Spectral Theory of Orthogonal Polynomials

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Szegő’s Theorem was proven by him in 1914 as a statement about Toeplitz Determinants as we discuss below.

In 1920–21, he rephrased it as a variational principle in OPUC. This (two-part) paper essentially invented the general theory of OPUC.

In these papers, Szegő assumed $d\mu$ was purely a.c. The addition of a singular continuous part is a discovery of Verblunsky in 1934–35 but his work was largely ignored and he didn’t get credit until about fifteen years ago when, in a different context, Killip and Simon rediscovered his proof and then his paper.
Szegő’s Theorem as a Variational Principle

\( \Phi_n \) has a variational form. Since \( \Phi_n = \text{Proj of } z^n \) onto the orthogonal complement of \( \{1, \ldots, z^{n-1}\} \),

\[
\| \Phi_n \| = \text{dist of } z^n \text{ to span of } \{1, \ldots, z^{n-1}\} \\
= \min\{\| P \| \mid P \text{ monic }, \deg P = n\} \\
= \min\{\| P \| \mid P(0) = 1, \deg P \leq n\}
\]

since \( P \) monic \( \iff \) \( P^*(0) = 1 \).

This implies \( \| \Phi_{n+1} \| \leq \| \Phi_n \| \) which is obvious from \( \| \Phi_n \| = \rho_0 \rho_1 \ldots \rho_{n-1} \) and \( \rho_j \leq 1 \).
Thus, clearly, \( \lim_{n \to \infty} \| \Phi_n \| \) exists and

\[
\lim_{n \to \infty} \| \Phi_n \| = \inf \{ \| P \| \mid P(0) = 1, P \text{ is a polynomial} \}
\]

**Szegő Theorem for OPUC.** Let

\[
d\mu = f(\theta) \frac{d\theta}{2\pi} + d\mu_s
\]

be an arbitrary probability measure. Then (NOTE THE SQUARE)

\[
\inf \{ \| P \|^2 \mid P(0) = 1, P \text{ is a polynomial} \} = \exp \left( \int \log f(\theta) \frac{d\theta}{2\pi} \right)
\]
Szegő’s Theorem as a Variational Principle

This innocuous-looking theorem will have remarkable consequences as we’ll see, in part because it has multiple equivalent forms.

Because $\int f(\theta) \frac{d\theta}{2\pi} < \infty$, the integral cannot diverge to $+\infty$, but it can to $-\infty$ in which case, we interpret $\exp(\ast \ast \ast)$ as $0$. Indeed, by Jensen’s inequality and the concavity of log, the integral is non-positive and the exponential in $[0,1]$ as it must be given that $\|\Phi_0\| = 1$.

One remarkable aspect of this theorem is that $d\mu_s$ doesn’t enter!

Before turning to the proof, we consider some equivalent forms and consequences.
Szegő’s Theorem as a Sum Rule

As we’ve seen, $\left\| \Phi_n \right\| = \rho_1 \ldots \rho_{n-1}$ so

$$\lim \|\Phi_n\|^2 = \prod_{j=0}^{\infty} (1 - |\alpha_j|^2)$$

**Szegő Theorem (Sum Rule Version).** If $d\mu = f(\theta) \frac{d\theta}{2\pi} + d\mu_s$, then

$$\sum_{j=0}^{\infty} \log(1 - |\alpha_j|^2) = \int \log(f(\theta)) \frac{d\theta}{2\pi}$$

This is a precursor of KdV sum rules. It is clearly equivalent to the variational form.
Szegő’s Theorem as a Sum Rule

**Corollary.**  \( \sum_{j=0}^{\infty} |\alpha_j|^2 < \infty \iff \int \log(f(\theta)) \frac{d\theta}{2\pi} > -\infty. \)

