INTEGRAL ESTIMATES FOR THE SPECTRAL SHIFT FUNCTION

A. B. Pushnitskiĭ

24/APR/1998

Abstract. The spectral shift function $\xi(\lambda)$ is considered for the pair of operators H_0, H_0+V, where H_0 is the Schrödinger operator with variable Riemannian metric and with electromagnetic field, and V is the operator of multiplication by the potential $V(x)$. For the integrals of type $\int \xi(\lambda) f(\lambda)\,d\lambda$, where $f(\lambda)$ is a weight, some estimates in terms of the integral characteristics of the potential V are obtained. These estimates are of asymptotically sharp order in λ and V; in a subsequent paper they will be applied to the problem of the asymptotics of $\xi(\lambda)$ in the large coupling constant limit.

§0. Introduction

0.1. Let H_0 and H be selfadjoint operators in a Hilbert space $\mathcal{H}$; we assume that their difference V is a trace-class operator:

$$V := H - H_0 \in \mathbf{S}_1(\mathcal{H}). \tag{0.1}$$

Then the following *Lifshits–Kreĭn trace formula* holds (see [21, 19]):

$$\operatorname{Tr}(\varphi(H) - \varphi(H_0)) = \int_{-\infty}^{\infty} \xi(\lambda; H, H_0)\varphi'(\lambda)\,d\lambda. \tag{0.2}$$

Here φ is any function of some function class, and $\xi(\lambda; H, H_0)$ is the *spectral shift function* (SSF) for the pair H_0, H, which is given by the *Kreĭn formula*

$$\xi(\lambda; H, H_0) = \frac{1}{\pi}\lim_{\varepsilon\to+0} \arg\det(I + V(H_0 - (\lambda + i\varepsilon)I)^{-1}) \quad \text{for a.e. } \lambda \in \mathbb{R}. \tag{0.3}$$

The branch of the argument in (0.3) is fixed by the condition

$$\arg\det(I + V(H_0 - zI)^{-1}) \to 0 \quad \text{as } \operatorname{Im} z \to +\infty.$$

1991 *Mathematics Subject Classification.* Primary 35Q40, 47B25.

Key words and phrases. Spectral shift function, Lifshits–Kreĭn trace formula, Schrödinger operator.

A historical review and a description of the modern state of the SSF theory can be found in [8, 36]. Among the most important facts of the SSF theory (see [19]) we mention the *Kreĭn inequality*

$$\int_{-\infty}^{\infty} |\xi(\lambda; H, H_0)|\, d\lambda \le \|V\|_{\mathbf{S}_1} \tag{0.4}$$

and the *monotonicity* of the SSF:

$$\pm V \ge 0 \quad \Rightarrow \quad \pm\xi(\lambda; H, H_0) \ge 0. \tag{0.5}$$

From (0.5) it follows that

$$\xi(\lambda; H_0 - V_-, H_0) \le \xi(\lambda; H_0 + V, H_0) \le \xi(\lambda; H_0 + V_+, H_0), \tag{0.6}$$

where $2V_\pm = |V| \pm V$.

0.2. In applications, instead of condition (0.1) it is usually possible to check the relation

$$h(H) - h(H_0) \in \mathbf{S}_1(\mathcal{H}), \tag{0.7}$$

where $h\colon \mathbb{R} \to \mathbb{R}$ is a sufficiently smooth monotone function. In this case, the SSF for the pair H_0, H is *defined* by the natural formula

$$\xi(\lambda; H, H_0) := \operatorname{sgn} h' \cdot \xi(h(\lambda); h(H), h(H_0)). \tag{0.8}$$

Of course, the trace formula (0.2) remains valid; only the class of admissible functions φ must be changed. As to relation (0.5), sometimes it can also be justified, but this problem is far from being trivial if the SSF is defined via (0.8). In [18] (see also [36, 8.10]), the implication (0.5) was proved for

$$h(\lambda) = (\lambda - \lambda_0)^{-k}, \quad \text{where } k > 0,\ \lambda_0 < \inf(\sigma(H_0) \cup \sigma(H)). \tag{0.9}$$

This is sufficient for most applications.

0.3. Let

$$H_0 = -\triangle \text{ in } L_2(\mathbb{R}^d),\ d \ge 1, \qquad V = V(x), \qquad H = H_0 + V. \tag{0.10}$$

If the potential V decays sufficiently fast at infinity, then the relation (0.7) can be verified for the functions h of the form (0.9). This makes it possible to define the SSF via (0.8); relations (0.5), (0.6) are valid in this case due to the results of [18].

Estimates for the SSF for the pair H_0, H as in (0.10) were studied in [35]. The potential V was assumed to satisfy the condition

$$|V(x)| \le (1 + |x|)^{-\rho}, \quad \rho > d.$$

The following estimate (see [35, Theorem 4.2]) was obtained:

$$\xi(\lambda; H_0 + \alpha V, H_0) \le C(\alpha^{(d/2)} + \alpha\lambda^{(d/2)-1}(|\log\lambda| + 1)), \quad \alpha > 0,\ \lambda \ge c; \tag{0.11}$$

for $V \ge 0$ it was improved up to the inequality

$$\xi(\lambda; H_0 + \alpha V, H_0) \le C\alpha\lambda^{(d/2)-1}(|\log\lambda| + 1), \quad \alpha > 0,\ \lambda \ge c. \tag{0.12}$$

Here C, c are some positive numbers independent of the spectral parameter λ and the *coupling constant* α, but possibly depending on V (this dependence was not studied in [35]).

Besides estimates (0.11), (0.12), there are many results concerning the asymptotics of $\xi(\lambda; H_0 + \alpha V, H_0)$ as $\lambda \to \infty$ for fixed α (high energy asymptotics; here the initial results were obtained in [9]) and as $\alpha \to \infty$ and $\lambda \to \infty$ with λ/α fixed (semiclassical asymptotics); see, e.g., [30] and the references therein.

0.4. The above results on the *pointwise* estimates and asymptotics for the SSF of the pair (0.10) are based on the fact that the spectrum of H_0 is absolutely continuous and that V is *smooth* with respect to H_0 in some sense. In this paper we use a different approach, considering the SSF entirely in the framework of the *trace class* perturbation theory. Starting with the Kreĭn inequality (0.4), we obtain *integral* estimates for the SSF. As an example, we present the following weighted estimate obtained for the SSF of the pair (0.10) for $d \geq 3$ (see Theorem 6.4(ii) and Corollary 6.8(ii) below and also (0.6)):

(0.13)
$$\int_0^\infty |\xi(\lambda; H_0 + V, H_0)| f(\lambda)\, d\lambda \\ \leq C_1 \int_0^\infty f(\lambda)\, d\lambda \int V_-^{d/2}(x)\, dx + +C_2 \int_0^\infty \lambda^{(d/2)-1} f(\lambda)\, d\lambda \int |V(x)|\, dx.$$

If $V \geq 0$, then, obviously, the first term on the right vanishes and we obtain

$$\int_0^\infty \xi(\lambda; H_0 + V, H_0) f(\lambda)\, d\lambda \leq C_2 \int_0^\infty \lambda^{(d/2)-1} f(\lambda)\, d\lambda \int V(x)\, dx. \tag{0.14}$$

Here $f = f(\lambda)$ is any nonnegative monotone decreasing function and C_1, C_2 are the constants independent of λ, V, f. The precise statements are given in Subsections 6.2 and 6.3.

Clearly, the pointwise estimates (0.11), (0.12) are stronger than the integral ones (0.13), (0.14) (up to constants and a logarithmic factor). However, our method of deriving integral estimates is "insensitive" to the specific nature of the spectrum of H_0. This allows us to consider a wide class of H_0's (the Schrödinger operators with variable metric and electromagnetic field; see Subsection 5.2 below) and to avoid imposing too restrictive conditions on V. Moreover, in contrast to (0.11), (0.12), estimates (0.13), (0.14) explicitly indicate the dependence on V. Also, we obtain some analogs of (0.13), (0.14) for $H_0 = (-\triangle)^l$ in $L_2(\mathbb{R}^d)$, $d \geq 1$; here l is not necessarily an integer.

We note that the properties of concavity and subadditivity with respect to V were studied in [12] for the quantities $\int_0^\lambda \xi(t; H_0 + \alpha V, H_0)\, dt$ and $\int_0^\infty e^{-t\lambda} \xi(\lambda; H_0 + \alpha V, H_0)\, d\lambda$, related to the pair (0.10) for $d = 3$.

0.5. Largely, the present paper is aimed at applications to the problem of asymptotics of the SSF in the large coupling constant limit (these applications themselves will be considered elsewhere). To a great extent, this has determined the style of presentation. Let $V \geq 0$; then for $\lambda < \inf \sigma(H_0)$ the SSF $\xi(\lambda; H_0 - \alpha V, H_0)$ is equal to minus the number $N_+(\lambda; H_0, \sqrt{V}, \alpha)$ of the eigenvalues of $H_0 - \alpha V$ located to the left from the point λ on the real axis (see, e.g., [8]). The asymptotic properties of $N_+(\lambda; H_0, \sqrt{V}, \alpha)$ as $\alpha \to \infty$ are well studied (see, e.g., [7] and the references therein). In particular, a description is known of the class of perturbations V (*regular* perturbations) such that the leading term of the asymptotics does not depend on λ. For this reason, in order to study the SSF in the large coupling constant limit for regular perturbations $V \geq 0$, it is natural to consider the quantity

$$\xi(\lambda; H_0 - \alpha V, H_0) + N_+(\Lambda_-; H_0, \sqrt{V}, \alpha) \tag{0.15}$$

with some fixed $\Lambda_- < \inf \sigma(H_0)$. Our initial estimates for $\xi(\lambda; H_0 - \alpha V, H_0)$ are formulated in terms of this quantity (see Theorems 6.6, 6.9). As a consequence, we obtain inequalities of the form (0.13) (see Corollary 6.8).

