constraints on the allowed singularities; the typical OPE of two (quasi-)primary fields \(\phi_i(z, \overline{z})\) and \(\phi_j(z, \overline{z})\) of left- and right-moving conformal dimensions \((h_i, \overline{h}_i)\) resp. \((h_j, \overline{h}_j)\) has the
where the sum is over all quasi-primary fields $\phi_k(z, \bar{z})$ with conformal dimensions $(h_k, \bar{h}_k)$, the $C_{ijk}$ are numerical coefficients (the OPE structure constants), and the dots refer to terms which are non-singular as $z \to w$.

In general, it is difficult to determine the allowed structure constants for arbitrary (quasi-)primary fields in a theory (including those of $\mathcal{W}$-algebra generators). From the relation between states and fields, it is clear that $C_{ijk}$ can be non-vanishing only if the conformal family of $\phi_k$ occurs in the fusion of the families of $\phi_i$ and $\phi_j$, i.e. if the fusion rule $N^k_{ij} \neq 0$. One can also show that the OPE of descendants is determined by those of the associated primary fields. The actual values of the $C_{ijk}$ (for three primary fields) are constrained by non-linear conditions like the sewing (or crossing) relations, see e.g. [6,24,26,59] and literature mentioned therein.

The OPEs involving the energy-momentum tensor $T(z)$, however, are universal:

$$
T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{1}{z-w}\partial_w T(w) + \ldots \quad (2.22)
$$

$$
T(z)\phi(w, \bar{w}) = \frac{h}{(z-w)^2}\phi(w, \bar{w}) + \frac{1}{z-w}\partial_w \phi(w, \bar{w}) + \ldots \quad (2.23)
$$

where $\phi(w, \bar{w})$ is a primary field.

From these relations, one can derive the Virasoro algebra and the commutator defining primary fields (2.17), respectively. The computation for the Virasoro algebra uses (2.20) and Cauchy’s integral formula

$$
\frac{1}{2\pi i} \oint_{C_w} dz \frac{f(z)}{(z-w)^n} = \frac{f^{(n-1)}(w)}{(n-1)!}
$$
(for $f$ holomorphic). Applied to (2.22), we have:

\[ [L_n, L_m] = \frac{1}{(2\pi i)^2} \oint_{C_o} dw \ w^{m+1} \oint_{C_w} dz \ z^{n+1} T(z) T(w) \]

\[ = \frac{1}{2\pi i} \oint_{C_o} dw \ w^{m+1} \left\{ \frac{c}{12} n(n^2 - 1)w^{n-2} + 2(n + 1)w^n T(w) \right\} \]

\[ + w^{n+1} \partial_w T(w) \]

\[ = \frac{1}{2\pi i} \oint_{C_o} dw \ w^{m+1} \left\{ \frac{c}{12} n(n^2 - 1)w^{n-2} + w^n \sum_{l \in \mathbb{Z}} (2n-l)w^{-l-2}L_l \right\} \]

\[ = \frac{c}{12} n(n^2 - 1)\delta_{m,-n} + \sum_{l \in \mathbb{Z}} (2n-l)L_l \delta_{m+n,l} \]

\[ = \frac{c}{12} n(n^2 - 1)\delta_{m,-n} + (n - m)L_{m+n} . \]

Once the coefficients in the OPE are known, they could, at least in principle, be used to compute \textbf{correlation functions} in the CFT. An $N$-point correlation function is the vacuum expectation value

\[ \mathcal{F}_{i_1, \ldots, i_N}(z_1, \bar{z}_1, \ldots, z_N, \bar{z}_N) = \langle 0, 0 | \phi_{i_1}(z_1, \bar{z}_1) \ldots \phi_{i_N}(z_N, \bar{z}_N) | 0, 0 \rangle \]

where the $\phi_{i_n}(z_n, \bar{z}_n)$ are (quasi-)primary fields, $| 0, 0 \rangle$ is the vacuum state (also referred to as the “in-vacuum” here), and $\langle 0, 0 |$ is its dual state, the “out-vacuum”. (We will drop the “0,0” in the following.)

One can show (see e.g. [6,26]) that correlation functions of descendants are determined by those of the corresponding primary fields (to see this, one inserts a $T(z)$ into the correlation function, then uses contour integration to obtain differential equations for the correlator of descendants). Furthermore, any singular vectors in the theory lead to differential equations on the correlation functions of primary fields, which are sometimes strong enough to determine those correlators up to normalisation [6]; this allows to express the correlators (2.30) as linear combinations of so-called conformal blocks. The normalisations are constrained by requiring associativity of the OPE, which leads to quadratic relations among the $C_{ijk}$ in (2.21).

For $N = 1, 2, 3$, the $z$-dependence of $N$-point correlation functions of primary fields is
completely determined by covariance under global conformal transformations (see [26] for an detailed derivation). One finds that one-point functions vanish unless the field is the identity operator,

\[ \langle \phi_i(z, \bar{z}) \rangle = \delta_{i,0} \]

where “0” denotes the vacuum sector. (The proof in particular uses translation invariance in both \( z \) and \( \bar{z} \), therefore it does not apply to bulk one-point functions in boundary CFTs – which can indeed be non-trivial for fields other than the identity.) For two-point functions, one has

\[ \langle \phi_i(z_1, \bar{z}_1) \phi_j(z_2, \bar{z}_2) \rangle = C_{i,j} \delta_{j,i} \]

where we have made left- and right-moving labels explicit, and where \( i^+ \) labels the conjugate sector of \( i \) (with \( h_{i^+} = h_i \)). Usually, primary fields are normalized so that \( C_{i,i} = 1 \).

The three-point function contains the OPE structure constants:

\[ \langle \phi_i(z_1, \bar{z}_1) \phi_j(z_2, \bar{z}_2) \phi_k(z_3, \bar{z}_3) \rangle = C_{ijk} \]

where we have for simplicity assumed that \( h = \bar{h} \) for all three primary fields, and where \( h_{ijk} = h_i + h_j - h_k \).

## 2.2 \( N = 1 \) and \( N = 2 \) Superconformal Field Theory

In string theory, the most important extensions of the Virasoro algebra are the \( N = 1 \) and \( N = 2 \) supersymmetric Virasoro algebras, which are used to build tachyon-free and space-time supersymmetric string backgrounds.

An \( N = 1 \) SCFT is a 2d CFT where the symmetry algebra \( \mathcal{W} \) contains \( N = 1 \) super-Virasoro algebra. This is generated by the Virasoro modes \( L_n, n \in \mathbb{Z} \) as before and
additional fermionic generators $G_r$, which satisfy

$$[L_m, G_r] = \left( \frac{m}{2} - r \right) G_{m+r}, \quad \{G_r, G_s\} = 2L_{r+s} + \frac{c}{3} \left( r^2 - \frac{1}{4} \right) \delta_{r,-s}.$$  

The commutator relation means that the $G_r$ form a primary field of dimension $\frac{3}{2}$, the anti-commutator means that $G_r$ are “supercharges”. Analogous relations hold for the right-movers. The mode index $r$ of $G_r$ can take either integer or half-odd integer values. Representations where $G_r$, $r \in \mathbb{Z}$, act form the so-called Ramond (or R-) sector of the theory, those where $G_r$, $r \in \mathbb{Z} + \frac{1}{2}$, act form the so-called Neveu-Schwarz (or NS-) sector.

For our purposes, $N = 2$ superconformal field theories are more important. Here, the symmetry algebra $W$ contains a copy of the $N = 2$ super-Virasoro algebra, generated by bosonic operators $L_n$, $J_n$, $n \in \mathbb{Z}$ and fermionic generators $G^\pm_n$, where again either $n \in \mathbb{Z}$ or $n \in \mathbb{Z} + \frac{1}{2}$. They satisfy (in addition to the Virasoro algebra):

$$[L_m, J_n] = -nJ_{m+n}, \quad [L_m, G^\pm_n] = \left( \frac{m}{2} - n \right) G^\pm_{m+n},$$

$$[J_m, J_n] = \frac{c}{3} m \delta_{m,-n}, \quad [J_m, G^\pm_n] = \pm G^\pm_{m+n},$$

$$\{G^-_m, G^+_n\} = \left( m^2 - \frac{1}{4} \right) \frac{c}{3} \delta_{m,-n} - (m - n) J_{m+n} + 2L_{m+n},$$

$$\{G^\pm_m, G^\pm_n\} = 0.$$  

Similarly for right-movers. The $J_n$ are the modes of an abelian (or U(1)) current, in particular the associated field $J(z) = \sum_n z^{-n-1} J_n$ has conformal dimension 1. The commutator of $J_n$ with $G^\pm_n$ states that the supercurrents $G^\pm(z) = \sum_n z^{-n-\frac{3}{2}} G^\pm_n$ have U(1) charge $\pm 1$.

Note that the $N = 2$ super-Virasoro algebra has an $N = 1$ super-Virasoro sub-algebra with

$$G_n := \frac{1}{\sqrt{2}} \left( G^+_n + G^-_n \right).$$  

(2.32)

Note also that the $N = 2$ super-Virasoro algebra admits a non-trivial automorphism
This is called the mirror automorphism and can be seen as the world-sheet “origin” of mirror symmetry of Calabi-Yau manifolds.

The operator product expansions equivalent to the $N = 2$ super-Virasoro commutation relations are

\[
T(z) T(w) = \frac{c/2}{(z - w)^4} + \frac{2}{(z - w)^2} T(w) + \frac{1}{z - w} \delta_w T(w) + \ldots,
\]

\[
T(z) G^\pm (w) = \frac{3/2}{(z - w)^2} G^\pm (w) + \frac{1}{z - w} \partial_w G^\pm (w) + \ldots,
\]

\[
T(z) J(w) = \frac{1}{(z - w)^2} J(w) + \frac{1}{z - w} \partial_w J(w) + \ldots,
\]

\[
J(z) J(w) = \frac{c/3}{(z - w)^2} + \ldots,
\]

\[
J(z) G^\pm (w) = \pm \frac{1}{z - w} G^\pm (w) + \ldots,
\]

\[
G(z)^+ G(w)^- = \frac{2c/3}{(z - w)^3} + \frac{2}{(z - w)^2} J(w) + \frac{1}{z - w} (2T(w) + \partial_w J(w)) + \ldots.
\]

A primary field $\phi(w, \bar{w})$ of the $N = 2$ super-Virasoro algebra has the operator product expansions

\[
T(z) \phi(w, \bar{w}) = \frac{h_\phi}{(z - w)^2} \phi(w, \bar{w}) + \frac{1}{z - w} \delta_w \phi(w, \bar{w}) + \ldots,
\]

\[
J(z) \phi(w, \bar{w}) = \frac{q_\phi}{z - w} \phi(w, \bar{w}) + \ldots,
\]

\[
G^\pm(z) \phi(w, \bar{w}) = \frac{1}{z - w} (G^\pm_{-1/2} \phi)(w, \bar{w}) + \ldots = \frac{1}{z - w} \tilde{\phi}^\pm(w, \bar{w}) + \ldots,
\]

where $\tilde{\phi}^\pm(w, \bar{w}) = (G^\pm_{-1/2} \phi)(w, \bar{w})$ is the superpartner of $\phi(w, \bar{w})$.

The maximal commuting subalgebra of the $N = 2$ super-Virasoro algebra consists of the zero modes $L_0$ and $J_0$, so in irreducible representations one can choose a basis of

\[
W_n \mapsto \Omega_M(W_n) \quad \text{with} \quad \Omega_M(L_n) = L_n, \quad \Omega_M(J_n) = -J_n, \quad \Omega_M(G^\pm_n) = G^\mp_n.
\]

(2.33)
simultaneous eigenstates,

\[ L_0 | \phi \rangle = h_{\phi} | \phi \rangle , \]
\[ J_0 | \phi \rangle = q_{\phi} | \phi \rangle , \]

where \( h_{\phi} \) is the conformal dimension and \( q_{\phi} \) the U(1)-charge of the state \( | \phi \rangle \).

In the NS-sector of a unitary theory, the two satisfy the inequality [56]

\[ h_{\phi} \geq \frac{|q_{\phi}|}{2} \tag{2.34} \]

which follows from

\[ \langle \phi | \{ G^{-}_{\frac{1}{2}}, G^{+}_{\frac{1}{2}} \} | \phi \rangle = (2L_0 - J_0) | \phi \rangle \geq 0 \]

together with the expression where \( G^{+} \) and \( G^{-} \) are swapped. States with \( h_{\phi} = \frac{q_{\phi}}{2} \) satisfy \( G^{+}_{\frac{1}{2}} | \phi \rangle = 0 \) and are called (left-moving) chiral primaries; one can show that they indeed correspond to primary fields [56]. States with \( h_{\phi} = -\frac{q_{\phi}}{2} \) satisfy \( G^{-}_{\frac{1}{2}} | \phi \rangle = 0 \) and are called (left-moving) anti-chiral primaries. Analogous statements hold for the right-moving conformal dimensions and U(1)-charges. Note that the notation "chiral" here comes from a different context and has nothing to do with holomorphic or anti-holomorphic \( z \)-dependence. One can have states which are \((c,c)\) – i.e. satisfy \( G^{-}_{\frac{1}{2}} | \phi \rangle = 0 \), as well as \((c,a)\)-states – those with \( G^{-}_{\frac{1}{2}} | \phi \rangle = 0 \), states which are chiral with respect to the left-moving super-Virasoro algebra and neither chiral nor anti-chiral with respect to the right-moving generators, etc. Note also that applying the mirror automorphism interchanges chiral and anti-chiral states.

In the R-sector, the analogous special states are those annihilated by \( G^{+}_{0} \) or \( G^{-}_{0} \); they satisfy \( h = \frac{c}{24} \), are called Ramond ground states and are related to chiral and anti-chiral states by spectral flow.

The extra relations that hold for chiral or anti-chiral primaries in the NS-sector have consequences for their operator product expansion, leading to the notion of the chiral ring (resp. anti-chiral ring). Consider the OPE of two chiral fields with U(1)-charges.
$q_1$ and $q_2$, thus with conformal dimensions $h_1 = 2q_1$ and $h_2 = 2q_2$. Since U(1)-
charges are conserved in an $N = 2$ SCFT and additive, only fields with U(1)-charge $q = q_1 + q_2$ can occur on the right hand side of the OPE. Due to the above inequality (2.34) (we assume unitarity here), the fields in the OPE all have conformal dimension $h \geq h_1 + h_2$. In particular, the OPE of two chiral primaries has no singular terms. Taking the limit where the two insertions points coincide projects the OPE onto those fields with $h = h_1 + h_2 = 2q$, i.e. onto chiral primaries. This introduces a multiplication among the set of chiral primaries (without any $z$-dependence) which is called the chiral ring. The structure constants of this ring are the OPE coefficients where all indices belong to chiral primaries.

### 2.3 Boundary Conditions

We give a very brief sketch of boundary CFT, which provides the tools for a world-sheet description of branes in string theory. (A brane is nothing but a conformal boundary condition, from the world-sheet point of view.) For more details on boundary CFT, see e.g. [13,66,67].

A boundary CFT is a CFT on the complex plane which has been restricted to the upper half-plane. To ensure consistency, one needs to impose boundary conditions on the fields. Introducing a boundary affects the symmetries of the theory, e.g. one loses invariance under translation perpendicular to the boundary. In order to preserve half of the symmetry algebra $\mathcal{W} \times \mathcal{W}$ that was present in the bulk theory (the CFT on the full plane), one imposes so-called gluing conditions which relate left- and right-moving $\mathcal{W}$- generators along the boundary:

$$T(z) = T(\bar{z}) \quad \text{and} \quad W(z) = \Omega W(\bar{z}) \quad \text{for} \quad z = \bar{z}. \quad (2.35)$$

Here $\Omega : \mathcal{W} \rightarrow \mathcal{W}$ is a local (pointwise) automorphism which leaves the energy-momentum tensor invariant: $\Omega T = T$. Apart from the case $\Omega = \text{id}$, the most important example from the point of view of topological branes is the mirror automorphism $\Omega = \Omega_M$ of the $N = 2$ super-Virasoro algebra, see eq. (2.33) above.
The choice of the gluing automorphism $\Omega$ fixes the boundary condition only partly, there are further non-linear constraints (Cardy’s conditions, see below, and associativity of OPEs) which need to satisfied. We label a boundary condition by $(\Omega, \tilde{\alpha})$, where $\tilde{\alpha}$ labels solutions to those consistency relations per fixed gluing $\Omega$.

Information on boundary conditions $(\Omega, \tilde{\alpha})$ can be encoded via boundary states $|\alpha\rangle_{\Omega}$. These are linear combinations of so-called Ishibashi states $|i\rangle\rangle_{\Omega}$, which in turn are (non-normalisable) combinations of states from Hilbert space of the bulk theory ($i$ denotes an irreducible $\mathcal{W}$-representation). The Ishibashi states [36,37] are constructed so as to satisfy the gluing conditions (2.35), which in terms of modes read

$$ (L_n - L_{-n}) | i \rangle\rangle_{\Omega} = 0, \quad (2.36) $$

$$ (W_n - (-1)^h \Omega \bar{W}_{-n}) | i \rangle\rangle_{\Omega} = 0. \quad (2.37) $$

Choosing an orthonormal basis, $|i,N\rangle$, for each $\mathcal{H}_i$ ($N$ labels the different elements), one can give a concrete formula for the Ishibashi state (at first for $\Omega = \text{id}$):

$$ |i\rangle\rangle = \sum_{N=0}^{\infty} |i,N\rangle \otimes U |i,N\rangle \quad (2.38) $$

where $U$ is an anti-unitary operator satisfying

$$ U \bar{W}_n = (-1)^h \bar{W}_n U. \quad (2.39) $$

One can show that $|i\rangle\rangle$ is unique up normalisation, which will be fixed by specifying an “inner product” for Ishibashi states below. Note that the components of $|i\rangle\rangle$ are elements of $\mathcal{H}_i \otimes \mathcal{H}_{i^+}$, where $i^+$ is the conjugate irrep of $i$.

When $\Omega$ is non-trivial, one obtains an $\Omega$-twisted Ishibashi state (satisfying (2.36)) as follows: First note that, given an irrep $\pi_i$ in $\mathcal{H}_i$, the $\mathcal{W}$-automorphismism can be used to induce another irrep $\pi_i^{\Omega} := \pi_i \circ \Omega$; this will be isomorphic to some unique irrep $\mathcal{H}_{\omega(i)}$ in the list of all $\mathcal{W}$-irreps; the isomorphism can be implemented by a unitary operator

$$ V_{\Omega} : \mathcal{H}_{\omega(i)} \rightarrow \mathcal{H}_i \quad (2.40) $$
which commutes with $U$ and satisfies $\pi_i \circ \Omega = Ad(V_\Omega) \circ \pi_{\omega(i)}$. The $\Omega$-twisted Ishibashi state is then given by the formula

$$| i \rangle\rangle_\Omega := (\text{id} \otimes V_\Omega) | i \rangle\rangle .$$

(2.41)

Components of $| i \rangle\rangle_\Omega$ are elements of $\mathcal{H}_i \otimes \mathcal{H}_{\omega^{-1}(i^+)}$.

A general boundary state for given gluing conditions $\Omega$ is a linear combination of Ishibashi states

$$| \alpha \rangle\rangle_\Omega = \sum_{j \in I^\Omega_W} B^j_\alpha | i \rangle\rangle_\Omega .$$

(2.42)

The set $I^\Omega_W$ labels the available Ishibashi states for the given gluing condition. One can show that the coefficients $B^j_\alpha$, which are related to one-point functions of primary bulk fields (see [14]), They are restricted by non-linear relations, the best-studied of which are the so-called Cardy constraints.

Using the language of string world-sheets, Cardy’s condition requires that the cylinder diagram can be interpreted and evaluated in two ways, namely as tree level closed string propagation between two branes (given in terms of boundary states) or as one-loop open string diagram (with two prescribed boundary conditions on the open string endpoints). The latter computation should result in an open string partition, which is a sum over characters of the open string symmetry algebra (in our cases a single copy of $\mathcal{W}$). The former computation gives the overlap of two boundary states, with the “closed string propagator” $\tilde{q}^{\frac{1}{2}(L_0 + \bar{L}_0 - c/12)}$ inserted in between. As an equation, Cardy’s condition reads

$$\Omega \langle \beta | \tilde{q}^{\frac{1}{2}(L_0 + \bar{L}_0 - c/12)} | \alpha \rangle \rangle = \sum_{i \in I_W} n^i_{\alpha \beta} \chi_i(q)$$

(2.43)

where $n^i_{\alpha \beta}$ is the multiplicity of the irrep $i$ in the open string partition function; if $q = \exp\{2\pi i \tau\}$ with $\Im(\tau) > 0$, then $\tilde{q} = \exp\{-2\pi i/\tau\}$. This leads to strong non-linear constraints on the coefficients $B^j_\alpha$ if one uses the inner product of Ishibashi states defined by

$$\Omega \langle \langle j | q^{L_0 - \bar{L}_0} | i \rangle \rangle = \delta_{i,j} \chi_i(q)$$

(2.44)
where \( \chi_i(q) \) is the conformal character of the irreducible representation \( i \), and if one exploits the behaviour of characters under modular transformations (2.14). Using those, Cardy’s conditions become

\[
\sum_{j \in \mathcal{I}_W} B_{\beta j}^+ B_{\alpha i}^j S_{j i} \chi_i(q) = \sum_{i \in \mathcal{I}_W} n_{\alpha \beta}^i \chi_i(q) .
\] (2.45)

As the \( n_{\alpha \beta}^i \) are positive integers, these constraints even affect the overall normalisation of boundary states.

In the special case of bulk theories with charge conjugate modular invariant partition function, i.e. with \( \mathcal{H} = \bigoplus_{i \in \mathcal{I}_W} \mathcal{H}_i \otimes \mathcal{H}_i^* \), where \( \mathcal{I}_W \) denotes the set of all possible irreducible representations of \( W \), Cardy [13] has given the following general solution, applicable for trivial gluing automorphism,

\[
B_{\alpha}^i = \frac{S_{\alpha i}}{\sqrt{S_0 i}}
\] (2.46)

where 0 denotes the vacuum representation and where the labels \( \alpha \) of the different Cardy boundary states run over the same set \( \mathcal{I}_W \) as the representation label \( i \).

For more general bulk partition functions, or for non-trivial gluing conditions, one cannot directly rely on Cardy’s formula (2.46), but has to carefully analyse the set of available Ishibashi states and the linear combinations that satisfy Cardy’s constraints.

The most important cases for our purposes are models with \( N = 2 \) conformal symmetry like minimal models and Gepner models (constructed from the former). Boundary states for these have been worked out in [8,11,65,67]. As stated above, one can choose two different types of gluing conditions for the \( N = 2 \) super-Virasoro algebra, due to the mirror automorphism; in particular, B-type branes satisfy the gluing conditions

\[
(L_n - \bar{L}_{-n})\langle B \rangle = 0 ,
\] (2.47)

\[
(J_n - \bar{J}_{-n})\langle B \rangle = 0 ,
\] (2.48)

\[
(G^+_{\tau} + i \eta \bar{G}^+_{-\tau})\langle B \rangle = 0 .
\] (2.49)
We refer to the papers cited above for concrete formulas, but note that from the boundary states, one can both compute open string partition functions and the so-called Witten index

\[ I_{\alpha\beta} = \text{Tr} \mathcal{M}_{\alpha\beta} (-1)^F q^{L_0 - \frac{c}{24}} \]  

which, in string language, counts the number of “bosonic” minus the number of “fermionic” Ramond ground states of the open string stretched between two branes described by the boundary conditions \( \alpha \) and \( \beta \). This quantity is the main tool to compare CFT boundary states with matrix factorizations.

### 2.4 Topological Twisting

In this section topologically twisted conformal field theories in both the bulk and boundary cases is reviewed. Topological field theories (TFTs) were introduced by Witten in 1988 [77, 78] and this was implemented for conformal field theories by Eguchi and Yang two years later [21]. The idea is that by taking an \( N = 2 \) SCFT and redefining the stress-energy tensor (topological twisting) one constructs a simplified field theory where we are left with only the chiral primary fields. The theory retains all information on supersymmetry preserving observables. We may choose our twist so as to preserve either A-type or B-type supersymmetry. Since we are interested in B-branes we will restrict ourselves here to a discussion of the B-twist. Further details can be found in [17,18,21,32,77,78].

In order to preserve B-type supersymmetry the topological twist

\[ T(z) \rightarrow T(z) + \frac{1}{2} \partial J(z) \]  

is performed\(^1\). Meaning that \( L_n \rightarrow L_n - \frac{1}{2} (n + 1) J_n \). The spin of the \( N = 2 \) supercurrents \( G^\pm(z) \) becomes \( \frac{3}{2} \mp \frac{1}{2} \) under this twist, and they are called \( G(z) \) (spin 2) and \( Q(z) \) (spin 1) in the topological theory. Inserting this into the \( N = 2 \) superconformal

---

\(^1\)Preservation of A-type supersymmetry comes from the twist \( T(z) \rightarrow T(z) - \frac{1}{2} \partial J(z) \)
algebra (2.31) the twisted algebra

\[
\begin{align*}
[L_n, L_m] &= (n - m) L_{m+n}, & [J_n, Q_m] &= Q_{n+m}, \quad (2.52) \\
[J_n, Q_m] &= -mQ_{n+m}, & [J_n, G_m] &= -G_{n+m}, \quad (2.53) \\
[J_n, G_m] &= (n - m) G_{m+n}, & [J_n, J_m] &= dn\delta_{n,-m}, \quad (2.54)
\end{align*}
\]

is obtained. Similarly for right-movers. All modes are now integer. The Virasoro algebra no longer has central charge, although, a central extension \( \hat{c} = \frac{\epsilon}{3} \) does appear in the algebra and is the background charge. For a level \( k \) minimal model, the background charge is \( d = \frac{k}{k+2} \).

In TFTs the only physical states are the chiral primaries so the Hilbert space of a TFT often only contains a finite number of states. By definition, the space of physical states is in one to one correspondence with the cohomology classes of \( Q = Q_0 + \overline{Q}_0 \)

\[
\mathcal{H}_{phys} = \ker(Q)/\text{Im}(Q). \quad (2.57)
\]

\( Q \) is a nilpotent symmetry satisfying \( Q^2 = 0 \).

The \( U(1) \) charge, \( q \), of chiral primary fields, \( \phi_i \), never exceeds the background charge

\[
0 \leq q \leq d. \quad (2.58)
\]

Unlike in the untwisted theory, all physical states \( |\phi_i\rangle \) are annihilated by \( L_0 \) as well as by \( L_n \) for \( n > 0 \) (cf. (2.10))

\[
L_n|\phi_i\rangle = 0 \quad \text{for } n \geq 0. \quad (2.59)
\]

For each physical state \( |\phi_i\rangle \) there are also the following observables which are descen-
dants of $\phi_i$

\begin{align*}
\phi_i^{(1,0)} &= [G_{-1}, \phi_i] \, dz , \\
\phi_i^{(0,1)} &= [G_{-1}, \phi_i] \, d\bar{z} , \\
\phi_i^{(1,1)} &= [G_{-1}, [G_{-1}, \phi_i]] \, dz \wedge d\bar{z} \\
&= [G_{-1}, [G_{-1}, \phi_i]] \, d\bar{z} \wedge dz .
\end{align*}

They satisfy the descent equations

\begin{align*}
\{ Q, \phi_i^{(1,0)} \} &= \partial \phi_i , & \{ \bar{Q}, \phi_i^{(0,1)} \} &= \bar{\partial} \phi_i , \\
\{ Q, \phi_i^{(0,1)} \} &= \partial \phi_i^{(0,1)} = d\phi_i^{(0,1)} , & \{ \bar{Q}, \phi_i^{(1,1)} \} &= \bar{\partial} \phi_i^{(1,0)} = d\phi_i^{(1,0)} .
\end{align*}

which follows from that fact that acting on the states with $L_{-1}$ gives $Q$-exact states

\begin{equation}
L_{-1}|\phi_i\rangle = Q(G_{-1}|\phi_i\rangle) .
\end{equation}

The complete list of BRST invariant observables is

\begin{equation}
\left\{ \phi, \oint dz \phi^{(1,0)}, \oint d\bar{z} \phi^{(0,1)}, \int dz \wedge d\bar{z} \phi^{(1,1)} \right\}
\end{equation}

Due to (2.66), bulk field correlation functions

\begin{equation}
\langle \phi_{i_1}, \ldots, \phi_{i_n} \rangle
\end{equation}

in TFT are independent of insertion point so are just constants and are symmetric under permutations of operators. They are also independent of the choice of $Q$-cohomology representative.

As mentioned previously, there exists a background charge $d$ so the charge restriction for non-zero correlation functions on a genus $g$ surface is

\begin{equation}
\sum_i q_i = d(1 - g) .
\end{equation}
Two- and three-point functions are

\[ \eta_{ij} := \langle \phi_i \phi_j \rangle, \quad C_{ijk} := \langle \phi_i \phi_j \phi_k \rangle \]  \hspace{1cm} (2.70)

and the bulk metric is

\[ \eta_{ij} = C_{0ij}. \]  \hspace{1cm} (2.71)

All correlation functions can be factorized into three point functions, for example

\[ \langle \phi_i \phi_j \phi_k \phi_l \rangle = C_{ijm} \eta_{mn} C_{nkl} . \]  \hspace{1cm} (2.72)

The \( C_{ijk} \) determine the operator product algebra

\[ \phi_i \phi_j = \sum_k C_{ij}^k \phi_k. \]  \hspace{1cm} (2.73)

More general amplitudes are given by

\[ C_{i_1 \ldots i_n} := \left\langle \phi_{i_{1_1}} \phi_{i_{2_1}} \phi_{i_{3_1}} \int \phi_{i_{4_1}}^{(1,1)} \ldots \int \phi_{i_{n_1}}^{(1,1)} \right\rangle \]  \hspace{1cm} (2.74)

where, since \( G_{-1} \) has \( U(1) \) charge \(-1\), charge conservation means

\[ \sum_i q_i - (n - 3) = d(1 - g). \]  \hspace{1cm} (2.75)

Amplitudes may be perturbed by integrated insertions of descendants \( \int \phi^{(1,1)} \)

\[ C_{i_1 \ldots i_n}(t) := \left\langle \phi_{i_{1_1}} \phi_{i_{2_1}} \phi_{i_{3_1}} \int \phi_{i_{4_1}}^{(1,1)} \ldots \int \phi_{i_{n_1}}^{(1,1)} \exp \left( \sum_p t_p \int \phi_p^{(1,1)} \right) \right\rangle \]  \hspace{1cm} (2.76)

\[ = \partial t_1 \ldots \partial t_n C_{i_1 \ldots i_n}(t)|_{t=0}, \quad \text{for } n \geq 3 \]  \hspace{1cm} (2.77)

where \( \partial_i := \frac{\partial}{\partial t_i} \). The metric is independent of the deformation parameters \( t \)

\[ C_{0ij}(t) = \eta_{ij} \]  \hspace{1cm} (2.78)

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and $\partial_i C_{jkl}(t)$ is symmetric in all indices so the deformed triple correlator can be integrated to the WDVV potential

$$C_{ijk}(t) = \partial_i \partial_j \partial_k F(t)$$ \hspace{1cm} (2.79)

which can (sometimes) be thought of as the effective prepotential of a Calabi-Yau compactification of the associated untwisted superstring theory. It satisfies the WDVV relations

$$\partial_i \partial_j \partial_m \eta^{mn} \partial_n \partial_k \partial_l F = \partial_i \partial_k \partial_m \eta^{mn} \partial_n \partial_j \partial_l F.$$ \hspace{1cm} (2.80)

We can also look at TFTs on world-sheets with boundaries. On insertion of a boundary we must have that the boundary conditions preserve the actions of $Q := Q_0 + \overline{Q}_0$ and $J := J_0 + \overline{J}_0$ and at the boundary

$$G(z) = \overline{G}(\overline{z}), \hspace{0.5cm} T(z) = T(\overline{z}), \hspace{0.5cm} \text{for } z = \overline{z}$$ \hspace{1cm} (2.81)

(cf. (2.35)).