A consequence of this is that \( d\mu_s \) can be more or less arbitrary while one still has \( \sum_{j=0}^{\infty} |\alpha_j|^2 < \infty \); for example, if \( \int d\mu_s = \eta < 1, (1 - \eta) \frac{d\theta}{2\pi} + d\mu_s = d\mu \) has \( \sum_{j=0}^{\infty} |\alpha_j(\mu)|^2 < \infty. \)

This is remarkable because we’ll see in a future lecture that \( \sum_{j=0}^{\infty} |\alpha_j| < \infty \Rightarrow d\mu \) is purely a.c. and \( \varepsilon < |f(\theta)| < \varepsilon^{-1} \) for some \( \varepsilon > 0 \) and all \( \theta. \)

It is also remarkable because it is not easy to construct operators with mixed spectrum and potential decay.
Given \( \{c_n\}_{n=-\infty}^{\infty} \), the corresponding \( N \times N \) Toeplitz matrix \( T_N(c) \) has the form

\[
\begin{pmatrix}
c_0 & c_1 & \ldots & c_{N-1} \\
c_{-1} & c_0 & \ldots & c_N \\
& \ddots & \ddots & \ddots \\
c_{-N+1} & c_{-N+2} & \ldots & c_0
\end{pmatrix}
\]

i.e., \( (T_N(c))_{ij} = c_{j-i} \). If \( \mu \) is a measure, we set \( c_j = \int e^{-ij\theta} d\mu(\theta) \) and write (\( \mu \) is called the symbol)

\[
D_N(\mu) = \det(T^{N+1}(\mu))
\]
Szegő’s Theorem and Toeplitz Determinant Asymptotics

Notice that in the $L^2(d\mu)$ inner product,

$$ (T_N)_{kj} = \langle e^{ik\theta}, e^{ij\theta} \rangle = \langle z^k, z^j \rangle $$

Writing $\Phi_N = z^N + \text{l.o.}$ and using sums of rows and columns, one sees that

$$ D_N(\mu) = \det(\langle \Phi_j, \Phi_k \rangle)_{0 \leq j, k \leq N} $$

$$ = \|\Phi_0\|^2 \cdots \|\Phi_N\|^2 $$
Szegő’s Theorem and Toeplitz Determinant Asymptotics

Since \( \| \Phi_j \| \downarrow \), one sees that

\[
\lim_{N \to \infty} D_N(\mu)^{1/N+1} = \lim_{N \to \infty} \| \Phi_N \|^2
\]

Thus,

**Toeplitz Determinant Form of Szegő’s Theorem.** For any \( \mu \),

\[
\lim_{N \to \infty} \frac{1}{N+1} \log D_N(\mu) = \int \log f(\theta) \frac{d\theta}{2\pi}
\]
Aside: It is known that if \( d\mu_s = 0 \) and

\[
\log(f(\theta)) \equiv \sum_{n=-\infty}^{\infty} \hat{L}_n e^{in\theta}
\]

and

\[
\sum_{n=1}^{\infty} n|\hat{L}_n|^2 < \infty
\]

then

\[
\log D_N(\mu) = (N + 1)\hat{L}_0 + \sum_{n=1}^{\infty} n|\hat{L}_n|^2 + o(1)
\]

This is the Strong Szegő Theorem. [OPUC1], Chap. 6 has many proofs of this.
When are Polynomials Dense in $L^2(\partial \mathbb{D}, d\mu)$?

By Weierstrass’ Theorem, for any $\mu$ of compact support on $\mathbb{R}$, the polynomials in $x$ are dense in $L^2(\mathbb{R}, d\mu)$.

But this is not true for $\partial \mathbb{D}$. Indeed, if $d\mu = \frac{d\theta}{2\pi}$, the closure of the polynomials are those functions in $L^2$ whose negative Fourier coefficient $\int e^{-in\theta} f(e^{i\theta}) \frac{d\theta}{2\pi} = 0$ for $n \leq -1$. On the other hand, we’ll see soon that if $\text{supp}(d\mu) \neq \partial \mathbb{D}$, the polynomials are dense.
When are Polynomials Dense in $L^2(\partial \mathbb{D}, d\mu)$?