0.6. Relation (0.6) shows that in order to obtain estimates for the SSF, we can restrict ourselves to perturbations of definite sign. We use the following approach. Suppose that a nonnegative operator V is factored as $V = G^*G$. In [25], the following new representation of the SSF was obtained (see Propositions 2.6, 2.7 below):

$$\xi(\lambda; H_0 \pm V, H_0) = \pm\mathcal{N}_\mp(\lambda; H_0, G). \tag{0.16}$$

Here $\mathcal{N}_\pm(\lambda; H_0, G)$ is the integral (see (2.4)) of the counting function for the eigenvalues of a certain family of compact operators (this family is related to G and to the resolvent of H_0). Formula (0.16) is of abstract nature. In applications it turns out that the existence of the quantities $\mathcal{N}_\pm$ can be established under somewhat more general and more natural conditions than the existence of the SSF (see Subsection 6.1). For this reason, we regard the quantities $\mathcal{N}_\pm$ as an initial object and define them independently of SSF. To analyze $\mathcal{N}_\pm$, we combine different approaches, employing, on the one hand, the straightforward analysis of the integral representation for $\mathcal{N}_\pm$ and, on the other hand, identity (0.16) together with estimate (0.4) for "intermediate" objects. This yields integral estimates for $\mathcal{N}_\pm(\lambda; H_0, G)$. The corresponding inequalities for the SSF are obtained by using Propositions 2.6 and 2.7, which give conditions sufficient for the validity of (0.16).

0.7. The paper is organized as follows. In §1 we collect the necessary notation and definitions. In §2 we define the quantities $\mathcal{N}_\pm(\lambda; H_0, G)$ and discuss their basic properties and relationship with the SSF and the "counting function" $N_\pm(\lambda; H_0, G, \alpha)$. Estimates for the quantities $\mathcal{N}_\pm$ are obtained in §3 and for the quantity (0.15) in §4. The results of §§2–4 are of abstract nature. In §5 we introduce the Schrödinger operator and the polyharmonic operator (in applications, they play the role of H_0) and present relevant information. Finally, in §6, on the basis of the results of §§3–4 we obtain integral estimates for $\mathcal{N}_\pm(\lambda; H_0, \sqrt{V})$, where H_0 is the Schrödinger or the polyharmonic operator and V is the operator of multiplication by the potential $V(x) \geq 0$.

0.8. Acknowledgments. The author is deeply grateful to his scientific advisor M. Sh. Birman, who introduced him into the SSF theory, for many valuable discussions and remarks. The author thanks V. S. Buslaev, who attracted his attention to the problem of integral estimates for the SSF. The author is grateful to A. Laptev and G. Rozenblum for consultations and to N. D. Filonov for useful remarks on the text of the paper. Finally, the author expresses his gratitude to the Department of Mathematics of the Royal Institute of Technology, Stockholm, for hospitality and to the ISF foundation for financial support.

§1. Notation and preliminaries

1.1. Notation.

1) The integrals in which the integration domain is not indicated explicitly are taken over $\mathbb{R}^d$. We denote $\omega_d := \operatorname{vol}\{x \in \mathbb{R}^d : |x| \leq 1\}$. The statements with double indices $\pm$ or $\mp$ should be understood as a pair of statements: in one of them *all* indices take the upper values, and in the other *all* indices take the lower values. In the statements involving upper estimates we assume that all quantities (norms and integrals) on the right-hand side are finite. A constant that appears for the first time in formula $(i.j)$ is denoted by $C_{i.j}$.

2) *Functions.* The spaces $L_p(\mathbb{R}^d)$, $L_{p,\,\mathrm{loc}}(\mathbb{R}^d)$ are defined in a usual way. Let $\mathbb{Q}^d = (0,1)^d \subset \mathbb{R}^d$. The space $l_1(\mathbb{Z}^d; L_2(\mathbb{Q}^d))$ consists of the functions $u \in L_2(\mathbb{R}^d)$ such that the functional

$$\|u\|_{l_1(L_2)} := \sum_{j\in\mathbb{Z}^d} \left(\int_{\mathbb{Q}^d+j} |u|^2\, dx \right)^{1/2},$$

is finite. For a real-valued function F we put $2F_\pm := |F| \pm F$. The characteristic function of a set M is denoted by χ_M.

3) *Operators.* In what follows, $\mathcal{H}$, $\mathcal{H}_1$, $\mathcal{H}_2$ are separable Hilbert spaces. By $\mathrm{Dom}\, A$, $\mathrm{Ran}\, A$, and $\mathrm{Ker}\, A$ we denote the domain, the range and the kernel of a linear operator A, respectively; $|A| := \sqrt{A^*A}$; I is the identity operator. For a selfadjoint operator A, the symbols $\sigma(A)$, $\rho(A) = \mathbb{C} \setminus \sigma(A)$, and $E_A(\delta)$ denote, respectively, the spectrum, the resolvent set, and the spectral measure of a Borel set $\delta \subset \mathbb{R}$; we put $2A_\pm := |A| \pm A$. The resolvent of a selfadjoint operator H_0 is denoted by $R_0(z) = (H_0 - zI)^{-1}$.

We denote by $\mathbf{S}_\infty(\mathcal{H}_1, \mathcal{H}_2)$ the space of compact operators acting from $\mathcal{H}_1$ into $\mathcal{H}_2$; $\mathbf{S}_\infty(\mathcal{H}) := \mathbf{S}_\infty(\mathcal{H}, \mathcal{H})$. For $T = T^* \in \mathbf{S}_\infty(\mathcal{H})$ and $s > 0$ we put $n_\pm(s, T) := \dim \mathrm{Ran}\, E_{T_\pm}(s, +\infty)$, and for $T \in \mathbf{S}_\infty(\mathcal{H}_1, \mathcal{H}_2)$ we put $n(s,T) := n_+(s^2, T^*T)$. If T_1, T_2 are compact selfadjoint operators, then

$$n_\pm(s_1 + s_2, T_1 + T_2) \le n_\pm(s_1, T_1) + n_\pm(s_2, T_2), \quad s_1, s_2 > 0 \tag{1.1}$$

(see, e.g., [3]); this estimate can be written in the form

$$n_\pm(s_1, T_1 + T_2) \ge n_\pm(s_1 + s_2, T_1) - n_\mp(s_2, T_2), \quad s_1, s_2 > 0. \tag{1.2}$$

For $1 \le p < \infty$, the Neumann–Schatten class $\mathbf{S}_p(\mathcal{H}_1, \mathcal{H}_2) \subset \mathbf{S}_\infty(\mathcal{H}_1, \mathcal{H}_2)$ is defined as the set of all compact operators T for which the norm

$$\|T\|_{\mathbf{S}_p} := \left(p \int_0^\infty s^{p-1} n(s, T)\, ds \right)^{1/p}$$

is finite. In particular, $\mathbf{S}_1$ is the trace class and $\mathbf{S}_2$ is the Hilbert–Schmidt class.

1.2. Auxiliary facts from perturbation theory. Let $\mathcal{H}$ be a "basic" and $\mathcal{K}$ an "auxiliary" Hibert space, H_0 a selfadjoint operator in $\mathcal{H}$, and G a linear operator acting from $\mathcal{H}$ into $\mathcal{K}$. Assume that

$$G \text{ is closed}, \qquad \mathrm{Dom}\, G \supset \mathrm{Dom}\, |H_0|^{1/2}, \qquad G(|H_0| + I)^{-1/2} \in \mathbf{S}_\infty(\mathcal{H}, \mathcal{K}). \tag{1.3}$$

For $z \in \rho(H_0)$ we introduce the operator

$$T(z; H_0, G) := (G(|H_0| + I)^{-1/2})(|H_0| + I)R_0(z)(G(|H_0| + I)^{-1/2})^*, \tag{1.4}$$

which is compact in $\mathcal{K}$. We shall write $T(z)$ in place of $T(z; H_0, G)$ if the choice of H_0, G is clear from the context. On the set $\mathrm{Dom}\, G^*$ (dense in $\mathcal{K}$) the operator $T(z)$ can be defined by a simpler formula:

$$T(z)\psi = GR_0(z)G^*\psi, \quad \psi \in \mathrm{Dom}\, G^*. \tag{1.5}$$

It is easy to check that for all $z \in \mathbb{C} \setminus \mathbb{R}$ the operator $I \pm T(z; H_0, G)$ has a bounded inverse. Let $V := G^*G$. Below we introduce the operator corresponding to the formal sum $H_0 \pm V$. For $z \in \mathbb{C} \setminus \mathbb{R}$, we define a bounded operator $R_\pm(z)$ in $\mathcal{H}$ by the formula

$$R_\pm(z) = R_0(z) \mp (GR_0(\overline{z}))^*(I \pm T(z; H_0, G))^{-1}(GR_0(z)). \tag{1.6}$$

Proposition 1.1 (see [14] or [36]). *The operator* $R_\pm(z)$ *coincides with the resolvent* (*evaluated at the point* z) *of an operator* $H_\pm = H_\pm(H_0, G)$ *selfadjoint in* $\mathcal{H}$ *and independent of* z. *For all* $z \in \rho(H_0) \cap \rho(H_\pm)$ *the operator* $I \pm T(z; H_0, G)$ *has a bounded inverse*, *and identity* (1.6) *is fulfilled for* $R_\pm(z) = (H_\pm - zI)^{-1}$. *If* H_0 *is lower semibounded*, *then* $H_\pm$ *coincides with the sum* $H_0 \pm V$ *in the form sense. If* V *is* H_0*-bounded with relative bound* $\gamma < 1$, *then* $H_\pm = H_0 \pm V$ *in the sense of the Kato–Rellich theorem.*

From (1.6) and (1.3) it follows that the difference $R_\pm(z) - R_0(z)$ is compact; consequently, the essential spectra of $H_\pm$ and H_0 coincide. It is easy to check that $H_\pm(H_0, G) = H_\pm(H_0, |G|)$.

