Fixed points in multi-field Landau–Ginzburg models

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The Parisi approach to critical behaviour, based on the Callan–Symanzik renormalisation group equation, allows one to work directly in the dimension of interest. This method is applied to two-dimensional $N=2$ Landau–Ginzburg models with general superpotentials. The relationship of singularity theory to critical behaviour is discussed, and some critical exponents (anomalous dimensions) computed, these being in agreement with those found earlier (for the case of one field) by the $\epsilon$-expansion technique. The issues of existence and stability of fixed points are addressed and the renormalisation group flow between two neighbouring single field models computed.

In a previous paper [1] we showed that the Landau–Ginzburg theory with the action (hamiltonian)

$$S = \int d^2x d^4\theta \phi^2 + \left[ \int d^2x d^2\theta \frac{G}{n!} \phi^n + \text{c.c.} \right]$$

(1)

possesses an infra-red stable fixed point at which it corresponds to the $n$th member of the minimal series of $N=2$ superconformal models. This was done using the Wilson–Fisher $\epsilon$-expansion [2] and confirmed a conjecture made in ref. [3]. (A subsequent derivation from an alternative point of view was given in ref. [4]). The $\epsilon$-expansion is not suitable, however, for application to general (non-homogeneous) $N=2$ potentials because terms with different powers of the fields will be associated with different critical dimensions.

Fortunately there is another technique, due to Parisi [5], which allows one to work directly in the dimension of interest, two in our case. The basic observation is that for systems in less than four dimensions the bare coupling, $u_0$, has dimension $m^{4-d}$ so that the dimensionless parameter which is used in the perturbation expansion, $u_0/m$, has the property that it diverges as the (renormalised) mass, $m$, goes to zero at the critical point. This problem can, however, be circumvented by a careful choice of the renormalised coupling defined through a normalisation condition. This (dimensionless) coupling can be shown to be well-defined even in the critical region where $m\to 0$. Anomalous dimensions of operators, which are related to the critical exponents, can be calculated with the mass non-zero by using the Callan–Symanzik version of the renormalisation group equation. In addition to recovering the homogeneous renormalisation group equation in the asymptotic region $m\to 0$ (i.e. $p/m\to \infty$), the C–S equation can also be used to relate the correlation functions to the correlation functions with one zero-momentum insertion of the operator $\phi^2$. In this way it is possible to calculate the correlation functions of interest at the critical point.

In this paper we use this approach in the context of two-dimensional $N=2$ models. We begin by using the Callan–Symanzik equation to derive the non-renormalisation theorem. This is an extension of the work of ref. [6] where this method was applied to the four-dimensional Wess–Zumino model. We consider first a model with a single scalar chiral superfield $X$ and action

$$S_0 = \int d^2x d^2\theta X_0 X_0$$

$$+ \left[ \int d^2x d^2\theta \sum_{p=2}^{n} \frac{g_0^{(p)}}{p!} X_0^p + \text{c.c.} \right],$$

(2)

where $X_0$ is the bare field, $g_0^{(2)} = m_0$ is the bare mass and the $g_0^{(p)}$, $p=3,\ldots,n$, are the bare couplings. The equation of motion for this model, integrated over
the chiral subspace, can be written in the form
\[ \int d^2x d^4\theta \left( \frac{\delta S_0}{\delta \bar{X}_0(x, \theta)} - \bar{D}^2 \bar{X}_0 - m_0 X_0 \right) \]
\[ = \sum_{p=3}^{n} g^{(p)} \frac{\partial}{\partial g^{(p-1)}} S_0. \]  
(3)

Using this formula and standard path-integral techniques it is easy to show that the same equation holds for the effective bare action, \( \Gamma_0 \), the generating functional of the one-particle irreducible vertices of the theory. We thus have
\[ \int d^2x d^4\theta \left( \frac{\delta \Gamma_0}{\delta \bar{X}_0(x, \theta)} - \bar{D}^2 \bar{X}_0 - m_0 X_0 \right) \]
\[ = \sum_{p=3}^{n} g^{(p)} \frac{\partial}{\partial g^{(p-1)}} \Gamma_0. \]  
(4)