**Theorem** (Kolmogorov-Krein). If $d\mu = f \frac{d\theta}{2\pi} + d\mu_s$, then the polynomials in $z$ are dense in $L^2(\partial \mathbb{D}, d\mu)$ if and only if

$$
\int \log f(e^{i\theta}) \frac{d\theta}{2\pi} = -\infty.
$$

They found this because this density result was relevant to their theory of prediction for stochastic processes.

Given Szegő’s Theorem, the proof is almost trivial for

$$
\inf_P \| z^{-1} - P \|^2_{L^2} = \inf_P \| 1 - zP \|^2_{L^2}
$$

$$
= \inf_{Q|Q(0)=1} \| Q \|^2_{L^2} = \exp(\int \log f \frac{d\theta}{2\pi})
$$
When are Polynomials Dense in $L^2(\partial \mathbb{D}, d\mu)$?

So $z^{-1} \in$ closure of polys $\iff \int \log f \frac{d\theta}{2\pi} = -\infty$.

Thus, if the integral is finite, $z^{-1} \notin$ closure of polys and thus, polynomials are not dense.

On the other hand, if $z^{-1} = \lim P_n$, then

$z^{-2} = \lim_{n \to \infty} P_n \left[ \lim_{m \uparrow \infty} P_m \right]$ so all polynomials in $z$ and $z^{-1}$ are in closure of polys and they are dense (by Weierstrass’ other density theory).

Krein used this to show (see [SzThm], p. 319) that on $\mathbb{R}$, if $d\rho = F dx + d\rho_\nu$, then $\{e^{i\alpha x}\}_{\alpha \geq 0}$ are dense in $L^2 \iff \int \frac{\log F(x)}{1+x^2} dx = -\infty$. This, in turn, implies that if $\int |x|^n d\rho(x) < \infty$, the moment problem is indeterminate if the integral is finite, for example,

$$d\rho(x) = e^{-|x|^\alpha} dx, \quad \alpha < 1$$
Strategy of the Proof

As with all good proofs of equalities, we’ll prove two inequalities. We’ll use “entropy term” for \( \exp \left[ \int \log f \frac{d\theta}{2\pi} \right] \) for reasons that will become clear soon.

The proof that \( \lim_{n \to \infty} \| \Phi_n^* \|^2 \) is an upper bound will be variational. We’ll show that for any polynomial with \( P(0) = 1 \), we have \( \| P \|^2 \geq \text{entropy term} \).
The lower bound on the entropy term will come from the fact that $\mu \mapsto$ entropy term is weakly upper-semicontinuous (usc), i.e., $\mu_n \to \mu \Rightarrow S(\mu) \geq \lim \sup S(\mu_n)$.

We’ll prove that $S(\mu) = \prod_{j=0}^{\infty} (1 - |\alpha_j|^2)^{1/2}$ for Bernstein–Szegő measures by direct calculation and then use this and usc to get the other inequality.
Lemma. For any polynomial $P$, with $P(0) \neq 0$, we have that
\[
\int \log |P(e^{i\theta})| \frac{d\theta}{2\pi} \geq \log |P(0)|
\]

Remark. One proof notes that $\log(P(z))$ is subharmonic.

Proof. If $\{z_j\}_{j=1}^m$ are zeros in $\mathbb{D}$, let
\[
Q(z) = \prod_{j=1}^m \frac{1 - \bar{z}_j z}{z - z_j} P(z)
\]

Then $\log Q(z)$ is analytic in $\mathbb{D}$, so
\[
\log |Q(0)| = \lim_{r \uparrow 1} \int \log |Q(re^{i\theta})| \frac{d\theta}{2\pi} = \int \log |Q(e^{i\theta})| \frac{d\theta}{2\pi} \\
= \int \log |P(e^{i\theta})| \frac{d\theta}{2\pi}
\]