Proposition 1.2 ([17, Lemma 1]). *Under conditions* (1.3), *for every* $\lambda \in \rho(H_0) \cap \mathbb{R}$ *we have*

$$\dim \operatorname{Ker}(H_\pm(H_0, G) - \lambda I) = \dim \operatorname{Ker}(T(\lambda; H_0, G) \pm I). \tag{1.7}$$

In particular, Proposition 1.2 implies that the eigenvalues of $H_\pm(H_0, \sqrt{\alpha} G)$, $\alpha > 0$, are monotone functions of α. For $\lambda \in \rho(H_0) \cap \mathbb{R}$ and $\alpha > 0$, the "counting function" $N_\pm(\lambda; H_0, G, \alpha)$ is defined as the number of the eigenvalues of $H_\mp(H_0, \sqrt{t} G)$ (counting multiplicities) that cross the point λ as t grows monotonically in the interval $(0, \alpha)$. Proposition 1.2 implies the following identity, known as the *Birman–Schwinger principle*:

$$N_\pm(\lambda; H_0, G, \alpha) = n_\pm(\alpha^{-1}; T(\lambda; H_0, G)), \quad \lambda \in \rho(H_0) \cap \mathbb{R}. \tag{1.8}$$

In this paper, the scope of applications is restricted to lower semibounded operators H_0 (the Schrödinger and the polyharmonic operator). In this case, $H_\pm$ can be defined as a form sum, which simplifies the arguments. However, in the abstract part of the paper we do not suppose that H_0 is lower semibounded (with the exception of §4, where this condition is dictated by the nature of the question).

We note that the requirement that G be closed is imposed only for the sake of simplicity. The content of the present section, as well as that of §§2–4, can be reformulated for the case of a nonclosable operator G. What matters is that the second and the third of conditions (1.3) be fulfilled.

§2. The functions $\mathcal{N}_\pm$

2.1. The definition of the functions $\mathcal{N}_\pm$. In this and the next two sections $\mathcal{H}$ is a "basic" and $\mathcal{K}$ an "auxiliary" Hilbert space, H_0 is a selfadjoint operator in $\mathcal{H}$, and G is an operator acting from $\mathcal{H}$ into $\mathcal{K}$ and satisfying conditions (1.3). For $z \in \rho(H_0)$ we use (1.4) to define operators $T(z; H_0, G)$ compact in $\mathcal{K}$; next, we put

$$A(z; H_0, G) := \operatorname{Re} T(z; H_0, G), \qquad K(z; H_0, G) := \operatorname{Im} T(z; H_0, G). \tag{2.1}$$

We shall write $A(z)$, $K(z)$ in place of $A(z; H_0, G)$, $K(z; H_0, G)$ if the choice of H_0, G is clear from the context. Suppose that for some $\lambda \in \mathbb{R}$ the pair of operators H_0, G satisfies the following condition.

Condition 2.1. *The limit*

$$\lim_{\varepsilon\to+0} T(\lambda+i\varepsilon;H_0,G) =: T(\lambda+i0;H_0,G) \tag{2.2}$$

exists in the operator norm, and

$$K(\lambda+i0;H_0,G)\in \mathbf{S}_1(\mathcal{K}). \tag{2.3}$$

Then we define

$$\mathcal{N}_\pm(\lambda;H_0,G) := \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{dt}{1+t^2} n_\pm(1;A(\lambda+i0)+tK(\lambda+i0)). \tag{2.4}$$

It is easily seen that condition (2.3) implies the convergence of the integral (2.4).

Lemma 2.2. *Fixing an open interval $\delta\subset\mathbb{R}$ and a number $\lambda\in\delta$, we put $G_\delta := GE_{H_0}(\delta)$. At the point λ, condition* 2.1 *is satisfied for the pair H_0, G if and only if it is satisfied for the pair H_0, G_δ.*

Proof. Let $\widetilde{G}_\delta := GE_{H_0}(\mathbb{R}\setminus\delta)$. Obviously, for every $z\in\rho(H_0)$ we have

$$T(z;H_0,G) = T(z;H_0,G_\delta) + T(z;H_0,\widetilde{G}_\delta).$$

It is clear that the limit $T(\lambda+i0;H_0,\widetilde{G}_\delta)$ exists in the operator norm and that $K(\lambda+i0;H_0,\widetilde{G}_\delta)=0$. This implies the assertion of the lemma. □

Lemma 2.3. *For any $\lambda\in\mathbb{R}$, Condition* 2.1 *is satisfied for H_0, G if and only if it is satisfied for H_0, $|G|$; moreover,*

$$\mathcal{N}_\pm(\lambda;H_0,G) = \mathcal{N}_\pm(\lambda;H_0,|G|). \tag{2.5}$$

Proof. Let $G=\Phi|G|$ be a polar decomposition of G; the operator Φ acts unitarily from $\overline{\operatorname{Ran}|G|}$ into $\overline{\operatorname{Ran} G}$. It is easy to check that

$$\Phi T(z;H_0,|G|) = T(z;H_0,G)\Phi, \quad z\in\rho(H_0).$$

This implies the first assertion of the lemma and the identity

$$\begin{aligned} &n_\pm(1;A(\lambda+i0;H_0,G)+tK(\lambda+i0;H_0,G)) \\ &\quad = n_\pm(1;A(\lambda+i0;H_0,|G|)+tK(\lambda+i0;H_0,|G|)). \end{aligned}$$

Integrating this relation in t with the weight $\pi^{-1}(1+t^2)^{-1}$, we arrive at (2.5). □

Lemma 2.4. *Let V_1, V_2 be selfadjoint operators in $\mathcal{H}$ such that $0\le V_1\le V_2$. Suppose that for some $\lambda\in\mathbb{R}$ Condition* 2.1 *is fulfilled for the pair H_0, $V_2^{1/2}$; then so it is for the pair H_0, $V_1^{1/2}$, and*

$$\mathcal{N}_\pm(\lambda;H_0,V_1^{1/2}) \le \mathcal{N}_\pm(\lambda;H_0,V_2^{1/2}).$$

Proof. We write $V_1 = (BV_2^{1/2})^*BV_2^{1/2}$, where $\|B\|\le 1$. Then

$$T(z;H_0,V_1^{1/2}) = BT(z;H_0,V_2^{1/2})B^*, \quad z\in\rho(H_0).$$

It follows that Condition 2.1 is satisfied for the pair H_0, $V_1^{1/2}$, and

$$\begin{aligned} &n_\pm(1;A(\lambda+i0;H_0,V_1^{1/2})+tK(\lambda+i0;H_0,V_1^{1/2})) \\ &\quad \le n_\pm(1;A(\lambda+i0;H_0,V_2^{1/2})+tK(\lambda+i0;H_0,V_2^{1/2})). \end{aligned}$$

Integrating this inequality in t with the weight $\pi^{-1}(1+t^2)^{-1}$, we get the desired result. □

2.2. Relationship between $\mathcal{N}_\pm$ and the "counting function" $N_\pm$. Let $\lambda \in \rho(H_0)\cap\mathbb{R}$. Comparing (1.8) and (2.4) and observing that $K(\lambda+i0; H_0, G) = 0$ and $A(\lambda + i0; H_0, G) = T(\lambda + i0; H_0, G)$, we arrive at the following statement.

Proposition 2.5. *For $\lambda \in \rho(H_0)\cap\mathbb{R}$ we have*

$$\mathcal{N}_\pm(\lambda; H_0, G) = N_\pm(\lambda; H_0, G, 1). \tag{2.6}$$

2.3. Relationship between $\mathcal{N}_\pm$ and the SSF. Here we formulate the results of [25] that will be used in what follows.

1) *Trace class perturbations.* Let H_0 be a selfadjoint operator in $\mathcal{H}$, and let $G \in \mathbf{S}_2(\mathcal{H}, \mathcal{K})$. It is well known that under these conditions for a.e. $\lambda \in \mathbb{R}$ the limits (2.2) exist in the Hilbert–Schmidt norm and that (2.3) is true (see [2] or [36]).