As in CFT there are now both bulk fields ($\phi_i$) and boundary fields ($\psi_a$) to consider. We also assume that $[Q, \psi_a] = 0$.

Defining $G := G_{-1} + \overline{G}_{-1}$

$$\{Q, G\} = L_{-1} + \overline{L}_{-1}$$ \hspace{1cm} (2.82)

means the descendants will be

$$\phi_i^{(1)} = \phi_i^{(1,0)} + \phi_i^{(0,1)}$$ \hspace{1cm} (2.83)

$$\phi_i^{(2)} = \phi_i^{(1,1)}$$ \hspace{1cm} (2.84)

$$\psi_a^{(1)} = [G, \psi_a]d\tau$$ \hspace{1cm} (2.85)
where $\tau$ is the world-sheet time coordinate. They satisfy the descent relations

\begin{align}
\{Q, \phi_{i}^{(1)}\} &= d\phi_{i}, \quad (2.86) \\
\{Q, \phi_{i}^{(2)}\} &= d\phi_{i}^{(1)}, \quad (2.87) \\
\{Q, \psi_{a}^{(1)}\} &= \left( \frac{d}{d\tau} \psi_{a} \right) d\tau. \quad (2.88)
\end{align}

On the boundary, the metric is given by

$$\omega_{ab} := \langle \psi_{a} \psi_{b} \rangle$$  \quad (2.89)

and is subject to the $\mathbb{Z}_2$-grading

$$\omega_{ab} = (-1)^{|a||b|} \omega_{ba}$$  \quad (2.90)

where $|a|$ is the degree of $\psi_{a}$.

Amplitudes are therefore of the form

$$\langle \psi_{a_{1}} \psi_{a_{2}} P \int \psi_{a_{3}}^{(1)} \cdots \int \psi_{a_{m-1}}^{(1)} \psi_{a_{m}} \int \phi_{i_{1}}^{(2)} \cdots \int \phi_{i_{n}}^{(2)} \rangle$$  \quad (2.91)

or

$$\langle \phi_{i_{1}} \psi_{a_{1}} P \int \psi_{a_{2}}^{(1)} \cdots \int \psi_{a_{m}}^{(1)} \int \phi_{i_{2}}^{(2)} \cdots \int \phi_{i_{n}}^{(2)} \rangle$$  \quad (2.92)

with path-ordering $P$.

For more information on boundary TFT and sewing relations, see in particular [32].

Strings attached to a single D-brane correspond to boundary preserving operators $\psi_{a}^{AA}$. Strings between two different branes correspond to boundary changing operators $\psi_{a}^{AB}$. We can treat both in the same way.
Landau-Ginzburg Models

In this chapter we explain (following [10]) how Landau-Ginzburg (LG) models provide us with a useful way of working with D-branes - via matrix factorizations of the LG-superpotential. In fact, each different factorization of the superpotential corresponds to a different D-brane (using the word “different” also means excluding the equivalence between branes and anti-branes). Preservation of either A-type or B-type supersymmetry on the boundary leads to A-branes respectively B-branes [60,75]. In the first section, following [22, 28], a brief description of a bulk LG model is given. In the second section we discuss variations of the action in a boundary LG model and show how this leads us to the notion of matrix factorizations. Finally, we look at topological boundary LG models where we pass to the cohomology of the BRST operator which corresponds to the brane’s open string spectrum.

3.1 Bulk Landau-Ginzburg Models

Landau-Ginzburg models provide an alternative approach towards \(N = 2\) superconformal field theories, beginning with a Lagrangian formulation of a \(N = 2\) supersymmetric Landau-Ginzburg theory and then allowing it to flow to the infrared fixed point, which gives an \(N = 2\) superconformal theory.

The world-sheet \(\Sigma\) is spanned by bosons \(x^0, x^1\) and fermions \(\theta^\pm, \bar{\theta}^\pm\). We introduce variables \(x^\pm = x^0 \pm x^1\) and \(y^\pm = x^\pm - i\theta^\pm \bar{\theta}^\pm\). The supercharges are given by

\[
Q_\pm = \frac{\partial}{\partial \theta^\pm} + i\bar{\theta}^\pm \frac{\partial}{\partial x^\pm}, \quad \overline{Q}_\pm = - \frac{\partial}{\partial \bar{\theta}^\pm} - i\theta^\pm \frac{\partial}{\partial x^\pm}
\]

and the covariant derivative is

\[
D_\pm = \frac{\partial}{\partial \theta^\pm} - i\bar{\theta}^\pm \frac{\partial}{\partial x^\pm}, \quad \overline{D}_\pm = - \frac{\partial}{\partial \bar{\theta}^\pm} + i\theta^\pm \frac{\partial}{\partial x^\pm}.
\]
They satisfy the supersymmetry algebra

\[ Q_\pm, \bar{Q}_\pm = -2i\partial_\pm, \quad D_\pm, \bar{D}_\pm = 2i\partial_\pm. \]  

(3.3)

A chiral superfield satisfying \( \bar{D}_\pm \Phi = 0 \) is introduced by

\[ \Phi(y^\pm, \theta^\pm) = \phi(y^\pm) + \theta^+\psi_+(y^\pm) + \theta^-\psi_-(y^\pm) + \theta^+\theta^- F(y^\pm) \] ;  

(3.4)

similarly, an anti-chiral superfield \( \bar{\Phi} \) is introduced which satisfies \( D_\pm \bar{\Phi} = 0 \). The variations of the component fields of \( \Phi \) can be found by applying \( \delta = \epsilon_+ Q_- - \epsilon_- Q_+ - \tau_+\bar{Q}_- + \tau_-\bar{Q}_+ \) to \( \Phi \) and rearranging by powers of \( \theta^\pm \). We also use the fact that \( \phi \) is a bosonic field so commutes with everything and the \( \psi_\pm \) are fermionic fields so are anti-commuting. The variations are

\[ \begin{align*}
\delta \phi &= \epsilon_+ \psi_- - \epsilon_- \psi_+, \\
\delta \psi_+ &= 2i\tau_- \partial_+ \phi + \epsilon_+ F, \\
\delta \psi_- &= 2i\tau_+ \partial_- \phi + \epsilon_- F,
\end{align*} \]

(3.5)

The bulk action of the \( N = 2 \) supersymmetric LG-model on a world-sheet \( \Sigma \) is given by

\[ S_\Sigma = \int_\Sigma d^2x \left[ -\partial^\mu \bar{\phi} \partial_\mu \phi + \frac{i}{2} \bar{\psi}_- (\bar{\partial}_0 + \bar{\partial}_1) \psi_- + \frac{i}{2} \bar{\psi}_+ (\bar{\partial}_0 - \bar{\partial}_1) \psi_+ \\
- \frac{1}{4} |W'|^2 - \frac{1}{2} W'' \bar{\psi}_+ \psi_- - \frac{1}{2} W'' \bar{\psi}_- \psi_+ \right] \]  

(3.6)

\( \Phi \) can be an \( n \)-component field (then the LG model has target space \( \mathbb{C}^n \)). Above, \( A\bar{\partial}_\pm B = A\partial_\pm B - (\partial_\pm A)B \) The polynomial \( W(\phi) \) is the LG superpotential, assumed to be homogeneous. The LG superpotential is left essentially unchanged by the renormalization group flow process. This means that we can use the LG superpotential as a way to differentiate between families of these theories and their RG fixed points.

\( W \) also enters the equation of motion \( F = -\frac{1}{2} \bar{W}(\bar{\phi}) \) for the auxiliary field \( F \), which
can therefore be eliminated from the theory. Other equations of motion are

\[
\Box \phi - \frac{1}{4} W'' \bar{W}' - \frac{1}{2} W''' \psi_+ \psi_- = 0, \\
\partial_- \psi_+ - \frac{1}{4} \bar{W}'' \bar{\psi}_- = 0, \\
\partial_+ \psi_- + \frac{1}{4} \bar{W}'' \bar{\psi}_+ = 0
\]

(3.7)

along with their complex conjugates (where one has to use \((\psi_+ \psi_-) = -\bar{\psi}_+ \bar{\psi}_-\) etc., since the \(\psi\) are anti-commuting variables).

### 3.2 Boundary Landau-Ginzburg Models

Before twisting, one can impose B-type or A-type supersymmetry transformations on the fields in the action (3.6), but only the B-conditions lead to matrix factorizations. However, if one does a topological twist to the bulk theory, i.e. \(T(z) \rightarrow T(z) \pm \frac{1}{2} \partial J(z)\), then one makes a choice (in the \(\pm\)), and after that only one type of boundary condition is consistent.

We show, again following the original paper [10], that in order for B-type supersymmetry to be conserved on a world-sheet with boundary the superpotential \(W\) must be factorizable into two polynomials \(E\) and \(J\) (or more generally into two polynomial matrices).

We take the world-sheet with boundary to be a strip, with boundaries at \(x_1 = 0, \pi\) being parametrised by \(x^0 \in \mathbb{R}\); the associated Grassmann variables are \(\theta^0 = \frac{1}{2}(\theta^- + \theta^+)\) and \(\bar{\theta}^0 = \frac{1}{2}(\bar{\theta}^- + \bar{\theta}^+)\). The preserved supercharge for B-type supersymmetry is \(Q = \bar{Q}_+ + \bar{Q}_-\). We introduce new variables \(\eta = \psi_+ + \psi_-\) and \(\theta = \psi_+ - \psi_-\). Setting
\[ \epsilon = \epsilon_+ = -\epsilon_- \text{ and } \delta = \epsilon Q - \tau Q \] the variations of the fields (3.5) are rewritten

\[ \begin{align*}
\delta \phi &= \epsilon \eta, & \delta \bar{\phi} &= -\overline{\epsilon \eta}, \\
\delta \eta &= -2i \epsilon \partial_0 \phi, & \delta \bar{\eta} &= -2i \epsilon \partial_0 \bar{\phi}, \\
\delta \theta &= 2i \epsilon \partial_1 \phi + \epsilon W'(\phi), & \delta \bar{\theta} &= 2i \epsilon \partial_1 \bar{\phi} + \tau W'(\phi).
\end{align*} \] (3.8)

On the boundary, the supercharges are

\[ \begin{align*}
\bar{Q} &= \frac{\partial}{\partial \theta^0} + i \theta^0 \frac{\partial}{\partial x^0}, & Q &= -\frac{\partial}{\partial \bar{\theta}^0} - i \bar{\theta}^0 \frac{\partial}{\partial x^0},
\end{align*} \] (3.9)

along with the derivatives

\[ \begin{align*}
\bar{D} &= \frac{\partial}{\partial \bar{\theta}^0} - i \bar{\theta}^0 \frac{\partial}{\partial x^0}, & D &= -\frac{\partial}{\partial \theta^0} + i \theta^0 \frac{\partial}{\partial x^0}.
\end{align*} \] (3.10)

In this new setting the superfield (3.4) splits into a chiral bosonic part \( \Phi' (D \Phi' = 0) \) and a fermionic part \( \Theta' \) which satisfies \( D \Theta' = -2i \partial_1 \Phi' \)

\[ \begin{align*}
\Phi' (y^0, \theta^0) &= \phi(y^0) + \theta^0 \eta(y^0) \\
\Theta' (y^0, \theta^0, \bar{\theta}^0) &= \theta(y^0) - 2 \theta^0 F(y^0) + 2i \bar{\theta}^0 \left[ \partial_1 \phi(y^0) + \theta^0 \partial_1 \eta(y^0) \right]
\end{align*} \] (3.11) (3.12)

where \( y^0 = x^0 - i \theta^0 \bar{\theta}^0 \).

Using the above expressions for the variations of fields, and the equations of motion, one finds that under B-type variation of the action (3.6) taken on the strip, boundary terms are left over, even in a free theory without superpotential. For \( W = 0 \), these boundary terms can be cancelled by adding the following extra piece to the action,

\[ S_{\partial \Sigma, \psi} = \frac{i}{4} \int dx^0 \left\{ \theta \eta - \bar{\eta} \partial_0 \phi \right\}\bigg|_0^\pi \] (3.13)

but when \( W \neq 0 \), variation of \( S_\Sigma + S_{\partial \Sigma, \psi} \) leads to

\[ \delta(S_\Sigma + S_{\partial \Sigma, \psi}) = \frac{i}{2} \int dx^0 \left\{ \epsilon \eta W' - \tau \eta W' \right\}\bigg|_0^\pi, \] (3.14)
a boundary term that cannot be cancelled as there is no combination in (3.8) which mixes barred and unbarred $\epsilon$ and $\eta$. In order to cancel this term, Warner suggested to introduce a new fermionic superfield on the boundary [75];

$$\Pi(y^0, \theta^0, \bar{\theta}^0) = \pi(y^0) + \theta^0 l(y^0) - \bar{\theta} p [E(\phi) + \theta^0 \eta(y^0) E'(\phi)].$$ \hfill (3.15)

This satisfies $D \Pi = E(\Phi')$ with some as yet unspecified function $E$, and its component fields have variations

$$\delta \pi = \epsilon l - \epsilon E(\phi), \quad \delta \bar{\pi} = \epsilon \bar{l} - \epsilon \bar{E}(\bar{\phi}),$$ \hfill (3.16)

$$\delta l = -2i\epsilon \partial_0 \pi + \epsilon \eta E'(\phi), \quad \delta \bar{l} = -2i\epsilon \partial_0 \bar{\pi} - \epsilon \bar{\eta} \bar{E}'(\bar{\phi}).$$ \hfill (3.17)

In the paper [10], and additional boundary action involving this fermion, along with some bosonic boundary potential $J$, was added to $S_{\Sigma} + S_{\partial \Sigma, \psi}$, namely

$$S_{\partial \Sigma} = -\frac{1}{2} \int dx^0 d^2 \theta \Pi \Pi'\bigg|^{\pi}_0 - \frac{i}{2} \int_{\partial \Sigma} dx^0 d\theta \Pi J(\Phi)_{\bar{p}=0}^{\pi} + c.c.$$ \hfill (3.18)

Written in components, this becomes

$$S_{\partial \Sigma} = \int dx^0 \left[ i \pi \partial_0 \pi - \frac{1}{2} J J - \frac{1}{2} \bar{E} E + i \frac{1}{2} \pi \eta J' + i \frac{1}{2} \pi \bar{E} J' - i \frac{1}{2} \eta \bar{E}' + i \frac{1}{2} \pi \eta E' \right] \bigg|^{\pi}_0$$ \hfill (3.19)

using the equation of motion $l = -iJ(\bar{\phi})$. With this, the boundary fermion transforms as

$$\delta \pi = -i\epsilon J(\phi) - \epsilon E(\phi), \quad \delta \bar{\pi} = i\epsilon J(\phi) - \epsilon \bar{E}(\bar{\phi}).$$ \hfill (3.20)

Applying $\delta$ to (3.19) gives the boundary term

$$\delta S_{\partial \Sigma} = -\frac{i}{2} \int_{\partial \Sigma} dx^0 \left\{ \epsilon \eta (\overline{EJ})' + \epsilon \eta (EJ)' \right\}$$ \hfill (3.21)

which cancels (3.14) if

$$W = EJ + \text{const.}$$ \hfill (3.22)

So the bulk LG superpotential (minus a possible constant which we will set to zero)
must be factorizable into two polynomials which are the model’s boundary potentials. One can generalize this by introducing several boundary fermions \( \pi_\alpha, \alpha = 1, \ldots, r \), obeying a Clifford algebra

\[
\{\pi_\alpha, \pi_\beta\} = \{\bar{\pi}_\alpha, \bar{\pi}_\beta\} = 0, \quad \{\pi_\alpha, \bar{\pi}_\beta\} = \delta_{\alpha\beta}
\]  

(3.23)

then the factorization condition involves \( r \) pairs of polynomials \( E_\alpha, J_\alpha \) in the \( \phi \)-variables and reads (see e.g. [1])

\[
\sum_{\alpha=1}^{r} E_\alpha J_\alpha = W .
\]  

(3.24)

Using some matrix representation of the Clifford algebra, this can be more compactly written as a matrix factorization of the Landau-Ginzburg potential \( W \), as

\[
E \cdot J = J \cdot E = W \cdot 1
\]  

(3.25)

where \( J = \sum_\alpha J_\alpha \pi_\alpha \) and \( E = \sum_\alpha E_\alpha \bar{\pi}_\alpha \) are viewed as \( 2^r \times 2^r \) matrices.

In the classical LG model, arbitrary boundary fields are given by polynomials in the \( \pi_\alpha \) and \( \bar{\pi}_\alpha \) with \( \phi \)-dependent coefficients. Choosing a matrix representation of the Clifford algebra, general boundary fermions can be written as matrices, too,

\[
\Phi = \begin{pmatrix} f_{00} & f_{10} \\ f_{01} & f_{11} \end{pmatrix}.
\]  

(3.26)

The block-structure indicates the natural \( \mathbb{Z}_2 \)-grading \( \sigma \) by the fermion number; we normally assume the Clifford matrices to be such that

\[
\sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]  

(3.27)

With this, the diagonal blocks of \( \Phi \) represent bosonic, the off-diagonal blocks fermionic
fields. $E$ and $J$ themselves can be combined into a matrix

$$Q = \begin{pmatrix} 0 & E \\ J & 0 \end{pmatrix}$$

(3.28)

in terms of which the condition for $N = 2$ supersymmetry in the boundary Landau-Ginzburg model simply becomes

$$Q^2 = W \cdot 1 .$$

The notation $Q$ is chosen because $E$ and $J$ indeed capture the action of the boundary supercharge on boundary fields,

$$Q_{\text{bdy}} = \sum \pi_\alpha E + \pi_\alpha J_\alpha .$$

(3.29)

On general boundary fields, the supercharge is understood to act by graded commutator, which we denote by $D_Q$ to avoid confusion with the matrix $Q$ introduced in (3.28):

$$D_Q \Phi = Q \Phi - \Phi^\sigma Q$$

(3.30)

where

$$\Phi^\sigma = \sigma \Phi \sigma .$$

(3.31)

It is easy to check that this $D_Q$ is a differential ($D_Q^2 = 0$). Generalization to situations with two different boundary conditions (thus different boundary potentials, or different matrices $Q$) on the two boundaries of the strip is straightforward: one uses

$$D_{Q_1, Q_2} \Phi = Q_1 \Phi - \Phi^\sigma Q_2$$

(3.32)

where $\Phi$ now describes what corresponds to open strings stretched between two different branes. One can also drop the restriction to matrices of size $2^r \times 2^r$, e.g. by filling up with sufficiently many zeroes (see [55] for an alternative approach).

Summarizing, boundary conditions respecting B-type $N = 2$ supersymmetry in
Landau-Ginzburg models are given by matrix factorizations $Q^2 = W \cdot 1$ of the LG potential $W$. Under RG flow to the infrared, they are expected to correspond to superconformal boundary states in the associated SCFT. In the string theory context, matrix factorizations correspond to D-branes.

3.3 Topological Boundary Landau-Ginzburg Models

One can pass to topological Landau-Ginzburg models in (partial) analogy to topological twisting performed in the case of $N = 2$ superconformal field theories. One has “BRST operators” (the supercharges) available in the Lagrangian LG model as well as in the SCFT, and following section 2.4 we can pass to cohomology classes of those supercharges, which are defined to correspond to the physical states of the topological model.

For bulk Landau-Ginzburg theories with potential $W(\phi)$ with $\phi$ taking values in $\mathbb{C}^n$, it is well known that the set of physical states (which corresponds to the set of chiral primary bulk fields) is isomorphic to the space $\mathbb{C}[x_1, \ldots, x_n]/(\partial_i W)$. This ring of polynomials even captures the OPE of chiral primaries in the LG-model (or in the associated $N = 2$ SCFT to which the LG-model flows). For example, for a minimal model with $W_{k+2}(\phi) = \frac{1}{k+2} \phi^{k+2}$, this gives the bulk chiral ring $\{1, \phi_1, \ldots, \phi_k\}$.

In the boundary case, one also has to use the boundary supercharges to obtain the ring of physical boundary fields of the topological Landau-Ginzburg theory: This ring is then given by the cohomology of the differential (3.30), or of (3.32) if one has two different boundary conditions. Again, the boundary chiral ring (the three-point functions) are captured by the multiplication of cohomology classes.

So in order to describe topological B-type branes in topological boundary Landau-Ginzburg models with potential $W$, one needs to find matrix factorizations $Q$ of $W$ and then to consider the cohomology of the associated differentials operators $\mathcal{D}_{Q_i Q_j}$ acting on matrices with polynomial entries. (Strictly speaking, one should use $\text{Ext}^\bullet$ rather than $H^\bullet$ to compute the open string spectrum, see appendix B.1.)

We add some assorted general remarks about matrix factorizations and their role in
topological Landau-Ginzburg models here.

First, different matrix factorizations may have identical physical content. In particular, two matrix factorizations \((E, J)\) and \((E', J')\) are said to be equivalent (or related by a “gauge transformation” see e.g. [35,74]) if there exist invertible matrices \(U, V \in \text{GL}(k, \mathbb{C}[x])\) with polynomial entries such that

\[
U JV^{-1} = J', \quad V EU^{-1} = E'
\]  

\[
\begin{pmatrix}
V & 0 \\
0 & U
\end{pmatrix}
\begin{pmatrix}
0 & E \\
J & 0
\end{pmatrix}
\begin{pmatrix}
V^{-1} & 0 \\
0 & U^{-1}
\end{pmatrix}
= \begin{pmatrix}
0 & E' \\
J' & 0
\end{pmatrix}.
\]  

A matrix factorization of the form

\[
Q = \begin{pmatrix}
0 & 1 \\
W & 0
\end{pmatrix}
\]  

is said to be trivial since the cohomology of \(D_Q\) is empty.

A direct sum of matrix factorizations

\[
Q = Q^A \oplus Q^B
\]  

with grading

\[
\sigma = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}
\]  

corresponds to a superposition of two boundary states (a “stack” of two branes). Its cohomology spectrum of is given by the cohomology of \(D_Q\) which acts on \(\Phi\) as

\[
D_Q\Phi = \begin{pmatrix}
D_{Q^A,Q^A} & D_{Q^A,Q^B} \\
D_{Q^B,Q^A} & D_{Q^B,Q^B}
\end{pmatrix}
\]  

where the mixed terms come from strings stretched between the branes \(A\) and \(B\) and are given by (3.32).
Using (3.33) any matrix factorization with a constant entry can be put into the form [27]

\[ Q = Q^{\text{trivial}} \oplus Q' \]  

and we reason that

\[ Q \cong Q' \]  

because the open string spectrum between any \( Q' \) and a trivial matrix factorization \( Q^{\text{trivial}} \) is also empty, and see that the open string spectrum of \( Q \) is equivalent to that of \( Q' \). So if a matrix factorization has a constant entry one really can ignore the trivial part and work with the smaller matrix.

The matrix factorization \((E, J)\) is equivalent to the shifted matrix factorization \((-J, -E)\) [1]. However, it is not equivalent to its anti-brane \((J, E)\) (see Appendix (B.2)).

We add some remarks on R-charges in the context of matrix factorizations, in particular on the application for computing the R-charges of the basis elements of the cohomology of \( D_Q \). For greater detail see for example [33–35, 74].

Superconformal invariance of the LG bulk requires the superpotential to be quasi-homogeneous and for physical reasons its R-charge is normalized to be 2.

\[ W(e^{\lambda q_i} x_i) = e^{2i\lambda} W(x_i) . \] (3.41)

This means that since \( Q^2 = W1 \), the R-charge of \( Q \) must be 1. But \( Q \) is a more complicated object than \( W \), and defining its R-charge involves another \( R \), which must satisfy the boundary equivalent of (3.41)

\[ e^{i\lambda R} Q(e^{i\lambda p_i} x_i) e^{-i\lambda R} = e^{i\lambda} Q(x_i) . \] (3.42)

One expects that if the entries of \( Q \) are homogeneous then \( R \) will be diagonal (the converse is also true). If \( Q \) has non-homogeneous entries it is assumed that \( R \) is diagonalizable [74]. The \( R \) matrix provides an additional grading on the space of
open strings
\[ e^{i\lambda R} \Phi(e^{i\lambda q} x_i) e^{-i\lambda R} = e^{i\lambda q} \Phi(x_i). \] (3.43)

If solutions \( R \) and \( R' \) to (3.42) exist for matrices \( Q \) and \( Q' \), strings between the two branes will have R-charges determined by
\[ e^{i\lambda R'} \Phi(e^{i\lambda q} x_i) e^{-i\lambda R} = e^{i\lambda q} \Phi(x_i). \] (3.44)

A diagonal solution \( R = \text{diag}(R_1, \ldots, R_{2r}) \) to (3.42) (for \( 2r \times 2r Q \)) will satisfy [74]
\[ R_j - R_{k+r} = 1 - \deg(E_{jk}), \] (3.45)
\[ R_{k+r} - R_j = 1 - \deg(J_{kj}). \] (3.46)

There are a number of ambiguities in defining the matrix, see [74] for a discussion. Here, let us just look at the most important example, namely the R-charges for the A-type minimal model.

We can compute the R-Charges for a minimal model with superpotential \( W = x^d \), matrix factorization
\[ Q = \begin{pmatrix} 0 & x^{d-n} \\ x^n & 0 \end{pmatrix} \] (3.47)
and a string spectrum of bosons, \( \phi_j \), and fermions, \( \alpha_i \), given by
\[ \phi_j = \begin{pmatrix} x^j & 0 \\ 0 & x^j \end{pmatrix} \] (3.48)
\[ \alpha_j = \begin{pmatrix} 0 & x^{n-j-i} \\ -x^{d-n-j-1} & 0 \end{pmatrix} \] (3.49)
for \( j = 0, \ldots, n - 1 \) (where we assume that \( n \leq \frac{d}{2} \)).

If the R-matrix is diagonal,
\[ R = \begin{pmatrix} R_1 & 0 \\ 0 & R_2 \end{pmatrix}, \] (3.50)
it is easy to see from (3.42) that
\[ R_1 - R_2 = \frac{2n}{d} - 1 = nq(x) - 1 = q(E) - 1 \] (3.51)
and
\[ R_2 - R_1 = q(J) - 1 \] (3.52)
since \( q(W) = 2 \) implies \( q(x) = 2/d \).

One can choose the R-matrix to be
\[ R = \begin{pmatrix} \frac{1}{2} - \frac{n}{d} & 0 \\ 0 & -\frac{1}{2} + \frac{n}{d} \end{pmatrix} \] (3.53)
which leads to the following charges for the boundary fields [10,33]
\[ q(\phi_j) = \frac{2j}{d}, \quad q(\alpha_j) = -\frac{2(j + 1)}{d} + 1. \]

These charges satisfy \( 0 \leq q \leq \frac{c}{3} \) just as in a unitary \( N = 2 \) SCFT [56].

Before we turn to concrete examples of matrix factorizations and to the Massey product algorithm to compute deformations, let us add some very brief remarks about the string theory and mathematics context, which may serve as additional motivation to study matrix factorizations and their deformations.

We have seen that matrix factorizations describe B-type supersymmetric boundary conditions in \( N = 2 \) (topological) Landau-Ginzburg models. As stated before, the RG-fixed points of these field theories are given by \( N = 2 \) superconformal field theories; e.g., a LG model with (homogeneous) superpotential \( W = x_1^{d_1} + \cdots + x_n^{d_n} \) corresponds to a tensor product of \( N = 2 \) minimal models with total central charge \( c = \sum_i \left( 3 - \frac{6}{d_i} \right) \). Choosing the \( d_i \) such that \( c = 9 \), tensoring with an \( N = 2 \) theory of four free bosons (and their fermionic superpartners), and performing a suitable orbifolding by the U(1)-charge, one obtains Gepner models which provide string compactifications to four dimensions. The tensor product of minimal models describes the internal sector of the string compactification and dictates the particle content of
the string background. The $N = 2$ world-sheet supersymmetry ensures target-space supersymmetry of the compactification.

On the other hand, a Landau-Ginzburg model with potential of the above type is related to a sigma models with target-space being the 3-dimensional Calabi-Yau manifold $X = \{ W = 0 \}$, in weighted projective space (see e.g. [28] for a review and for references). One can express the fields of the $N = 2$ super-Virasoro algebra in terms of (classical) sigma-model fields and geometric data of the Calabi-Yau target, and moreover one can relate the Dolbeault cohomology groups $H^{1,1}(X)$ and $H^{2,1}(X)$ to the chiral and anti-chiral rings made up from $(c, c)$ and $(c, a)$ primaries of conformal dimension $\frac{1}{2}$. The OPE coefficients in the chiral ring, which can be computed already in the topological version of these models, also show up in the low-energy effective space-time physics of that Calabi-Yau string compactification, as Yukawa couplings. Also, the forms in $H^{1,1}(X)$ and $H^{2,1}(X)$ parametrize the geometric moduli of $X$.