But, \[|Q(0)| = \prod_{j=1}^{m} |z_j|^{-1} |P(0)| \geq |P(0)|.\]
For any polynomial, \( P \), with \( P(0) \neq 0 \), \( d\mu = f \frac{d\theta}{2\pi} + d\mu_s \), we have

\[
\int |P(e^{i\theta})|^2 d\mu(\theta) \geq \int |P(e^{i\theta})|^2 f(\theta) \frac{d\theta}{2\pi}
\]

\[
= \int \exp \left[ 2 \log |P(e^{i\theta})| + \log (f(\theta)) \right] \frac{d\theta}{2\pi}
\]

\[
\geq \exp \left( \int 2 \log \left( |P(e^{i\theta})| \frac{d\theta}{2\pi} \right) \exp \left( \int \log f \frac{d\theta}{2\pi} \right) \right)
\]

(by Jensen)

\[
\geq |P(0)|^2 \exp \left( \int \log |f(\theta)| \frac{d\theta}{2\pi} \right)
\]

by the Lemma. Thus

\[
\inf_{P | P(0)=1} \int |P(e^{i\theta})|^2 d\mu \geq \exp(\int \log(f(\theta))) \frac{d\theta}{2\pi}
\]
Upper Bound

One can also get a variational upper bound to complete the proof. The idea is to consider the function

\[ D(z) = \exp \left( \int \frac{e^{i\theta} + z}{e^{i\theta} - z} \log(f(\theta)) \frac{d\theta}{4\pi} \right) \]

Formally, and we’ll see later that \( D \) is actually in \( H^2(\mathbb{D}) \) and has boundary values, \( D(e^{i\theta}) = \lim_{r \to \infty} D(re^{i\theta}) \) exists for a.e. \( \theta \) and \( |D(e^{i\theta})|^2 = f(\theta) \).

If \( d\mu_s = 0 \), we have \( P(z) = D(0)/D(z) \) has \( P(0) = 0 \) and

\[
\int |P(z)|^2 d\mu = D(0)^2 \int f(\theta)^{-2} \left[ f(\theta) \frac{d\theta}{2\pi} \right] = D(0)^2
\]

\[
= \exp \left( \int \log(f(0)) \frac{d\theta}{2\pi} \right)
\]
$P$ isn’t a polynomial but one can approximate by polynomials. Handling $d\mu_s$ is a separate issue, but it can be done (see [OPUC1], Section 2.5 and [SzThm], Section 2.12).
Suppose $\alpha_j = 0$ for $j \geq N$. Then, we’ve seen that

$$d\mu = f(\theta)\frac{d\theta}{2\pi}, \quad f(\theta) = |\varphi_N^*(e^{i\theta})|^{-2}$$

Thus,

$$\log f(\theta) = -2 \log |\varphi_N^*(e^{i\theta})| = \log \|\Phi_N^*\|^2 - 2 \log |\Phi_N^*(e^{i\theta})|$$

Since $\Phi_N^*(z)$ is analytic in a nbhd of $\overline{\mathbb{D}}$, so is $\log(\Phi_N^*(z))$, so

$$\int \frac{d\theta}{2\pi} \log |\Phi_N^*(e^{i\theta})| = \log |\Phi_N^*(0)| = 0$$

Thus,

$$\int \log f(\theta)\frac{d\theta}{2\pi} = \log \|\Phi_N^*\|^2 = \log \prod_{j=0}^{N-1} (1 - |\alpha_j|^2)^{1/2}$$

proving Szegő’s Theorem in this case.
The Szegő Integral as an Entropy

Given two prob. measures on $\partial \mathbb{D}$, we define their relative entropy by

$$S(\mu \mid \nu) = \begin{cases} -\infty & \text{if } \mu \text{ is not } \nu\text{-a.c.} \\ - \int \log \left( \frac{d\mu}{d\nu} \right) d\mu & \text{if } \mu \text{ is } \nu\text{-a.c.} \end{cases}$$