Proposition 2.6 ([25]). *If H_0 is a selfadjoint operator in $\mathcal{H}$ and $G \in \mathbf{S}_2(\mathcal{H}, \mathcal{K})$, then*

$$\xi(\lambda; H_0 \pm G^*G, H_0) = \pm\mathcal{N}_\mp(\lambda; H_0, G) \quad \textit{for a.e. } \lambda \in \mathbb{R}. \tag{2.7}$$

2) *Relatively nuclear perturbations.* Let H_0 be a selfadjoint lower semibounded operator in $\mathcal{H}$,

$$H_0 = H_0^* \text{ in } \mathcal{H}, \quad -\infty < \inf\sigma(H_0). \tag{2.8}$$

Suppose that the operator $G\colon \mathcal{H} \to \mathcal{K}$ satisfies conditions (1.3), and for some $m > 0$ we have

$$GR_0^m(\mu) \in \mathbf{S}_2(\mathcal{H}, \mathcal{K}), \quad \mu < \inf\sigma(H_0). \tag{2.9}$$

Then for a.e. $\lambda \in \mathbb{R}$ Condition 2.1 is fulfilled for the pair H_0, G (see, e.g., Corollary 3.8 below). Let $H_\pm = H_\pm(H_0, G)$ (see Proposition 1.1). In order to define the SSF for the pair H_0, $H_\pm$, assume additionally that for some $k > 0$ and some $\lambda_0 < \inf(\sigma(H_0)\cup\sigma(H_\pm))$ we have

$$(H_\pm - \lambda_0 I)^{-k} - (H_0 - \lambda_0 I)^{-k} \in \mathbf{S}_1(\mathcal{H}). \tag{2.10±}$$

Relation (2.10) allows us to define the SSF via (0.8) with $h(\lambda) = (\lambda - \lambda_0)^{-k}$.

Proposition 2.7 ([25]). *Under conditions* (1.3), (2.8)–(2.10) *we have*

$$\xi(\lambda; H_\pm, H_0) = \pm\mathcal{N}_\mp(\lambda; H_0, G) \quad \textit{for a.e. } \lambda \in \mathbb{R}. \tag{2.11}$$

Finally, we mention two conditions ensuring (2.10).

Proposition 2.8. (i) *If conditions* (1.3) *and* (2.9) *are true with $m = 1$, then* (2.10) *with $k = 1$ is true for all $\lambda_0 \in \rho(H_0)\cap\rho(H_\pm)$.*

(ii) *Under conditions* (1.3) *and* (2.8), *if for some $m > 0$ we have*

$$(GR_0^{1/2}(\mu))^*(GR_0^m(\mu)) \in \mathbf{S}_1(\mathcal{H}), \quad \mu < \inf\sigma(H_0), \tag{2.12}$$

then (2.10) *is true for all integers $k > m + 1/2$ provided that the absolute value of $\lambda_0 < \inf(\sigma(H_0)\cup\sigma(H_\pm))$ is sufficiently large.*

Statement (i) follows directly from the definition (1.6). Statement (ii) was proved in [28, Theorem XI.12].

§3. Estimates for $\mathcal{N}_\pm$

In this section, H_0 is a selfadjoint operator in $\mathcal{H}$ and G is an operator acting from $\mathcal{H}$ into $\mathcal{K}$ and satisfying conditions (1.3).

3.1. Monotonicity estimates. For $\lambda \in \mathbb{R}$ we put

$$E(\lambda) := E_{H_0}(-\infty, \lambda), \qquad G_-(\lambda) := GE(\lambda), \qquad G_+(\lambda) := GE_{H_0}(\lambda, +\infty). \tag{3.1}$$

Lemma 3.1. *Let λ, $\lambda_\pm$ be numbers satisfying $\pm(\lambda - \lambda_\pm) > 0$. Then Condition* 2.1 *is fulfilled for H_0, $G_\pm(\lambda_\pm)$ if and only if it is fulfilled for H_0, G, and*

$$\mathcal{N}_\pm(\lambda; H_0, G) \le \mathcal{N}_\pm(\lambda; H_0, G_\pm(\lambda_\pm)). \tag{3.2$\pm$}$$

Proof. The first assertion follows from Lemma 2.2. For definiteness, we prove (3.2−). It is easily seen that

$$A(\lambda + i0; H_0, G) \ge A(\lambda + i0; H_0, G_-(\lambda_-)),$$
$$K(\lambda + i0; H_0, G) = K(\lambda + i0; H_0, G_-(\lambda_-)).$$

This implies (3.2−). □

Corollary 3.2. *For any $\lambda \in \mathbb{R}$ we have the estimate*[1]

$$\int_{\pm(t-\lambda)\ge 0} \mathcal{N}_\pm(t; H_0, G)\, dt \le \|G_\pm(\lambda)\|^2_{\mathbf{S}_2}. \tag{3.3$\pm$}$$

Proof. Using Lemma 3.1, Proposition 2.6, and inequality (0.4), we obtain

$$\begin{aligned}
&\int_{\pm(t-\lambda)\ge 0} \mathcal{N}_\pm(t; H_0, G)\, dt \\
&\quad \le \int_{\pm(t-\lambda)\ge 0} \mathcal{N}_\pm(t; H_0, G_\pm(\lambda))\, dt = \int_{\pm(t-\lambda)\ge 0} |\xi(t; H_0 \mp G^*_\pm(\lambda) G_\pm(\lambda), H_0)|\, dt \\
&\quad \le \int_{-\infty}^{\infty} |\xi(t; H_0 \mp G^*_\pm(\lambda) G_\pm(\lambda), H_0)|\, dt \\
&\quad \le \|G^*_\pm(\lambda) G_\pm(\lambda)\|_{\mathbf{S}_1} = \|G_\pm(\lambda)\|^2_{\mathbf{S}_2}. \quad \square
\end{aligned}$$

Lemma 3.3. *Let $f\colon \mathbb{R} \to [0, \infty)$ be a function such that $\pm f$ is monotone nondecreasing. Then*

$$\int_{-\infty}^{\infty} \mathcal{N}_\pm(\lambda; H_0, G) f(\lambda)\, d\lambda \le \|G\sqrt{f(H_0)}\|^2_{\mathbf{S}_2}. \tag{3.4$\pm$}$$

Proof. For definiteness, we consider the case of the lower signs. First, we assume that $f(\lambda) \to 0$ as $\lambda \to +\infty$. We denote

$$\lambda_{\max} := \sup \operatorname{supp} f \le +\infty. \tag{3.5}$$

[1] We remind the reader that in the upper estimates all quantities on the right-hand side are assumed to be finite.

Since $G\sqrt{f(H_0)} \in \mathbf{S}_2$, it follows that $G_-(\lambda) \in \mathbf{S}_2$ for $\lambda < \lambda_{\max}$. We fix two numbers R_1, R_2 such that $R_1 < R_2 < \lambda_{\max}$ (from what follows it will be seen that if f is bounded, we can take $R_1 = -\infty$). For $\lambda \in (R_1, R_2)$, we put $S(\lambda) := \int_{R_1}^{\lambda} \mathcal{N}_-(t; H_0, G)\, dt$ (since $G_-(\lambda) \in \mathbf{S}_2$, inequality (3.3−) shows that this integral converges). Integrating by parts, we get

$$\begin{aligned} &\int_{R_1}^{R_2} \mathcal{N}_-(\lambda; H_0, G)(f(\lambda) - f(R_2))\, d\lambda \\ &\qquad = \int_{R_1}^{R_2} (f(\lambda) - f(R_2))\, dS(\lambda) = -\int_{R_1}^{R_2} S(\lambda)\, df(\lambda) \\ &\qquad \le -\int_{R_1}^{R_2} \|G_-(\lambda)\|^2_{\mathbf{S}_2}\, df(\lambda), \end{aligned} \tag{3.6}$$

where at the last step we have used inequality (3.3−). Next, we integrate by parts once again to obtain

$$\begin{aligned} &-\int_{R_1}^{R_2} \|G_-(\lambda)\|^2_{\mathbf{S}_2}\, df(\lambda) \\ &= -\operatorname{Tr}(G_-(R_2) \int_{R_1}^{R_2} E(\lambda)\, d(f(\lambda) - f(R_2))G_-^*(R_2)) \\ &= \operatorname{Tr}(G_-(R_2)(f(R_1) - f(R_2))E(R_1)G_-^*(R_2) \\ &\quad + \operatorname{Tr}(G_-(R_2)(f(H_0) - f(R_2)I)E_{H_0}(R_1, R_2)G_-^*(R_2)) \\ &\le \operatorname{Tr}(G_-(R_2)f(H_0)E(R_1)G_-^*(R_2)) + \operatorname{Tr}(G_-(R_2)f(H_0)G_-^*(R_2)) \\ &\le \|G\sqrt{f(H_0)}E(R_1)\|^2_{\mathbf{S}_2} + \|G\sqrt{f(H_0)}\|^2_{\mathbf{S}_2}. \end{aligned} \tag{3.7}$$

Combining (3.6) and (3.7), we see that

$$\int_{R_1}^{R_2} \mathcal{N}_-(\lambda; H_0, G)(f(\lambda) - f(R_2))\, d\lambda \le \|G\sqrt{f(H_0)}\|^2_{\mathbf{S}_2} + \|G\sqrt{f(H_0)}E(R_1)\|^2_{\mathbf{S}_2}.$$

Now, letting $R_1 \to -\infty$ and $R_2 \to \lambda_{\max}$, we arrive at (3.4−).