The link between (anti-)chiral primaries and Calabi-Yau topology leads to the idea of mirror symmetry: At CFT-level, interchanging chiral and anti-chiral primaries can simply be achieved by applying the mirror automorphism (which leaves the CFT invariant); at Calabi-Yau level, however, the target manifold is changed. Accordingly, the original mirror symmetry conjecture states that to a Calabi-Yau manifold $X$ there should exist another Calabi-Yau manifold $\tilde{X}$, called the mirror of $X$, such that $H^{1,1}(X) \cong H^{2,1}(\tilde{X})$ (and vice versa) and such that the effective physics of the two string compactifications is the same.

Adding branes to such string compactifications enriches the mirror symmetry picture. In the world-sheet approach, branes correspond to A-type or B-type $N = 2$ superconformal boundary conditions, which are again interchanged by the mirror automorphism. In a target space picture, B-type branes on $X$ are realised as holomorphic submanifolds, while A-type branes correspond to special Lagrangian submanifolds (or more generally co-isotropic submanifolds) of $X$. Looking at additional structure coming with branes, one is led to identifying B-branes in a Calabi-Yau manifold $X$ with objects in the derived category of locally free sheaves (the objects are then bounded complexes of locally free sheaves and morphisms are described by the Ext-groups between them), often denoted $D^b(\text{coh}(X))$. A-type branes in $X$ are subsumed in the
so-called Fukaya category $Fuk(X)$ – and the homological mirror symmetry conjecture
of Kontsevich’s states that there is an equivalence between $\mathcal{D}^b(\text{coh}(X))$ and $Fuk(\check{X})$.
Matrix factorizations, which offer a lot of concrete computational power, become
relevant in this rather abstract mathematical context thanks to a theorem by Orlov
[61,62] which establishes an equivalence between the category of matrix factorizations
of a Landau-Ginzburg potential $W$ and, roughly, the category $\mathcal{D}^b(\text{coh}(X))$ of B-type
branes on $X = \{ W = 0 \}$. A rather explicit description of the equivalence can be
found in [2,3].
We will neither use the category theoretic language nor talk about target-space as-
pects in this thesis, but these short remarks should show that following results may
be of wider interest for string theory and mathematics.
Examples

In this chapter, we collect examples of matrix factorizations for minimal models and tensor products thereof. These models are arguably the most important ones from the string theory point of view (as they are the building blocks of Gepner models), and in fact not many other matrix factorizations have been studied in the physics literature – notable exceptions being the elliptic curve (i.e., the SCFT with central charge $c = 3$ and corresponding Landau-Ginzburg potential $x^3 + y^3 + z^3 + axyz$), see [27, 38] and references therein, and some (bulk) deformations of Gepner models, see in particular [5, 9]. The examples just mentioned also belong to the few where matrix factorizations with continuous parameters are known; in particular, the minimal model and the linear matrix factorizations listed in the following do not have continuous parameters (apart from those associated with the cone construction, obviously). As we will see in Chapter 8, the explicit deformations obtained using the Singular program provide families of matrix factorization for tensor products of minimal models.

4.1 Minimal Models

Minimal models are rational conformal field theories (of central charge $3p/(p+2)$, $p \in \mathbb{N}$) whose state spaces are formed from finitely many irreducible representations of the $N = 2$ super-Virasoro algebra. There are different ways of combing left- and right-moving sectors, leading to an ADE classification of modular invariant bulk partition functions. The Landau-Ginzburg models associated to these CFTs differ accordingly. We look at the Landau-Ginzburg potentials of A-, D-, and E-type minimal models and give general forms for their “fundamental” matrix factorizations. (For minimal models, all matrix factorizations have been found to be equivalent to finite direct sums of these “fundamental” ones [8, 15, 31].) Also included are tables listing the numbers of bosons and fermions in the open string spectrum of some examples as calculated
by functions in the Singular library file described in Chapter 6. These dimensions provide the main tool to establish a correspondence between matrix factorizations on the one hand and superconformal boundary states on the other: The “intersection form” (or Witten index)

\[ I_{Q_1, Q_2}^{\text{CFT}} \equiv \text{tr}_{H_{Q_1, Q_2}} q^{E_0 - \frac{c}{24}} (-1)^F \]

\[ I_{Q_1, Q_2}^{\text{LG}} \equiv \dim \text{Ext}^0(Q_1, Q_2) - \dim \text{Ext}^1(Q_1, Q_2) \] (4.1)

can be computed in the CFT, where it counts (with fermion numbers) the open string Ramond ground states; by the spectral flow automorphism, those are related to chiral primary open string states in the Neveu-Schwarz sector – i.e. precisely the physical states after topological twisting, counted as difference of cohomology dimensions. Many details on the matching between CFT boundary states and matrix factorizations are given in particular in [7, 8, 22].

We will not list explicit cohomology representatives here, as the Singular program yields those among its output data.

The superpotentials for ADE models are given by (see [41] and references therein)

\[ A_{d-2} : \quad W = x^d \quad \text{for} \quad d \geq 3 \] (4.2)

\[ D_{d+1} : \quad W = x^d - xy^2 \quad \text{for} \quad d \geq 3 \] (4.3)

\[ E_6 : \quad W = x^3 + y^4 \] (4.4)

\[ E_7 : \quad W = x^3 + xy^3 \] (4.5)

\[ E_8 : \quad W = x^3 + y^5 \] (4.6)

Apart from choosing a modular invariant partition function for the minimal models, one also has two different GSO projections at one’s disposal; the “other GSO projection” leads to an added \( z^2 \) term in the LG superpotential. The matrix factorizations for these superpotentials can be found, for example, in [15, 39], and we will also consider an example of a deformation of a matrix factorization for the second GSO projection of \( E_7 \) in section 7.7.

Next we look at the matrix factorizations for each of these models.
4.1.1 A-Model

The A-model is described by superpotentials of the form

\[ W_A = x^d. \]  \hfill (4.7)

Matrix factorizations are of the form

\[ Q = \begin{pmatrix} 0 & x^{d-n} \\ x^n & 0 \end{pmatrix}, \]  \hfill (4.8)

for \( 0 \leq n \leq d/2 \). The matrix factorizations with \( n > d/2 \) can be found by using the shift equivalence (3.3).

The cohomology of \( D_Q \) for the above matrix factorization will comprise \( n \) bosons and \( n \) fermions, explicit representatives have appeared in many papers, e.g. [10,33,41]. Higher rank matrix factorizations can be found by taking direct sums of the above \( 2 \times 2 \) matrices.

4.1.2 D-Model

The D-model is described by superpotentials of the form

\[ W_D = x^d - xy^2 \]  \hfill (4.9)

where \( d = k/2 + 1 \) and \( k \) must be even. The different matrix factorizations are [8]

Rank 1 factorizations:

\[ R_0 : \quad E = x^{d-1} - y^2, \quad J = x. \]  \hfill (4.10)

This factorization’s spectrum has 2 bosons and no fermions. If \( n \) is even there are
the additional factorizations:

\[ R_+ : \quad E = x^{d-1} + y, \quad J = x(x^{d-1} - y), \quad (4.11) \]
\[ R_- : \quad E = x(x^{d-1} + y), \quad J = x^{d-1} - y. \quad (4.12) \]

Here, between the same branes the bosonic spectrum has dimension \( n/2 + 1 \) and a zero dimensional fermionic spectrum. Between \( R_+ \) and \( R_- \) there are \( n/2 + 1 \) bosons and \( n/2 \) fermions.

Rank 2 factorizations:

\[ S_l : \quad E = \begin{pmatrix} x^l & x y \\ -y & -x^{d-l} \end{pmatrix}, \quad J = \begin{pmatrix} x^{d-l} & x y \\ -y & -x^l \end{pmatrix} \quad (4.13) \]

where \( l = 1, \ldots, d - 1 \). These factorizations have \( 2l \) bosons and \( 2l \) fermions. An example using the factorization \( S_1 \) is computed in section 7.2.

\[ T_l : \quad E = \begin{pmatrix} x^l & y \\ -y & -x^{d-1-l} \end{pmatrix}, \quad J = \begin{pmatrix} x^{d-l} & x y \\ -x y & -x^{l+1} \end{pmatrix} \quad (4.14) \]

where \( l = 0, \ldots, d - 1 \). Here the spectrum consists of \( 2l + 2 \) bosons and \( 2l \) fermions. An example using the factorization \( T_2 \) is computed in section 7.3.

4.1.3 E-Model

\( E_6 \)

\( E_6 \) is described by the superpotential

\[ W_{E_6} = x^3 + y^4. \quad (4.15) \]

The different matrix factorizations \([39,44]\) are given below and the dimensions of their cohomologies are given in Table 4.1

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Rank 2 factorizations:

\[ E_1 = J_5 = \begin{pmatrix} x & y \\ y^3 & -x^2 \end{pmatrix}, \quad J_1 = E_5 = \begin{pmatrix} x^2 & y \\ y^3 & -x \end{pmatrix}, \quad (4.16) \]

\[ E_6 = \begin{pmatrix} x & y^2 \\ y^2 & -x^2 \end{pmatrix}, \quad J_6 = \begin{pmatrix} x^2 & y^2 \\ y^2 & -x \end{pmatrix}. \quad (4.17) \]

Rank 3 factorization:

\[ E_2 = J_4 = \begin{pmatrix} x^2 & -xy & y^2 \\ y^3 & x^2 & -xy \\ -xy^2 & y^3 & x^2 \end{pmatrix}, \quad J_2 = E_4 = \begin{pmatrix} x & y & 0 \\ 0 & x & y \\ y^2 & 0 & x \end{pmatrix}. \quad (4.18) \]

Rank 4 factorization:

\[ E_3 = \begin{pmatrix} x & y^2 & 0 & 0 \\ y^2 & -x^2 & 0 & 0 \\ 0 & -xy & x^2 & y^2 \\ y & 0 & y^2 & -x \end{pmatrix}, \quad J_3 = \begin{pmatrix} x^2 & y^2 & 0 & 0 \\ y^2 & -x & 0 & 0 \\ 0 & -y & x & y^2 \\ xy & 0 & y^2 & -x^2 \end{pmatrix}. \quad (4.19) \]

<table>
<thead>
<tr>
<th>Rank</th>
<th>Factorization</th>
<th>Number of Bosons</th>
<th>Number of Fermions</th>
</tr>
</thead>
<tbody>
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<td>2</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>2 or 4</td>
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<td>6</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>12</td>
<td>12</td>
</tr>
</tbody>
</table>

Table 4.1: Cohomology dimensions of matrix factorizations for \( E_6 \)

\( E_7 \)

\( E_7 \) is described by the superpotential

\[ W_{E_7} = x^3 + xy^3. \quad (4.20) \]
The different matrix factorizations [39,44] are given below and the dimensions of their cohomologies are given in Table 4.2

Rank 1 factorization:

\[ E_1 = x, \quad J_1 = x^2 + y^3. \quad (4.21) \]

Rank 2 factorizations:

\[ E_2 = \begin{pmatrix} x^2 & y^2 \\ xy & -x \end{pmatrix}, \quad J_2 = \begin{pmatrix} x & y^2 \\ xy & -x^2 \end{pmatrix}, \quad (4.22) \]

\[ E_6 = \begin{pmatrix} x^2 & y \\ xy^2 & -x \end{pmatrix}, \quad J_6 = \begin{pmatrix} x & y \\ xy^2 & -x^2 \end{pmatrix}, \quad (4.23) \]

\[ E_7 = \begin{pmatrix} x^2 & xy \\ xy^2 & -x^2 \end{pmatrix}, \quad J_7 = \begin{pmatrix} x & y \\ y^2 & -x \end{pmatrix}. \quad (4.24) \]

Rank 3 factorizations:

\[ E_3 = \begin{pmatrix} x^2 & -y^2 & -xy \\ xy & x & -y^2 \\ xy^2 & xy & x^2 \end{pmatrix}, \quad J_3 = \begin{pmatrix} x & 0 & y \\ -xy & x^2 & 0 \\ 0 & -xy & x \end{pmatrix}, \quad (4.25) \]

\[ E_5 = \begin{pmatrix} y & 0 & x \\ -x & xy & 0 \\ 0 & -x & y \end{pmatrix}, \quad J_5 = \begin{pmatrix} xy^2 & -x^2 & -x^2 y \\ xy & y^2 & -x^2 \\ x^2 & xy & xy^2 \end{pmatrix}. \quad (4.26) \]

Rank 4 factorization:

\[ E_4 = \begin{pmatrix} x & y & -y & 0 \\ y^2 & -x & 0 & -y \\ 0 & 0 & x^2 & xy \\ 0 & 0 & xy^2 & -x^2 \end{pmatrix}, \quad J_4 = \begin{pmatrix} x^2 & xy & y & 0 \\ xy^2 & -x^2 & 0 & y \\ 0 & 0 & x & y \\ 0 & 0 & y^2 & -x \end{pmatrix}. \quad (4.27) \]
<table>
<thead>
<tr>
<th>Rank</th>
<th>Factorization</th>
<th>Number of Bosons</th>
<th>Number of Fermions</th>
</tr>
</thead>
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</tr>
<tr>
<td>4</td>
<td>4</td>
<td>8</td>
<td>8</td>
</tr>
</tbody>
</table>

Table 4.2: Cohomology dimensions of matrix factorizations for $E_7$

$E_8$

$E_8$ is described by the superpotential

$$W_{E_7} = x^3 + y^5. \quad (4.28)$$

The different matrix factorizations [39,44] are given below and the dimensions of their cohomologies are given in Table 4.3

Rank 2 factorizations:

$$E_1 = \begin{pmatrix} x^2 & y \\ y^4 & -x \end{pmatrix}, \quad J_1 = \begin{pmatrix} x & y \\ y^4 & -x^2 \end{pmatrix}, \quad (4.29)$$

$$E_7 = \begin{pmatrix} x & y^2 \\ y^3 & -x^2 \end{pmatrix}, \quad J_7 = \begin{pmatrix} x^2 & y^2 \\ y^3 & -x \end{pmatrix}. \quad (4.30)$$

Rank 3 factorizations:

$$E_2 = \begin{pmatrix} y^4 & xy^3 & x^2 \\ -x^2 & y^4 & xy \\ -xy & -x^2 & y^2 \end{pmatrix}, \quad J_2 = \begin{pmatrix} y & -x & 0 \\ 0 & y & -x \\ x & 0 & y^3 \end{pmatrix}, \quad (4.31)$$

$$E_8 = \begin{pmatrix} y^4 & xy^2 & x^2 \\ -x^2 & y^3 & xy \\ -xy^2 & -x^2 & y^3 \end{pmatrix}, \quad J_8 = \begin{pmatrix} y & -x & 0 \\ 0 & y^2 & -x \\ x & 0 & y^2 \end{pmatrix}. \quad (4.32)$$
Rank 4 factorizations:

\[ E_3 = \begin{pmatrix} 0 & x^2 & -y^2 & 0 \\ -x^2 & xy & 0 & -y^3 \\ 0 & -y^2 & -x & 0 \\ y^2 & 0 & y & -x \end{pmatrix}, \quad J_3 = \begin{pmatrix} y & -x & 0 & y^3 \\ x & 0 & -y^3 & 0 \\ -y^2 & 0 & -x^2 & 0 \\ 0 & -y^2 & -xy & -x^2 \end{pmatrix}, \quad (4.33) \]

\[ E_6 = \begin{pmatrix} x^2 & y^2 & 0 & xy \\ y^3 & -x & -y^2 & 0 \\ 0 & 0 & x & y^2 \\ 0 & 0 & y^3 & -x^2 \end{pmatrix}, \quad J_6 = \begin{pmatrix} x & y^2 & 0 & y \\ y^3 & -x^2 & -xy^2 & 0 \\ 0 & 0 & x^2 & y^2 \\ 0 & 0 & y^3 & -x \end{pmatrix}. \quad (4.34) \]

Rank 5 factorization:

\[ E_4 = \begin{pmatrix} y & -x & 0 & 0 & 0 \\ x & 0 & 0 & y^2 & 0 \\ -y^2 & 0 & -x^2 & 0 & -y^3 \\ 0 & -y^2 & 0 & x & 0 \\ 0 & 0 & y^2 & y & -x \end{pmatrix}, \quad (4.35) \]

\[ J_4 = \begin{pmatrix} y^4 & x^2 & 0 & -xy^2 & 0 \\ -x^2 & xy & 0 & -y^3 & 0 \\ 0 & -y^2 & -x & 0 & y^3 \\ -xy^2 & y^3 & 0 & x^2 & 0 \\ -y^3 & 0 & -y^2 & xy & -x^2 \end{pmatrix}. \quad (4.36) \]
Rank 6 factorization:

\[
E_5 = \begin{pmatrix}
    y^4 & xy^2 & x^2 & 0 & 0 & xy \\
    -x^2 & y^3 & xy & -x & 0 & 0 \\
    -xy^2 & -x^2 & y^3 & 0 & -xy & 0 \\
    0 & 0 & 0 & y & -x & 0 \\
    0 & 0 & 0 & 0 & y^2 & -x \\
    0 & 0 & 0 & x & 0 & y^2
\end{pmatrix}, \quad (4.37)
\]

\[
J_5 = \begin{pmatrix}
    y & -x & 0 & 0 & 0 & -x \\
    0 & y^2 & -x & xy & 0 & 0 \\
    x & 0 & y^2 & 0 & xy & 0 \\
    0 & 0 & 0 & y^4 & xy^2 & x^2 \\
    0 & 0 & 0 & -x^2 & y^3 & xy \\
    0 & 0 & 0 & -xy^2 & -x^2 & y^3
\end{pmatrix}. \quad (4.38)
\]

<table>
<thead>
<tr>
<th>Rank</th>
<th>Factorization</th>
<th>Number of Bosons</th>
<th>Number of Fermions</th>
</tr>
</thead>
<tbody>
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<td>2</td>
<td>1 7</td>
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<td>3 6</td>
<td>12 14</td>
<td>12 14</td>
</tr>
<tr>
<td>5</td>
<td>4 6</td>
<td>20 30</td>
<td>20 30</td>
</tr>
<tr>
<td>6</td>
<td>5 5</td>
<td>30 30</td>
<td>30 30</td>
</tr>
</tbody>
</table>

Table 4.3: Cohomology dimensions of matrix factorizations for $E_8$
4.2 Tensor Products of Minimal Models

For applications in string theory, it is important to study tensor products of minimal models, as these appear in the Gepner construction, which provides a large set of string backgrounds with target space supersymmetry. In CFT, one can construct (maximally symmetric) boundary states for these tensor product theories [65,67]. In this section, we list all the matrix factorizations known for tensor products of minimal Landau-Ginzburg models.

4.2.1 Tensor Product Matrix Factorizations

The following is in fact a general method to obtain a matrix factorization for a superpotential \( W(x_1, x_2) = W_A(x_1) + W_B(x_2) \) from matrix factorizations of \( W_A(x_1) \) and of \( W_B(x_2) \). If the two factorizations are given by

\[
Q_A = \begin{pmatrix} 0 & E_A \\ J_A & 0 \end{pmatrix}, \quad Q_B = \begin{pmatrix} 0 & E_B \\ J_B & 0 \end{pmatrix}
\]

then their tensor product is defined to be [1]

\[
Q^\otimes := \begin{pmatrix} 0 & 0 & E_A \otimes 1_B & 1_A \otimes J_B \\ 0 & 0 & 1_A \otimes E_B & -J_A \otimes 1_B \\ J_A \otimes 1_B & 1_A \otimes J_B & 0 & 0 \\ 1_A \otimes E_B & -E_A \otimes 1_B & 0 & 0 \end{pmatrix}.
\]

It is easy to see that \((Q^\otimes)^2 = W_A(x_1) + W_B(x_2)\), and the open string spectrum of \(Q^\otimes\) matches up with that associated to tensor product boundary conditions. One may also perform successive tensor products to obtain matrix factorizations of superpotentials with more than two variables. These tensor products can easily be computed using the function “mftensor” written for Singular and described in (6.2.13).
4.2.2 Linear Matrix Factorizations

For a superpotential \( W = x^d + y^d \) one can form a set of linear rank 1 matrix factorizations of the form [1]

\[
E = \prod_{i \in I} (x - \eta_i y), \quad J = \prod_{i \in I^c} (x - \eta_i y) \tag{4.41}
\]

where \( I \subset \{0, \ldots, d-1\} \), \( I^c \) the complement of \( I \), and \( \eta \) is a primitive \( d^{th} \) root of \(-1\).

Linear rank 1 matrix factorizations do not have any fermions in their cohomology so they cannot be deformed. We can, however, deform direct sums of linear rank 1 matrix factorizations - see section 7.4 for an example of this.

More generally, for a superpotential \( W = x_1^d + \cdots + x_n^d \), there exists a special class of matrix factorizations [4, 22] which are unique, indecomposable and linear in the \( x_i \). They are \( d^n \times d^n \) matrices where \( \gamma = \left[ \frac{n-1}{2} \right] \), made up of products of

\[
\alpha_i = x_1 + \xi^i \alpha_{d,n}, \tag{4.42}
\]

where \( \xi \) is a primitive \( d^{th} \) root of unity and the \( \alpha_{d,n} \) are described recursively: To this end, one introduces \( d \times d \) matrices

\[
(\epsilon_1)_{ij} = \xi^{i-1} \delta_{i,j-1}, \quad (\epsilon_2)_{ij} = \xi^{i-1} \delta_{i,j}, \quad (\epsilon_3)_{ij} = \delta_{i,j-1}, \tag{4.43}
\]

where all Kronecker deltas are understood modulo \( d \), as well as the number

\[
\mu_n = \begin{cases} 
1 & d \text{ odd} \\
\eta & d \text{ even and } \left[ \frac{n-1}{2} \right] \text{ even} \\
\eta^{-1} & d \text{ even and } \left[ \frac{n-1}{2} \right] \text{ odd}
\end{cases} \tag{4.44}
\]
where \( \eta \) is a primitive \( d \)th root of \(-1\) and \( \eta^2 = \xi \). Using these, one defines

\[
\alpha_{d,1} = 0, \quad \alpha_{d,2} = \mu_2 x_2, \\
\alpha_{d,n+2} = \epsilon_2 \otimes \alpha_{d,n} + \epsilon_3 \otimes \mu_{n+2} x_{n+1} \mathbb{1} + \epsilon_1 \otimes x_{n+2} \mathbb{1},
\]

where the \( \mathbb{1} \) are of the same size as \( \alpha_{d,n} \). Since all the \( \alpha_i \) commute, we can form matrix factorizations

\[
E = \prod_{i \in I} \alpha_i, \quad J = \prod_{i \in I^c} \alpha_i
\]

where \( I \subset \{0, \ldots, d-1\} \) and \( I^c = \{0, \ldots, d-1\}\backslash I \) as in the rank 1 case. Note that there are some equivalences among this class of matrix factorizations, see [22] for some more details. For an example of this see section 7.4.

For a superpotential \( W = x_1^{d_1} + \cdots + x_n^{d_r} \), different matrix factorizations can be found by forming tensor products of various linear matrix factorizations (which include minimal model matrix factorizations as a special case). In [22], evidence (using the Witten index) was gathered indicating that these matrix factorizations account for all rational boundary states in Gepner models as constructed in [65, 67]; see also [7] for earlier work connecting rank 1 factorizations and transposition branes. Indeed, it seems that there are more matrix factorizations available than rational Gepner model boundary states.
Massey Products

Massey products have been around in mathematics since the 1960s [20, 58] in a far more general context then is necessary for us. Then in [51, 52] Massey products were used in obstruction theory building on the work of [69]. This was continued in [23, 53, 70–72] reviews can be found in [63, 68] It was not until 2006 that Massey products were used to find deformations of matrix factorizations [46]. Since then more work on the subject has been carried out in [45,47,48].

5.1 Computing Deformations of Matrix Factorizations Using Massey Products

In the following the notation $\vec{u}^{\vec{n}} = u_{1}^{n_{1}}u_{2}^{n_{2}} \cdots u_{d}^{n_{d}}$ is used.

We would like to be able to deform the matrix factorization, $Q$, by fermions (which correspond to representatives of the even cohomology of $D_{Q}$). We also want the deformed matrix factorization, $Q_{def}$, to square to the same LG superpotential (multiplied by an identity matrix) as the original matrix factorization. One way of doing this is to compute the formal moduli (or hull) of a deformation functor related to $Q$. This allows us to find obstructions to the deformations of the matrix factorization. The obstructions are polynomials $f_{j}$, $j = 1, \ldots, |H^{even}(D_{Q})|$, which must be set to zero in order to ensure $Q_{def}^{2} = Q^{2}$. If a non-trivial common zero locus of the polynomials $f_{j}$ exists, then we are able to construct a new matrix factorization i.e. $Q_{def} \neq Q$. It should also be that the $f_{j}$ span the same space as the polynomials $\partial_{i}W$ for $i = 1, \ldots, |H^{odd}(D_{Q})|$ allowing $W$ to be computed using the $f_{j}$, where $W$ is the effective superpotential of the brane corresponding to $Q$.

We begin by giving a short overview of the general situation, then concentrate on how the algorithm described by Laudal [51,52] and Siqveland [70] for computing the
deformation hull is applied to matrix factorizations. A simple example is given in
detail to illustrate the procedure. For the general algorithm please refer to [51,52,70].
This method was first applied to matrix factorizations for D-branes in [46].

For mathematical definitions see Appendix A.

**Theorem 5.1.1.** Let $A$ be a $k$-algebra and $E$ an $A$-module. Let $\mathcal{L}$ be the category of
local artinian $k$-algebras with residue field $k$ and consider the deformation functor
\[ \text{Def}_E : \mathcal{L} \rightarrow \text{sets} \]
given by
\[ \text{Def}_E(S) = \left\{ (E, \theta) \mid E \text{ is an } A \otimes_k S - \text{module, flat over } S, \text{ and } E \otimes_S k \overset{\theta}{\cong} E \right\} / \cong \]
then according to Schlessinger [69] there exists a hull $H_E$ of $\text{Def}_E$.

**Definition 5.1.2.** A hull, $H_E$ is a complete local $k$-algebra with a smooth morphism
$\text{Mor}(H_E, -) \rightarrow \text{Def}_E$ such that $\text{Mor}(H_E, k[s]/(s^2)) \rightarrow \text{Def}_E(k[s]/(s^2))$ is a bijection.

**Theorem 5.1.3.** [52] Let $A$ be a $k$-algebra. Given an $A$-module $E$, the formal moduli,
or hull, $H$ of $E$ is determined by the Massey products of $\text{Ext}^1_A(E, E)$. In fact
\[ H \simeq k[[s_1, \ldots, s_d]]/(f_1, \ldots, f_r) \]
where
\[ f_j = \sum_{l=2}^{\infty} \sum_{\vec{n} \in B'_l} t_j \langle \vec{s}^*, \vec{n} \rangle \vec{s}^{\vec{n}} \]

Notation: $\{s_1, ..., s_d\}$ is a basis of $\text{Ext}^1_A(E, E)^*$; $\{t_1, ..., t_r\}$ is a basis of $\text{Ext}^2_A(E, E)^*$;
$\{s_1^*, ..., s_d^*\}$ and $\{t_1^*, ..., t_r^*\}$ are the dual bases of $\text{Ext}^1_A(E, E)$ and $\text{Ext}^2_A(E, E)$. $B'_l$ is a set of vectors and is defined in (5.17).

We fix a free resolution $L_\bullet$ of $E$ with differential $d_i : L_i \rightarrow L_{i-1}$. 
\[ \cdots \xrightarrow{d_4} L_3 \xrightarrow{d_3} L_2 \xrightarrow{d_2} L_1 \xrightarrow{d_1} L_0 \to E \to 0 \]  

(5.1)

In the case of matrix factorizations, \( D_Q \) is a differential on \( C := \mathbb{C}[x_1, \ldots, x_d]/W \) and we have (for \( E \cdot J = W \cdot 1 \))

\[
\begin{align*}
    d_i &= \begin{cases} 
        E, & \text{i odd} \\
        J, & \text{i even}
    \end{cases} \\
\end{align*}
\]

\[
\cdots \xrightarrow{J} C^r \xrightarrow{E} C^r \xrightarrow{J} C^r \xrightarrow{E} C^r \to \text{coker}(E) \to 0
\]  

(5.2)

We want to lift this to the ring \( S = \mathbb{C}[\vec{u}]/(f_1, \ldots, f_r) \) (the \( f_j \) are obstructions to the lifting of \( (C^r, d_i) \) from \( C^r \otimes \mathbb{C}[\vec{u}] \) to \( \mathbb{C}[\vec{u}] \)) so that the following diagram commutes

\[
\begin{array}{cccccccccccc}
0 & \leftarrow & (\text{coker}(E))_S & \leftarrow & C \otimes S & \xrightarrow{E(S)} & C \otimes S & \xrightarrow{J(S)} & C \otimes S & \xrightarrow{E(S)} & \cdots \\
& & & & & & & & & & \downarrow & \downarrow & \downarrow \\
0 & \leftarrow & \text{coker}(E) & \leftarrow & C & \xrightarrow{E} & C & \xrightarrow{J} & C & \xrightarrow{E} & \cdots
\end{array}
\]

this means that \( (\text{coker}(E))_S \in \text{Def}_Q \) is a lifting of \( \text{coker}(E) \) to \( S \).

This lift needs to be performed in several smaller steps, so as to be able to exploit the following theorem on obstructions.

**Definition 5.1.4.** A morphism \( \pi' : S' \to S \) is called small if it satisfies \( m_{S'} \cdot \ker \pi' = 0 \).

Where \( m_{S'} \) is the maximal ideal in \( S' \).

**Theorem 5.1.5.** [69, 70] Let \( \pi' : S' \to S \) be a small morphism in \( \underline{l} \). Let \( \mathcal{E}_S \in \text{Def}_S(S) \) correspond to the lifting \( \{L_* \otimes_k, S, d_i(S)\} \) of \( L_* \) to \( S \). Then there is a unique defined obstruction

\[
o(\mathcal{E}_S, \pi') \in \text{Ext}^2_A(\mathcal{E}, \mathcal{E}) \otimes_k I
\]

given in terms of the 2-cocycle \( o \in \text{Hom}^\bullet_A(L_*, L_*) \otimes_k I \), such that \( o(\mathcal{E}_S, \pi') = 0 \) if and only if \( \mathcal{E}_S \) may be lifted to \( S' \).

A natural first lifting would be to a ring containing elements of first order in \( u_i \).

\( S_1 = \mathbb{C}[[\vec{u}]]/m^2 \) is chosen.
For the next step we cannot usually lift directly to $S$ from $S_1$ since the morphism $\pi : S \to S_1$ is not usually small ($m \cdot \ker \pi = m^3/(f_1, \ldots, f_r)$).