For example, $S(gd\nu \mid d\nu) = - \int g \log(g) d\nu$

Usually $\nu$ is fixed and we vary $\mu$. 
The Szegő Integral as an Entropy

We claim that

$$S\left(\frac{d\theta}{2\pi} \mid f \frac{d\theta}{2\pi} + d\mu_s\right) = \int \log(f(\theta)) \frac{d\theta}{2\pi}$$

For $\mu$ is $\nu$-a.c. iff $f(\theta) \neq 0$ for $\frac{d\theta}{2\pi}$-a.e. $\theta$. If $f(\theta) = 0$ on a positive Lebesgue measure set, the integral is $-\infty$, so both sides are $-\infty$.

If $f(\theta) \neq 0$ for a.e. $\theta$, $\frac{d\mu}{d\nu} = f^{-1} \chi_S$ where $\chi_S$ is a set with $d\mu_s(S) = 0$ and $|S| = 1$. Clearly

$$-\int \log\left(\frac{d\mu}{d\nu}\right) = \int \log(f(\theta)) \frac{d\theta}{2\pi}$$
Here is a basic fact which we’ll make plausible but not formally prove (but see Section 2.2 of [SzThm]).

**Theorem.** Let $\mathcal{E}(\partial\mathbb{D})$ be the continuous strictly positive functions on $\partial\mathbb{D}$. Then

$$S(\mu \mid \nu) = \inf_{f \in \mathcal{E}(\partial\mathbb{D})} S(f; \mu, \nu)$$

where

$$S(f; \mu, \nu) = \int f(x) \, d\nu(x) - \int 1 + \log(f(x)) \, d\mu$$

**Proof.** If $d\mu = gd\nu$ with $g \in \mathcal{E}(\partial\mathbb{D})$, then

$$S(g; gd\nu, \nu) = 1 - 1 - \int \log(g(x)) \, d\mu = S(gd\nu \mid \nu)$$

By an approximation argument (and control of $d\mu_s$) one obtains

$$S(\mu \mid \nu) \geq \inf S$$
Let’s prove $S(f; \mu, \nu) \geq S(\mu | \nu)$ in case $d\mu_s = 0$ so

$$d\nu = g^{-1} d\mu$$

so that

$$S(f; \mu, \nu) = \int Q_{g(x)}(f(x)) \, d\mu(x)$$

where

$$Q_{b}(x) = xb^{-1} - 1 - \log x$$

Then

$$Q'_{b}(x) = b^{-1} - x^{-1}, \quad Q''_{b}(x) = x^{-2} \geq 0$$

so $Q_b$ is convex, $Q'_{b}(b) = 0$, so $Q_{b}(x) \geq Q_{b}(b)$, i.e.,

$$Q_{b}(x) \geq - \log(b)$$

Thus

$$S(f; \mu, \nu) \geq - \int \log(g(x)) \, d\mu(x) = S(\mu | \nu)$$
Variational Principle for $S$

For each fixed $f$ in $\mathcal{E}(\partial\mathbb{D})$, $S(f; \mu, \nu)$ is linear and weakly continuous so the inf is concave and weakly usc, i.e.

**Theorem.** $S(\mu | \nu)$ is jointly converse and jointly weakly usc in $\mu$ and $\nu$.

**Corollary.** Define $Sz(\mu) = \int \log f \frac{d\theta}{2\pi}$ if $d\mu = f \frac{d\theta}{2\pi} + d\mu_s$. Then $\mu \mapsto Sz(\mu)$ is weakly usc.
Let $\mu$ have Verblunsky coefficients, $\{\alpha_n\}_{n=0}^{\infty}$. Let $\mu_n$ be the Bernstein–Szegő approximation.

We’ve proven above that

$$Sz(\mu_n) = \prod_{j=0}^{n-1} \rho_j^2$$

By weak use

$$Sz(\mu) \geq \lim_{n \to \infty} Sz(\mu_n) = \prod_{j=0}^{\infty} \rho_j^2$$

which is the other inequality that we needed to prove.