Suppose that $\lim_{\lambda\to+\infty} f(\lambda) =: f(\infty) \neq 0$. Then $G\sqrt{f(H_0)} \in \mathbf{S}_2$ implies $G \in \mathbf{S}_2$. Hence, by Proposition 2.6 and inequality (0.4), we have

$$\int_{-\infty}^{\infty} \mathcal{N}_-(\lambda; H_0, G)\, d\lambda \le \|G^*G\|_{\mathbf{S}_1} = \|G\|^2_{\mathbf{S}_2}. \tag{3.8}$$

The first part of the proof shows that

$$\begin{aligned} &\int_{-\infty}^{\infty} \mathcal{N}_-(\lambda; H_0, G)(f(\lambda) - f(\infty))\, d\lambda \\ &\qquad \le \|G\sqrt{f(H_0) - f(\infty)I}\|^2_{\mathbf{S}_2} \\ &\qquad = \operatorname{Tr}(G(f(H_0) - f(\infty)I)G^*) = \|G\sqrt{f(H_0)}\|^2_{\mathbf{S}_2} - f(\infty)\|G\|^2_{\mathbf{S}_2}. \end{aligned}$$

Combining this inequality with (3.8), we arrive at (3.4−). □

The following result of [4] is very close to Lemma 3.3.

Proposition 3.4. *Suppose that* $H_0 = H_0^*$ *in* $\mathcal{H}$, $V = V^* \in \mathbf{S}_1(\mathcal{H})$, *and* $H = H_0 + V$. *Let* $f(\lambda)$, $\lambda \in \mathbb{R}$, *be a continuous*[2] nonnegative monotone nonincreasing function. Then

$$\operatorname{Tr}(f(H)V) \le \int_{-\infty}^{\infty} \xi(\lambda; H, H_0) f(\lambda)\, d\lambda \le \operatorname{Tr}(f(H_0)V).$$

It is easy to check that for trace-class perturbations of definite sign and for bounded functions f Proposition 3.4 and Lemma 3.3 are equivalent.

In a certain important special case, estimates of the form (3.4−) for the SSF can be proved straightforwardly, without using $\mathcal{N}_-$.

Proposition 3.5. *Let* $H_0 \ge 0$ *be a selfadjoint operator in* $\mathcal{H}$. *Suppose that* G *satisfies* (1.3) *and that* $GR_0(-1) \in \mathbf{S}_2(\mathcal{H}, \mathcal{K})$. *We put* $H_+ = H_+(H_0, G)$. *Then for any* $\lambda_0 < 0$ *relation* (2.10+) *is true with* $k = 1$; *thus, the SSF for the pair* H_0, H_+ *is well defined. Moreover, we have*

$$\int_{-\infty}^{\infty} (\lambda - \lambda_0)^{-2} \xi(\lambda; H_+, H_0)\, d\lambda \le \|GR_0(\lambda_0)\|^2_{\mathbf{S}_2}, \quad \lambda_0 < 0. \tag{3.9}$$

Proof. Relation (2.10+) with $k = 1$ follows from (1.6). On the basis of (2.10+), the SSF $\xi(\lambda; H_+, H_0)$ can be defined via (0.8) (with $H = H_+$ and $h(\lambda) = (\lambda - \lambda_0)^{-1}$). Applying (0.4) and taking into account the fact that $T(\lambda_0; H_0, G) \ge 0$, we find

$$\begin{aligned}
&\int_{-\infty}^{\infty} (\lambda - \lambda_0)^{-2} \xi(\lambda; H_+, H_0)\, d\lambda \\
&\quad \le \|(H_+ - \lambda_0 I)^{-1} - (H_0 - \lambda_0 I)^{-1}\|_{\mathbf{S}_1} \\
&\quad = \|(GR_0(\lambda_0))^*(I + T(\lambda_0; H_0, G))^{-1}(GR_0(\lambda_0))\|_{\mathbf{S}_1} \\
&\quad \le \|GR_0(\lambda_0)\|^2_{\mathbf{S}_2}. \quad \square
\end{aligned}$$

3.2. Partition of the spectrum. Let $\delta_1, \delta_2 \subset \mathbb{R}$ be Borel sets such that $\delta_1 \cap \delta_2 = \varnothing$ and $\sigma(H_0) \subset \delta_1 \cup \delta_2$. We put $G_j = GE_{H_0}(\delta_j)$, $j = 1, 2$.

Lemma 3.6. *If for some* $\lambda \in \mathbb{R}$ *Condition* 2.1 *is fulfilled for the pairs of operators* H_0, G_j, $j = 1, 2$, *then so is it for the pair* H_0, G, *and*

(3.10±)
$$\mathcal{N}_\pm(\lambda; H_0, G) \le \mathcal{N}_\pm(\lambda; H_0, (1-\theta)^{-1/2} G_1) + \mathcal{N}_\pm(\lambda; H_0, \theta^{-1/2} G_2), \quad \theta \in (0,1),$$
(3.11±)
$$\mathcal{N}_\pm(\lambda; H_0, G) \ge \mathcal{N}_\pm(\lambda; H_0, (1+\theta)^{-1/2} G_1) - \mathcal{N}_\mp(\lambda; H_0, \theta^{-1/2} G_2), \quad \theta > 0.$$

Proof. Clearly, for all $z \in \rho(H_0)$ we can write

$$\begin{aligned}
T(z; H_0, G) &= T(z; H_0, G_1) + T(z; H_0, G_2), \\
A(z; H_0, G) &= A(z; H_0, G_1) + A(z; H_0, G_2), \\
K(z; H_0, G) &= K(z; H_0, G_1) + K(z; H_0, G_2),
\end{aligned} \tag{3.12}$$

[2]It is easily seen that the condition of continuity of f can be lifted.

which implies the first assertion of the lemma. Using (3.12) and (1.1), we obtain

$$\begin{aligned} &n_\pm(1; A(\lambda+i0;H_0,G)+tK(\lambda+i0;H_0,G)) \\ &\quad\le n_\pm(\theta; A(\lambda+i0;H_0,G_1)+tK(\lambda+i0;H_0,G_1)) \\ &\qquad + n_\pm(1-\theta; A(\lambda+i0;H_0,G_2)+tK(\lambda+i0;H_0,G_2)). \end{aligned}$$

Integrating this inequality in t with the weight $\pi^{-1}(1+t^2)^{-1}$, we arrive at (3.10). Relation (3.13) can be proved in a similar way, by using (1.2) instead of (1.1). □

Remark 3.7. We sketch another proof of Lemma 3.6, which clarifies the role of the fact that the perturbation is of definite sign. In order to prove (3.10), first we observe that, as is easily checked, for any $\theta \in (0,1)$ we have

$$V = G^*G \le (1-\theta)^{-1}G_1^*G_1 + \theta^{-1}G_2^*G_2 = \widetilde{G}^*\widetilde{G},$$

where the operator $\widetilde{G} = (1-\theta)^{-1/2}G_1 \oplus \theta^{-1/2}G_2$ acts from $\mathcal{H}$ to $\mathcal{K}\oplus\mathcal{K}$. Here the condition $V \ge 0$ is essential. Next, we use Lemma 2.4 (for $V_1 = V$, $V_2 = \widetilde{G}^*\widetilde{G}$) and Lemma 2.3 to obtain

$$\begin{aligned} \mathcal{N}_\pm(\lambda;H_0,G) &\le \mathcal{N}_\pm(\lambda;H_0,(\widetilde{G}^*\widetilde{G})^{1/2}) = \mathcal{N}_\pm(\lambda;H_0,\widetilde{G}) \\ &= \mathcal{N}_\pm(\lambda;H_0,(1-\theta)^{-1/2}G_1) + \mathcal{N}_\pm(\lambda;H_0,\theta^{-1/2}G_2). \end{aligned}$$

The proof of (3.11) is similar. Also, we note that there is a version of Lemma 3.6 for a partition of the spectrum into an arbitrary finite number of sets δ_j.

Corollary 3.8. *Suppose that*

$$GE_{H_0}(\delta) \in \mathbf{S}_2(\mathcal{H},\mathcal{K}) \tag{3.13}$$

for some open interval $\delta \subset \mathbb{R}$. Then for a.e. $\lambda \in \delta$ Condition 2.1 *is fulfilled for the pair H_0, G, and*

$$\mathcal{N}_\pm(\lambda;H_0,G) \in L_{1,\mathrm{loc}}(\delta). \tag{3.14}$$

In particular, if (3.13) *is true for any bounded interval $\delta \subset \mathbb{R}$, then*

$$\mathcal{N}_\pm(\lambda;H_0,G) \in L_{1,\mathrm{loc}}(\mathbb{R}).$$

Proof. For definiteness, we consider the case of the upper signs. In (3.10+) we take $\theta = 1/2$, $\delta_1 = \delta$, $\delta_2 = \mathbb{R}\setminus\delta$. By (3.13), Proposition 2.6, and inequality (0.4), we have

$$\mathcal{N}_+(\lambda;H_0,\sqrt{2}G_1) = -\xi(\lambda;H_0-2G_1^*G_1,H_0) \in L_1(\mathbb{R}).$$

At the same time, it is easy to check that $T(\lambda+i0;H_0,G_2)$ is selfadjoint and that $\mathcal{N}_+(\lambda;H_0,\sqrt{2}G_2) = n_+(1;2T(\lambda+i0;H_0,G_2))$ is a monotone nonincreasing $\mathbb{Z}_+$-valued function of $\lambda \in \delta$, possibly unbounded near $\sup\delta$. It follows that $\mathcal{N}_+(\lambda;H_0,\sqrt{2}G_2) \in L_{1,\mathrm{loc}}(\delta)$, which yields (3.14). □

We note that for the SSF there is no statement similar to Corollary 3.8. Namely, operators H_0, G and an interval δ can be constructed such that the hypotheses of Corollary 3.8 are true, but $\varphi(H_\pm) - \varphi(H_0) \notin \mathbf{S}_1$ for some $\varphi \in C_0^\infty(\delta)$. Thus, the SSF for the pair $H_\pm$, H_0 cannot be defined on the interval δ in any reasonable sense.