We proceed in steps and by increasing powers of $u_i$ introduce auxiliary rings $S'_n = \mathbb{C}[\vec{u}]/(m^{n+1} + m(f^n_1, \ldots, f^n_r))$ and $S_n = \mathbb{C}[\vec{u}]/(m^{n+1} + (f^n_1, \ldots, f^n_r))$ which contain only terms up to order $n$ and are constructed in such a way that the maps $\pi'_n : S'_n \to S_n$ are always small.

\[
\ker \pi'_n+1 = (m^{n+1} + (f^n_j))/((m^{n+2} + m(f^n_j)))
\]

\[
= (f^n_j)/m(f^n_j) \oplus m^{n+1}/(m^{n+2} + m^{n+1} \cap m(f^n_j))
\]

\[
m \cdot \ker \pi'_n + 1 = 0
\]

We have also chosen $S_n = S'_n/(f^n_1, \ldots, f^n_r)$ so that if we can lift from $S_n$ to $S'_{n+1}$ then we can also lift from $S_n$ to $S_{n+1}$. The polynomials $f^n_j$ are the obstructions to lifting of $(\text{coker}(E))$ to the ring $S'_n$.

Repeating these liftings order by order will eventually lead us to the lifting of $(\text{coker}(E))$ to $S$. Laudal’s structure theorem says that

$$H \cong \lim_{\leftarrow k} S_{2+k} = \mathbb{C}[\vec{u}] / (f_1, \ldots, f_r)$$

where $f_j = \lim_{k} f^{2+k}_j$

### 5.2 Algorithm

In the following, for ease of notation, define

$$\text{Ext}^*_A(Q, Q) := \text{Ext}^*_A(\text{coker}E, \text{coker}E)$$

66
We are interested in finding the deformed matrix factorization

\[ Q_{\text{def}} = Q + \sum_{\vec{m}} \alpha_{\vec{m}} \vec{u}^{\vec{m}} \]  

(5.6)

\[ Q_{\text{def}}^2 - W \cdot 1 = \sum_j f_j \phi_j \]  

(5.7)

The method is as follows:

In our case we will associate one parameter \( u_i \) per “generator of deformations”, i.e. basis elements of \( \text{Ext}_A^\ast(Q,Q) \).

In the case of matrix factorizations [22] (see also appendix B.1):

\[ H_{\text{even}}(D_Q) \cong \text{Ext}_A^2(Q,Q) \]  

(5.8)

\[ H_{\text{odd}}(D_Q) \cong \text{Ext}_A^1(Q,Q) \]  

(5.9)

So first we use this compute \( \text{Ext}_A^1(Q,Q) \) and \( \text{Ext}_A^2(Q,Q) \) and choose \( \alpha_{e_i}, i = 1, \ldots, d \) respectively \( \phi_j, j = 1, \ldots, r \) to represent their bases.

We also have the dual bases

\[ s_i \in \text{Hom}(\text{Ext}_A^1(Q,Q),A), \quad t_i \in \text{Hom}(\text{Ext}_A^2(Q,Q),A) \]  

(5.10)

\[ s_i(\alpha_{e_j}) = \delta_{ij}, \quad t_i(\phi_j) = \delta_{ij}. \]  

(5.11)

On lifting to \( S_1 \) we choose a basis \( \bar{B}_1 \) so that \( \vec{u}^{\vec{n}}, \) for \( \vec{n} \in \bar{B}_1, \) is a basis for \( S_1, \) we will have \( \bar{B}_1 = \{e_1, \ldots, e_d\}. \)

Let \( \varphi_1 : H \to H/m^2, \varphi_1(x_i) = u_i. \)

\[ S'_2 = \mathbb{C}[u_1 \ldots u_d]/m^3 \]  

\[ \xrightarrow{\pi'_2} \]

\[ H \xrightarrow{\varphi_1} S_1 = \mathbb{C}[u_1 \ldots u_d]/m^2 \]

We want to find the least ideal, \( a, \) which allows us to lift \( \text{coker}(E)_{\varphi_1} \) to \( S'_2/a = S_2 \) so that we have \( S_2/a \cong H/m^3. \) This least ideal will be the obstructions at order 2, i.e. it will be generated by \( (f_1^2, \ldots, f_r^2). \)
This is done by first finding a basis for \( \ker \pi' \) which allows us to find a vector basis \( B'_2 \) which contains all vectors \( \vec{n} \) such that \( \vec{u}^{\vec{n}} \) is an element of the chosen basis for \( \ker \pi' \).

Now we compute the Massey products at order 2. This means that for each \( \vec{n} \in B'_2 \) we compute the following:

\[
y(\vec{n}) = \sum_{\vec{m}_1, \vec{m}_2} \alpha_{\vec{m}_1} \circ \alpha_{\vec{m}_1},
\]

The Massey products \( y(\vec{n}) \) provide us with the obstructions to the lifting to \( S_2 \) as follows:

\[
f^2_j = \sum_{\vec{n} \in B'_2} t_j(y(\vec{n})) \vec{u}^{\vec{n}}
\]

coker(\( E_{\varphi_1} \)) can now be lifted to \( S_2 \).

We choose a basis \( B_2 \subseteq B'_2 \) such that \( \vec{u}^{\vec{n}} \), for \( \vec{n} \in B_2 \), is a basis for \( \ker \pi_2 \), then \( \vec{u}^{\vec{n}} \), for \( \vec{n} \in B_2 \cup \hat{B}_1 \) is a basis for \( S_2 \).

In \( S_2 \) there is a unique relation for every \( n \) with \( |\vec{n}| \leq 2 \)

\[
\vec{u}^{\vec{n}} = \sum_{\vec{m} \in B_2} \beta_{\vec{n}, \vec{m}} \vec{u}^{\vec{m}}
\]

The coefficients \( \beta_{\vec{n}, \vec{m}} \) are computed then used along with the Massey products to define new cochains \( \alpha_{\vec{m}} \in \text{Hom}_{\Lambda}(L_\bullet, L_\bullet) \) satisfying

\[
\{ Q, \alpha_{\vec{m}} \} = -\sum_{\vec{n} \in B'_2} \beta_{\vec{n}, \vec{m}} y(\vec{n})
\]
We find $\ker \pi_3' = a \oplus I_3$ and a vector basis corresponding to a basis for $I_3$ is computed.
Together with the $f_j^2$, this forms a basis for $S_3'$. Computing Massey products at orders
greater than 2 is slightly trickier as we need to take the obstructions into account.
There is a unique relation in $S_3'$ for every $\vec{n}$ with $|\vec{n}| \leq 3$
\[
\vec{u}\vec{n} = \sum_{\vec{m} \in B_3'} \beta_{\vec{n},\vec{m}}^{(3)} \vec{u}\vec{m} + \sum_j \beta_{\vec{n},j}^{(3)} f_j^2
\]
(5.15)
The Massey products are also defined differently at orders greater than 2 and the
above coefficients $\beta_{\vec{n},\vec{m}}^{(3)}$ are needed. For each $\vec{n} \in B_3'$ to be
\[
y(\vec{n}) = \sum_{|\vec{m}| \leq 3} \sum_{\vec{m}_1+\vec{m}_2=\vec{m}} \beta_{\vec{m},\vec{n}}^{(3)} \alpha_{\vec{m}_1} \circ \alpha_{\vec{m}_1}.
\]
(5.16)
The process continues: compute obstructions; lift to $S_3$; find basis for $S_3$; find co-
coefficients $\beta_{\vec{n},\vec{m}}^{(3)}$; find new cochains; find basis for $S_3'$; find coefficients $\beta_{\vec{n},\vec{m}}^{(4)}$; compute
Massey products at order 4 and so on.

At order $k + 1$:
\[
S_{k+1}' \xrightarrow{\pi_{k+1}'} \pi_{k+1}' \xrightarrow{\varphi_k} S_{k+1} \xrightarrow{\pi_{k+1}} S_k \xrightarrow{\varphi_{k-1}} S_{k-1}
\]
We now want to lift from $S_{k-1}$ to $S_{k+1}' = k[u_i]/(m^{k+2} + m(f_j^k))$ and do this by finding
a vector basis, $B_{k+1}'$ related to a monomial basis for
\[
\ker \pi_{k+1}' = (f_j^k)/m(f_j^k) \oplus I_{k+1}
\]
(5.17)
for
\[
I_{k+1} = m^{k+1}/(m^{k+2} + m^{k+1} \cap (f_j^k))
\]
(5.18)
The split illustrates the fact that $S_{k+1}'$ has, as a basis, the polynomials $f_j^k$ and the
monomials $\vec{u}^\vec{n}$ for $\vec{n} \in \tilde{B}'_{k+1}$. Where

$$\tilde{B}'_{k+1} = \tilde{B}_k \cup B'_{k+1}$$ (5.19)

So a unique polynomial in $S'_{k+1}$ for every $\vec{n}$ with $|\vec{n}| \leq k + 1$ is

$$\vec{u}^\vec{n} = \sum_{\vec{m} \in B'_{k+1}} \beta_{\vec{n},\vec{m}}^{(k+1)} \vec{u}^\vec{m} + \sum_j \beta_{\vec{n},j}^{(k+1)} f_j^k$$ (5.20)

We note the constants $\beta_{\vec{n},\vec{m}}^{(k+1)}$. The obstructions to this lifting are found by computing the Massey product

$$y(\vec{n}) = \sum_{|\vec{m}| \leq k+1} \sum_{\vec{m}_1 + \vec{m}_2 = \vec{m}} \beta_{\vec{m},\vec{n}}^{(k+1)} \alpha_{\vec{m}_1} \circ \alpha_{\vec{m}_1}$$ (5.21)

for each $\vec{n} \in B_{k+1}$. The obstructions are

$$f_j^{k+1} = \sum_{\vec{n} \in B'_{k+1}} t_j(y(\vec{n})) \vec{u}^\vec{n}$$ (5.22)

Now there exists a lifting from $S_k$ to $S_{k+1}$ where

$$S_{k+1} = k[u_i]/(m^{k+2} + (f_j^{k+1})).$$ (5.23)

Finding a vector basis, $B_{k+1} \subseteq B'_{k+1}$ related to a monomial basis for

$$\ker \pi_{k+1} = m^{k+1}/(m^{k+2} + (f_j^{k+1}))$$ (5.24)

where $\pi_{k+1} : S_{k+1} \to S_k$ i.e. elements of $S_{k+1}$ of order $k + 1$ and setting

$$\tilde{B}_{k+1} = B_{k+1} \cup \tilde{B}_k$$ (5.25)

gives a basis $\vec{u}^\vec{n}$ for $\vec{n} \in \tilde{B}_{k+1}$ for $S_{k+1}$. We determine the coefficients $\beta_{\vec{n},\vec{m}}$ from the
unique polynomial in $S_k$ for each $\vec{n}$ with $|\vec{n}| \leq k + 1$

$$\tilde{u}^{\vec{n}} = \sum_{\vec{m} \in B_{k+1}} \beta_{\vec{n}, \vec{m}} \tilde{u}^{\vec{m}}$$  \hspace{1cm} (5.26)

Then for each $\vec{m} \in B_k$ we solve

$$\{Q, \alpha_{\vec{m}}\} = -\sum_{l=2}^{k+1} \sum_{\vec{n} \in B_l'} \beta_{\vec{n}, \vec{m}} y(\vec{n})$$  \hspace{1cm} (5.27)

for $\alpha_{\vec{m}}$.

At this stage the deformation of the matrix factorization can be written:

$$Q^2_{\text{def}} - W \cdot 1 = \sum_j f_j^{k+1} \phi_j + \text{higher order terms}$$  \hspace{1cm} (5.28)

$$= \sum_{|\vec{m}| \leq k+1} \sum_{\vec{m}_1, \vec{m}_2 \in \bar{B}_k} \alpha_{\vec{m}_1} \alpha_{\vec{m}_2} \tilde{u}^{\vec{m}} + \{Q, \tilde{\alpha}\} + \text{higher order terms}$$  \hspace{1cm} (5.29)

For most examples this process terminates (it is possible that in some cases e.g. the quintic the obstructions may be power series rather than polynomials).

Defining $\mathcal{O}(A)$ to be the order in $x_i$ of a matrix $A$ (taken position-wise) it is obvious that we must have $\mathcal{O}(\alpha_{\vec{e}_i}) < \mathcal{O}(Q)$. Then, if a Massey product at order 2 is trivial in the image of $D_Q$ we must find some matrix $\alpha_{\vec{n}}$ with $|\vec{n}| = 2$ which satisfies

$$\mathcal{O}(Q) + \mathcal{O}(\alpha_{\vec{n}}) = \mathcal{O}(\alpha_{\vec{e}_i}) + \mathcal{O}(\alpha_{\vec{e}_j})$$  \hspace{1cm} (5.30)

so that

$$\mathcal{O}(\alpha_{\vec{n}}) < \mathcal{O}(\alpha_{\vec{e}_i})$$  \hspace{1cm} (5.31)

for all fermions $\alpha_{\vec{e}_i}$. Similarly, the order in $x_i$ of the $\alpha_{\vec{n}}$ will be lower than all of the $\alpha_{\vec{m}}$ used to compute $y(\vec{n})$. This means that there will come a point when there will be no more Massey products trivial in the cohomology of $D_Q$ meaning that no new cochains $\alpha_{\vec{n}}$ can occur. This leads to the Massey products soon becoming necessarily zero and the process will terminate.
It can be seen that in order to deform the matrix factorization and have $Q_{\text{def}}^2 = Q^2$ the obstructions must be set to zero. The obstructions are polynomials in the $u_i$, and often it is possible to find a non-trivial vanishing locus for the $f_j$ (i.e. not $u_i = 0 \forall i \in \{1, \ldots, d\}$) (some examples can be found in chapter 7). Some of these deformed matrix factorizations can be shown to be equivalent to other known matrix factorizations.

5.3 Simple Examples of Deformations and of Massey Products

Here we collect some examples of deformations of increasing complexity. While no Massey products nor obstructions are needed in the first two cases, that of the cone construction and that of deformation by a fermion with trivial cohomology class, Massey products do show up later in the examples that follow. In some of the latter, the obstructions can still be computed directly, without resorting to the algorithm above.

5.3.1 Trivial deformations

We start with a matrix factorization, $Q$, such that $Q^2 = W.1$, then deform it by a fermion $\psi := [Q, \phi]$ where $\phi$ is some bosonic matrix. This means that $\psi$ is trivial in $\text{Ext}^1(Q, Q)$, so according to the general theorems mentioned before deformations by $\psi$ should not be obstructed. Indeed, taking the deformed matrix factorization to be

$$Q_{\text{def}} = Q - \sum_{n \geq 1} (-u)^n \phi^{n-1} \psi$$

for some deformation parameter $u \in \mathbb{C}$, one can compute that again

$$Q_{\text{def}}^2 = W.1$$
5.3.2 Cone Construction

Let $Q^A$ and $Q^B$ be two matrix factorizations of the same superpotential $W$, and let $\psi \in Ext^1(Q_1,Q_2)$ be a fermionic morphism. Then

$$
\begin{pmatrix}
Q^A & u \psi \\
0 & Q^B
\end{pmatrix}
$$

is also a matrix factorization of $W$ for any $u \in \mathbb{C}$.

5.3.3 A General Result For A-type Minimal models

For superpotential $W = x^n$, $n > 2$, the choice of matrix factorization

$$
Q = \begin{pmatrix}
0 & x^{n-1} \\
x & 0
\end{pmatrix}
$$

always leads to an obstruction of the form

$$
(-1)^{n+1}u_1^n
$$

Proof.

Take $\Phi \in Mat(2r, \mathbb{C}[x])$

$$
\Phi = \begin{pmatrix}
a & b \\
c & d
\end{pmatrix}
$$

Then

$$
D_Q\Phi = \begin{pmatrix}
x(x^{n-2} + b) & x^{n-1}(d - a) \\
x(a - d) & x(b + cx^{n-2})
\end{pmatrix}
$$

and it can be seen that the basis of $Ext^*_A(Q,Q)$ contains one boson and one fermion. Choose

$$
\phi_1 = \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}, \quad \alpha_{(1)} = \begin{pmatrix}
0 & -x^{n-2} \\
1 & 0
\end{pmatrix}
$$

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Compute the Massey product at order 2:

\[ y(2) = \alpha^2_{(1)} = -x^{n-2}\phi_1 \]

So there are no obstructions to lift to \( S'_2 \) and we skip straight to finding a matrix \( \alpha_{(2)} \) satisfying

\[ D_Q\alpha_{(2)} = -y(2) \]

The only possible choice is

\[ \alpha_{(2)} = \begin{pmatrix} 0 & x^{n-3} \\ 0 & 0 \end{pmatrix} \]

Then the third order Massey product is

\[ y(3) = \{\alpha_{(1)}, \alpha_{(2)}\} = x^{n-3}\phi_1 \]

and

\[ \alpha_{(3)} = \begin{pmatrix} 0 & -x^{n-4} \\ 0 & 0 \end{pmatrix} \]

Obviously, this will continue as long as for order \( k, n - k > 0 \). Then in general

\[ y(k) = \{\alpha_{(1)}, \alpha_{(k-1)}\} = (-1)^{k+1}x^{n-k}\phi_1 \]

and

\[ \alpha_{(k)} = \begin{pmatrix} 0 & (-1)^kx^{n-(k+1)} \\ 0 & 0 \end{pmatrix} \]

When \( k = n \)

\[ y(n) = \{\alpha_{(1)}, \alpha_{(n-1)}\} = (-1)^{n+1}\phi_1 \]

and since \( y(n) \in \text{Ext}^2_A(Q, Q) \) there is the obstruction

\[ f^n_1 = (-1)^{n+1}u^n_1 \]

The process terminates here as the set \( B_n \) is empty and so there will be no more
Massey products at higher order. The final obstruction is

\[ f_1 = f_1^n = (-1)^{n+1} u_1^n \]

Another way of seeing that it terminates here is to check directly that \( Q_{def}^2 = x^n \cdot 1 + f_1 \phi_1 \), which is straightforward since

\[
Q_{def} = \begin{pmatrix}
0 & x^{n-1} + \sum_{k=1}^{n-1} (-1)^k x^{n-(k+1)} u_1^k \\
x + u_1 & 0
\end{pmatrix}
\]

Generalization To Tensor Products of A-type Minimal models

We can look at a matrix factorization of a LG superpotential of the form

\[
W = x^n + y^n
\]

as a tensor product of two minimal models with superpotentials \( W_1 = x^n \) and \( W_2 = y^n \) with \( n > 2 \). A matrix factorization can then be given by

\[
Q = \begin{pmatrix}
0 & 0 & y & x^{n-1} \\
0 & 0 & x & -y^{n-1} \\
-y^{n-1} & x^{n-1} & 0 & 0 \\
x & -y & 0 & 0
\end{pmatrix}
\]

which leads to an obstruction of the form

\[
(-1)^{n+1} (u_1^n + u_2^n)
\]

In particular, the obstruction has a non-trivial zero-locus.

**Proof.**

Take \( \Phi \in \text{Mat}(4r, \mathbb{C}[x, y]) \)

\[
\Phi = \begin{pmatrix}
a & b \\
c & d
\end{pmatrix}
\]
where \( a, b, c \in \text{Mat}(4r, \mathbb{C}[x, y]) \) Then after computing \( D_Q \Phi \) it can be seen that the basis of \( \text{Ext}^* A(Q, Q) \) contains two bosons and two fermions. Choose

\[
\phi_1 = \begin{pmatrix}
0 & x^{n-2} & 0 & 0 \\
-y^{n-2} & 0 & 0 & 0 \\
0 & 0 & 0 & -x^{n-2}y^{n-2} \\
0 & 0 & 1 & 0
\end{pmatrix}
\] (5.39)

\[
\phi_2 = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\] (5.40)

\[
\alpha_{[1,0]} = \begin{pmatrix}
0 & 0 & 0 & -x^{n-2} \\
0 & 0 & 1 & 0 \\
0 & -x^{n-2} & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix}
\] (5.41)

\[
\alpha_{[0,1]} = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & y^{n-2} \\
-y^{n-2} & 0 & 0 & 0 \\
0 & -1 & 0 & 0
\end{pmatrix}
\] (5.42)

The non-zero second order Massey products are

\[
y(2, 0) = -x^{n-2} \phi_2
\] (5.43)

\[
y(0, 2) = y^{n-2} \phi_2
\] (5.44)

So there are no obstructions and we must find matrices \( \alpha_{[2,0]} \) and \( \alpha_{[0,2]} \) satisfying

\[
D_Q \alpha_{[2,0]} = -y(2, 0)
\] (5.45)

\[
D_Q \alpha_{[0,2]} = -y(0, 2)
\] (5.46)
We must have

\[ \alpha_{[2,0]} = \begin{pmatrix}
0 & 0 & 0 & x^{n-3} \\
0 & 0 & 0 & 0 \\
x^{n-3} & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} \quad (5.47) \]

\[ \alpha_{[0,2]} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & -y^{n-3} \\
y^{n-3} & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} \quad (5.48) \]

The non-zero Massey products at order 3 are

\[ y(3, 0) = x^{n-3} \phi_2 \quad (5.49) \]
\[ y(0, 3) = y^{n-3} \phi_2 \quad (5.50) \]

so we must choose

\[ \alpha_{[3,0]} = \begin{pmatrix}
0 & 0 & 0 & -x^{n-4} \\
0 & 0 & 0 & 0 \\
x^{n-4} & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} \quad (5.51) \]

\[ \alpha_{[0,3]} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & y^{n-4} \\
y^{n-4} & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} \quad (5.52) \]

This continues so that at order \( k \) (\( n - k > 0 \)) the only non-zero Massey products are

\[ y(k, 0) = (-1)^{k+1} x^{n-k} \phi_2 \quad (5.53) \]
\[ y(0, k) = (-1)^{k+1} y^{n-k} \phi_2 \quad (5.54) \]
which leads to sets

$$
\alpha_{[k,0]} = (-1)^k \begin{pmatrix}
0 & 0 & 0 & x^{n-(k+1)} \\
0 & 0 & 0 & 0 \\
n & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
$$

(5.55)

$$
\alpha_{[0,k]} = (-1)^k \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & -y^{n-(k+1)} \\
y^{n-(k+1)} & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
$$

(5.56)

When \( k = n \)

$$
y(n, 0) = \{\alpha_{[1,0]}, \alpha_{[n-1,0]}\} = (-1)^{n+1} \phi_2
$$

(5.57)

$$
y(0, n) = \{\alpha_{[0,1]}, \alpha_{[0,n-1]}\} = (-1)^{n+1} \phi_2
$$

(5.58)

and since \( y(n, 0), y(0, n) \in \text{Ext}^2_A(Q, Q) \) the obstructions are

$$
f_1 = 0
$$

(5.59)

$$
f_2 = (-1)^{n+1} (u_1^n + u_2^n)
$$

(5.60)

and the process terminates here. The deformed matrix factorization is

$$
Q_{\text{def}} = Q + \sum_k (-1)^k \begin{pmatrix}
0 & 0 & 0 & x^{n-(k+1)} u_1^k \\
0 & 0 & 0 & -y^{n-(k+1)} u_2^k \\
y^{n-(k+1)} u_2^k & x^{n-(k+1)} u_1^k & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
$$

(5.61)

I conjecture that for a superpotential of the form

$$
W = x_1^n + \cdots + x_m^n
$$

(5.62)
with a matrix factorization which is the tensor product of the matrix factorizations

\[ Q_i = \begin{pmatrix} 0 & x_i^{n-1} \\ x_i & 0 \end{pmatrix} \] (5.63)

one will always obtain a single obstruction

\[ (-1)^n(u_1^n + \cdots + u_m^n) \] (5.64)

Note that there is a dependence on the representatives chosen for the cohomology of \( D_Q \) which may affect coefficients of the deformation parameters in the obstructions.

### 5.3.4 Minimal Model Example

Consider the matrix factorization

\[ Q = \begin{pmatrix} 0 & x^2 \\ x^3 & 0 \end{pmatrix} \]

of the Landau-Ginzburg potential \( W = x^5 \). This is the example used in [46] to illustrate the Massey product algorithm; we repeat it partly to allow for a comparison of conventions.

On computing the cohomology of our choice of \( L_\bullet \) and renaming the basis representatives we have

\[ X_1 = \alpha_{(1,0)} = \begin{pmatrix} 0 & x \\ -x^2 & 0 \end{pmatrix}, \quad X_2 = \alpha_{(0,1)} = \begin{pmatrix} 0 & 1 \\ -x & 0 \end{pmatrix}, \] (5.65)

\[ Y_1 = \phi_1 = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}, \quad Y_2 = \phi_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \] (5.66)

So in this case \( d = 2, r = 2 \).

We choose a monomial basis for \( S_1, \{1, u_1, u_2\} \) and use it to determine the corre-
sponding vector “basis” \( \vec{B}_1 = \{(0,0), (1,0), (0,1)\} \). We also have

\[
\ker \pi_2' = I_2 = m^2/m^3.
\]

We choose a basis \( \{u_1^2, u_1 u_2, u_2^2\} \) for \( I_2 \) and so the corresponding vector basis is \( B_2' = \{(2, 0), (1, 1), (0, 2)\} \). We set \( \vec{B}_2' = \vec{B}_1 \cup B_2' \).

Now we compute the Massey products at order 2. This gives

\[
y(2, 0) = \alpha_{(1,0)}^2 = -x^2 \phi_1 \tag{5.67}
\]
\[
y(1, 1) = \alpha_{(1,0)} \cdot \alpha_{(0,1)} + \alpha_{(0,1)} \cdot \alpha_{(1,0)} = -2x \phi_1 \tag{5.68}
\]
\[
y(0, 2) = \alpha_{(0,1)}^2 = -\phi_1 \tag{5.69}
\]

We then set (using 5.11)

\[
f_1^2 = \sum_{\vec{n} \in B_2'} y_1(y(\vec{n})) \vec{u}^{\vec{n}} = -u_2^2 \tag{5.70}
\]
\[
f_2^2 = \sum_{\vec{n} \in B_2'} y_2(y(\vec{n})) \vec{u}^{\vec{n}} = 0 \tag{5.71}
\]

We choose a monomial basis \( \{u_1^2, u_1 u_2\} \) for \( \ker \pi_2 = m^2/(m^3 + (f_1^2, f_2^2)) \). The corresponding vector basis is \( B_2 = \{(2, 0), (1, 1)\} \subseteq B_2' \). We set \( \vec{B}_2 = \vec{B}_1 \cup B_2 = \{(0,0), (1,0), (0,1), (2,0), (1,1)\} \).

There is a unique relation in \( S_2 \) for every \( n \) with \( |\vec{n}| \leq 2 \)

\[
\vec{u}^{\vec{n}} = \sum_{\vec{m} \in B_2} \beta_{\vec{n}, \vec{m}} \vec{u}^{\vec{m}} \tag{5.72}
\]

From this we see that \( \beta_{(0,2), \vec{m}} = 0 \) and \( \beta_{\vec{n}, \vec{m}} = \delta_{\vec{n}, \vec{m}} \) for \( \vec{n} \neq (0,2) \). Then for every \( \vec{m} \in B_2 \) we choose a cochain \( \alpha_{\vec{m}} \in \text{Hom}_A(L^\bullet_*, L^\bullet_*) \) satisfying

\[
\{Q, \alpha_{\vec{m}}\} = -\sum_{\vec{n} \in B_2'} \beta_{\vec{n}, \vec{m}} y(\vec{n}) \tag{5.73}
\]
We choose

\[ \alpha(2,0) = \begin{pmatrix} 0 & 0 \\ x & 0 \end{pmatrix} \]

\[ \alpha(1,1) = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix} \]

It can be seen from the diagram above that

\[ \ker \pi' = (f_1^2, f_2^2)/(\mathfrak{m}(f_1^2, f_2^2)) \oplus I_3 \]

We find a basis \{u_1^3, u_2^3u_2\} for \( I_3 = \mathfrak{m}^3/(\mathfrak{m}^4 + \mathfrak{m}(f_1^2, f_2^2)) \). Its corresponding vector basis is \( B'_3 = \{(3,0), (2, 1)\} \). We have assumed that for \( \vec{n} \in B'_3 \), \( \vec{u}_n = u_k \vec{u}^n \) for some \( \vec{u} \in B_2 \).

We set \( \vec{B}'_3 = \vec{B}_2 \cup B'_3 = \{(0, 0), (1, 0), (0, 1), (2, 0), (1, 1), (3, 0), (2, 1)\} \).