Differentiating (4) with respect to \( X_0 \), evaluating at \( X_0=0 \) and using the fact that the supersymmetry Ward Identities imply that \( \Gamma^{(1,0)}_0(0; m_0, g_0)=0 \) we find
\[ \Gamma^{(2,0)}_0(0; m_0, g_0)=m_0 A^{(2,0)}, \]  
(5)

where
\[ (2\pi)^2 \delta \left( \sum_{p} \right) \Gamma^{(n,m)}_0(p, \theta; m_0, g_0) \]
\[ = \frac{\delta}{\delta X_0(p_1, \theta)} \cdots \frac{\delta}{\delta X_0(p_n, \theta)} \frac{\delta}{\delta \bar{X}_0(p_{n+1}, \theta)} \cdots \]
\[ \times \frac{\delta}{\delta \bar{X}_0(p_{n+m}, \theta)} \Gamma_0 \]  
(6)

and
\[ A^{(n,0)}=\bar{D}^2 \delta^{(4)}(\theta_1 - \theta_2) \bar{D}^2 \delta^{(4)}(\theta_1 - \theta_3) \cdots \]
\[ \times \bar{D}^2 \delta^{(4)}(\theta_1 - \theta_p). \]  
(7)

If we differentiate (4) with respect to \( X_0(p-1) \) times we find
\[ \Gamma^{(p,0)}_0(0; m_0, g_0)=g^{(p)} A^{(p,0)}, \quad \text{if } 3 \leq p \leq n, \]
\[ =0, \quad \text{if } p>n. \]  
(8)

To derive the Callan–Symanzik equation it is necessary to introduce also the effective action for 1PI graphs with one insertion of the operator \( \frac{1}{2} X^2 \) evaluated at zero momentum (or, equivalently, integrated over the chiral superspace). In the bare theory this operator can be defined by
\[ \Gamma^{(n,m)}_0(p, \theta; m_0, g_0) := \frac{\partial}{\partial m_0} \Gamma^{(n,m)}_0(p, \theta; m_0, g_0). \]  
(9)

From (7) and (9) it follows that
\[ \Gamma^{(p,0)}_0(0; m_0, g_0) = 1, \quad \text{if } p=2; \]
\[ =0, \quad \text{if } p \geq 3. \]  
(10)

The transition to the renormalised theory is through multiplicative renormalisation constants and this allows us to conclude that \( \Gamma^{(p,0)}, p>n, \) and \( \Gamma^{(p,0)}, p \geq 3 \) also vanish at zero momentum. The Callan–Symanzik equation is
\[ m \frac{\partial}{\partial m} + \sum_{p=3}^{n} \beta^{(p)} \frac{\partial}{\partial g^{(p)}} - (r+s)\gamma \Gamma^{(r,s)}(p, \theta; m, g) \]
\[ = \alpha m \Gamma^{(r,s)}(p, \theta; m, g), \]  
(11)
where \( \gamma \) is the anomalous dimension of the field \( X \) and \( \beta^{(p)} = \frac{\partial}{\partial m} g^{(p)}, \) \( 3 \leq p \leq n, \) are the beta functions. The normalisation conditions on the chiral renormalised vertex functions are taken to be
\[ \Gamma^{(2,0)}(0; m, g) = m A^{(2,0)}, \]  
\[ \Gamma^{(p,0)}(0; m, g) = m g^{(p)} A^{(p,0)}, \quad 3 \leq p \leq n, \]  
\[ \Gamma^{(2,0)}(0; m, g) = 1. \]  
(12)