§4. Estimates for $\mathcal{N}_+ - N_+$

The content of this section is aimed at applications to the asymptotics of SSF in the large coupling constant limit. The operator H_0 is assumed to be nonnegative; we consider the function $\mathcal{N}_+(\lambda; H_0, G)$. For $\lambda < \inf\sigma(H_0)$, the asymptotic behavior of $\mathcal{N}_+(\lambda; H_0, \sqrt{t}G) = N_+(\lambda; H_0, G, t)$ as $t \to \infty$ is well studied in applications. For this reason, we estimate the difference

$$Q(\lambda; \Lambda_-, \alpha) := \mathcal{N}_+(\lambda; H_0, G) - N_+(\Lambda_-; H_0, G, \alpha),$$

where $\Lambda_- < \inf\sigma(H_0)$ is fixed, and $\alpha > 0$ is some auxiliary parameter. Let

$$Q_\pm(\lambda; \Lambda_-, \alpha) := (Q(\lambda; \Lambda_-, \alpha))_\pm.$$

We keep the notation introduced in (3.1) and set $\widetilde{E}(\lambda) = I - E(\lambda)$.

4.1. Local estimates.

Lemma 4.1. *Suppose that conditions* (1.3) *are satisfied for the operators* $H_0 \geq 0$ *and* G. *Then for any* $\theta \in (0,1)$ *and any* $\Lambda_- \leq \lambda_- < \lambda_+ < \Lambda_+$, $\Lambda_- < \inf\sigma(H_0)$, *we have*

$$\int_{\lambda_-}^{\lambda_+} Q_+\Big(\lambda, \Lambda_-, \frac{\Lambda_+ - \Lambda_-}{(1-\theta)(\Lambda_+ - \lambda_+)}\Big)\, d\lambda \leq \theta^{-1}\|GE_{H_0}(\lambda_-, \Lambda_+)\|^2_{\mathbf{S}_2}. \tag{4.1}$$

Proof. We apply estimate (3.10+) for $\delta_1 = [\Lambda_+, +\infty)$, $\delta_2 = (-\infty, \Lambda_+)$, $\lambda \in (\lambda_-, \lambda_+) \subset \delta_2$, obtaining

$$\mathcal{N}_+(\lambda; H_0, G) \leq \mathcal{N}_+(\lambda; H_0, (1-\theta)^{-1/2}G\widetilde{E}(\Lambda_+)) + \mathcal{N}_+(\lambda; H_0, \theta^{-1/2}G_-(\Lambda_+)). \tag{4.2}$$

We dominate the first summand on the right in (4.2). To do this, first we observe that for $\lambda \leq \lambda_+$ we have the following simple relation:

$$R_0(\lambda)\widetilde{E}(\Lambda_+) \leq \frac{\Lambda_+ - \Lambda_-}{\Lambda_+ - \lambda_+} R_0(\Lambda_-).$$

This implies that

$$T(\lambda + i0; H_0, G\widetilde{E}(\Lambda_+)) \leq \frac{\Lambda_+ - \Lambda_-}{\Lambda_+ - \lambda_+} T(\Lambda_-; H_0, G), \tag{4.3}$$

which yields the estimate

$$\begin{aligned}\mathcal{N}_+(\lambda; H_0, (1-\theta)^{-1/2}G\widetilde{E}(\Lambda_+)) &= n_+(1; (1-\theta)^{-1}T(\lambda + i0; H_0, G\widetilde{E}(\Lambda_+)) \\ \leq n_+\Big(1; \frac{\Lambda_+ - \Lambda_-}{(1-\theta)(\Lambda_+ - \lambda_+)} T(\Lambda_-; H_0, G)\Big) &= N_+\Big(\Lambda_-; H_0, G, \frac{\Lambda_+ - \Lambda_-}{(1-\theta)(\Lambda_+ - \lambda_+)}\Big)\end{aligned}$$

for any $\lambda \leq \lambda_+$. Substituting this in (4.2), we find:

$$Q_+\Big(\lambda; \Lambda_-, \frac{\Lambda_+ - \Lambda_-}{(1-\theta)(\Lambda_+ - \lambda_+)}\Big) \leq \mathcal{N}_+(\lambda; H_0, \theta^{-1/2}G_-(\Lambda_+)).$$

Integrating the latter inequality in λ and using (3.3+), we see that

$$\begin{aligned}&\int_{\lambda_-}^{\lambda_+} Q_+\Big(\lambda, \Lambda_-, \frac{\Lambda_+ - \Lambda_-}{(1-\theta)(\Lambda_+ - \lambda_+)}\Big)\, d\lambda \\ &\quad\leq \int_{\lambda_-}^{\lambda_+} \mathcal{N}_+(\lambda; H_0, \theta^{-1/2}G_-(\Lambda_+))\, d\lambda \leq \int_{\lambda_-}^{\infty} \mathcal{N}_+(\lambda; H_0, \theta^{-1/2}G_-(\Lambda_+))\, d\lambda \\ &\quad\leq \theta^{-1}\|GE_{H_0}(\lambda_-, \Lambda_+)\|^2_{\mathbf{S}_2}. \quad \square\end{aligned}$$

Lemma 4.2. *Under the assumptions of Lemma* 4.1, *for any* θ_1, $\theta_2 > 0$ *and any* $\Lambda_- \le \lambda_- < \lambda_+$, $\Lambda_- < \inf \sigma(H_0)$, *we have*

$$
\begin{aligned}
&\int_{\lambda_-}^{\lambda_+} Q_-(\lambda, \Lambda_-, (1+\theta_1+\theta_2)^{-1})\, d\lambda \\
&\qquad \le \theta_1^{-1}(\lambda_+ - \lambda_-)\|G_-(\lambda_+)R_0^{1/2}(\Lambda_-)\|^2_{\mathbf{S}_2} + \theta_2^{-1}\|G_-(\lambda_+)\|^2_{\mathbf{S}_2}.
\end{aligned} \tag{4.4}
$$

Proof. We apply (3.11+) with $\delta_1 = [\lambda_+, +\infty)$, $\delta_2 = (-\infty, \lambda_+)$, $\lambda \in (\lambda_-, \lambda_+) \subset \delta_2$, $\theta = \theta_2$; this yields

$$
\mathcal{N}_+(\lambda; H_0, G) \ge \mathcal{N}_+(\lambda; H_0, (1+\theta_2)^{-1/2} G\widetilde{E}(\lambda_+)) - \mathcal{N}_-(\lambda; H_0, \theta_2^{-1/2} G_-(\lambda_+)). \tag{4.5}
$$

In order to get a lower estimate of the first summand on the right in (4.5), first we observe that for $\lambda \in (\lambda_-, \lambda_+)$ we can write

$$
\widetilde{E}(\lambda_+)R_0(\lambda) \ge \widetilde{E}(\lambda_+)R_0(\Lambda_-) = R_0(\Lambda_-) - E(\lambda_+)R_0(\Lambda_-),
$$

which implies that

$$
\begin{aligned}
&T(\lambda + i0; H_0, G\widetilde{E}(\lambda_+)) \\
&\qquad \ge T(\Lambda_-; H_0, G\widetilde{E}(\lambda_+)) = T(\Lambda_-; H_0, G) - T(\Lambda_-; H_0, G_-(\lambda_+)).
\end{aligned}
$$

Recalling (1.2), we deduce that

$$
\begin{aligned}
\mathcal{N}_+(\lambda; H_0, (1+\theta_2)^{-1/2} G\widetilde{E}(\lambda_+)) &= n_+(1+\theta_2, T(\lambda + i0; H_0, G\widetilde{E}(\lambda_+))) \\
&\ge n_+(1+\theta_2+\theta_1; T(\Lambda_-; H_0, G)) - n_+(\theta_1; T(\Lambda_-; H_0, G_-(\lambda_+))) \\
&= N_+(\Lambda_-; H_0, G, (1+\theta_1+\theta_2)^{-1}) - n_+(\theta_1; T(\Lambda_-; H_0, G_-(\lambda_+))).
\end{aligned} \tag{4.6}
$$

The second summand on the right in (5.6) will be estimated as follows:

$$
\begin{aligned}
&n_+(\theta_1; T(\Lambda_-; H_0, G_-(\lambda_+))) \\
&\qquad \le \theta_1^{-1}\|T(\Lambda_-; H_0, G_-(\lambda_+))\|_{\mathbf{S}_1} = \theta_1^{-1}\|G_-(\lambda_+)R_0^{1/2}(\Lambda_-)\|^2_{\mathbf{S}_2}.
\end{aligned} \tag{4.7}
$$