There is a unique relation in \( S'_3 \) for every \( \vec{n} \) with \( |\vec{n}| \leq 3 \)

\[ \vec{u}_n = \sum_{\vec{m} \in \vec{B}'_3} \beta'_{\vec{n}, \vec{m}} \vec{u}_m + \sum_j \beta'_{\vec{n}, j} f_j^2 \quad (5.74) \]

Then, as we will only be using \( \beta'_{\vec{n}, \vec{m}} \) for \( \vec{m} \in B'_3 \), we record the following non-zero \( \beta'_{\vec{n}, \vec{m}} \):

\[ \beta'_{\vec{n}, \vec{m}} = \delta_{\vec{n}, \vec{m}} \quad \text{for } \vec{m} \in B'_3 \]

The Massey products at order 3 are

\[ \gamma(3,0) = \{ \alpha(2,0), \alpha(1,0) \} \]

\[ = x \varphi_1 \]

\[ \gamma(2,1) = \{ \alpha(2,0), \alpha(0,1) \} + \{ \alpha(1,1), \alpha(1,0) \} \]

\[ = 3 \varphi_1 \]
From this we find

\begin{align*}
    f^3_1 &= f^2_1 + \sum_{\vec{n} \in B'_3} y_1(y(\vec{n})) \vec{u}^{\vec{n}} = -u^2_2 + 3u^2_1u_2 \\
    f^3_2 &= f^2_2 + \sum_{\vec{n} \in B'_3} y_2(y(\vec{n})) \vec{u}^{\vec{n}} = 0
\end{align*}

Now consider the diagram

\[ S'_3 = \mathbb{C}[u_1, u_2]/(m^5 + m(f^3_1, f^3_2)) \]

A monomial basis for \( \ker \pi_3 \) is \( \{u_3^3, u_1^2u_2\} \). Its vector basis is \( B_3 = \{(3, 0), (2, 1)\} \subseteq B'_3 \), \( \tilde{B}_3 = \tilde{B}_2 \cup B_3 = \{(0, 0), (1, 0), (0, 1), (2, 0), (1, 1), (3, 0), (2, 1)\} \). Then \( \vec{u}^{\vec{n}} \vec{u}^{\vec{m}} \) is a monomial basis for \( S_3 \), in which there is a unique relation for every \( \vec{n} \) with \( |\vec{n}| \leq 3 \)

\[ \vec{u}^{\vec{n}} = \sum_{\vec{m} \in B_3} \beta_{\vec{n}, \vec{m}} \vec{u}^{\vec{m}} \]  

(5.75)

This gives us

\[ \beta_{(0,2),(2,1)} = 3 \]

\[ \beta_{\vec{n}, \vec{m}} = \delta_{\vec{n}, \vec{m}} \quad \text{for all other choices of } \vec{n}, \vec{m} \]

Now we choose \( \alpha_{\vec{m}} \) for \( \vec{m} \in B_3 \).

\[ \{Q, \alpha_{\vec{m}}\} = -\sum_{l=2}^3 \sum_{\vec{n} \in B'_l} \beta_{\vec{n}, \vec{m}} y(\vec{n}) \]  

(5.76)
We choose
\[
\alpha_{(3,0)} = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}
\]

For (2, 1) we have two terms on the right-hand side of 5.76 to consider.
\[
\{Q, \alpha_{(2,1)}\} = -y(2, 1) - 3y(0, 2) = 0
\]

So we choose
\[
\alpha_{(2,1)} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}
\]

Now choose a basis \(\{u_1^4, u_3^1\} for I_4 = \mathfrak{m}^4/(\mathfrak{m}^5 + \mathfrak{m}(f_3^1, f_3^2))\) with vector basis \(B'_4 = \{(4,0), (3,1)\}\) chosen so that for \(\vec{n} \in B'_4\), \(\vec{u}^\vec{n} = u_k \vec{w}^\vec{n}\) for some \(\vec{m} \in B_3\). Set \(B'_4 = B_3 \cup B'_4 = \{(0, 0), (1, 0), (0, 1), (2, 0), (1, 1), (3, 0), (2, 1), (4, 0), (3, 1)\}\).

There is a unique relation in \(S'_4\) for every \(\vec{n}\) with \(|\vec{n}| \leq 4\)
\[
\vec{u}^\vec{n} = \sum_{\vec{m} \in B'_4} \beta^{(4)}_{\vec{n}, \vec{m}} \vec{w}^\vec{m} + \sum_{j} \beta^{(4)}_{\vec{n}, j} f_{j}^3
\]  
(5.77)

For \(\vec{m} \in B'_4\) we have the only non-zero coefficients
\[
\beta^{(4,0),(4,0)} = 1
\]  
(5.78)
\[
\beta^{(3,1),(3,1)} = 1
\]  
(5.79)
\[
\beta^{(1,2),(3,1)} = 3
\]  
(5.80)

We use these to compute the Massey products for \(\vec{n} \in B'_4\)
\[
y(\vec{n}) = \sum_{|\vec{m}| \leq 4} \sum_{\vec{m}_1 + \vec{m}_2 = \vec{n}} \beta^{(4)}_{\vec{m}_1, \vec{m}_2} \alpha \vec{m}_1 \circ \alpha \vec{m}_1.
\]  
(5.81)
We obtain
\[ y(4,0) = \alpha_{(2,0)}^2 + \{\alpha_{(3,0)}, \alpha_{(1,0)}\} \]
\[ = -\phi_1 \]
\[ y(3,1) = \{\alpha_{(3,0)}, \alpha_{(0,1)}\} + \{\alpha_{(1,1)}, \alpha_{(2,0)}\} + 3\{\alpha_{(1,1)}, \alpha_{(0,1)}\} \]
\[ = 5\phi_2 \]

From this we find the obstructions
\[ f_4^1 = f_1^3 + \sum_{\vec{n} \in B'_4} y_1(y(\vec{n}))\vec{u}^{\vec{n}} = -u_2^2 + 3u_1u_2 - u_4^2 \]
\[ f_4^2 = f_2^3 + \sum_{\vec{n} \in B'_4} y_2(y(\vec{n}))\vec{u}^{\vec{n}} = 5u_1^3u_2 \]

We find a monomial basis, \{u_4^1\}, for the kernel of the map \( \pi_4 : S_4 \mapsto S_3 \). Its vector basis is \( B_4^\prime = \{(4,0)\} \subseteq B_4^\prime, \bar{B}_4 = \bar{B}_3 \cup B_4 = \{(0,0), (1,0), (0,1), (2,0), (1,1), (3,0), (2,1), (4,0)\} \). Then \( \{\vec{u}^{\vec{n}}\}_{\vec{n} \in \bar{B}_4} \) is a monomial basis for \( S_4 \), in which there is a unique relation for every \( \vec{n} \) with \(|\vec{n}| \leq 4\)

\[ \vec{u}^{\vec{n}} = \sum_{\vec{m} \in B_4} \beta_{\vec{n},\vec{m}}\vec{u}^{\vec{m}} \tag{5.82} \]

There are no non-zero choices for \( \beta_{\vec{n},\vec{m}} \) here, so there will be no new matrices \( \alpha_{\vec{m}} \) at this order.

We skip straight to choosing a basis \{u_5^1\} for \( I_5 = \mathfrak{m}^5/(\mathfrak{m}^6 + \mathfrak{m}(f_1^4, f_2^4)) \) with vector basis \( B_5^\prime = \{(5,0)\} \) chosen so that for \( \vec{n} \in B_5^\prime, \vec{u}^{\vec{n}} = u_k\vec{u}^{\vec{m}} \) for some \( \vec{m} \in B_4 \).

Set \( \bar{B}_5^\prime = \bar{B}_4 \cup B_5^\prime = \{(0,0), (1,0), (0,1), (2,0), (1,1), (3,0), (2,1), (4,0), (5,0)\} \).

There is a unique relation in \( S_5^\prime \) for every \( \vec{n} \) with \(|\vec{n}| \leq 5\)

\[ \vec{u}^{\vec{n}} = \sum_{\vec{m} \in B_5^\prime} \beta_{\vec{n},\vec{m}}^{(5)}\vec{u}^{\vec{m}} + \sum_j \beta_{\vec{n},j}^{(5)}f_j^4 \tag{5.83} \]
For $\vec{m} \in B'_5$ we have the only non-zero coefficients

$$\beta'_{(5,0),(5,0)} = 1$$
$$\beta'_{(1,2),(5,0)} = -1$$

This allows us to find the only Massey product at order 5

$$y(5,0) = \{\alpha_{(3,0)}, \alpha_{(2,0)}\} - \{\alpha_{(1,1)}, \alpha_{(0,1)}\} = -2\phi_2$$

Giving the obstructions

$$f^5_1 = f^4_1 + \sum_{\vec{n} \in B'_5} y_1(y(\vec{n}))u^{\vec{n}} = -u_2^2 + 3u_1^2u_2 - u_1^4$$
$$f^5_2 = f^4_2 + \sum_{\vec{n} \in B'_5} y_2(y(\vec{n}))u^{\vec{n}} = 5u_1^3u_2 - 2u_1^5$$

We find a monomial basis, \{u_5^\vec{n}\}, for the kernel of the map $\pi_5 : S_5 \rightarrow S_4$. Its vector basis is $B_5 = \{(5,0)\} \subseteq B'_5$, $\bar{B}_5 = \bar{B}_4 \cup B_5 = \{(0,0), (1,0), (0,1), (2,0), (1,1), (3,0), (2,1), (4,0), (5,0)\}$. Then \{u^{\vec{n}}\}_{\vec{n} \in B_5}$ is a monomial basis for $S_5$, in which there is a unique relation for every $\vec{n}$ with $|\vec{n}| \leq 5$

$$\vec{u}^{\vec{n}} = \sum_{\vec{m} \in B_5} \beta_{\vec{n},\vec{m}}u^{\vec{m}}$$

Again, there are no non-zero choices for $\beta_{\vec{n},\vec{m}}$ here, so there will be no new matrices $\alpha_{\vec{m}}$ at this order.

We choose a basis \{u_6^\vec{n}\} for $I_6 = m^6/(m^7 + m^6 \cap m(f^5_1, f^5_2))$ with vector basis $B'_6 = \{(6,0)\}$ chosen so that for $\vec{n} \in B'_6$, $\vec{u}^{\vec{n}} = u_6\vec{u}^{\vec{m}}$ for some $\vec{m} \in B_5$. Set $\bar{B}_6' = \bar{B}_5 \cup B'_6 = \{(0,0), (1,0), (0,1), (2,0), (1,1), (3,0), (2,1), (4,0), (5,0), (6,0)\}$.

There is a unique relation in $S'_6$ for every $\vec{n}$ with $|\vec{n}| \leq 6$

$$\vec{u}^{\vec{n}} = \sum_{\vec{m} \in B'_6} \beta_{\vec{n},\vec{m}}u^{\vec{m}} + \sum_j \beta'_{\vec{n},j}f^5_j$$
For \( \tilde{m} \in B'_6 \) we have the non-zero coefficients

\[
\begin{align*}
\beta^{(6)}_{(6,0),(6,0)} &= 1 & (5.88) \\
\beta^{(6)}_{(0,3),(6,0)} &= \frac{1}{5} & (5.89) \\
\beta^{(6)}_{(2,2),(6,0)} &= \frac{1}{5} & (5.90) \\
\beta^{(6)}_{(4,1),(6,0)} &= \frac{2}{5} & (5.91) \\
\beta^{(6)}_{(2,2),(6,0)} &= \frac{2}{5} & (5.92)
\end{align*}
\]

Using these coefficients we find that all Massey products at order 6 are zero matrices and so there are no new obstructions or \( \alpha_{\tilde{m}} \) at this order. In fact, we find that \( B'_7 \) is empty so we cannot compute any more Massey products (If \( B'_7 = \emptyset \), then \( B_l = \emptyset \) and \( B'_l = \emptyset \ \forall \ l \geq 7 \)). This means that the process is over and the final obstructions are

\[
\begin{align*}
f_1 &= -u_2^2 + 3u_1^2u_2 - u_1^4 \\
f_2 &= 5u_1^4u_2 - 2u_1^5
\end{align*}
\]

The deformed matrix factorization is:

\[
Q_{\text{def}} = Q + \begin{pmatrix} 0 & x u_1 + u_2 \\ -x^2 u_1 - x u_2 + x u_1^2 + 2 u_1 u_2 - u_1^3 & 0 \end{pmatrix}
\]

and

\[
Q_{\text{def}}^2 = W.1 + \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}
\]

where

\[
p = -x u_2^2 + 3x u_1^2 u_2 - x u_1^4 + 2 u_1 u_2^2 - u_1^3 u_2
\]

To obtain a factorization \( Q_{\text{def}} \) of the original LG polynomial, \( u_1 \) and \( u_2 \) must be chosen such that \( p = 0 \). It can be seen that \( f_1 \) can be written as a polynomial in \( u_1^2 \).
and $f_2$ is a polynomial in $u_2$. The non-trivial solution to $f_2 = 0$ is

$$u_2 = \frac{5}{2} u_1^2$$

which sets

$$f_1 = \frac{1}{4} u_1^4$$

so there are no non-trivial solutions to $f_1, f_2 = 0$ in this case and we cannot construct a deformed matrix factorization. See 7.2 for an example of a matrix factorization which can be deformed. Also see section 8.1 where the effective superpotential is calculated for this example.
It can easily be seen that computing obstructions for any but the simplest examples is both tedious and prone to error, especially when the matrices involved are large. For this reason it is necessary, if we want to take advantage of this method, to write a program allowing a computer to do the hard work for us. The computer algebra package, Singular [29], is useful for this purpose as it allows one to make computations in polynomial rings and comes with a variety of useful libraries and procedures. The function “mod_versal” from the library “deform.lib” can be used will compute the obstruction polynomials with the following input:

```plaintext
LIB"deform.lib";
int p=printlevel;
printlevel=1;
ring r=0,(ring variables),dp;
ideal i=W
matrix m[r][r]=E (or J)
list L=mod_versal(m,i);
```

The output is accessed by typing

```plaintext
def Qx=L[2];
setring Qx;
print(Ms);
print(Js);
```

However, this is extremely limited for our purposes as there is not enough output to be able to make any connection between the variables in the polynomials returned and the fermions in our brane configuration. This connection is necessary in order to be able to write down a deformed matrix factorization. We would, in fact, need to
run through the entire calculation by hand to be able to obtain this information. The polynomials calculated by hand may not be the same - it is only necessary that the ideals generated by our polynomials and by the polynomials output by “mod_versal” be the same.

With this in mind, a new library file for Singular was written which allows the obstructions to be calculated by the method of Massey products described in (5.1) such that all information from this process is available.

The following section begins by giving an overview of the method of using Massey products to compute the obstruction polynomials. It follows the process set out in (5.1) but is written in such a way as to clarify the order of individual tasks completed by the computer. The following section then describes in detail the main functions appearing in the library file “MF.lib” which contribute to the process of computing the obstruction polynomials for a given matrix factorization via Massey products as described in (5.1),(5.2) and (6.1). The actual code is given in [19].

6.1 Computational Algorithm

This explains, in pseudocode, how starting from a BRST operator $Q$ satisfying $Q^2 = W.1$, where $W$ is the superpotential, we can compute the deformation hull and the deformed matrix factorization.

Firstly, we construct a basis for the cohomology of $D_Q$ using the functions “simple bosonic” (see 6.2.2) and “simple fermionic” (see 6.2.3).

Once a basis has been constructed for both the odd and even cohomology of $Q$ they are label by $\alpha_\vec{e}_i$ for elements of the odd (fermionic) cohomology and by $\phi_j$ for elements of the even (bosonic) cohomology.

Then by [52], [70] we begin to compute the Massey products and obstructions.

The notation $(f_j^k)$ means the ideal generated by the polynomials $f_j$ (only the non-zero ones), with

\[ j \in \{1, 2, \ldots, \dim_k(\Ext^2(D_Q, D_Q))\} \]

at order $k$.

- Starting at order $k = 1$, $f_j^1 = 0$. 

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• Set $\bar{B}_1 = \{ \vec{n} \in \mathbb{N}^d | |\vec{n}| \leq 1 \}$, $B_1 = \{ \vec{n} \in \mathbb{N}^d | |\vec{n}| = 1 \}$.

* Pick a monomial basis for $I_{k+1} = m^{k+1} / (m^{k+2} + m^{k+1} \cap \bar{m}(f_j))$. Use these $\vec{u} \vec{n}$ to find an equivalent vector basis $B'_{k+1}$ such that for $\vec{n} \in B'_{k+1}$, $\vec{u} \vec{n}$ is of the form $u_i \vec{u} \vec{m}$ for some $\vec{m} \in B_k$. Set $\bar{B}'_{k+1} = \bar{B}_k \cup B'_{k+1}$.

• There is a unique relation in $S_{k+1}' = C[\vec{u}]/(\bar{m}^{k+2} + \bar{m}(f_k))$ for every $\vec{n}$ with $|\vec{n}| \leq k + 1$:

$$\vec{u} \vec{n} = \sum_{\vec{m} \in \bar{B}'_{k+1}} \vec{u} \vec{m} \beta'_{\vec{n} \vec{m}} + \sum_j f_j \beta'_j \vec{n}$$

Using the $f_j$ solve for $\beta'_{\vec{n} \vec{m}}$, $\beta'_j$.

• For each $\vec{n} \in B'_{k+1}$ compute the Massey product $y(\vec{n})$, which is given by:

$$y(\vec{n}) = \sum_{|\vec{m}| \leq k+1} \sum_{\vec{m}, \in B_k} \beta'_{\vec{n} \vec{m}} \alpha_{\vec{m}_1} \alpha_{\vec{m}_2}$$

• Set $f_{j+1} = f_j + \sum_{\vec{n} \in B'_{k+1}} \phi_j \vec{y}(\vec{n}) \vec{u} \vec{n}$.

• Pick a monomial basis for ker $\pi_{k+1} = m^{k+1} / (m^{k+2} + (f_{j+1}))$ and find an equivalent vector basis $B_{k+1}$ such that $B_{k+1} \subseteq B'_{k+1}$. Set $\bar{B}_{k+1} = \bar{B}_k \cup B_{k+1}$.

• There is a unique relation in $S_{k+1} = S_{k+1}' / (f_{j+1})$ for every $\vec{n}$ with $|\vec{n}| \leq k + 1$:

$$\vec{u} \vec{n} = \sum_{\vec{m} \in \bar{B}_{k+1}} \beta'_{\vec{n} \vec{m}} \vec{u} \vec{m}$$

Using the $f_{j+1}$ solve for $\beta'_{\vec{n} \vec{m}}$.

• For each $\vec{m} \in B_{k+1}$ choose $\alpha_{\vec{m}} \in \text{Hom}_{B_{k+1}}^1 (L_\bullet, L_\bullet)$ such that

$$\{Q, \alpha_{\vec{m}}\} = -\sum_{l=0}^{k-1} \sum_{\vec{n} \in B_{k+l+1}} \beta'_{\vec{n} \vec{m}} \vec{y}(\vec{n})$$

• $k \rightarrow k + 1$ and return to [*].
6.2 Description of Main Functions in “MF.lib”

6.2.1 MFcohom_def

Parameters: (matrix Q, int b_or_d)

This is the main procedure which is called from Singular. It takes in a matrix Q which should satisfy $Q^2 = W$ as its first argument and an integer which should be 0 if only the basis elements of the cohomology of Q are required or 1 to compute deformations as well. It should be noted that the working ring should never contain more variables than appear in Q, else the process of finding basis elements will never end.

The procedure first notes the start time then checks whether Q is on or off block-diagonal. At the moment the code is not fully set up to handle Q being block-diagonal and will print a warning so Q should be an off-block-diagonal matrix.

The program then uses the procedures “simplebosonic” 6.2.2 and “simplefermionic” 6.2.3 to compute the even and odd bases of the cohomology. Representatives of these bases are returned giving a list of fermions and bosons. I will be calling the even basis elements $\phi_i$ (bosons) and the odd ones $\alpha_{\vec{n}}$ (fermions). A set of unit vectors is set up to be the $\vec{n}$ that label the $\alpha_{\vec{n}}$. A matrix $\sigma$ is found which gives the grading for Q. For Q block-off-diagonal the grading will be given by

$$
\sigma = \begin{pmatrix}
1_{r \times r} & 0_{r \times r} \\
0_{r \times r} & -1_{r \times r}
\end{pmatrix},
$$

and for block-diagonal Q the grading is given by

$$
\sigma = \begin{pmatrix}
1_{2r \times 2r} & 0_{2r \times 2r} & 0_{2r \times 2r} & 0_{2r \times 2r} \\
0_{2r \times 2r} & -1_{2r \times 2r} & 0_{2r \times 2r} & 0_{2r \times 2r} \\
0_{2r \times 2r} & 0_{2r \times 2r} & 1_{2r \times 2r} & 0_{2r \times 2r} \\
0_{2r \times 2r} & 0_{2r \times 2r} & 0_{2r \times 2r} & -1_{2r \times 2r}
\end{pmatrix}
$$

if Q is a $2r \times 2r$ matrix.

A $2r^2 \times 2r^2$ matrix, $D^+$ (as in [15]), is then found (using the function “dplus” 6.2.4).
It is constructed so as to have the equivalent action, on vectors related to fermionic objects, to $Q$ on bosonic matrices. It maps fermions to bosons. The vectors related to bosons and fermions are defined as follows,

$$
\begin{pmatrix}
  a & 0 \\
  0 & d
\end{pmatrix}
\rightarrow
(a_{11}, a_{12}, \ldots, a_{rr}, d_{11}, d_{12}, \ldots, d_{rr})^T \tag{6.1}
$$

$$
\begin{pmatrix}
  0 & b \\
  c & 0
\end{pmatrix}
\rightarrow
(b_{11}, b_{12}, \ldots, b_{rr}, c_{11}, c_{12}, \ldots, c_{rr})^T \tag{6.2}
$$

Functions exist in the program to move from matrices to vectors and back again for both bosons and fermions. $D^+$ will be used later on in the algorithm along with Singular’s function “reduce” in several places. The main use will be to put Massey products into normal form which can be compared with the normal form of a module with columns being vectors corresponding to the $\phi_i$ (see 6.2.5).

Next the procedure “masseyalg” 6.2.5 is called.

Finally the time is compared with the start time so that a run time for the whole process can be printed out.

This procedure has no return value.

### 6.2.2 simplebosonic

Parameters: (matrix f, matrix g, matrix fs, matrix gs)

This is a procedure written by Corina Baciu based on a different procedure written by Nils Carqueville which computes a basis for the bosonic cohomology of $Q$. If

$$
Q = \begin{pmatrix}
  0 & f \\
  g & 0
\end{pmatrix}
$$

one should use simplebosonic($f$, $g$, $f$, $g$) to find a list of suitable $\phi_i$. The fact that the procedure takes in four variables rather than just the two we use here enables one to find the fermions and bosons between two different branes.

This procedure finds ker($D^+$)/Im($D^-$). $D^-$ is similar to $D^+$ but maps bosons to
fermions.
The image module is also returned and will be used later under the name $Dp$.
The above list is returned to “MFcohom_def” 6.2.1.

6.2.3 simplefermionic

Parameters: (matrix f, matrix g, matrix fs, matrix gs)

This procedure is similar to “simplebosonic” (6.2.2) but it returns a fermions cohomology basis for $Q$. Using the same form of $Q$, given above, the syntax for using this function is

$$\text{simplefermionic}(g,f,-f,-g);$$

The procedure computes $\ker(D^-)/\text{Im}(D^+)$ and returns a list of fermions to “MFcohom_def” 6.2.1.

6.2.4 dplus

Parameters: (matrix f,g,f',g')

This procedure was written by Corina Baciu and it returns a matrix $D^+$ defined above [15]. It takes in four matrices which define two matrix factorizations:

$$Q = \begin{pmatrix} 0 & f \\ g & 0 \end{pmatrix}$$

$$Q' = \begin{pmatrix} 0 & f' \\ g' & 0 \end{pmatrix}$$

As mentioned previously $D^+$ has the equivalent action, on vectors related to fermionic
objects, as $Q$ does on bosonic matrices. It is defined by

\[
\begin{pmatrix}
  g'_{11} & 0 & 0 & g'_{1r} & 0 & 0 \\
  f^T & 0 & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 \\
  0 & \cdots & 0 & 0 & 0 & g'_{11} & 0 & 0 & g'_{1r} \\
  0 & \cdots & 0 & \vdots & \vdots & \vdots & & & & \\
  g'_r & 0 & 0 & g'_{rr} & 0 & 0 \\
  0 & 0 & f^T & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
  0 & 0 & 0 & 0 & 0 & g'_r & 0 & 0 & g'_{rr} \\
  f'_{11} & 0 & 0 & f'_{1r} & 0 & 0 \\
  0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 \\
  0 & 0 & f'_{11} & 0 & 0 & f'_{1r} \\
  \vdots & \vdots & \vdots & & & & 0 & \cdots & 0 \\
  f'_r & 0 & 0 & f'_r & 0 & 0 \\
  0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
  0 & 0 & f'_{r1} & 0 & 0 & f'_{rr}
\end{pmatrix}
\]

i.e.

\[
\left( \oplus_{i=1}^{r} f^T \otimes 1_{1 \times r} \quad g' \otimes 1_{1 \times r} \quad f' \otimes 1_{1 \times r} \quad \oplus_{i=1}^{r} g^T \right)
\]

This $2r^2 \times 2r^2$ matrix is returned by the procedure.

6.2.5 masseyalg

Parameters: (list alpha, list alphav, list phi, matrix Q, matrix Dp, matrix sigma)

The arguments taken in by this procedure are the list of $\alpha_i$, the list of their corresponding vectors, the list of $\phi_i$, the matrix $D^+$ (called $Dp$), and the grading matrix $\sigma$.

The “basering” i.e. the ring defined by the user in Singular before MFcohom_def is called is given the name $r$.

The first thing this procedure does is to check the size of the $\alpha_i$ and $\phi_i$ lists. If there
are either no bosons or no fermions an error message will be printed and the program will return nothing. The lists of $\alpha_i$ and $\phi_i$ are printed.

Then there is the option to enter a filename so that information can be saved to a file external to Singular. Instructions are printed to the screen and one should enter something similar to “example.txt” (including the “”). To ignore this option just press return. There are couple of occasions when this is of particular use: firstly, when running a computation remotely one will often want to use the “screen” command in Linux but given its limited buffer it is unlikely that all useful information will be available when checking the example’s progress; secondly, when computing an example with large matrices and/or a large fermionic basis there is a real risk that Singular will experience memory problems and the computation may not have a chance to finish. Having information printed to a separate file (which is readable during the computation) means that there is no chance of losing it and there is no need to check the computation regularly.

There is then an option to deform the matrix factorization by only a subset of the fermions and to keep only a subset of bosons. When the list bosons and fermions was printed above they were each given a number which can then be typed into Singular to choose that element. Other options are “AB” to keep all bosons or just pressing return to keep all bosons and fermions.

Next, a new ring “uring” is created with variables $u_j$ for $j \in \{1, \ldots, |\phi|\}$, where $|\phi|$ is the size of the bosonic cohomology of $Q$. The ordering of “uring” is set to “rp” (reverse lexicographical ordering [29]) so that the program behaves correctly when computing the $B_k, B'_k$ bases (i.e. replaces highest order terms in the $f_j^k$ by the lower order ones). Any minimal polynomial defined with the ring r is carried over into uring.

After the above is set up we enter a loop where the algorithm explained above in 6.1 is followed. We start at $N = 2$. The order of the method is

1. Set ring uring.

2. Initialize polynomial deformations $f_j^N$ and find all vectors with length $N$, call this set allvec.
3. Find the complement of the set $B'_N(5.17)$ using procedure “Ivecex” 6.2.10. Use this to find the complement of the set $\bar{B}'_N (5.19)$.

4. If $|\bar{B}'_N| = |allvec|$, else skip to 5.
   (a) Find $\beta'_{\bar{n}, \bar{m}}$ with $\bar{n} \leq N$ and $\bar{m} \in B'_N$ using the procedure “fbp” 6.2.9.
   (b) Set ring $r$.
   (c) Compute Massey products using procedure “MP” 6.2.6 and print out the ones not equal to the zero matrix.
   (d) Use procedure “matrix2vector” to put Massey products in the vector form described above in 6.2.1. Use the Singular procedure “reduce” to put the vector into normal form with respect to $D^+$. Convert this reduced vector back into a matrix using “vector2matrix”.
   (e) Find out whether or not the reduced Massey product is non-trivial in the cohomology of $D_Q$. Check whether or not the Massey product can be written as a linear combination of the basis matrices $\phi_j$ using procedure “lcombom” 6.2.12 and find the relevant coefficients.
   (f) Construct the deformations, $f^N_j$, using (5.22).

5. Find the complement of the set $B_N (5.24)$ using procedure “Kpvecex” 6.2.11. Use this to find the complement of the set $\bar{B}_N (5.25)$.

6. Set ring uring.

7. Find $\beta_{\bar{n}, \bar{m}}$ with $\bar{n} \in B'_l$ for $l \leq N$ and $\bar{m} \in B_N$ using the procedure “fb” 6.2.8.

8. Find $M(\bar{m}) = \sum_{\bar{n} \in B'_l} \beta_{\bar{n}, \bar{m}} y(\bar{n})$, i.e. - the right-hand side of (5.27)

9. Choose $\alpha_{\bar{m}}$ for $\bar{m} \in B_N$ using the procedure “liftsolve” 6.2.7.

10. If no new matrices $\alpha_{\bar{m}}$ have been found, increase a counter, else, set the counter to zero.

11. If the counter is equal to $\frac{N}{2}$, break the loop as the process has finished, else, increase $N$ by one and return to 4a.
At the end of this procedure various pieces of information about the matrix factorization and its deformations are printed out (mostly in latex format). The information printed out is the superpotential; the matrix factorization, $Q$; the number of fermions and bosons; the order at which the calculation ended; the obstruction polynomials, $f_j$; the deformed matrix factorization and the deformed matrix factorization squared. MFcohom_def returns a list of four strings each containing different pieces of information in latex format. The first entry gives the representatives of the cohomology of $Q$, which were calculated at the beginning; the second entry contains the obstruction polynomials; the third entry contains $Q_{def}$; and the fourth has $(Q_{def})^2$. This returned list is especially useful when computing an example remotely using ssh as the printed output normally takes up more lines than can be accessed in the terminal. Displaying the results one part at a time and at the user’s request means that nothing is lost after a calculation.

### 6.2.6 MP

Parameters: (list L,list Lv,int order,int vector_length,list bpc,
list bbc,list beta,list allvec)

This is the procedure which computes all Massey products at a given order. It takes the list of $\alpha_{\vec{n}}$; their associated vectors; the order of the calculation; the vector length (equal to the number of $\phi_i$ in the cohomology of $Q$); the sets $(B'_{order})^c$ and $(\bar{B}_{order})^c$, which are the complements of the sets $B'_{order}$ and $\bar{B}_{order}$ defined in (5.17) and (5.25); the list of coefficients $\beta'_{\vec{r},\vec{m}}$ given in (5.20); and a list of all vectors whose order (sum of entries) is less than or equal to the order of the calculation.

Using this information MP computes the Massey products following (5.21).

### 6.2.7 liftsolve

Parameters: (matrix Dp,matrix m)

This is the procedure which chooses a matrix $\alpha_{\vec{m}}$ which satisfies $\{Q,\alpha_{\vec{m}}\}_\pm = -M(\vec{m})$ (computed at stage 8 in 6.2.5). It takes as arguments the image matrix (module) $Dp$ and the Massey product sum M.
M must first be converted to the vector form described above in (6.1) Then one of Singular’s built in functions, “lift”, can be used as follows

\texttt{lift(Dp,-Mv);} \\

where \( Mv \) is the vector form of \( M \). “lift” solves, for \( T \), the equation

\[ m.T = sm.1 \]

where \( m \) is a module and \( sm \) is a sub-module of \( m \) [29].

Our solution \( \alpha_{\bar{m}} \) is then found by converting the result obtained by using “lift” back into matrix form.

The matrix \( \alpha_{\bar{m}} \) is returned.

6.2.8 fb

Parameters:(list f,list Bbc,list av,int order)

This is the procedure which finds the coefficients \( \beta_{\bar{n},\bar{m}} \) which solve the relation (5.26) in \( S_{\text{order}} \). Its arguments are a list of the polynomial obstructions \( f_{j_{\text{order}}} \); a list of the vectors which do not belong to the set \( \bar{B}_{\text{order}} \); a list of all vectors with length less than or equal to \( \text{order} \); and the order of the calculation.