Combining the fact that \( \Gamma^{(p,0)}, p>n, \) and \( \Gamma^{(p,0)}, p \geq 2, \) vanish at zero momentum with the Callan–Symanzik equation and the normalisation conditions we find that
\[ \alpha = 1 - 2\gamma \]  
(13)
and
\[ \beta^{(p)} = g^{(p)} (-1 + py). \]  
(14)

In addition, the Callan–Symanzik equation for the vertex function \( \Gamma^{(1,1)} \) can be used to calculate \( \gamma \) directly without one having to calculate the wave-function renormalisation constant itself. From the equations for the beta-functions it is apparent that there will be a fixed point when all the couplings except \( g^{(n)} \) are zero and at this fixed point the anomalous dimension of the field \( X \) is \( 1/n \) in agreement with the result for the \( X^n \) potential found in ref. [1]. A more complete account of the present work can be found in ref. [7] which includes an explicit one-loop cal-
calculation of $\gamma$ from the Callan-Symanzik equation. Thus the Parisi method confirms the result found using the $\epsilon$-expansion. We refer the reader to ref. [1] for further details of the identification of this model with the minimal series model with central charge $(1-2n)$.

We now consider the case with many fields $X_i$, $i=1, \ldots, n$. The Landau-Ginzburg action is

$$S = \int d^2x \, d^2\theta \, X_i \bar{X}_i + \left[ \int d^2x \, d^2\theta \, m(Q+V) + \text{c.c.} \right]$$

+ counterterms, \hspace{1cm} (15)

where the mass terms $Q$ can be chosen to be

$$Q = \frac{1}{2} X_i X_j$$

and the interaction superpotential $V$ has the form

$$V = \sum_{p=3}^{p=N} \sum_{i_1, \ldots, i_p} g_{i_1, \ldots, i_p} X_{i_1} \ldots X_{i_p},$$

i.e. it is a general polynomial in the fields of degree $N$. Following the same procedure as before we find

$$\beta_{i_1, \ldots, i_p} = -g_{i_1, \ldots, i_p} + \sum_{k=1}^{n} \sum_{q} \gamma_{i_k} g_{i_1, \ldots, i_{k-1}, \ldots, i_p},$$

where the matrix of anomalous dimensions $\gamma_{ij}$ need not necessarily be diagonal. At a fixed point we therefore have

$$g_{i_1, \ldots, i_p} - \sum_{k=1}^{n} \sum_{q} \gamma_{i_k} g_{i_1, \ldots, i_{k-1}, \ldots, i_p} = 0.$$ \hspace{1cm} (19)

Applying these results to the superpotential

$$V = g_1 X^2 Y + g_2 Y^{k-1},$$

we find that the matrix of anomalous dimensions is diagonal and that

$$\gamma_X = \frac{1}{2} - \frac{1}{k-1}, \quad \gamma_Y = \frac{1}{k-1}.$$ \hspace{1cm} (21)

The fields $X$ and $Y$ can be identified with two of the chiral primary fields of the $D_k$ series of $N=2$ minimal models.

For a general potential equation (18) can be rewritten in the form

$$\sum_{i} X_i \gamma_{i} \frac{\partial}{\partial X_i} V + \sum \beta \frac{\partial}{\partial g} V = 0.$$ \hspace{1cm} (22)

Since $\gamma_{ij}$ is a symmetric matrix we can diagonalise it by an orthogonal transformation of the fields after which (22) becomes simply

$$V = \sum_i \gamma_i X_i \frac{\partial}{\partial X_i} V,$$ \hspace{1cm} (23)

at the fixed point where $\beta=0$ for all of the couplings. Eq. (23) implies that $V$ is a quasi-homogeneous polynomial which obeys the scaling law

$$V(\lambda^{n_1} X_1, \lambda^{n_2} X_2, \ldots) = \lambda^2 V(X_i).$$ \hspace{1cm} (24)

This result was conjectured by Vafa and Warner [8]. (Quasi-homogeneous polynomials are common in statistical mechanics, see e.g. ref. [9]).