Substituting (4.6) and (4.7) in (4.5), we obtain

$$
\begin{aligned}
&Q_-(\lambda, \Lambda_-, (1+\theta_1+\theta_2)^{-1}) \\
&\qquad \le \theta_1^{-1}\|G_-(\lambda_+)R_0^{1/2}(\Lambda_-)\|^2_{\mathbf{S}_2} + \mathcal{N}_-(\lambda; H_0, \theta_2^{-1/2} G_-(\lambda_+)).
\end{aligned}
$$

We integrate this inequality in λ and estimate the second summand on the right in accordance with (0.4) to get

$$
\begin{aligned}
&\int_{\lambda_-}^{\lambda_+} Q_-(\lambda, \Lambda_-, (1+\theta_1+\theta_2)^{-1})\, d\lambda \\
&\qquad \le \theta_1^{-1}(\lambda_+ - \lambda_-)\|G_-(\lambda_+)R_0^{1/2}(\Lambda_-)\|^2_{\mathbf{S}_2} + \int_{\lambda_-}^{\lambda_+} \mathcal{N}_-(\lambda; H_0, \theta_2^{-1/2} G_-(\lambda_+))\, d\lambda \\
&\qquad \le \theta_1^{-1}(\lambda_+ - \lambda_-)\|G_-(\lambda_+)R_0^{1/2}(\Lambda_-)\|^2_{\mathbf{S}_2} + \theta_2^{-1}\|G_-(\lambda_+)\|^2_{\mathbf{S}_2}. \quad \square
\end{aligned}
$$

4.2. Weighted estimates.

Lemma 4.3. *Under the assumptions of Lemma* 4.1, *let* $\Lambda_- < \inf \sigma(H_0)$, *and let* $f\colon (\Lambda_-, \infty) \to [0, \infty)$ *be a monotone nonincreasing function such that* $f(\lambda) \to 0$ *as* $\lambda \to \infty$. *Then for any* θ_1, $\theta_2 \in (0, 1)$ *we have*

$$\begin{aligned}\int_{\Lambda_-}^{\infty} & Q_+(\lambda; \Lambda_-, (1-\theta_1)^{-1}(1-\theta_2)^{-1}) f(\lambda)\, d\lambda \\ & \leq \theta_1^{-1} \|G\sqrt{f(\theta_2 H_0 + \Lambda_-(1-\theta_2))}\|_{\mathbf{S}_2}^2.\end{aligned} \tag{4.8}$$

Proof. Replacing H_0 and $f(x)$ by $H_0 + \Lambda_-$ and $f(x - \Lambda_-)$, we see that it suffices to consider the case where $\Lambda_- = 0$. We restore the notation (3.5). The relation $G\sqrt{f(\theta_2 H_0)} \in \mathbf{S}_2$ implies that $G_-(\lambda/\theta_2) \in \mathbf{S}_2$ for $\lambda < \lambda_{\max}$. Taking $\Lambda_- = \lambda_- = 0$, $\theta = \theta_1$ and $\lambda_+/\Lambda_+ = \theta_2$ in (4.1), we get

$$\int_0^{\lambda} Q_+(t; 0, (1-\theta_1)^{-1}(1-\theta_2)^{-1})\, dt \leq \theta_1^{-1}\|G_-(\lambda/\theta_2)\|_{\mathbf{S}_2}^2, \quad \lambda < \lambda_{\max}. \tag{4.9}$$

Let $S(\lambda)$ denote the left-hand side of (4.9). As in the proof of Lemma 3.3, we integrate by parts and use (3.7) to obtain

$$\begin{aligned}\int_0^R & Q_+(\lambda; 0, (1-\theta_1)^{-1}(1-\theta_2)^{-1})(f(\lambda) - f(R))\, d\lambda = \int_0^R (f(\lambda) - f(R))\, dS(\lambda) \\ & = -\int_0^R S(\lambda)\, df(\lambda) \leq -\theta_1^{-1}\int_0^R \|G_-(\lambda/\theta_2)\|_{\mathbf{S}_2}^2\, df(\lambda) \\ & \leq \theta_1^{-1}\|G\sqrt{f(\theta_2 H_0)}\|_{\mathbf{S}_2}^2,\end{aligned}$$

where $R < \lambda_{\max}$. Letting $R \to \lambda_{\max}$, we arrive at (4.8). $\square$

Lemma 4.4. *Under the assumptions of Lemma* 4.1, *let* $f\colon (0, \infty) \to [0, \infty)$ *be a monotone nonincreasing integrable function*, *and let* $F(\lambda) = \int_\lambda^\infty f(t)\, dt$. *Then for all* $\Lambda_- < 0$, θ_1, $\theta_2 > 0$ *we have*

$$\begin{aligned}\int_0^\infty & Q_-(\lambda, \Lambda_-, (1+\theta_1+\theta_2)^{-1}) f(\lambda)\, d\lambda \\ & \leq \theta_1^{-1}\|GR_0^{1/2}(\Lambda_-)\sqrt{F(H_0)}\|_{\mathbf{S}_2}^2 + (\theta_1^{-1} + \theta_2^{-1})\|G\sqrt{f(H_0)}\|_{\mathbf{S}_2}^2.\end{aligned} \tag{4.10}$$

Proof. First, assume that f is bounded. Taking $\lambda_- = 0$ in (4.4), we obtain

$$\begin{aligned}\int_0^\lambda & Q_-(t, \Lambda_-, (1+\theta_1+\theta_2)^{-1})\, dt \\ & \leq \theta_1^{-1}\lambda\|G_-(\lambda)R_0^{1/2}(\Lambda_-)\|_{\mathbf{S}_2}^2 + \theta_2^{-1}\|G_-(\lambda)\|_{\mathbf{S}_2}^2.\end{aligned} \tag{4.11}$$

Since $G\sqrt{f(H_0)} \in \mathbf{S}_2$, the right-hand side of (4.11) is finite for $\lambda < \lambda_{\max}$ (see (3.5)); we denote by $S(\lambda)$ the left hand side of this inequality. As in the proof of

Lemma 4.3, we integrate by parts and use (4.11) to obtain

(4.12)
$$\begin{aligned}\int_0^R & Q_-(\lambda,\Lambda_-,(1+\theta_1+\theta_2)^{-1})(f(\lambda)-f(R))\,d\lambda\\ &= \int_0^R (f(\lambda)-f(R))\,dS(\lambda) = -\int_0^R S(\lambda)\,df(\lambda)\\ &\leq -\theta_1^{-1}\int_0^R \lambda\|G_-(\lambda)R_0^{1/2}(\Lambda_-)\|^2_{\mathbf{S}_2}\,df(\lambda) - \theta_2^{-1}\int_0^R \|G_-(\lambda)\|^2_{\mathbf{S}_2}\,df(\lambda),\end{aligned}$$

where $R<\lambda_{\max}$. We estimate the first summand on the right in (4.12). As in (3.7), we get

$$\begin{aligned}-\int_0^R \lambda\|G_-(\lambda)R_0^{1/2}(\Lambda_-)\|^2_{\mathbf{S}_2}\,df(\lambda) &= -\operatorname{Tr}(G_-(R)R_0(\Lambda_-)\int_0^R \lambda E(\lambda)\,df(\lambda)G_-^*(R))\\ &= -\operatorname{Tr}(G_-(R)R_0(\Lambda_-)\int_0^R E(\lambda)\,d(F(\lambda)+\lambda f(\lambda))G_-^*(R))\\ &= \operatorname{Tr}(G_-(R)R_0(\Lambda_-)(F(H_0)+H_0f(H_0)-F(R)I-RF(R)I)G_-^*(R))\\ &\leq \|GR_0^{1/2}(\Lambda_-)\sqrt{F(H_0)}\|^2_{\mathbf{S}_2} + \|G\sqrt{f(H_0)}H_0^{1/2}R_0^{1/2}(\Lambda_-)\|^2_{\mathbf{S}_2}\\ &\leq \|GR_0^{1/2}(\Lambda_-)\sqrt{F(H_0)}\|^2_{\mathbf{S}_2} + \|G\sqrt{f(H_0)}\|^2_{\mathbf{S}_2}.\end{aligned}$$

Substituting this inequality and (3.7) in (4.12), we see that

$$\begin{aligned}\int_0^R & Q_-(\lambda;\Lambda_-,(1+\theta_1+\theta_2)^{-1})(f(\lambda)-f(R))\,d\lambda\\ &\leq \theta_1^{-1}\|GR_0^{1/2}(\Lambda_-)\sqrt{F(H_0)}\|^2_{\mathbf{S}_2} + (\theta_1^{-1}+\theta_2^{-1})\|G\sqrt{f(H_0)}\|^2_{\mathbf{S}_2}.\end{aligned}$$

Letting $R\to\lambda_{\max}$, we arrive at (4.10). Now, suppose that f is unbounded (i.e., f has a singularity at $x=0$). We pick a sequence of bounded integrable functions f_n such that $f_{n+1}(x)\geq f_n(x)$ and $f_n(x)\to f(x)$ for any $x>0$. Using inequality (4.10) for f_n and passing to the limit as $n\to\infty$, we obtain the desired result. □