The returned list of the \( \beta_{\bar{n},\bar{m}} \) will have the format (for the \( i \)th \( \beta_{\bar{n},\bar{m}} \))

\[
\begin{array}{ll}
[1]: & \bar{n} \\
[2]: & \bar{m} \\
[3]: & \beta_{\bar{n},\bar{m}} \\
\end{array}
\]

Firstly, we change to a local ordering “ds” which is a negative degree reverse lexicographical ordering i.e. if \( \deg(\bar{u}^\bar{a}) = n_1 + n_2 + \cdots + n_j \), then \( \bar{u}^\bar{a} < \bar{v}^\bar{m} \iff \deg(\bar{u}^\bar{a}) > \deg(\bar{v}^\bar{m}) \) or \( \deg(\bar{u}^\bar{a}) = \deg(\bar{v}^\bar{m}) \) and \( \exists 1 \leq s \leq j : n_j = m_j, \ldots, n_{s+1} = m_{s+1}, n_s > m_s \) [29]. This means that the Singular function “reduce” will behave in the required way described below. The ordering will revert to its previous setting at the end of the procedure.
The next thing to do is to set up the ideal

\[ I = (m^{\text{order}+1} + (f_j^{\text{order}})) \]

since we which to work in the ring \( S_{\text{order}} \) (5.23).

Next a \( 1 \times t \) matrix, \( A \), is set up with \( \vec{u}^{\vec{n}} \), such that \( \vec{n} \in \vec{B}_{\text{order}} \), as its entries. For every \( \vec{u}^{\vec{n}} \) as above we must have \( \beta_{\vec{n},\vec{m}} = \delta_{\vec{n},\vec{m}} \) so the list \( \vec{n}, \vec{n}, 1 \) is added to the list to be returned by the program.

We are now interested in the case where \( \vec{n} \notin \vec{B}_{\text{order}} \). In this case we need to find out how \( \vec{u}^{\vec{n}} \) can be rewritten in our ring \( S_{\text{order}} \). The best function in Singular to use here is “reduce” which will be used to put \( \vec{u}^{\vec{n}} \) into normal form with respect to the ideal \( I \). As previously described, it is important for lower order monomials to be rewritten as polynomials of higher order. This cannot be achieved in a global ordering as the lowest order term of a polynomial will never be its leading monomial. This is why it was necessary to change to a local ordering at the start of the procedure. Higher order terms should either stay the same or be set to zero by the use of the “reduce” function. If the rewritten form (i.e. the right-hand side of (5.26)) is non-zero we then solve for \( \beta_{\vec{n},\vec{m}} \). The function “lift” is again implemented, this time to solve, for \( \beta^{\text{mat}} \), the equation

\[ \text{matrix}(\vec{u}^{\vec{n}}) = A \cdot \beta^{\text{mat}} \]

where \( \beta^{\text{mat}} \) is a \( t \times 1 \) matrix. If we call the \( 1 \times 1 \) matrix \( \vec{u}^{\vec{n}} \) umat, the syntax is

```
reduce(lift(A,umat),std(syz(A)));
```

Then the values can be read off and put into a list in the same format as before: \( \vec{n}, A[1,s], \beta^{\text{mat}}[s, 1] \) for \( s = 1, \ldots, t \), and this is added to the list to be returned by the procedure. This is only done for non-zero \( \beta_{\vec{n},\vec{m}} \).

Using

```
reduce(lift(A,umat),std(syz(A)));
```

rather than

```latex
lift(A,umat);
```

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ensures that we obtain non-polynomial coefficients. This is because $\text{syz}(A)$; computes the first syzygy of $A$ i.e. the module of relations of the generators of $A$ [29].

When this has been repeated for all $\vec{n}$ in the list $\mathbf{av}$, the list of $\beta_{\vec{n},\vec{m}}$ is returned.

6.2.9 fbp

Parameters: (list $f$, list $B\mathbf{pbc}$, list $\mathbf{av}$, int $\text{order}$)

This procedure is similar to “fb” but a little more complicated. It returns the coefficients $\beta'_{\vec{n},\vec{m}}$ from (5.20). It takes in the list of polynomial obstructions $f_{\text{order} - 1}$; a list of the vectors which do not belong to the set $B'_{\text{order}}$; a list of all vectors with length less than or equal to $\text{order}$; and the order of the calculation.

The format for the returned list of coefficients $\beta'_{\vec{n},\vec{m}}$ will be the same as for “fb”.

Again, the monomial ordering is changed to “ds”.

The ideal which gets set up is

$$I = (m^{\text{order}} + m(f_{\text{order} - 1}))$$

This time the $1 \times t$ matrix, $A$, is set up with not just $\vec{n}^\vec{u}$, such that $\vec{n} \in B'_{\text{order}}$, as its entries, but also with the polynomials $f_{\text{order} - 1}$. This is because together they make up a basis for $S'_{\text{order}}$.

The next difference is that since we are only interested in $\beta'_{\vec{n},\vec{m}}$ with $|\vec{m}| = \text{order}$ because of (5.21), we will only be recording those $\beta'_{\vec{n},\vec{m}}$ satisfying this.

Singular’s function “lift” is again used in the same way to solve for the $\beta'_{\vec{n},\vec{m}}$.

6.2.10 Ivecex

Parameters: (int $\text{order}$, list $f$, ideal $\mathbf{ul}$, list $\mathbf{vl}$)

The values taken in by this procedure are: the order of the calculation; the list of deformations $f_{\text{order} - 1}$; the ideal $m^{\text{order}}$; and the list of all vectors $\vec{n}$ such that $|\vec{n}| = \text{order}$. $\mathbf{ul}$ and $\mathbf{vl}$ have the same order so $\mathbf{ul}[i] = \vec{u}|i|$ $i$ A list of vectors which have length $\text{order}$ but are not in $B'_{\text{order}}$ is returned. This is determined by looking at the definition of $I_{\text{order}}$ given in (5.18).
Firstly, it computes the ideal \((m_{\text{order}+1} + m_{\text{order}} \cap m(f_{\text{order}}^{-1}))\) then the function “reduce” is used with each member of ul and the ideal defined above. If 
\[
\text{reduce}(ul[i],\text{ideal});
\]
returns zero, then vl[i] is added to a list which is returned at the end of the procedure.

6.2.11 Kvecex

Parameters: (int order,ideal modi,ideal ul,list vl)

This procedure returns a list of vectors of length \(order\) which do not belong to the set \(B_{\text{order}}\). It takes in the order of the calculation \((order)\); the ideal modi = std\((m_{\text{order}+1} + f_{\text{order}})\) (std computes the standard basis of the ideal); the ideal \(m_{\text{order}}\); and the list of all vectors \(\vec{n}\) such that \(|\vec{n}| = order\).

The rest of the computation is identical to that in “Ivecex” 6.2.10 - it checks which monomials in ul “reduce” to zero and adds their corresponding vectors to a list which is returned at the end of the procedure.

6.2.12 lcombom

Parameters: (matrix m,list l,module Dp)

This is a procedure which determines whether or not the matrix m can be written as a linear combination of the matrices in the list l. A 1 is returned if true and a 0 is returned if false. If true, a list of the coefficients of the members of l is also returned.

Firstly m and the members of l are converted to their vector form (as described previously in 6.2.5), then a module lm is formed by setting its columns to be the vectors from l.

It is possible that a matrix which is non-trivial in the fermionic cohomology of Q is not a linear combination of the chosen representatives \(\phi_j\). Therefore, both m and lm are put into their normal forms by using e.g.

\[
\text{reduce}(m,Dp);
\]

A similar method to that used in “fb” 6.2.8 and “fbp” 6.2.9 is then used to find the coefficients relating the \(\phi_j\) to m:
\textbf{reduce(lift(lm,mv),std(syz(lm)))};}

where \textnormal{mv} is the vector form of \textnormal{m}.
A list of the non-zero coefficients and the position of the corresponding $\phi_j$ is formed and is returned along with either 1 or 0.

6.2.13 \textbf{mftensor}

Parameters: (matrix $f,g,f',g'$)

This function takes in four matrices representing two matrix factorizations and computes the tensor product brane as in (4.40) with

$$Q_A = \begin{pmatrix} 0 & f \\ g & 0 \end{pmatrix}, \quad Q_B = \begin{pmatrix} 0 & g' \\ f' & 0 \end{pmatrix}$$

If the tensor product brane is

$$Q^\otimes = \begin{pmatrix} 0 & E^\otimes \\ J^\otimes & 0 \end{pmatrix}$$

then the function returns a list $E, J$. 
Some Results

Here we collect some results given by the program for various Landau-Ginzburg superpotentials $W$ and matrix factorizations $Q$. In all examples, the deformed matrix factorization $Q_{\text{def}}$ as a function of the deformation parameters $u_i$ are given, along with obstruction polynomials $f_j(u_i)$. In some examples, additional details like explicit expressions for the chosen cohomology basis and some Massey products are listed, as a reminder that the program provides all these data in its output.

Moreover, some cases display interesting vanishing loci of the obstructions, sometimes with different branches, and one can check whether $Q_{\text{def}}$ at some selected point on a branch is equivalent to some known matrix factorization of the original $W$; again, these checks are performed using Singular, with the help of the “Conedims” function. From this one can read of boundary RG flow patterns.

At the moment the program is unable to compute the effective superpotential $W$ so this part has been done by hand where possible following the procedure given in section 8.1.

Deformation results for numerous other examples are available. Somewhat disappointingly, computer memory constraints make it impossible, at present, to compute deformations of “geometrically interesting” branes like the D0-brane on the quintic. We hope that partial deformations (turning on only some of the possible boundary fermions) can be computed with the program. Also see the next chapter for alternative methods to obtain some special deformations of branes on the quintic.

The general Singular input for any example begins with setting up the ring in which we wish to work

\[
\text{ring } r = (0, n), (x_1, \ldots, x_k), (c, dp);
\]
\[
\text{minpoly} = f(n);
\]
Zero in the first bracket means that we are working in the rational numbers. \( n \) is an optional parameter, the minimal polynomial is also optional and \( f(n) \) is some function of \( n \). For example, if we wanted to include the imaginary numbers we would need \( n = i \) so the function \( f(n) \) would be \( f(n) = n^2 + 1 \).

In the second bracket are the variables adjoined to \( Q \). \( k \) must be exactly equal to the number of variables appearing in the superpotential, \( W \), otherwise representatives for the cohomology of the matrix factorization, \( Q \), will not be calculated correctly.

In the third bracket \( c \) essentially means that vectors are displayed as \([v_1, v_2, \ldots, v_m]\) rather than \( v_1 \ast gen(1) + v_2 \ast gen(2) + \cdots + v_m \ast gen(m)\); \( dp \) is the global degree reverse lexicographical ordering defined as follows:

Let \( \deg(x^\alpha) = \alpha_1 + \cdots + \alpha_n \), then \( x^\alpha < x^\beta \Leftrightarrow \deg(x^\alpha) < \deg(x^\beta) \) or \( \deg(x^\alpha) = \deg(x^\beta) \) and \( \exists 1 \leq i \leq n : \alpha_n = \beta_n, \ldots, \alpha_{i+1} = \beta_{i+1}, \alpha_i > \beta_i \).

Singular allows to choose global orderings.

Then the \( r \times r \) matrices \( E, J \) are entered, which make up the \( 2r \times 2r \) matrix \( Q \):

matrix \( E[r][r] = E_{11}, \ldots, E_{1r}, \ldots, E_{r1}, \ldots, E_{rr} \);

matrix \( J[r][r] = J_{11}, \ldots, J_{1r}, \ldots, J_{r1}, \ldots, J_{rr} \);

matrix \( Q[2r][2r] = \text{blockmat}(0, J, E, 0) \);

Finally, to compute the obstructions,

\[
\text{list } l = \text{MFcohom}_{\text{def}}(Q, 1);
\]

is entered.

It is advisable to include the line “option(noloadLib,noredefine);” before loading the library file or starting any computations to prevent a huge amount of unnecessary output.
7.1 A Simple Minimal Model Example with $E = J$

Here we use Singular to compute the deformed matrix factorization for a simple example.

```plaintext
ring r = 0, x, (c, dp);
W = x^6
Q = \begin{pmatrix}
    0 & x^3 \\
    x^3 & 0
\end{pmatrix}
```

The bosonic and fermionic cohomologies are 3-dimensional, which makes computations by hand lengthier than in the $W = x^5$ minimal model example with 2-dimensional cohomologies reviewed in Chapter 4.

The string spectrum is

```
\alpha[1, 0, 0] = \begin{pmatrix}
    0 & x^2 \\
    -x^2 & 0
\end{pmatrix}, \quad \alpha[0, 1, 0] = \begin{pmatrix}
    0 & x \\
    -x & 0
\end{pmatrix}, \quad \alpha[0, 0, 1] = \begin{pmatrix}
    0 & 1 \\
    -1 & 0
\end{pmatrix},
```

```
\phi_1 = \begin{pmatrix}
    x^2 & 0 \\
    0 & x^2
\end{pmatrix}, \quad \phi_2 = \begin{pmatrix}
    x & 0 \\
    0 & x
\end{pmatrix}, \quad \phi_1 = \begin{pmatrix}
    1 & 0 \\
    0 & 1
\end{pmatrix}.
```

The obstructions computed by the program are

```
f_1 = -2u_1u_3 - u_2^2 + 3u_1^2u_2 - u_1^4
f_2 = -2u_2u_3 + \frac{3}{2}u_1u_2^2 + \frac{1}{2}u_1^3u_2 - \frac{1}{2}u_1^5
f_3 = -u_3^2 + \frac{3}{2}u_1^2u_2^2 + \frac{1}{2}u_1^4u_2
```

The deformed matrix factorization is

```
Q_{def} = Q + \begin{pmatrix}
    0 & x^2u_1 + xu_2 + u_3 \\
    -x^2u_1 - xu_2 + xu_1^2 - u_3 + 2u_1u_2 - u_1^3 & 0
\end{pmatrix}
```
\[ Q_{\text{def}}^2 = W + \left( \begin{array}{cc} p_1 & 0 \\ 0 & p_1 \end{array} \right) \]

\[ p_1 = (-2x^2u_1u_3 - u_2^2 + 3u_1^2u_2 - u_1^4) x^2 + (-2u_2u_3 + u_1^2u_3 + 2u_1u_2^2 - u_1^3u_2) x 
- u_3^2 + 2u_1u_2u_3 - u_1^3u_3 \]

So for us to have \( Q_{\text{def}}^2 = Q^2 \), we must set \( p_1 = 0 \), which yields three polynomial equations for the \( u_i \).

The only non-trivial solution to \( p_1 = 0 \) given by Maple is quite complicated

\[ u_1 = \frac{1}{z} \text{RootOf}(3A^4 - 2Az^2 + Z^2) \]

\[ u_2 = A \]

where

\[ A = \text{RootOf}(-3Z^3z^2 + 3Z^6 + z^4) \] (7.1)

For this example it would not be easy to explore non-trivial deformations of \( Q \).

### 7.2 D-model \( W = x^4 - xy^2 \)

This example is \( S_1 \) from (4.13) and we go through it thoroughly. The following information is put into Singular

\[
\text{ring } r = 0, (x, y), (c, dp); \\
Q = \begin{pmatrix}
0 & 0 & x & xy \\
0 & 0 & -y & -x^3 \\
x^3 & xy & 0 & 0 \\
-y & -x & 0 & 0
\end{pmatrix}
\]
then calling the library file LIB “MF.lib” and running list $L = \text{MFcohom} \_\text{def}(Q,1)$ we obtain the open string spectrum

\[
\alpha_{[1,0]} = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & x^2 \\
-x^2 & 0 & 0 & 0 \\
0 & -1 & 0 & 0
\end{pmatrix}, \quad \alpha_{[0,1]} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 1 & y \\
y & 0 & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix}
\]

(7.2)

\[
\phi_1 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
x^2 & y & 0 & 0 \\
0 & 0 & y & 0 \\
0 & 0 & -1 & 0
\end{pmatrix}, \quad \phi_2 = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

(7.3)

At order 2 the non-zero Massey products are

\[
y(1,1) = -y_1
\]

(7.4)

\[
y(2,0) = -x^2_1
\]

(7.5)

Since these are both trivial in the cohomology of $D_Q$ there are no obstructions at this order and therefore no non-trivial $\beta_{\vec{n},\vec{m}}$. We move straight to finding $\alpha_{\vec{m}}$ satisfying

\[
\{Q, \alpha_{\vec{m}}\} = -y(\vec{m})
\]

(7.6)

The program chooses

\[
\alpha_{[1,1]} = \begin{pmatrix}
0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad \alpha_{[2,0]} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & -x \\
x & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

(7.7)

There are no non-trivial $\beta'_{\vec{n},\vec{m}}$ so move on the order 3. The non-zero Massey products
are

\[ y(1, 2) = -1 = -\phi_2 \quad (7.8) \]
\[ y(3, 0) = x1 \quad (7.9) \]

This time we have a Massey product which is non-trivial in the cohomology. An obstruction will come from the Massey product \( y(1, 2) \)

\[ f^3_2 = -u_1 u^2_2 \quad (7.10) \]

This gives \( \beta_{(1,2),\vec{m}} = 0 \). For the other Massey product we need to find a matrix \( \alpha_{[3,0]} \) satisfying

\[ \{Q, \alpha_{[3,0]}\} = -y([3,0]) \quad (7.11) \]

and the program chooses

\[
\alpha_{[3,0]} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} \quad (7.12)
\]

We have \( \beta'_{(2,2),\vec{m}} = 0 \) and \( \beta'_{(1,3),\vec{m}} = 0 \). Non-zero Massey products at order 4 are

\[ y(4, 0) = -1 = -\phi_2 \quad (7.13) \]

which adds a new term to the obstruction

\[ f^4_2 = -u_1 u^2_2 - u^4_1 \quad (7.14) \]

I will ignore the \( \beta_{\vec{n},\vec{m}} \) at this order as they will not be used. The \( \beta'_{\vec{n},\vec{m}} \) are equal to zero for the following \( \vec{n} \): (1, 3), (2, 2), (2, 3), (1, 4), (3, 2). \( \beta'_{(5,0),(2,2)} = -1 \) and \( \beta'_{(4,1),(1,3)} = -1 \).

The only possible Massey products at order 5 are \( y(5, 0) \) and \( y(4, 1) \) but they are both
zero. There will not be any way of calculating Massey products at order 6 because the only possible ones are \( y(5, 0) \) and \( y(4, 1) \) but \( \alpha_{\{3,0\}} \) squares to zero and no combination of \( \alpha_{\bar{\pi}} \) can combine to get \( \bar{\pi}_1 + \bar{\pi}_2 = (3, 2) \) or \( \bar{\pi}_1 + \bar{\pi}_2 = (2, 3) \).

The final obstructions are

\[
\begin{align*}
  f_1 &= 0 \\
  f_2 &= -u_1 u_2^2 - u_1^4
\end{align*}
\]

The deformed matrix factorization is

\[
Q_{\text{def}} = Q + 
\begin{pmatrix}
  0 & 0 & u_1 & -u_1 u_2 \\
  0 & 0 & u_2 & x^2 u_1 - x u_1^2 + y u_2 + u_1^3 \\
-x^2 u_1 + x u_1^2 - y u_2 - u_1^3 & -u_1 u_2 & 0 & 0 \\
u_2 & -u_1 & 0 & 0
\end{pmatrix}
\]

\[
Q_{\text{def}}^2 = W_1 + 
\begin{pmatrix}
  -u_1 u_2^2 - u_1^4 & 0 & 0 & 0 \\
  0 & -u_1 u_2^2 - u_1^4 & 0 & 0 \\
  0 & 0 & -u_1 u_2^2 - u_1^4 & 0 \\
  0 & 0 & 0 & -u_1 u_2^2 - u_1^4
\end{pmatrix}
\]

Insisting on \( Q_{\text{def}}^2 = Q^2 \) means we must choose either \( u_1 = 0 \) or \( u_2 = \sqrt{-u_1^4} \) which gives

\[
Q_{\text{def}} \bigg|_{u_1 = 0} = Q + 
\begin{pmatrix}
  0 & 0 & 0 & 0 \\
  0 & 0 & u_2 & y u_2 \\
  -y u_2 & 0 & 0 & 0 \\
u_2 & 0 & 0 & 0
\end{pmatrix}
\]

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(i.e., deforming by $\alpha[0,1]$ alone is unobstructed) or

$$Q_{\text{def}}|_{u_2=-\sqrt{u_1}} = Q + \begin{pmatrix} 0 & 0 & u_1 & -iu_1^2\sqrt{u_1} \\ 0 & 0 & iu_1\sqrt{u_1} & p \\ -p & -iu_1^2\sqrt{u_1} & 0 & 0 \\ iu_1\sqrt{u_1} & -u_1 & 0 & 0 \end{pmatrix}$$

where

$$p = (x^2 - xu_1 + iy\sqrt{u_1 + u_1^2})u_1.$$  

Using the function “Conedims” to check equivalence to known matrix factorizations of $W$ yields the following result: The choices $u_1 = 0$, $u_1 = -1$ and $u_2 = 1$, and $u_1 = 1$ and $u_2 = i$ all lead to a trivial matrix factorization i.e. there are no bosons or fermions in the deformed matrix factorizations’ spectra. It is reasonable to assume that this feature applies to the entire branches $u_1 = 0$ or $u_2 = \sqrt{-u_1^2}$, which would mean that any deformation of the D-model factorization $S_1$ leads to an annihilation of the brane.

### 7.3 D-model $W = x^5 - xy^2$

This is the $T_2$ example listed in (4.14).

**ring** $r = 0, (x,y),(c,dp)$;

$$Q = \begin{pmatrix} 0 & 0 & x^2 & y \\ 0 & 0 & -y & -x^2 \\ x^3 & xy & 0 & 0 \\ -xy & -x^3 & 0 & 0 \end{pmatrix}$$

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The open string spectrum is

$$\alpha_{[1,0,0,0]} = \begin{pmatrix} 0 & 0 & 0 & x \\ 0 & 0 & x & 0 \\ 0 & x^2 & 0 & 0 \\ x^2 & 0 & 0 & 0 \end{pmatrix}$$ \quad \alpha_{[0,1,0,0]} = \begin{pmatrix} 0 & 0 & x & 0 \\ 0 & 0 & 0 & x \\ -x^2 & 0 & 0 & 0 \\ 0 & -x^2 & 0 & 0 \end{pmatrix}, \quad (7.15)$$

$$\alpha_{[0,0,1,0]} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & x & 0 & 0 \\ x & 0 & 0 & 0 \end{pmatrix}$$ \quad \alpha_{[0,0,0,1]} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -x & 0 & 0 & 0 \\ 0 & -x & 0 & 0 \end{pmatrix}, \quad (7.16)$$

$$\phi_1 = \begin{pmatrix} x^2 & 0 & 0 & 0 \\ 0 & x^2 & 0 & 0 \\ 0 & 0 & x^2 & 0 \\ 0 & 0 & 0 & x^2 \end{pmatrix}$$ \quad \phi_2 = \begin{pmatrix} 0 & x & 0 & 0 \\ x & 0 & 0 & 0 \\ 0 & 0 & 0 & -x \\ 0 & 0 & -x & 0 \end{pmatrix}, \quad (7.17)$$

$$\phi_3 = \begin{pmatrix} y & 0 & 0 & 0 \\ 0 & y & 0 & 0 \\ 0 & 0 & y & 0 \\ 0 & 0 & 0 & y \end{pmatrix}$$ \quad \phi_4 = \begin{pmatrix} x & 0 & 0 & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & x \end{pmatrix}, \quad (7.18)$$

$$\phi_5 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$ \quad \phi_6 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (7.19)$$
The obstructions are

\[ f_1 = -2u_2u_4 + 2u_1u_3 + u_2^3 - u_1^2u_2 \]
\[ f_2 = 0 \]
\[ f_3 = 0 \]
\[ f_4 = -u_4^2 - u_1^2u_4 + u_3^2 + u_1u_2u_3 + \frac{1}{2}u_2^4 - \frac{1}{2}u_1^2u_2^2 \]
\[ f_5 = 0 \]
\[ f_6 = 0 \]

The deformed matrix factorization is

\[
Q_{\text{def}} = Q + \begin{pmatrix}
0 & 0 & xu_2 + u_4 & xu_1 + u_3 \\
0 & 0 & xu_1 + u_3 & p_1 \\
xp_1 & xu_2 + xu_3 & 0 & 0 \\
x^2u_1 + xu_3 & -x^2u_2 - xu_4 & 0 & 0
\end{pmatrix}
\]

(7.20)

\[ p_1 = xu_2 + u_4 - u_2^2 + u_1^2 \]

and its square is

\[
Q_{\text{def}}^2 = W_1 + \begin{pmatrix}
p_2 & 0 & 0 & 0 \\
0 & p_2 & 0 & 0 \\
0 & 0 & p_2 & 0 \\
0 & 0 & 0 & p_2
\end{pmatrix}
\]

\[ p_2 = (-2u_2u_4 + 2u_1u_3 + u_2^3 - u_1^2u_2)x^2 + (-u_4^2 + u_2^2u_4 - u_1^2u_4 + u_3^2)x\]
Some solutions to $p_2 = 0$ are

\[
\begin{align*}
&u_2, u_3, u_4 = 0 & u_1 = u_2, u_3 = u_4 \\
u_2, u_3 = 0, u_3^2 = u_1^2 & u_3, u_4 = 0, u_1^2 = u_2^2 \\
u_1, u_2 = 0, u_3^2 = u_4^2 & u_1 = \pm u_2, u_3 = -iu_4^2, u_2 = \mp iu_4 \\
u_1 = \pm u_2, u_3 = 2u_4 - 3, u_4 = 3 \text{ or } 1
\end{align*}
\]

Without deforming the matrix factorization there are 6 bosons and 4 fermions in the cohomology of $Q$. However, if we move on to non-trivial solutions of the obstruction zero-locus we find that this changes and the matrix factorization has “flowed” to another factorization. See table 7.1 for the dimensions of the cohomologies of various deformed matrix factorizations which were computed in Singular.

<table>
<thead>
<tr>
<th>$u_1$</th>
<th>$u_2$</th>
<th>$u_3$</th>
<th>$u_4$</th>
<th>Number of Bosons</th>
<th>Number of Fermions</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>6</td>
<td>4</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>42</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>2u_4 - 3</td>
<td>$u_1^2 - 3u_4 + 3 = 0$</td>
<td>2</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 7.1: Cohomology dimensions for deformed permutation brane (7.20)

The deformed matrix factorization given by the example $u_1 = 1, u_2 = 1, u_3 = 0, u_4 = 0$ was found, using the “Conedims” function of Carqueville in Singular, to be equivalent to the matrix factorization $T_1$ (4.14). In the same way it was found that $u_1 = 1, u_2 = 2, u_3 = 2u_4 - 3, u_1^2 - 3u_2 + 3 = 0$ gives a matrix factorization equivalent to $R_0$ (4.10).

This means that starting from the $T_2$ factorization of the D-model superpotential $W = x^5 - xy^2$ we can deform to either $T_1$ or $R_0$. 

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7.4 Linear Matrix Factorization $W = x^k + y^k$

For $k = 5$ the following information is input to Singular

\[
\text{ring } r = (0, n), (x, y), (c, dp);
\]
\[
\text{minpoly} = n^4 - n^3 + n^2 - n + 1;
\]

where $n$ has the same role as $\eta$ above. As a single rank one linear matrix factorization has no fermions in its spectrum, we consider a matrix factorization which is the direct sum of two such factorizations,

\[
Q = \begin{pmatrix}
0 & 0 & x - ny & 0 \\
0 & 0 & 0 & x + n^2y \\
W/(x - ny) & 0 & 0 & 0 \\
0 & W/(x + n^2y) & 0 & 0
\end{pmatrix}
\]

There are 2 fermions and 8 bosons between the two branes.

The obstructions are found to be

\[
\begin{aligned}
f_1 &= (n^3 - 2n^2 - 2n + 1)u_1u_2 \\
f_2 &= (-n^3 - 3n^2 + 2n - 1)u_1u_2 \\
f_3 &= 0 \\
f_4 &= 0 \\
f_5 &= 0 \\
f_6 &= 0 \\
f_7 &= 0 \\
f_8 &= 0
\end{aligned}
\]
The deformed matrix factorization is

\[ Q_{\text{def}} = Q + \begin{pmatrix} 0 & 0 & 0 & u_2 \\ 0 & 0 & u_1 & 0 \\ 0 & p_3 & 0 & 0 \\ p_4 & 0 & 0 & 0 \end{pmatrix} \]  

(7.21)

where

\[ p_3 = (-x^3 + (n^2 - n)x^2y - (n - 1)xy^2 - n^2y^3)u_2 \]

\[ p_4 = (-x^3 + (n^2 - n)x^2y - (n - 1)xy^2 - n^2y^3)u_1 \]

\[ Q_{\text{def}}^2 = W_1 + \begin{pmatrix} p_5 & 0 & 0 & 0 \\ 0 & p_5 & 0 & 0 \\ 0 & 0 & p_5 & 0 \\ 0 & 0 & 0 & p_5 \end{pmatrix} \]

where

\[ p_5 = (-x^3 + (n^2 - n)x^2y - (n - 1)xy^2 - n^2y^3)u_1u_2 \]

So in order to find deformed matrix factorizations we must find a solution to \( p_5 = 0 \) which is equivalent to setting either \( u_1 \) or \( u_2 \) equal to zero. Therefore, we have a completely free choice for either \( u_1 \) or \( u_2 \) in \( Q_{\text{def}} \) as long as the other parameter is set to zero. The resulting \( Q_{\text{def}} \) are simply the cones of the two deforming fermions

\[ \alpha_{(1,0)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -x^3 + (n^2 - n)x^2y + (-n + 1)xy^2 - n^2y^3 & 0 & 0 & 0 \end{pmatrix} \]

\[ \alpha_{(0,1)} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -x^3 + (n^2 - n)x^2y + (-n + 1)xy^2 - n^2y^3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \]
7.5 Deformed Brane and Anti-Brane Linear MF $W = x^5 + y^5$

Now we look at the deformed matrix factorization of a brane-anti-brane configuration i.e. $J_P = E_O$ and $E_P = J_O$. We deformation it by $x + ny$.