In order to ensure that a system is at its critical point one must impose normalisation conditions. No matter what system is being discussed the effective (super) potential $W_{\text{eff}}$ must satisfy

$$\frac{\partial^2}{\partial X_i \partial X_j} W_{\text{eff}} \bigg|_{X_i=0} = 0,$$ \hspace{1cm} (25)

corresponding to the fact that the correlation length goes to infinity at the critical point. The equation of motion for constant fields is

$$\frac{\partial}{\partial X_i} W_{\text{eff}} = 0.$$ \hspace{1cm} (26)

Thus in the language of singularity theory $W_{\text{eff}}$ is a function which has a degenerate critical singularity. This connection between critical phenomena and singularity theory is well-known and there is a large literature on it [10]. More recently it has also been discussed in the context of $N=2$ conformal models [8].

The basic notions in singularity theory have correspondences in critical phenomena. For example, given a degenerate superpotential $W$ the dimension of the quotient ring $\mathbb{C}[X_i]/\{\partial W\}$ is called the multiplicity of $W$. The basis functions of this ring are often identified with the relevant operators of the theory. Another concept is that of modality, which is a measure of the complexity of a singularity. A singularity of modality 0 corresponds to a fixed point in critical phenomena while a singularity of modality 1 corresponds to a line of fixed points, i.e. the existence of a marginal operator in the theory. The classification of the simplest types of singularities has been carried out.
by Arnold and his co-workers [11,12]; the singularities of modalities 0 and 1 are listed, for example, in ref. [11], p. 73 and pp. 76–78.

It is important to remember that in critical phenomena quantitatively correct answers for the critical exponents are in general only obtained by going beyond the tree-level approximation, i.e. the path integral for the partition function must be evaluated in some computational scheme; in the language of quantum field theory the quantum fluctuations are important. Consequently, one is entitled to ask which degenerate singular (super)potentials (in the action used to define the path-integral) actually give rise to stable fixed points in the quantum theory. In this context the $d=2$, $N=2$ Landau-Ginzburg models are special because of the non-renormalisation theorem. As we have seen equation (23) is a necessary condition for an $N=2$ superpotential $V$ to give rise to a fixed point. This equation relates the anomalous dimensions and the couplings, but since the former are functions of the latter, it can be expressed as a set of algebraic relations between the couplings. Although there are always solutions of these relations it is not guaranteed that a given solution corresponds to an infra-red stable fixed point.

We remark further that not all singularities are related to $(N=2)$ critical behaviour. There are many examples of singularities which are not quasi-homogeneous (starting at modality one) and such functions cannot be used as $N=2$ Landau-Ginzburg potentials in view of eq. (23). The correspondences between singularity theory and critical behaviour outlined above can be explicitly demonstrated in $N=2$ models with only one field rather easily as we shall see below. In particular one can show that infrared stable fixed points exist and that the relevant operators are indeed those of the quotient ring. This is more complicated in general models and will be discussed elsewhere [7]. For $N=1$ supersymmetric and non-supersymmetric models the connection between singularity theory and critical behaviour is much harder to demonstrate.

We now return to the single field model with action (2). There are clearly several fixed points as can be seen from the expression for the $\beta$-functions given in eq. (14). If $g^{(k)} \neq 0$, then all the other couplings must be zero in order to get to the fixed point at which the potential is $X^k$ (corresponding to the $(k-1)$th member of the $A_k$ series of minimal models). One can calculate at one loop to find the fixed point $g^*$, and directly calculate $\gamma(g^*)$ which is found to be $1/k$ in agreement with the all orders result found above. The anomalous dimensions of the other chiral operators in the theory $X^p$, $p=2, \ldots, k-2$ are readily found to be $p/k$ to all orders using the techniques discussed earlier and this set of operators can indeed be identified with the quotient ring for the potential $X^k$.

To examine the stability of this model at the fixed point $g^*=\delta_{p,n}g^{(n)}(g^*)$ we compute the matrix of derivatives of the $\beta$-functions with respect to the couplings. To all orders in perturbation theory we find

$$\frac{\partial g^{(p)}}{\partial g^{(s)}(g^*)} = g^{(p)}(g^{(s)}(g^*)^{(p)} \frac{\partial g^{(s)}}{\partial g^{(s)}(g^*)}, \quad p \neq 2,$$

$$\frac{\partial \beta^{(s)}}{\partial g^{(s)}(g^*)} = 1 - \gamma^* = \left(1 - \frac{s}{n}\right), \quad s \neq n,$$

$$\frac{\partial \beta^{(n)}}{\partial g^{(n)}(g^*)} = \delta_{p,n}^{(n)}(g^*) \frac{\partial g^{(n)}}{\partial g^{(n)}}(g^*).$$

The eigenvalues of the matrix $a_{rs} = \partial \beta^{(r)}/\partial g^{(s)}|_{g^*}$ control the stability of the fixed point behaviour. Examining the form of $a_{rs}$ we see that its eigenvalues are $a_{rs}, r=1, \ldots, n$ to all orders in perturbation theory. We observe that $a_{rs} < 0$ for $s=1, \ldots, (n-2)$ corresponding to the relevant operators $X^p$ discussed above, while $a_{rs} > 0$ for $s=n+1, n+2, \ldots$ corresponding to the irrelevant operators. While $\gamma$ is on general grounds a positive quantity it is not immediately clear that $a_{nn}$ is positive. The anomalous dimension is computed from $\beta^{(1,1)}$ at zero momentum and is found to be of the form

$$\gamma = \frac{1}{2} \sum_{p=3}^{\infty} (g^{(p)})^2 c_p,$$

(28)

where $c_p$ are positive numbers. At one loop therefore $a_{nn}$ is a diagonal matrix and $a_{nn} = 2 > 0$. Consequently we have seen that the $A_k$ series of singularities correspond to fixed points which are infra-red stable with relevant operators which correspond to the quotient ring of the singularity.

Finally we consider the renormalisation group flow from one fixed point to another. For the case of a single field we can consider the potential

$$V = gX^n + hX^{n-2},$$

(29)
and examine the flow in \( (g, h) \) space. The \( \beta \)-functions are

\[
\beta_g = g(-1 + n \gamma), \quad \beta_h = h[-1 + (n-2) \gamma].
\]

The two non-trivial fixed points are \((g^*, 0)\) with \(g^* = 1/n\) and \((0, h^*)\) with \(h^* = 1/(n-2)\). To all or-
ders in perturbation theory \(a_{n-2,n} < 0\) so that the fixed
point \((g^*, 0)\) is infra-red unstable in the \(h\) direction
while the \(a_{n-2,n} > 0\) so that the fixed point \((0, h^*)\) is
infra-red stable in both directions. Thus flow from
the former to the latter is inevitable. At one loop
\[\gamma = \frac{1}{2} c_2 g^2 + c_{n-2} h^2\]
and one can verify the existence of
the fixed points, \(a_{nr} = 0, r \neq s\) and \(a_{nr} > 0, r = n-2, n\).
Integrating the equations \(d g(t)/dt = \beta_r(g(t))\) we can
find the trajectories for the flows between the fixed
points. This is easily seen to occur at the one-loop
level. The stability of the fixed points for a general
potential and the flows between them will be dis-
cussed elsewhere.

The arguments of this paper and those of ref. [1]
allow the calculation of the anomalous dimensions of
the chiral operators and the identification of the con-
formal theory with the Landau–Ginzburg theory at
the fixed point. These arguments and results will be
independent of the kinetic terms provided that the
latter induce no non-linear renormalisations of the
fields which would compliciate the form of the
Callan–Symanzik equation. On the other hand the
identification of the irrelevant primary fields would
presumably be sensitive to the precise form of the ki-
netic terms. If one could solve this problem it would
establish the precise way in which the Gepner con-
jecture [13] is realised.

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