§5. Preliminary facts on differential operators

5.1. Operators dominated by the "free Hamiltonian". In this subsection H_0 is an operator such that

(5.1) $$H_0 = H_0^* \geq 0 \text{ in } \mathcal{H} = L_2(\mathbb{R}^d), \quad d\geq 1,$$

and

(5.2) $$|e^{-H_0t}\psi| \leq Me^{\beta\Delta t}|\psi|, \quad t>0,\ \psi\in\mathcal{H},$$

for some constants $M>0$, $\beta>0$. Below we shall often use the following obvious observation. Let

(5.3) $$\eta(\lambda) = \int_0^\infty e^{-t\lambda}\,d\mu(t), \quad \text{where } \mu \text{ is a finite measure on } (0,\infty).$$

Then (5.2) implies the inequality

$$|\eta(H_0)\psi| \leq M\eta(\beta(-\triangle))|\psi|, \quad \psi \in \mathcal{H}. \tag{5.4}$$

In particular, taking $d\mu(t) = e^{-t\gamma}t^{m-1}dt$ with $m > 0$, $\gamma > 0$, we obtain

$$|(H_0 + \gamma I)^{-m}\psi| \leq M(\beta(-\triangle) + \gamma I)^{-m}|\psi|, \quad \psi \in \mathcal{H}. \tag{5.5}$$

The following two propositions are obvious generalizations of the corresponding statements in [1].

Proposition 5.1. *Let $V \geq 0$ be the operator of multiplication by a measurable function in $\mathcal{H}$; suppose V is $(-\triangle)$-form compact. Then V is H_0-form compact.*

Proof. We argue as in [1]. Inequality (5.5) implies the pointwise estimate

$$|\sqrt{V}(H_0 + I)^{-1/2}\psi| \leq M\sqrt{V}(\beta(-\triangle) + I)^{-1/2}|\psi|, \quad \psi \in \mathcal{H}.$$

By assumption, the operator on the right-hand side of the above relation is compact; hence (see [1, Theorem 2.2]), so is the operator on the left-hand side. □

Sufficient conditions for the $(-\triangle)$-form compactness of a multiplication operator can be formulated in various terms; see, e.g., [22, 5].

Proposition 5.2. (i) *Let F be the operator of multiplication by a function $F \in L_2(\mathbb{R}^d)$, and let η be a function of the form* (5.3)*. Then*

$$\|F\eta(H_0)\|^2_{\mathbf{S}_2} \leq (2\pi)^{-d}\omega_d \frac{d}{2} M^2 \beta^{-d/2} \int_0^\infty \lambda^{d/2-1}\eta^2(\lambda)\, d\lambda \int |F(x)|^2\, dx; \tag{5.6}$$

in particular,

$$FR_0^m(-1) \in \mathbf{S}_2(\mathcal{H}), \quad m > d/4. \tag{5.7}$$

(ii) *Let F be the operator of multiplication by a function $F \in l_1(\mathbb{Z}^d; L_2(\mathbb{Q}^d))$. Then*

$$FR_0^m(-1) \in \mathbf{S}_1(\mathcal{H}), \quad m > d/2. \tag{5.8}$$

Proof. (i) By (5.4), we have

$$\|F\eta(H_0)\|_{\mathbf{S}_2} \leq \|F\|_{L_2}\|\eta(H_0)\|_{2,\infty} \leq \|F\|_{L_2} M \|\eta(\beta(-\triangle))\|_{2,\infty},$$

where $\|\cdot\|_{2,\infty}$ denotes the norm of an operator acting from $L_2(\mathbb{R}^d)$ into $L_\infty(\mathbb{R}^d)$. Calculating the norms on the right-hand side of the above inequality, we obtain (5.6).

Statement (ii) was proved in [1, Theorem 2.12] in the case of a magnetic Schrödinger operator H_0 (see also [32, Theorem B.9.2]). The proof presented in [1] can be carried over to the general case without any modifications. □

5.2. The Schrödinger operator. As a basic example of an operator of the form (5.1), (5.2), we consider the Schrödinger operator with variable metric and electromagnetic field. In $\mathbb{R}^d$, $d \geq 1$, we fix a real $(d \times d)$-matrix-valued function $g = g(x)$ such that

$$g_- \mathbf{1} \leq g(x) \leq g_+ \mathbf{1}, \qquad 0 < g_- \leq g_+ < \infty \tag{5.9}$$

(here and in what follows $\mathbf{1}$ denotes the unit $(d \times d)$–matrix), a real magnetic vector-potential $\mathbf{A} = \mathbf{A}(x)$ with

$$|\mathbf{A}| \in L_{2,\mathrm{loc}}(\mathbb{R}^d), \tag{5.10}$$

and a scalar electric potential $U = U(x)$ with

$$U \in L_{1,\mathrm{loc}}(\mathbb{R}^d), \quad U \geq 0. \tag{5.11}$$

We introduce the quadratic form

$$h^{(0)}_{g,\mathbf{A},U}[u,u] = \sum_{k,j=1}^{d} \int g_{kj}(x)\Big(-i\frac{\partial u}{\partial x_k} - \mathbf{A}_k u\Big)\overline{\Big(-i\frac{\partial u}{\partial x_j} - \mathbf{A}_j u\Big)}\,dx + \int U|u|^2\,dx \tag{5.12}$$

on the domain $d[h^{(0)}_{g,\mathbf{A},U}] = C_0^\infty(\mathbb{R}^d)$.

Proposition 5.3. *Under conditions* (5.9)–(5.11), *the form* $h^{(0)}_{g,\mathbf{A},U}$ *is closable, and the domain of its closure* $h_{g,\mathbf{A},U}$ *is given by the formula*

$$d[h_{g,\mathbf{A},U}] = \{u \in L_2(\mathbb{R}^d) \mid i\nabla u + \mathbf{A}u \in L_2(\mathbb{R}^d),\ U^{1/2}u \in L_2(\mathbb{R}^2)\}, \tag{5.13}$$

where the expression ∇u *is understood in the distribution sense. The corresponding selfadjoint operator* $H_0 = H_0(g, \mathbf{A}, U)$ *satisfies conditions* (5.1), (5.2) *with some constants* $M > 0$, $\beta > 0$.

For the flat metric ($g_{ij}(x) \equiv \delta_{ij}$) Proposition 5.3 is well known (see, e.g., [34] and the references therein); in this case we have $M = \beta = 1$. Essentially, in the above form Proposition 5.3 was proved in [11]. Some comments on this point can be found in the Appendix.

Proposition 5.4. *Let* $d \geq 3$, *let* $H_0 = H_0(g, \mathbf{A}, U)$ *be a selfadjoint operator corresponding to the form* $h_{g,\mathbf{A},U}$, *and let* V *be the operator of multiplication by a function* $V(x) \geq 0$, $V \in L_{d/2}(\mathbb{R}^d)$. *Then* V *is* H_0*-form compact, and*

$$N_+(\lambda; H_0, \sqrt{V}, 1) \leq C_{5.14}(d) g_-^{-d/2} \int V^{d/2}(x)\,dx, \quad \lambda < 0. \tag{5.14}$$

Proof. Obviously, $H_0(g, \mathbf{A}, U) \geq g_- H_0(1, \mathbf{A}, 0)$. For the operator $H_0(1, \mathbf{A}, 0)$ the following magnetic variant of the Cwikel–Lieb–Rozenblum estimate is known (see [20], and [31, p. 168]):

$$N_+(\lambda; H_0(1, \mathbf{A}, 0), \sqrt{V}, 1) \leq C(d) \int V^{d/2}(x)\,dx, \quad \lambda < 0.$$

This implies (5.14). □

We note that in [23] an estimate of the form (5.14) was proved under conditions on $\mathbf{A}$ and g much more general than (5.9), (5.10). Concerning estimates of $N_+(\lambda; H_0, \sqrt{V}, 1)$ for $d = 2$, see [27].

5.3. The polyharmonic operator. Let

$$H_0 = (-\triangle)^l, \quad l > 0 \text{ in } \mathcal{H} = L_2(\mathbb{R}^d),\ d \geq 1; \tag{5.15}$$

here l is not necessarily an integer. We denote $\varkappa := d/(2l)$.

Proposition 5.5. (i) *Let F be the operator of multiplication by a function $F \in L_2(\mathbb{R}^d)$, and let $f\colon (0,\infty) \to [0,\infty)$ be a function such that $\int_0^\infty \lambda^{\varkappa-1} f^2(\lambda)\, d\lambda < \infty$. Then*

$$\|Ff(H_0)\|^2_{\mathbf{S}_2} \leq (2\pi)^{-d}\omega_d \varkappa \int_0^\infty \lambda^{\varkappa-1} f^2(\lambda) d\lambda \int |F(x)|^2\, dx; \tag{5.16}$$

in particular,

$$FR_0^m(-1) \in \mathbf{S}_2(\mathcal{H}), \quad m > \varkappa/2. \tag{5.17}$$

(ii) *Let F be the operator of multiplication by a function $F \in l_1(\mathbb{Z}^d; L_2(\mathbb{Q}^d))$. Then*

$$FR_0^m(-1) \in \mathbf{S}_1(\mathcal{H}), \quad m > \varkappa. \tag{5.18}$$

Proof. