CM331A Special Relativity and Electromagnetism: Problem Sheet III

0." Please send an email to `andreas.recknagel@kcl.ac.uk` if you haven't done so by now.

1. Read the derivation of the Coulomb field (and of the Coulomb potential) given in the handout (pp. 23 ff). Then use the same method to compute the electric field of a uniformly charged ball of radius R, i.e. for the static charge density ($\rho_0 = \text{const}$)

$$\rho(\vec{x}') = \begin{cases} \rho_0 & \text{for } |\vec{x}'| \leq R \ , \\ 0 & \text{for } |\vec{x}'| > R \ . \end{cases}$$

(Note: To obtain $\vec{E}(\vec{x})$ outside the ball, i.e. for $|\vec{x}| > R$, one doesn't require new calculations – why?)
Then, give the electric field $\vec{E}(\vec{x})$ induced by a charged ball without core, i.e. for

$$\rho(\vec{x}') = \begin{cases} 0 & \text{for } |\vec{x}'| < R_1 \ , \\ \rho_0 & \text{for } R_1 \leq |\vec{x}'| \leq R_2 \ , \\ 0 & \text{for } |\vec{x}'| > R_2 \ . \end{cases}$$

in the region $|\vec{x}| > R_2$ (i.e. outside of the ball). (Note: This follows without real computation from the first part and the superposition principle.)

2. Compute the electric field of an infinite plate (coinciding with the x-y-plane) that carries a constant surface charge density σ. (This means that a piece of the plate with area A carries the charge $A\,\sigma$, and a volume V whose intersection with the plate has area A contains the same amount of charge.)
To do this, first use the symmetry of the situation (translations; field lines) to determine the direction of $\vec{E}$ and to see that $\vec{E}(\vec{x}) = \vec{E}(z)$. Then apply Gauss' law (in the integral form) to compute $|E(z)|$; the surface S_V used in Gauss' law should be chosen in such a way that the symmetry can be exploited and that the surface integral becomes simple (spheres are not useful here).
In addition, use the superposition principle to compute the electric field of two such planes, parallel, at a distance d, and carrying opposite constant surface charges σ resp. $-\sigma$. (This is an infinite capacitor.)

3. The Dirac delta distribution: Any continuous function $g(x)$ can be viewed as a distribution by the following definition of its action on a testfunction $f \in \mathcal{S}$:

$$g[f] := D_g[f] = \int_{-\infty}^{\infty} f(x)\, g(x)\, dx$$

(Recall that testfunctions are smooth and have compact support, i.e. vanish outside of a finite subset of $\mathbb{R}$; this ensures that the integral exists.) In slight abuse of notation (but employed all the time in physics), we have also written $\delta(x)$ for the Dirac delta 'function', which actually is not a function but a distribution defined by

$$\delta[f] := f(0) \quad ;$$

the abuse enables us to write this in the form

$$\delta[f] = \int_{-\infty}^{\infty} f(x)\, \delta(x)\; dx \quad .$$

(Note that also $\int_{-R_1}^{R_2} f(x)\, \delta(x)\; dx = f(0)$ for any $R_1 > 0$ and $R_2 > 0$ since the 'function' $\delta(x)$ vanishes away from $x = 0$.) Use this function notation and substitution of integration variables to prove the formula

$$\delta(a\, x) = \frac{1}{|a|}\, \delta(x) \quad \text{for any real} \;\; a \neq 0$$

given in the lecture.

While δ is not an ordinary function, one can approximate it by a sequence of smooth functions: Let

$$\delta_n(x) := n\, e^{-\pi\, n^2\, x^2} \quad \text{for} \;\; n = 1, 2, \ldots \quad .$$

First show that the δ_n satisfy the same normalisation condition as δ, namely

$$\int_{\mathbb{R}} \delta_n(x)\; dx = 1 \quad \text{for all} \;\; n \; .$$

Sketch the graphs of some δ_n and observe what happens as n grows.
Note that the $\delta_n(x)$ go to zero very fast when $|x|$ grows; the faster, the larger n is. Use this fact and the mean value theorem of integral calculus, to argue that

$$\lim_{n \to \infty} \delta_n[f] = \delta[f]$$

for any testfunction $f \in \mathcal{S}$. (A rigorous proof requires to make trickier estimates, but you should be able to find the idea of the argument.)

This shows that the highly singular delta 'function' can be approximated by a sequence of perfectly nice smooth functions. There are other possible choices for sequences that do the same job, e.g.

$$\tilde{\delta}_n(x) := \frac{1}{\pi}\, \frac{1/n}{x^2 + (1/n)^2} \quad \text{or} \quad \hat{\delta}_n(x) := \frac{1}{\pi}\, \frac{\sin nx}{x} \quad .$$