The information input into Singular is

$$r = (0, n), (x, y), (c, dp);$$

$$\text{minpoly} = n^4 - n^3 + n^2 - n + 1;$$

$$Q = \begin{pmatrix}
0 & 0 & x - ny & 0 \\
0 & 0 & x + ny & p_1 \\
p_1 & 0 & 0 & 0 \\
-x - ny & x - ny & 0 & 0
\end{pmatrix}$$

$$p_1 = x^4 + nx^3y + n^2x^2y^2 + n^3xy^3 + n^4y^4$$

The obstruction polynomials are found to be

$$f_1 = 0$$

$$f_2 = -\frac{5}{8}nu_1u_2^4 - 5n^3u_1^3u_2^2 + 2u_1^5$$
The deformed matrix factorization is

$$Q_{\text{def}} = Q + \begin{pmatrix} 0 & 0 & -2nu_1 & p_2 \\ 0 & 0 & u_2 & p_3 \\ p_3 & -p_2 & 0 & 0 \\ -u_2 & -2nu_1 & 0 & 0 \end{pmatrix}$$

(7.22)

$$p_2 = -5n^4 y^3 u_1 - \frac{5}{2} n^3 y^2 u_1 u_2 - 15n^4 y^2 u_1^2 - \frac{5}{4} n^2 y u_1 u_2^2 + 5n^3 y u_2^2 u_2$$

(7.23)

$$p_3 = 2n x^3 u_1 + 4n^2 x^2 y u_1 + 6n^3 x y u_2 + 3n^4 y^3 u_1 + 4n^2 x^2 u_2^2$$

(7.24)

$$+ 12n^3 x y u_1^2 + \frac{5}{2} n^3 y^2 u_1 u_2 + 9n^4 y^2 u_1^2 + 8n^3 x u_1^3$$

$$- \frac{5}{4} n^2 y u_1 u_2^2 + 5n^3 y u_1^2 u_2 + 7n^4 y u_1^3 + \frac{5}{8} n u_1 u_2^3 - \frac{5}{4} n^2 u_1^2 u_2$$

$$+ \frac{15}{2} n^3 u_1^3 u_2 + n^4 u_1$$

$$Q_{\text{def}}^2 = W_1 + \begin{pmatrix} p_4 & 0 & 0 & 0 \\ 0 & p_4 & 0 & 0 \\ 0 & 0 & p_4 & 0 \\ 0 & 0 & 0 & p_4 \end{pmatrix}$$

$$p_4 = -\frac{5}{8} n u_1 u_2^4 - 5n^3 u_1^3 u_2^2 + 2u_1^5$$

Solutions to $p_4 = 0$ are $u_1 = 0$ and $u_2 = \pm \sqrt{-4n^2 \pm \sqrt{-\frac{96}{5} n^{-1} u_1}}$. Since the use of square roots in Singular is particularly complicated we will not explore particular solutions to this deformed matrix factorization.
7.6 Deformation of Deformed Brane and Anti-Brane Example

For this example we take the deformed matrix factorization (7.5) with $u_1 = 0$, $u_2 = J_P = x - ny$ as defined in the previous example.

$$\text{ring } r = (0, n), (x, y), (c, dp);$$

$$\text{minpoly} = n^4 - n^3 + n^2 - n + 1;$$

$$W = x^5 + y^5$$

$$Q = \begin{pmatrix}
0 & 0 & x - ny & 0 \\
0 & 0 & 2x & p_1 \\
p_1 & 0 & 0 & 0 \\
-2x & x - ny & 0 & 0 \\
\end{pmatrix}$$

$$p_1 = x^4 + nx^3y + n^2x^2y^2 + n^3xy^3 + n^4y^4$$

Fermions and bosons are given by

$$\alpha_{[1,0]} = \begin{pmatrix}
0 & 0 & -2n & -5n^4y^3 \\
0 & 0 & 0 & s_1 \\
s_1 & 5n^4y^3 & 0 & 0 \\
0 & -2n & 0 & 0 \\
\end{pmatrix},$$

$$\alpha_{[0,1]} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
\end{pmatrix}.$$
\[ \phi_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 2n^2y^3 & s_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2n^4 & s_2 \end{pmatrix}, \]

\[ \phi_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \]

where

\[ s_1 = 2nx^3 + 4n^2x^2y + 6n^3xy^2 - 2n^4y^3 \]
\[ s_2 = n^4x^3 - x^2y - nx^2y - 2n^2y^3 \]

The obstructions are

\[ f_1 = 0 \]
\[ f_2 = -\frac{5}{8} nu_1u_2^4 - 5n^2u_1^2u_2^3 - 20n^3u_1^3u_2^2 - 40n^4u_1^4u_2 + 32u_1^5 \]

The deformed matrix factorization is

\[ Q_{def} = Q + \begin{pmatrix} 0 & 0 & -2nu_1 & p_2 \\ 0 & 0 & u_2 & p_3 \\ p_3 & -p_2 & 0 & 0 \\ -u_2 & -2nu_1 & 0 & 0 \end{pmatrix} \]
\[ p_2 = -5n^4y^3u_1 + \frac{5}{2}n^3y^2u_1u_2 - 10n^4y^2u_1^2 - \frac{5}{4}n^2yu_1u_2^2 - 20n^4yu_1^3 + \frac{5}{8}nu_1^3u_2^3 \\
+ \frac{5}{2}n^2u_1^2u_2^2 + 10n^3u_1^3u_2 \\
\]
\[ p_3 = 2nx^3u_1 + 4n^2x^2yu_1 + 6n^3xy^2u_1 - 2n^4y^3u_1 + 4n^2x^2u_1^2 + 12n^3xyu_1^2 \\
+ 5n^3y^2u_1u_2 + 4n^4y^2u_1^2 + 8n^3xu_1^3 - \frac{5}{2}n^2yu_1u_2^2 - 8n^4yu_1^3 + \frac{5}{4}nu_1u_2^3 \\
+ 5n^2u_1^2u_2^2 + 20n^3u_1^3u_2 + 16n^4u_1^4 \\
\]
\[ Q_{def}^2 = W1 + \begin{pmatrix} p_4 & 0 & 0 \\
0 & p_4 & 0 \\
0 & 0 & p_4 \\
0 & 0 & 0 
\end{pmatrix} \]
\[ p_4 = \frac{5}{8}nu_1u_2^4 - 5n^2u_1^3u_2^2 - 20n^3u_1^3u_2 - 40n^4u_1^4u_2 + 32u_1^5 \]

\( p_4 = 0 \) has the obvious solution \( u_1 = 0 \), and other branches along the roots of a quartic polynomial expressing \( u_1 \) as a function of \( u_2 \). Along the \( u_1 = 0 \) branch, the deformed matrix factorization is trivial as expected.

The other solutions are
\[ u_1 = \frac{u_2}{4}(-2 + n - n^2 + n^3) \quad (7.25) \]
\[ u_1 = \frac{u_2}{4}(-1 + 2n - n^2 + n^3) \quad (7.26) \]
\[ u_1 = \frac{u_2}{4}(-1 + n - 2n^2 + n^3) \quad (7.27) \]
\[ u_1 = \frac{u_2}{4}(-1 + n - n^2 + 2n^3) \quad (7.28) \]

Applying the above solutions for various values of \( u_2 \) results in the deformed matrix factorization again being trivial.
7.7 An $E_7$ Matrix Factorization

The superpotential

$$W = xy^3 + x^3 + z^2 \quad (7.29)$$

is for “the other GSO projection” of $E_7$. Its simplest matrix factorization is

$$Q = \begin{pmatrix}
0 & 0 & z & x \\
0 & 0 & y^3 + x^2 & -z \\
z & x & 0 & 0 \\
y^3 + x^2 & -z & 0 & 0
\end{pmatrix}$$

The cohomology of $Q$ contains 3 bosons and 3 fermions.

The obstructions are

$$\begin{align*}
f_1 &= -3u_1^{10} + 6u_1^5u_2 - u_2^2 - 2u_1u_3 \\
f_2 &= 3u_1^{14} - 12u_1^9u_2 + 12u_1^4u_2^2 - 2u_2u_3 \\
f_3 &= 6u_1^3u_2 - 12u_1^8u_2^2 + 8u_1^3u_2^3 - u_3^2
\end{align*} \quad (7.30)$$

$$\begin{align*}
f_1 &= -3u_1^{10} + 6u_1^5u_2 - u_2^2 - 2u_1u_3 \\
f_2 &= 3u_1^{14} - 12u_1^9u_2 + 12u_1^4u_2^2 - 2u_2u_3 \\
f_3 &= 6u_1^3u_2 - 12u_1^8u_2^2 + 8u_1^3u_2^3 - u_3^2
\end{align*} \quad (7.31)$$

The deformed matrix factorization is

$$Q_{\text{def}} = Q + \sum \bar{u} \bar{\alpha} \bar{u} \bar{\alpha}$$

$$= Q + \begin{pmatrix}
0 & 0 & p_2 & p_3 \\
0 & 0 & p_1 & p_2 \\
-p_2 & p_3 & 0 & 0 \\
p_1 & -p_2 & 0 & 0
\end{pmatrix} \quad (7.32)$$
where

\[ p_1 = -xyu_1^2 + y^2u_1^4 + xu_1^6 - 2xu_1u_2 - 2yu_1^8 + 4yu_1^3u_2 + u_1^{12} - 4u_1^7u_2 + 4u_1^2u_2^2 \] (7.35)
\[ p_2 = y^2u_1 + yu_2 + u_3 \] (7.36)
\[ p_3 = yu_1^2 - u_1^6 + 2u_1u_2 \] (7.37)

On squaring the deformed matrix factorization we obtain a matrix proportional to the unit matrix

\[ Q_{def}^2 = (W + p) \cdot 1 \]

where

\[ p = (-3u_1^{10} + 6u_1^7u_2 - u_2^2 - 2u_1u_3)y^2 + (3u_1^4 - 12u_1^3u_2 + 12u_1^4u_2^2 - 2u_2u_3)y \\
- u_1^{18} + 6u_1^{13}u_2 - 12u_1^8u_2^2 + 8u_1^3u_2^3 - u_3^2 \] (7.38)

This example has no non-trivial moduli (according to Mathematica).

### 7.8 Sum of Two Generalized Transposition Branes

In this example the superpotential \( W = y^6 + x^3 \) is factorized into

\[ Q = \begin{pmatrix}
0 & 0 & -ny^2 + x & 0 \\
0 & 0 & 0 & y^2 + x \\
(n - 1)y^4 + nxy^2 + x^2 & 0 & 0 & 0 \\
0 & y^4 - xy^2 + x^2 & 0 & 0
\end{pmatrix} \] (7.39)

Such matrix factorizations are obtained from linear rank 1 factorizations of \( x^3 + \tilde{y}^3 \) with \( \tilde{y} = y^2 \), see [16, 25].
The $Q$ from above has 4 bosons and 8 fermions. The non-zero obstructions are

\begin{align*}
    f_1 &= (-n - 1)u_2u_3 + (-n - 1)u_1u_4 \\
    f_2 &= (-2n + 1)u_2u_3 + (-2n + 1)u_1u_4 \\
    f_3 &= (-1/3 n - 1/3)u_1^2u_2^2 + (-n - 1)u_3u_4 \\
    f_5 &= (-2/3 n + 1/3)u_1^2u_2^2 + (-2n + 1)u_3u_4
\end{align*}

The deformed matrix factorization is

\[
Q_{\text{def}} = Q + \begin{pmatrix}
    0 & 0 & (\frac{1}{3}n + \frac{1}{3})u_1u_2 & yu_2 + u_4 \\
    0 & 0 & nyu_1 + nu_3 & -(\frac{1}{3}n + \frac{1}{3})u_1u_2 \\
    -p_1 & p_2 & 0 & 0 \\
    p_3 & p_1 & 0 & 0
\end{pmatrix}
\]  \hspace{1cm} (7.40)

where

\begin{align*}
    p_1 &= (\frac{1}{3}n - \frac{2}{3})y^2u_1u_2 + (\frac{1}{3}n + \frac{1}{3})xu_1u_2 \\
    p_2 &= -(n - 1)y^3u_2 - xyu_2 - (n - 1)y^3u_4 - xu_4 \\
    p_3 &= y^3u_1 - nxyu_1 + y^2u_3 - nxu_3
\end{align*}

This squares to

\[Q_{\text{def}}^2 = (W + p) \cdot 1\]  \hspace{1cm} (7.41)

with

\[p = (u_2u_3 + u_1u_4)(y^3 - nxy) + (\frac{1}{3}u_1^2u_2^2 + u_3u_4)(y^3 - nx)\]

$Q_{\text{def}}^2 = Q^2$ can be solved by $u_1 = u_3 = 0$ or by $u_1 = u_2 = u_4 = 0$ or by $u_1 \neq 0, u_4 = -\frac{u_2u_3}{u_1}, u_2 = \frac{3u_1^2}{u_4}$. We now travel along these branches of solutions and find that by deforming the matrix factorization one obtains non-equivalent matrix factorizations depending on the values of the deformation parameters $u_1, u_2, u_3, u_4$. The results for the cohomology dimensions are summarized in table 7.2 and show that already in this simple example there is a rather interesting pattern of
RG flows. One could now compare these dimensions to the ones for (direct sums of) known factorizations of $y^6 + x^3$, and employ the function “Conedims” to establish equivalence as demonstrated in previous examples.
Dealing with Obstructions

In this chapter we take a look at what can be done with the obstruction polynomials once they have been calculated. Firstly, the method for computing the effective superpotential $W_{\text{eff}}$ by integrating certain combinations of the obstructions is explained. This is followed by the computation of $W_{\text{eff}}$ for the minimal model example we went through in detail back in section 5.3.4.

The next section introduces the idea of replacing deformation parameters by nilpotent matrices which increases the rank of the deformed matrix factorization. We find that some of the resultant matrix factorizations are equivalent to new matrix factorizations. The final section of this chapter discusses the treatment of deformation parameters on an equal footing to that of the $x_i$. In this way the deformed matrix factorization becomes a factorization of a different superpotential.

8.1 Computing the Effective Superpotential

The effective superpotential is defined as

$$W_{\text{eff}} = \sum_{n,a_1,\ldots,a_n} B_{a_1\ldots a_n} u_{a_1} \cdots u_{a_n}$$  \hspace{1cm} (8.1)

where

$$B_{a_1\cdots a_n} = \langle \psi_{a_1} \cdots \psi_{a_3} \int G_{-1} \psi_{a_4} \cdots \int G_{-1} \psi_{a_n} \rangle.$$  \hspace{1cm} (8.2)

Correlation functions are only non-zero if the sum of charges of the fields inside is equal to the background charge $q_B = \hat{c} := \frac{c}{5}$. If the charge of $\psi_{a_i}$ is given by $q_{a_i}$ and since $G_{-1}$ has charge $-1$, $B_{a_1\cdots a_n}$ will be non-zero if

$$\sum_{a_i} q_{a_i} = q_B + n - 3.$$  \hspace{1cm} (8.3)
Since deformation terms in correlators turn up as exponentials of \( u_a \int \partial^2 G^{-1} \psi_a \) the degree of \( u_a \) must be given by \( \deg(u_a) = -q(G^{-1} \psi_a) = 1 - q_a \). Returning to (8.1), the \( u \)-degree of \( \mathcal{W}_{\text{eff}} \) is a constant

\[
\deg(\mathcal{W}_{\text{eff}}) = \sum_{a_i} \deg(u_{a_i}) = 3 - \hat{c}.
\]

Now we know what the \( u \)-degree of \( \mathcal{W}_{\text{eff}} \) is we will assume that we have already completed the Massey product algorithm and calculated the charges of the deformation parameters \( u_i \) using the charge of the \( \phi_j \) (calculated using the method given near the end of section 3.3) and \( q(W) = 2 \).

Since

\[
\{ u_i \mid \partial_i \mathcal{W}_{\text{eff}} = 0 \} = \{ u_i \mid f_j = 0 \}
\]

we can form quasi-homogeneous combinations, \( g_i \), of the \( f_j \) such that

\[
g_i = \partial_i \mathcal{W}_{\text{eff}}
\]

and

\[
\deg(g_i) = \deg(\mathcal{W}_{\text{eff}}) - \deg(u_i)
\]

The form of the \( g_i \) is given by

\[
g_i = \sum_{\deg(f_j) + n \deg(u_k) = \deg(g_i)} a_{ij} u_n^k f_j
\]

where the relations between the constants \( a_{ij} \in \mathbb{C} \) are determined by requiring \( \partial_1 \ldots \partial_s \mathcal{W}_{\text{eff}} \) (\( s = |\text{Ext}^1(Q, Q)| \)) to be that same for whichever \( \partial_i \mathcal{W}_{\text{eff}} \) we swap for \( g_i \).

Upon appropriate integration (such that the resultant polynomial has degree \( \deg(\mathcal{W}_{\text{eff}}) \)) the \( g_i \) give us the effective superpotential \( \mathcal{W}_{\text{eff}} \) (up to a constant).

This process is made clearer by the following example.
Example

Here, as an example we look at the deformation of the minimal model example given earlier in subsection 5.3.4 where the superpotential was $W = x^5$, the matrix factorization

$$Q = \begin{pmatrix} 0 & x^2 \\ x^3 & 0 \end{pmatrix} \quad (8.9)$$

and the bosonic basis

$$\phi_1 = \begin{pmatrix} x \\ 0 \end{pmatrix}, \quad \phi_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (8.10)$$

the obstructions were found to be

$$f_1 = -u_2^2 + 3u_1^2u_2 - u_1^4$$
$$f_2 = 5u_1^3u_2 - 2u_1^5$$

By (3.53), the R-charge matrix is

$$R = \begin{pmatrix} \frac{1}{10} & 0 \\ 0 & -\frac{1}{10} \end{pmatrix} \quad (8.11)$$

and we have

$$q(\phi_1) = \frac{2}{5}, \quad q(\phi_2) = 0 \quad (8.12)$$

Since $Q_{def}^2 = W1 + \sum_j f_j \phi_j$ we must have

$$\deg(f_1) = 2 - q(\phi_1) = \frac{8}{5}, \quad \deg(f_2) = 2 - q(\phi_2) = 2 \quad (8.13)$$

which means that

$$\deg(u_1) = \frac{2}{5}, \quad \deg(u_2) = \frac{4}{5} \quad (8.14)$$

Now we are looking to find polynomials $g_1, g_2$ in the ideal generated by $(f_1, f_2)$ which are equal to $\partial_1 W_{eff}$ and $\partial_2 W_{eff}$. 127
The degree of the effective superpotential is

\[ \text{deg}(W_{\text{eff}}) = 3 - \frac{3}{5} = \frac{12}{5} \]  

which means the charges of the \( g_i \) need to be

\[ \text{deg}(g_1) = \frac{12}{5} - \frac{2}{5} = 2 \]  

\[ \text{deg}(g_2) = \frac{12}{5} - \frac{4}{5} = \frac{8}{5}. \]  

The combinations of the \( f_j \) with the correct charge are therefore

\[ g_1 = a_{11}u_1 f_1 + a_{12}f_2 \]  

\[ g_2 = a_{21}f_1. \]  

Differentiating \( g_1 \) and \( g_2 \) to polynomials of \( u \)-degree \( \frac{6}{5} \) (i.e. \( \partial_1 \partial_2 W_{\text{eff}} \)) allows us to make a choice for the coefficients. We get

\[ a_{11} = -3 a_{21} \]  

\[ b_{12} = a_{21}. \]  

which will only lead to a unique choice for \( W_{\text{eff}} \) up to a constant. The following integrations should be performed in order to obtain \( W_{\text{eff}} \)

\[ \int g_1 du_1 = \int g_2 du_2 = W_{\text{eff}} \]  

which lead to the effective superpotential

\[ W_{\text{eff}} = a_{21} \left( \frac{1}{6} u_1^6 - u_1^4 u_2 + \frac{3}{2} u_1^2 u_2^2 - \frac{1}{3} u_2^3 \right). \]
8.2 Boosted Matrix Factorizations

Rather than simply computing at the critical locus of the obstruction polynomials to obtain a deformed matrix factorization of the original Landau-Ginzburg potential $W(x_k)$, there are other ways to get rid of the obstruction term $\sum_j f_j(u_i)\phi_j$. Recalling that the only properties of $u_i$ used in the computation were that they commute among each other and with the $x$-variables, one can imagine them to be elements of a more abstract ring with these properties, e.g. matrices.

The obstructions are at least quadratic in the $u_i$, so substituting suitable nilpotent matrices (of some size $t$) for the $u_i$ will make the obstructions disappear. If one replaces, at the same time, each $x_k$ by $x_k \cdot 1_r$, one arrives at a new matrix factorization of $W(x_k)$, with size “boosted” up by a factor $t$ from the starting $Q$. Typically, the new boosted factorization will include some constant entries, which can be eliminated (as trivial summands), and one may end up with a new matrix factorization rank between the original one and $t$ times that.

Let us demonstrate the procedure in a very simple minimal example first. We start with

$$Q(x) = \begin{pmatrix} 0 & x \\ x^4 & 0 \end{pmatrix}$$

which has one fermion in its cohomology. The deformed matrix factorization is (see 6.3.3)

$$Q_{\text{def}}(x,u) = \begin{pmatrix} 0 & x+u \\ x^4 - x^3u + x^2u^2 - xu^3 + u^4 & 0 \end{pmatrix}$$

and squares to $Q_{\text{def}}(x,u)^2 = x^5 + u^5$. Treating $u$ as a number, the vanishing locus of the obstruction is trivial, and deforming does not yield a new matrix factorization of $W(x) = x^5$. On the other hand, substituting $u$ by the matrix

$$N_u = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$
one has that the “boosted” $4 \times 4$ matrix

$$Q_b(x) := Q_{\text{def}}(x \cdot 1_2, N_u)$$

satisfies $Q_b^2(x) = W(x)$, and one can check that there it is equivalent (via elementary row and column transformations) to

$$Q_b \simeq \begin{pmatrix} 0 & W \\ 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & x^2 \\ x^3 & 0 \end{pmatrix} . \quad (8.24)$$

One can delete the trivial summand and obtains a new matrix factorization of $x^5$.

One could have used larger nilpotent matrices $N_u$ instead of the $2 \times 2$ matrix from above, as long as $N_u^5 = 0$ is satisfied. In the present example, however, all these $N_u$ lead to the same result (8.24) for the boosted factorization. Indeed, the original $Q(x)$ and $Q_b(x)$ provide already all fundamental branes for $W(x) = x^5$.

We present another example for this procedure, with the $E_6$-superpotential $W = y^4 + x^3 - z^2$. We choose the matrix factorization

$$Q_5 = \begin{pmatrix} 0 & 0 & -y^2 - z & x \\ 0 & 0 & x^2 & y^2 - z \\ -y^2 + z & x & 0 & 0 \\ x^2 & y^2 + z & 0 & 0 \end{pmatrix} \quad (8.25)$$

from [15].

Applying the program gives us the deformed matrix factorization

$$Q_{\text{def}}(x, y, z; u_1, u_2) = Q + \begin{pmatrix} 0 & 0 & p_1 & yu_1 + u_2 \\ 0 & 0 & p_2 & -p_1 \\ p_1 & yu_1 + u_2 & 0 & 0 \\ p_2 & -p_1 & 0 & 0 \end{pmatrix} \quad (8.26)$$
where

\[ p_1 = \frac{1}{2}yu_1^2 + \frac{3}{2}u_1 u_2 + \frac{1}{8}u_1^6 \]  
\[ p_2 = -xyu_1 + y^2 u_1^2 - xu_2 + 2yu_1 u_2 + u_2^2 \]  

(8.27)  
(8.28)

and squaring \( Q_{def} \) gives

\[ Q_{def}^2 = W.1 + \begin{pmatrix} p_3 & 0 & 0 & 0 \\ 0 & p_3 & 0 & 0 \\ 0 & 0 & p_3 & 0 \\ 0 & 0 & 0 & p_3 \end{pmatrix} \]  
(8.29)  

(8.30)

\[ p_3 = 3yu_1 u_2^2 + \frac{3}{2} y u_1^5 u_2 + \frac{1}{8} y u_1^9 + u_2^3 + \frac{9}{4} u_1^4 u_2 + \frac{3}{8} u_1^8 u_2 + \frac{1}{64} u_1^{12} \]  
(8.31)

Using Maple, one sees that there are 3 branches of solutions to \( p_3 = 0 \), none of which are at all pleasant. In all 3 cases \( u_1 \) is a free parameter and \( u_2 \) is a function of both \( y \) and \( u_1 \) (setting deformation parameters to be functions of the original variables \( x_k \) is an area covered in section 8.3). They are

\[ u_2 = \left( \frac{A}{8} - \frac{8B}{A^{1/3}} - \frac{3u_1^3}{4} - y \right) u_1 \]  
(8.32)  

\[ u_2 = \left( -\frac{A^{1/3}}{16} - \frac{4B}{A^{1/3}} - \frac{3u_1^3}{4} - y \pm \frac{i}{2} \sqrt{3} \left( \frac{A^{1/3}}{8} + \frac{8B}{A^{1/3}} \right) \right) u_1 \]  
(8.33)

where

\[ A = -148u_1^9 - 512yu_1^6 - 768y^2 u_1^3 - 512y^3 \]  
\[ + 4\sqrt{-3u_1^{18} + 64u_1^{15} y - 320u_1^{12} y^2 - 768u_1^9 y^3 - 1024u_1^6 y^4} \]  
\[ B = -\frac{7}{16} u_1^6 - y u_1^3 - y^2 \]  
(8.34)  
(8.35)  
(8.36)

However, taking \( u_1 \) and \( u_2 \) to be nilpotent matrices as above we see that there are
much simpler ways of ensuring \( p_3 = 0 \). Defining

\[
N_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad N_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}
\]

and using the “conedims” function we see that we can generate at least part of the full set of MFs for the \( E_6 \) polynomial starting from one MF, deforming it, and then making nilpotent substitutions for the deformation parameters:

\[
\begin{align*}
Q_{5, \text{def}}(0, N_2) &\simeq Q_6 \\
Q_{5, \text{def}}(N_2, 0) &\simeq Q_3 \\
Q_{5, \text{def}}(N_2, N_2) &\simeq Q_6 \\
Q_{5, \text{def}}(N_3, 0) &\simeq Q_2 \\
Q_{5, \text{def}}(N_3, N_3) &\simeq Q^{\text{trivial}}
\end{align*}
\]

where it is understood that \( x_i \to x_i \otimes 1 \) when substituting matrices for the deformation parameters. The numbering of the factorizations again comes from [15].

### 8.3 Exploiting Relations Between Different \( W_{LG} \)

There is one remarkable feature shared by the deformed matrix factorizations presented so far (and in fact every example computed by the program): they satisfy

\[
(Q_{\text{def}}(x_k, u_i))^2 = (W(x_k) + p(x_k, u_i)) \cdot 1
\]

for some polynomial \( p(x_k, u_i) \) depending on the original variables \( x_k \) and the deformation parameters \( u_i \). In other words, the \( Q_{\text{def}} \) encountered so far are, for arbitrary \( u \), themselves matrix factorizations of some polynomial \( W(x_k) + p(x_k, u_i) \). In some cases all chosen representatives of the bosonic cohomology are already proportional to the unit matrix, so \( p(x_k, u_i) = \sum_j f_j(u_i) \phi_j \) must of course share this property. In other examples, however, the bosonic representatives have many off-diagonal entries
and the property (8.43) is due to some “miraculous” cancellations. It was suggested to us (by R.O. Buchweitz in a private conversation with A. Recknagel) that it may well be possible to identify conditions on the original factorization $Q$ which guarantee the behaviour in (8.43), but no precise theorem exists to our knowledge.

Leaving such mathematical issues aside, there are still two lessons that can be drawn from (8.43). First, in the examples studied one sees that $p(x_k, u_i)$ contains only $x_k$-powers that occur in the bulk chiral ring of the LG-potential $W(x_k)$, meaning that $p(x_k, u_i)$ can actually be regarded as a bulk deformation of the original bulk potential. For deformations with these properties, one can therefore lift any obstructions against boundary deformations by simultaneously turning on the bulk deformation $p(x_k, u_i)$; the boundary deformation parameters $u_i$ determine the appropriate bulk deformation, via the $u_i$-polynomials showing up in $p(x_k, u_i)$.

The other way to look at deformations satisfying (8.43) is to simply forget that, physically, the $u_i$ had a very different meaning than the $x_k$. At the level of a matrix factorization (an equation between matrices with polynomial entries), there is no fundamental difference. Then equation (8.43) simply gives a matrix factorization for another polynomial $W_{ext}(x_k, u_i) := W(x_k) + p(x_k, u_i)$.

For example, if we take the matrix factorization $E = x^{n-1}, J = x$ of $W = x^n$ as given in subsection 5.3.3, then the deformed matrix factorization will square to

$$\left(Q_{def}(x, u_1)\right)^2 = \left( x^n + (-1)^{n+1} u_1^n \right) \cdot 1$$

(8.44)

and so by choosing $u_1 = y$ (or $u_1 = -y$ depending on $n$) we obtain a matrix factorization of $W' = x^n + y^n$.

Another example: in the $E_7$ example given in section 7.7 we can set set $u_1, u_2 = 0$ and $u_3 = z$ in the deformed factorization which gives a deformed matrix factorization

$$Q' = \begin{pmatrix}
0 & 0 & 2z & x \\
0 & 0 & y^3 + x^2 & 0 \\
0 & x & 0 & 0 \\
y^3 + x^2 & -2z & 0 & 0 \\
\end{pmatrix}$$

(8.45)
of the two-variable $E_7$ superpotential given in (4.20) and $z$ can act as a free parameter. Taking different values of $z$ (including polynomials in $x$ and $y$) leads to differently sized fermionic and bosonic cohomologies, although, as this is just a direct sum of two (deformed) rank 1 branes (4.21), none of the new matrix factorizations are equivalent to any of the fundamental branes listed in section 4.1.3.

Perhaps more interestingly, one learns that matrix factorizations of “extended” polynomials $W_{\text{ext}}(x_k, u_i)$ can sometimes be regarded as special deformations of simpler polynomials $W(x_k) = W_{\text{ext}}(x_k, u_i = 0)$. In fact, one can prove the following result (for notational convenience, we restrict to a single “extra” variable $u$ here, but generalization to several $u_i$ is straightforward):

Let $W_{\text{ext}}(x_k, u)$ be a polynomial such that

$$\partial_u W_{\text{ext}}(x_k, u)|_{u=0} = 0$$

(i.e. all $u$-terms in $W_{\text{ext}}$ are at least quadratic) and let $Q_{\text{ext}}(x_k, u)$ be a matrix factorization of $W_{\text{ext}}(x_k, u)$. Define $W(x_k) := W_{\text{ext}}(x_k, u = 0)$ and $Q(x_k) := Q_{\text{ext}}(x_k, u = 0)$, and set $\psi := \partial_u Q_{\text{ext}}(x_k, u)|_{u=0} = 0$.

