# A DESCRIPTION OF MY RESEARCH INTERESTS WRITTEN FOR A NON-SPECIALIST

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Abstract. This text is written predominantly for final year mathematics undergraduate students considering applying for a PhD in mathematics.

## 1. Spectral perturbation theory of self-adjoint operators in a Hilbert space

This heading describes my research interests in broadest terms. First consider a Hermitian matrix $A$ in a Euclidean space $\mathbb{C}^N$. To perform a *spectral analysis* of this matrix means to find its eigenvalues $\lambda_j$ and the corresponding eigenvectors $\psi_j \in \mathbb{C}^N$, $j = 1, \dots, N$:

$$A\psi_j = \lambda_j \psi_j. \tag{1}$$

Now suppose we change the matrix $A$ by adding another Hermitian matrix $V$; so the "new" matrix is $\widetilde{A} = A+V$. The matrix $\widetilde{A}$ will have its own set of eigenvalues $\widetilde{\lambda}_j$ and eigenvectors $\widetilde{\psi}_j$. Thus, changing the matrix, we change its spectral decomposition. When we go from $A$ to $\widetilde{A}$, the eigenvalues *shift* and the eigenvectors *rotate*. One of the main questions addressed by spectral perturbation theory is the description of these shifts and rotations depending on the perturbation $V$.

Of course, the answer depends on our assumptions on $A$ and $V$. One often assumes that $V$ is in some sense "small" or "weak". "Small" usually means that an appropriate norm of $V$ is small. "Weak" is a more subtle notion and can mean many things, but in this context it can mean, for example, that the matrix $V$ is sparse (i.e. the matrix elements of $V$ contain many zeros) or that it has some special structure (i.e. it is a rank one matrix).

Of course, the more advanced version of the above set-up is the situation when instead of Hermitian matrices in the finite dimensional Euclidean space $\mathbb{C}^N$ one considers self-adjoint operators in an infinite dimensional Hilbert space. In this case, the situation is much more complicated, mainly due to the fact that instead of a sequence of eigenvalues separated from each other, a much more complex spectrum can occur: one can have intervals of continuous spectrum, eigenvalues that are dense on some intervals, etc. A good background reading on spectral theory of self-adjoint operators is given by Sections VI and VII of the book [8].

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Even in the finite-dimensional setting, attempting to describe the spectral theory of *all* possible matrices is a problem of transcendental difficulty. More tractable problems arise if one considers a particular class of matrices/operators. For example (although this is not a subject of my research), much is known about the spectral theory of *Jacobi matrices*, i.e. three-diagonal Hermitian matrices of the form

$$J = \begin{pmatrix} b_1 & a_1 & 0 & 0 & \dots \\ a_1 & b_2 & a_2 & 0 & \dots \\ 0 & a_2 & b_3 & a_3 & \dots \\ 0 & 0 & a_3 & b_4 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

with $a_j > 0$ for all $j$. Another example is given by Hankel and Toeplitz matrices, see the last section.

## 2. Spectral shift function theory

Let us return to the situation with two Hermitian matrices $A$, $A'$. Suppose we would like to compare the two sets of eigenvalues $\{\lambda_j\}_{j=1}^N$ and $\{\widetilde{\lambda}_j\}_{j=1}^N$. One way of doing this is to introduce the *eigenvalue counting function*

$$N(\lambda; A) = \operatorname{card}\{j : \lambda_j < \lambda\},$$
$$N(\lambda; \widetilde{A}) = \operatorname{card}\{j : \widetilde{\lambda}_j < \lambda\}$$

(here $\operatorname{card} X$ stands for the number of elements of $X$), and then look at the difference

$$\xi(\lambda; \widetilde{A}, A) = N(\lambda; A) - N(\lambda; \widetilde{A}).$$

The function $\xi(\lambda; \widetilde{A}, A)$ gives information about the shifts of eigenvalues of $\widetilde{A}$ relatively to the eigenvalues of $A$. Thus, it is called the *spectral shift function.*

Again, a more sophisticated definition of spectral shift function involves pairs of self-adjoint operators in a Hilbert space rather than pairs of matrices. A standard survey in the spectral shift function theory is [2]; see also [1] for an easy introduction.

My main contribution to this field is a certain representation for the spectral shift function, see [1]. I no longer actively work on this topic as such, although spectral shift function often (unexpectedly!) turns up in various aspects of my research.

## 3. Scattering theory

Scattering theory is a multi-faceted subject. One facet is the study of wave propagation phenomena in physics. The waves could be, for example, acoustic waves, water waves, or electromagnetic waves. My own interests are mainly related to the quantum mechanical waves which are not really waves but mathematical objects (wave functions $\psi$) describing the motion of subatomic particles. These waves are governed by the Schrödinger equation

$$i\frac{\partial}{\partial t}\psi = -\Delta\psi + V\psi,$$

where $V$ is the potential of the external electric field. This external electric field is assumed to be localised in space, i.e. $V(x) = 0$ for all sufficiently large $|x|$. This corresponds to waves-particles coming from infinity, propagating through empty space, hitting some obstacle (represented by $V$) and being scattered away to infinity by this obstacle.

One of the methods of analysing solutions to this equation is to look at wave functions with a certain fixed energy $\lambda$. These wave functions satisfy the stationary Schrodinger equation

$$-\Delta\widetilde{\psi} + V\widetilde{\psi} = \lambda\widetilde{\psi}. \tag{2}$$

In the absence of an electric field, we have $V = 0$ and the equation becomes

$$-\Delta\psi = \lambda\psi. \tag{3}$$

This is the equation of a quantum particle with energy $\lambda$ moving in an empty space. The analysis usually involves comparison of the solutions $\psi$ and $\widetilde{\psi}$ corresponding to *the same value of* $\lambda$.

This equations (2) and (3) look like the eigenvalue equation (1) with the "matrices" $A = -\Delta$ and $\widetilde{A} = -\Delta + V$. This connection leads to another facet of scattering theory. It turns out that the values of $\lambda$ for which the above problem makes sense lie in the *continuous spectrum* of the operators $A$ and $\widetilde{A}$. One of the tasks of mathematical scattering theory is the comparison of eigenvectors of two self-adjoint operators $A$, $\widetilde{A}$ corresponding to the same eigenvalue $\lambda$ from the continuous spectrum. Here one assumes that $\widetilde{A} = A + V$, where $V$ is in some sense a "weak" perturbation of $A$. Informally speaking, mathematical scattering theory studies the rotations (in Hilbert space) of eigenvectors corresponding to the continuous spectrum. In rigourous terms, this is performed through the analysis of *wave operators, scattering operator, scattering matrix,* and *spectral shift function.* See, for example, [9] for an introduction to the main concepts of scattering theory.

My own interests in this field are in establishing a transparent relationship between the scattering matrix the geometry of Hilbert space (rotation of eigenvectors).

## 4. Hankel and Toeplitz operators

An (infinite) *Hankel matrix* is a matrix of the form

$$\Gamma(a) = \{a_{i+j}\}_{i,j\geq 0} = \begin{pmatrix} a_0 & a_1 & \cdots \\ a_1 & a_2 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix},$$

where $\{a_j\}_{j\geq 0}$ is a given sequence of complex numbers. The *symbol* of a Hankel matrix is the function on the unit circle in the complex plane, defined by

$$a(z) = \sum_{j=0}^{\infty} a_j z^j, \quad |z| = 1$$

(assuming that the series converges in appropriate sense).

An (infinite) *Toeplitz matrix* is a matrix of the form

$$T(a) = \{a_{i-j}\}_{i,j\geq 0} = \begin{pmatrix} a_0 & a_1 & \dots \\ a_{-1} & a_0 & \dots \\ \vdots & \vdots & \ddots \end{pmatrix},$$

where $\{a_j\}_{j\in\mathbb{Z}}$ is a given (doubly infinite) sequence of complex numbers. The *symbol* of a Toeplitz matrix is the function on the unit circle in the complex plane, defined by

$$a(z) = \sum_{j=-\infty}^{\infty} a_j z^j, \quad |z| = 1$$

(assuming convergence).

$\Gamma(a)$ and $T(a)$ are usually understood as linear operators on the Hilbert space $\ell^2(\mathbb{Z}_+)$, $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$. In this context, the terms *Hankel operator* and *Toeplitz operator* are used.

The central question of the spectral theory of Hankel operators is this:

*How are the spectral properties of* $\Gamma(a)$ *(eigenvalue distribution, location of the continuous spectrum etc) related to the properties of the function* $a(z)$*?*

A similar question can be asked for Toeplitz operators.

A very accessible introduction to the theory of Hankel and Toeplitz operators can be found in the books [4] and [10]; see also my lecture notes [7] for a very short introduction. For more in-depth exposition, see e.g. [6, 5, 3].

By mapping an arbitrary sequence $\{f_j\}_{j=0}^{\infty} \in \ell^2(\mathbb{Z}_+)$ to the function

$$f(z) = \sum_{j=0}^{\infty} f_j z^j, \quad |z| = 1,$$

one obtains a unitary map from $\ell^2(\mathbb{Z}_+)$ onto the Hardy space $H^2(\mathbb{T})$. Thus, $\Gamma(a)$ and $T(a)$ are unitarily equivalent to operators on the Hardy space. On the other hand, functions from the Hardy space have a canonical analytic extension into the unit disc $|z| < 1$. This provides a natural link between the theory of Hankel and Toeplitz operators and complex analysis. As a consequence, the spectral theory of these operators is a beautiful combination of some aspects of functional analysis, complex analysis and harmonic analysis.

I have become interested in this topic around 2014 and currently I spend most of my time working on it. My interests in this area are motivated by the intuition coming from spectral theory of differential operators. Broadly speaking, I am aiming to transfer some of the results and the techniques from spectral theory of differential operators to spectral theory of Hankel and Toeplitz operators. Analogies between these two areas of analysis are not always obvious but invariably are stimulating.

Here is a partial list of topics that I am interested in:

- Spectral analysis of Hankel operators with continuous spectrum (perturbation theory, scattering theory)
- Spectral asymptotics for compact Hankel operators
- Spectral asymptotics for compact Toeplitz operators on Bergman spaces
- Inverse problems for Hankel operators
- Weighted Hankel operators

## References

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