Then one has, from differentiating $Q_{\text{ext}}^2 = W_{\text{ext}}$,

$$\{ Q(x_k) , \psi \} = 0$$

i.e. $\psi$ is in the fermionic cohomology of $Q(x_k)$. If $\psi$ is a non-trival cohomology element (this needs to be checked case by case), then $Q_{\text{ext}}(x_k, u)$ is the deformation of $Q(x_k)$ by $\psi$, with all higher order terms in $u$ already included. The obstruction against deforming with $\psi$ is given by $W_{\text{ext}}(x_k, u) - W(x_k)$.

This remark obviously gives a much faster way to obtain (special) deformations than the computation of Massey products. The main difficulties in exploiting the method for concrete examples is, given some matrix factorization $Q(x_k)$ of interest, finding matrix factorizations $Q_{\text{ext}}(x_k, u_i)$ such that $Q_{\text{ext}}(x_k, u_i = 0)$ coincides with the given $Q(x_k)$, and such that the obstruction $W_{\text{ext}}(x_k, u_i) - W(x_k)$ still has an interesting zero locus in the $u_i$. 

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As an example, let us exploit the above to write down special deformations of the matrix factorization that corresponds to the D0-brane on the quintic. In [1, 7], this was identified as the matrix factorization

\[ Q(x_k) = Q_{mm}(x_1) \otimes Q_{mm}(x_2) \otimes Q_{mm}(x_3) \otimes Q_{lin}(x_4, x_5) \]

where the

\[ Q_{mm}(x) := \begin{pmatrix} 0 & x \\ x^4 & 0 \end{pmatrix} \]

are \( W(x) = x^5 \) minimal model matrix factorizations and where

\[ Q_{lin}(x_4, x_5) = \begin{pmatrix} 0 & x_4 - \eta x_5 \\ E & 0 \end{pmatrix} \quad \text{with} \quad E := (x_4^5 + x_5^5)/(x_4 - \eta x_5) \]

is a rank one linear matrix factorization of \( x_4^5 + x_5^5 \) (\( \eta \) is a fifth root of \(-1\)).

At the present state of the (programming and memory storage) art, the Singular program is not able to run the complete Massey product algorithm for this example. On the other hand, one sees that linear and minimal model factorizations are related by

\[ Q_{lin}(x, u)|_{u=0} = Q_{mm}(x) \]

which allows to recognize

\[ Q_{ext}(x_k, u_i) = Q_{lin}(x_1, u_1) \otimes Q_{lin}(x_2, u_2) \otimes Q_{lin}(x_3, u_3) \otimes Q_{lin}(x_4, x_5) \]

as a special three-parameter deformation of the D0-brane on the quintic, the zero-locus of the obstruction being \( u_1^5 + u_2^5 + u_3^5 = 0 \).
Conclusions and Open Questions

As we have seen, the implementation of the Massey product algorithm in Singular leads to faster and more reliable constructions of deformed matrix factorizations. The ease with which this can be done allows for greater analysis and comparison of results for different matrix factorizations. In conjunction with the “Conedims” function of Carqueville, we were able to find several deformation “flows” from one matrix factorization to another of the same superpotential.

One considerable problem we have when looking at deformed matrix factorizations is that it is impossible for us to check all values on the different branches of the zero-locus. So, when we have parameter families of MFs, how can we see whether or not they are all equivalent? It could in principle be that there is an invertible matrix $U(u_i)$ such that $Q_{\text{def}}(u_i) = U(u_i)\tilde{Q}U(u_i)^{-1}$ for some $\tilde{Q}$, at least along some sub-branch of the zero locus where the obstructions vanish. (The “Conedims” program has in some cases shown the equivalence of $Q_{\text{def}}(u_i^*)$ at certain values of $u_i$ to other, already known matrix factorizations, but that program can’t deal with free parameters $u_i$).

It should be rewarding to pursue the ideas sketched in Section 8.3 further. They may hint at deeper relations between the boundary theories associated to different bulk Landau-Ginzburg theories. At the very least, they can be exploited further to write down, in a straightforward manner, boundary deformations of matrix factorizations that are too complicated for the program to deal with.

Similarly, there may be an underlying mathematical construction which “explains” why the nilpotent-boosting recipe introduced in Section 8.2 yields a number of fundamental branes starting from deformations of a single one.

There are also several ways in which the programs in the library file “MF.lib” could be improved - the most important being to reduce its memory requirements.
It would be extremely useful be able to use the program to compute deformations of a matrix factorization of the quintic. However, this is a memory intensive calculation and as mentioned previously the process may not terminate. Luckily, we can still access all information calculated so far even if the process has not finished or has run out of memory. In this way we can recover partial information on the obstructions. It would also be interesting, when a calculation takes a long time or runs into memory problems, to deform the matrix factorization by only a subset of the fermions. This is an option which has already been implemented in the program. Keeping only the lowest order fermions can significantly reduce computation time although the returned obstructions are polynomials in only those deformation parameters related to the retained fermions. For example if we choose to deform only by $\alpha_{\epsilon_j}$, the returned obstructions would be

\[ f'_j = f_j \bigg|_{u_k = 0, \ u_k \neq u_i} \text{ for } j = 1, \ldots, r \]  

(9.1)

**Improvements to be Made to the Library File**

A list of desirable improvements and additions to the program is given:

- Allow for block-diagonal $Q$

- Improve the efficiency. Program slows substantially when handling larger matrices. Perhaps there are overlooked optimised functions already existing in Singular which would help. For particularly large or long examples Singular runs out of memory and gives up.

- I would like to automate the process of computing the effective superpotential. This may be best implemented as a separate function.

- It would be helpful to generalize the process so that it adequately handles strings stretched between different branes

- It would be very useful to be able to somewhat automate the process of computing examples. For example to write a file (or script) with various examples written in Singular code so that on completion of one example the next could
begin. This would mean that several examples could be calculated with very little effort.
Appendices
Homological Algebra

Here we review a few mathematical concepts which will be necessary in the formulation and understanding of matrix factorisations for D-branes [30, 50, 76]. This is followed by an index of definitions (page 148) to make finding a particular definition easier.

Definition A.0.1. A category $\mathcal{A}$ consists of objects $\text{Ob}(\mathcal{A})$ and morphisms $\text{Mor}(A, B)$ for $A, B \in \text{Ob}(\mathcal{A})$. Composition of morphisms is defined by

$$\text{Mor}(B, C) \times \text{Mor}(A, B) \rightarrow \text{Mor}(A, C)$$

The following axioms must also hold:

1. $\text{Mor}(A, B)$ and $\text{Mor}(A', B')$ are disjoint unless $A = A'$ and $B = B'$, then they are equal.

2. For each $A \in \text{Ob}(\mathcal{A})$ there is a left and right identity $id_A \in \text{Mor}(A, A)$ for elements of $\text{Mor}(A, B)$ respectively $\text{Mor}(B, A)$ for all $B \in \text{Ob}(\mathcal{A})$.

3. Composition is associative.

Definition A.0.2. Let $R$ be a unital ring. A left module over $R$ (or left $R$-module) $M$ is an abelian group, usually written additively, together with an operation of $R$ on $M$ (viewing $R$ as a multiplicative monoid - multiplication is associative) such that for all $a, b \in R$ and $x, y \in M$ we have

$$(a + b)x = ax + bx \text{ and } a(x + y) = ax + ay$$

A right $R$-module is defined similarly.
Definition A.0.3. Let $M$ be a module over the ring $R$ and let $S$ be a subset of $M$. The set of all linear combinations of $S$

$$\sum_{x \in S} a_x x, \quad a_x \in R$$

is a submodule $N$ of $M$. It is said that $N$ is the submodule generated by $S$ and $S$ is a set of generators for $N$.

A module is finite over $A$ if it has a finite number of generators.

Definition A.0.4. Let $R$ be a ring and $M$ an $R$-module. $M$ is free if it admits a basis or the zero module. By admitting a basis we mean that there exists a non-empty set, $I$, and for each $i \in I$ let $R_i = R$, viewed as an $R$-module. Then $M$ decomposes as

$$M = \bigoplus_{i \in I} R_i$$

and the basis elements of $M$ are elements $e_i$ with $i^{th}$ component the unit of $R_i$ and all other components are zero.

Definition A.0.5. A module $P$ is projective if there exists a module $M$ such that $P \oplus M$ is free. Another definition is that for a ring, $R$, the functor $M \mapsto \text{Hom}_R(P, M)$ is exact.

Definition A.0.6. Let $R$ be a ring and $Q$ a module then $Q$ is injective if the functor $M \mapsto \text{Hom}_R(M, Q)$ is exact.

Definition A.0.7. Let $E$ be a finite free module over a commutative ring $R$. The dual module $E^\vee$ of $E$ is the module $\text{Hom}(E, R)$ whose elements are functionals. So a functional on $E$ is an $R$-linear map $f : E \to R$. If $x \in E$ and $f \in E^\vee$, we denote $f(x)$ by $\langle x, f \rangle$.

Let $\{x_i\}_{i \in I}$ be a basis of $E$, then for each $i \in I$ there exists a unique functional such that $f_i(x_j) = \delta_{ij}$.

Definition A.0.8. A left ideal $i$ in a ring $R$ is a subset of $R$ which is a subgroup of the additive group of $R$, such that $Ri = i$. A right ideal requires $iR = i$. A two-sided
ideal is a subset which is both a left and a right ideal and is usually just called an ideal.

**Definition A.0.9.** Let \( m \) be an ideal in \( R \), then \( m \) is a maximal ideal if \( m \neq R \) and if there is no ideal \( i \neq R \) containing or equal to \( m \).

**Definition A.0.10.** Let \( R \) be a ring and \((a_1, \ldots, a_m)\) be the set of elements of \( R \) which can be written in the form \( x_1a_1 + \cdots + x_na_n \) with \( x_i \in R \). This set of elements is a left ideal and \( a_1, \ldots, a_m \) are called generators of the left ideal.

**Definition A.0.11.** A ring \( R \) is called a local ring if it is commutative and has a unique maximal ideal, \( m \).

**Definition A.0.12.** Let \( R \) be a ring and \( I \) an ideal with disjoint powers. A sequence \( \{a_n\} \) in \( R \) is Cauchy if given some power \( I^v \) there exists an integer \( N \) such that for all \( m, n \geq N \) we have \( a_m - a_n \in I^v \). \( R \) is said to be complete in the \( I \)-adic topology if every Cauchy sequence converges.

**Definition A.0.13.** A complete local ring is a local ring which is complete in the \( m \)-adic topology.

**Definition A.0.14.** Let \( A \) be a commutative ring. Let \( E, F \) be modules. A bilinear map \( g : E \times E \to F \) is a map such that given \( x \in E \), the map \( y \mapsto g(x, y) \) is \( A \)-linear and vice versa.

**Definition A.0.15.** Let \( R \) be a commutative ring. An \( R \)-algebra is an additive abelian group which has the structure of both a ring and an \( R \)-module. Ring multiplication is \( R \)-bilinear. It can be constructed by equipping an \( R \)-module, \( E \), with a non-zero bilinear map, \( g : E \times E \to E \), such that

\[
x(yz) = (xy)z, \quad \exists 1 \in E, \quad 1x = x1 = x \quad \forall x, y, z \in E.
\]

**Definition A.0.16.** A graded \( A \)-algebra \( K_\ast \) is a family \( \{K_p, p \geq 0\} \) of \( R \)-modules with a bilinear product \( K_p \otimes_R K_q \to K_{p+q} \) and an element \( 1 \in K_0 \) making \( K_0 \) and \( \oplus K - p \) into associative \( R \)-algebras with unit.
$K_\ast$ is graded-commutative if for every $a \in K_p$, $b \in K_q$ we have $a \cdot b = (-1)^{pq} b \cdot a$.

A **differential graded algebra** (DG-algebra) is a graded $R$-algebra $K_\ast$ with a map $d : K_p \to K_{p-1}$ satisfying:

\[
\begin{align*}
\text{(A.1)} & \quad d^2 = 0 \\
\text{(A.2)} & \quad d(a \cdot b) = d(a) \cdot b + (-1)^p a \cdot d(b), \text{ for } a \in K_p
\end{align*}
\]

**Definition A.0.17.** Given $R$-modules $A, B, C$ and $R$-module homomorphisms, $f, g$, the sequence

\[ A \xrightarrow{f} B \xrightarrow{g} C \]

is said to be **exact** (at $B$) if $\ker(g) = \text{im}(f)$ i.e. if $gf = 0$.

**Definition A.0.18.** A **(finite) complex** is a sequence of homomorphisms of modules

\[ 0 \to E^0 \xrightarrow{d^0} E^1 \xrightarrow{d^1} \cdots \xrightarrow{d^n} E^n \to 0 \]

such that $d^{i+1} \circ d^i = 0$ for all $i$.

$Z^n(C) = \ker(d^n)$ is the module of **$n$-cycles** of $C$. $B^n(C) = \text{im}(d^{n+1})$ is the module of **$n$-boundaries** of $C$. The **homology** $H^i$ of the complex is

\[ H^i := Z^n/B^n \]

**Definition A.0.19.** Let $\mathcal{A}$ be a category. A **double complex** in $\mathcal{A}$ is a family $\{C_{p,q}\}$ of objects of $\mathcal{A}$ together with maps

\[ d^n : C_{p,q} \to C_{p-1,q} \text{ and } d^v : C_{p,q} \to C_{p,q-1} \]

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such that \(d^n \circ d^n = d^v \circ d^v = 0\). A lattice is formed

\[
\begin{array}{ccccccc}
\vdots & \vdots & \vdots & \downarrow & \downarrow & \downarrow & \downarrow \\
\quad \downarrow & C_{p-1,q+1} & \xleftarrow{d^n} & C_{p,q+1} & \xleftarrow{d^n} & C_{p+1,q+1} & \quad \downarrow \\
\quad \downarrow & d^v & \downarrow & d^v & \downarrow & d^v & \\
\quad \downarrow & C_{p-1,q} & \xleftarrow{d^n} & C_{p,q} & \xleftarrow{d^n} & C_{p+1,q} & \quad \downarrow \\
\quad \downarrow & d^v & \downarrow & d^v & \downarrow & d^v & \\
\quad \downarrow & C_{p-1,q-1} & \xleftarrow{d^n} & C_{p,q-1} & \xleftarrow{d^n} & C_{p+1,q-1} & \quad \downarrow \\
\quad \downarrow & d^v & \downarrow & d^v & \downarrow & d^v & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{array}
\]

which anti-commutes. Each row or column of this lattice forms a complex with maps \(d^n\) respectively \((-1)^p d^v_p : C_{p,q} \to C_{p,q-1}\) (because of anti-commutativity). The total complex is defined to be

\[
\text{Tot}(C)_n = \prod_{p+q=n} C_{p,q}
\]

with map \(d = d^n + d^v : \text{Tot}(C)_n \to \text{Tot}(C)_{n-1}\) such that \(d \circ d = 0\).

**Definition A.0.20.** Let \(R\) be a ring, then a **cochain complex** \(C\) of \(R\)-modules is a family \(\{C^n\}\) of \(R\)-modules, with maps \(d^n : C^n \to C^{n+1}\) such that \(d \circ d = 0\).

\[
\begin{align*}
\cdots & \xrightarrow{d^{n-2}} C_{n-1} \xrightarrow{d^{n-1}} C_n \xrightarrow{d^n} C_{n+1} \xrightarrow{d^{n+1}} \cdots \\
\end{align*}
\]

\(Z^n(C) = \ker(d^n)\) is the module of **n-cocycles** of \(C\). \(B^n(C) = \im(d^{n-1}) \subseteq C^n\) is the module of **n-coboundaries** of \(C\). The \(n\)th **cohomology** module of \(C\) is defined as

\[
H^n(C) = Z^n / B^n
\]

**Definition A.0.21.** Let \(M\) be a module. A **resolution** of \(M\) is an exact sequence

\[
\to E_n \to E_{n-1} \to \cdots \to E_0 \to M \to 0
\]

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Conventional notation defines

\[ E_M : E_n \to E_{n-1} \to \cdots \to E_0 \]
\[ E : E_n \to E_{n-1} \to \cdots \to E_0 \to 0 \]

The resolution is \textbf{free} if \( E_i \) is free for all \( i \geq 0 \).

The resolution is \textbf{projective} if \( E_i \) is projective for all \( i \geq 0 \).

An \textbf{injective} resolution of \( M \) is an exact sequence

\[ 0 \to M \to I^0 \to I^1 \to I^2 \to \cdots \]

such that each \( I^n \) is injective for all \( n \geq 0 \).

The resolution is \textbf{finite} if \( E_i ( \text{ or } E^i) = 0 \) for all but a finite number of indices \( i \).

**Definition A.0.22.** Let \( \mathcal{A}, \mathcal{B} \) be categories. A \textbf{covariant functor} \( F \) of \( \mathcal{A} \) into \( \mathcal{B} \) is a rule which associates an object \( F(A) \in \text{Ob}(\mathcal{B}) \) to each \( A \in \text{Ob}(\mathcal{A}) \) and a morphism \( F(f) : F(A) \to F(B) \) to each morphism \( f : A \to B \) such that

1. For all \( A \in \text{Ob}(\mathcal{A}) \), \( F(\text{id}_A) = \text{id}_{F(A)} \).
2. If \( f : A \to B \) and \( g : B \to C \) are morphisms in \( \mathcal{A} \),
   \[ F(g \circ f) = F(g) \circ F(f) \].

**Definition A.0.23.** A covariant additive functor \( F \) is called \textbf{left exact} if it transforms an exact sequence into an exact sequence.

**Definition A.0.24.** Let \( I_M \) be the injective resolution of an object \( M \),

\[ 0 \to M \to I^0 \to I^1 \to I^2 \to \cdots \]

Now let \( I \) be the complex

\[ 0 \to I^0 \to I^1 \to I^2 \to \cdots \]
The **right-derived functor** $R^nF$ is defined by

$$R^nF(M) = H^n(F(I))$$

**Definition A.0.25.** Let $R$ be a ring and $E$ be a fixed $R$-module. The functor $M \mapsto \text{Hom}(E,M)$ is exact. So given an exact sequence $0 \rightarrow M' \rightarrow M \rightarrow M''$, the sequence $0 \rightarrow \text{Hom}(E,M') \rightarrow \text{Hom}(E,M) \rightarrow \text{Hom}(E,M'')$ is exact. Its right-derived functor is denoted by $\text{Ext}^n(E,M)$.

**Definition A.0.26.** A left $R$-module, $E$, is called **flat** if the functor $\otimes_R E$ is exact. Projective modules are examples of flat modules.

**Definition A.0.27.** Let $g$ be a map $g : \tilde{X} \rightarrow X$. A **lifting** of a map $f : Y \rightarrow X$ to $\tilde{X}$ is a map $\tilde{f} : Y \rightarrow \tilde{X}$ such that $g\tilde{f} = f$.

![Diagram](image)

**Definition A.0.28.** Obstruction theory is a procedure for defining a sequence of cohomology classes that are obstructions to finding a solution to a lifting problem. A lifting of a map $f : Y \rightarrow X$ to $\tilde{X}$ exists if and only if the **obstruction class** $\omega_n \in H^{n+1}$ is zero.

**Definition A.0.29.** A commutative ring $A$ is said to be **Noetherian** if every ideal is finitely generated.

**Definition A.0.30.** Let $R$ be a ring and $E$ an $R$-module. We say that $E$ is **Artinian** if it satisfies the descending chain condition on submodules, that is a sequence

$$E_1 \supset E_2 \supset E_3 \cdots$$

must stabilize: there exists an integer $N$ such that if $n \geq N$ then $E_n = E_{n+1}$.

**Definition A.0.31.** [69] Let $\Lambda$ be a complete Noetherian local ring with residue field $k$ and $C$ a category of Artin local $\Lambda$-algebras with residue field $k$. A morphism $F \rightarrow G$
of functors is **smooth** if for any surjection $B \to A$ in $C$, the morphism

$$F(B) \to F(A) \times_{G(A)} G(B)$$  \hspace{1cm} (A.3)$$

is surjective.
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Additional Calculations

B.1 Equivalence of Ext\(^{\bullet}(Q, Q)\) and H\(^{\bullet}(Q)\) for Matrix Factorizations

Here we illustrate the equivalence between H\(^{\bullet}(Q)\) and Ext\(^{\bullet}(\text{coker}(E), \text{coker}(E))\) for matrix factorizations. We will need the following definition (which is equivalent to the one given in appendix A)

**Definition B.1.1.** Given modules $M, N$ over a ring $S$ and their projective resolutions, Ext\(^{i}(M, N)\) is defined to be the $i^{th}$ cohomology of the complex

$$0 \rightarrow \text{Hom}_S(M_0, N) \rightarrow \text{Hom}_S(M_1, N) \rightarrow \cdots$$  \hspace{1cm} (B.1)

For $i = 0$

$$\text{Ext}^0(M, N) = \text{Hom}_S(M, N).$$  \hspace{1cm} (B.2)

The situation for matrix factorizations is as follows: Beginning with two $2r \times 2r$ matrix factorizations $(E, J)$ and $(\tilde{E}, \tilde{J})$, we associate modules over $C$, $P = \text{coker}(E)$ resp. $\tilde{P} = \text{coker}(\tilde{E})$, and their free resolutions

$$\cdots \rightarrow C^r \xrightarrow{E} C^r \xrightarrow{J} C^r \xrightarrow{E} P \rightarrow 0$$  \hspace{1cm} (B.3)

$$\cdots \rightarrow \tilde{C}^r \xrightarrow{\tilde{E}} C^r \xrightarrow{J} \tilde{C}^r \xrightarrow{\tilde{E}} \tilde{P} \rightarrow 0$$  \hspace{1cm} (B.4)

Recall that matrix factorizations

$$Q = \begin{pmatrix} 0 & E \\ J & 0 \end{pmatrix}$$  \hspace{1cm} (B.5)
act on boundary fields

\[ \Phi = \begin{pmatrix} f_{00} & f_{01} \\ f_{10} & f_{11} \end{pmatrix}, \quad (B.6) \]

where \( f_{ij} \in \text{Hom}_C(C^r, C^r) \), as

\[ D_{Q, \tilde{Q}} = Q \Phi \pm \Phi \tilde{Q}. \quad (B.7) \]

It can be seen that

\[ \ker^{even}(D_{Q, \tilde{Q}}) = \{ f_{00}, f_{11} | f_{00} \tilde{E} = Ef_{11} \} \quad (B.8) \]

and

\[ \text{im}^{even}(D_{Q, \tilde{Q}}) \simeq \text{Hom}_C(C^r, C^r) \circ \tilde{J} + E \circ \text{Hom}_C(C^r, C^r). \quad (B.9) \]

By passing to maps in \( \text{Hom}_C(C^r, P) \) we simplify the computation of the cohomology which can now be realized as the quotient

\[ \{ f_{00} | f_{00} \tilde{E} = 0 \} / \text{Hom}_C(C^r, P) \circ \tilde{J} \quad (B.10) \]

which is obviously equal to

\[ H^{2i}(\text{Hom}_C(C^r, C^r)). \quad (B.11) \]

We find a similar result for the odd part of the cohomology of \( D_{Q, \tilde{Q}} \) so that we obtain our result

\[ H^{even}(Q) = \text{Ext}^{2i}_C(P, \tilde{P}), \quad H^{odd}(Q) = \text{Ext}^{2i-1}_C(P, \tilde{P}) \quad (B.12) \]

for \( i > 0 \).
B.2 Equivalence of Brane and Shifted Brane Spectra in A-type Minimal Models

We take a matrix factorization with $E = x^{d-n}$ and $J = x^n$:

$$Q = \begin{pmatrix} 0 & x^{d-n} \\ x^n & 0 \end{pmatrix}.$$  

acting on the graded space $\mathcal{P} = \mathcal{P}_0 \oplus \mathcal{P}_1$.

To find the shifted brane of $(E, J)$ we exchange $E$ and $J$ and reverse the grading on the vector space i.e. $(E, J)[1] = (-J, -E)$. If the shifted brane of $Q$ is called $\hat{Q}$, then

$$\hat{Q} = \begin{pmatrix} 0 & -x^n \\ -x^{d-n} & 0 \end{pmatrix}.$$  

Simple calculations give

$$H^{0,1}(\mathcal{D}_Q) = \left\{ \begin{pmatrix} x^i & 0 \\ 0 & x^i \end{pmatrix}, \begin{pmatrix} 0 & -x^{i+d-2n} \\ x^i & 0 \end{pmatrix} \mid i = 0, 1, \ldots, n-1 \right\}$$

$$H^{0,1}(\mathcal{D}_{\hat{Q}}) = \left\{ \begin{pmatrix} x^i & 0 \\ 0 & x^i \end{pmatrix}, \begin{pmatrix} 0 & x^i \\ -x^{i+d-2n} & 0 \end{pmatrix} \mid i = 0, 1, \ldots, n-1 \right\}$$

With the reversed grading for the anti-brane, $\mathcal{P}' = \mathcal{P}_1 \oplus \mathcal{P}_0$, it can be seen that in both cases the bosonic spectrum is the same. For the fermions, in both cases the $x^{i+d-2n}$ term takes $\mathcal{P}_1$ to $\mathcal{P}_0$ and the $x^i$ term takes $\mathcal{P}_0$ to $\mathcal{P}_1$. In other words, the string spectrum for each brane is the same and we can view the two factorizations as being equivalent. This justifies taking $n$ to be in the restricted range $0 < n \leq [d/2]$. 

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Quintic Example

In this example we look at a matrix factorization for the superpotential

\[ W = x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5 \]  
\[ (C.1) \]

The matrix factorization is built via a tensor product (4.40) of two rank 1 linear matrix factorizations (4.41) and one A-type minimal model brane (4.8).

\[
E = \begin{pmatrix} x_1 - nx_2 & x_3 + n^2 x_4 & x_5^2 & 0 \\
 p_2 & -p_1 & 0 & x_5^2 \\
x_5^3 & 0 & -p_1 & -x_3 - n^2 x_4 \\
x_5^3 & -p_2 & x_1 - nx_2 & 
\end{pmatrix} 
\]

\[
J = \begin{pmatrix} p_1 & x_3 + n^2 x_4 & x_5^2 & 0 \\
p_2 & -x_1 + nx_2 & 0 & x_5^2 \\
x_5^3 & 0 & -x_1 + nx_2 & -x_3 - n^2 x_4 \\
0 & x_5^3 & -p_2 & p_1 & 
\end{pmatrix} 
\]

where

\[ p_1 = x_1^4 + nx_1^3 x_2 + n^2 x_1^2 x_2^2 + n^3 x_1 x_2^3 + n^4 x_2^4 \]  
\[ (C.2) \]

\[ p_2 = x_3^4 - n^2 x_3^3 x_4 + n^4 x_3^2 x_4^2 + nx_3 x_4^3 - n^3 x_4^4 \]  
\[ (C.3) \]

Singular input:

```
ring r=(0,n),(x1,x2,x3,x4,x5),(c,dp);minpoly=n^4-n^3+n^2-n+1;
option(noloadLib,noredefine);
LIB"MF.lib";
poly jp=x1-n*x2;poly jo=x3+n^2*x4;
```


```plaintext
poly ep=(x1^5+x2^5)/jp;poly eo=(x3^5+x4^5)/jo;
matrix qp[2][2]=0,jp,ep,0;matrix qo[2][2]=0,jo,eo,0;
list qL1=mftensor(jp,ep,jo,eo);
list qL2=mftensor(qL1[1],qL1[2],x5^2,x5^3);
matrix q=blockmat(0,qL2[1],qL2[2],0);
list l=MFcohom_def(q,1);
The bosonic and fermionic cohomologies are both 32 dimensional.
At order 2 the non-zero obstructions are:

\[ f_2^1 = -2u_4u_{32} - 2u_9u_{31} - 2u_8u_{30} - 2u_{14}u_{28} - 2u_{13}u_{26} - 2u_{15}u_{25} \]
\[ - 2u_{20}u_{23} - 2u_{19}u_{21} \]
\[ f_2^2 = -2u_8u_{32} - 2u_{14}u_{31} - 2u_{13}u_{30} - 2u_{23}u_{28} - 2u_{19}u_{26} - 2u_{20}u_{25} \]
\[ f_3^2 = -2u_9u_{32} - 2u_{15}u_{31} - 2u_{14}u_{30} - 2u_{20}u_{28} - 2u_{23}u_{26} - 2u_{21}u_{25} \]
\[ f_5^2 = -2u_{13}u_{32} - 2u_{23}u_{31} - 2u_{19}u_{30} - 2u_{25}u_{28} \]
\[ f_6^2 = -2u_{14}u_{32} - 2u_{20}u_{31} - 2u_{23}u_{30} - u_8^2 - 2u_{25}u_{26} \]
\[ f_7^2 = -2u_{15}u_{32} - 2u_{21}u_{31} - 2u_{20}u_{30} - 2u_{26}u_{28} \]
\[ f_{10}^2 = -2u_{19}u_{32} - 2u_{25}u_{31} \]
\[ f_{11}^2 = -2u_{20}u_{32} - 2u_{26}u_{31} - 2u_{28}u_{30} \]
\[ f_{12}^2 = -2u_{21}u_{32} - 2u_{26}u_{30} \]
\[ f_{16}^2 = -2u_{23}u_{32} - 2u_{28}u_{31} - 2u_{25}u_{30} \]
\[ f_{17}^2 = -2u_{25}u_{32} - u_{31}^2 \]
\[ f_{18}^2 = -2u_{26}u_{32} - u_{30}^2 \]
\[ f_{22}^2 = -2u_{28}u_{32} - 2u_{30}u_{31} \]
\[ f_{24}^2 = -2u_{31}u_{32} \]
\[ f_{27}^2 = -2u_{30}u_{32} \]
\[ f_{29}^2 = -u_{32}^2 \]
```

It took several days for the computation to reach this stage and after much longer

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it still runs with no memory problems. We should have enough information now to compute the degrees of the deformation parameters and so have an idea of what the final deformations might be. The degrees would also tell us whether or not these polynomials will change at higher orders.
References


[68] F. Rosay. Sur quelques points de la théorie des déformations dérivée.


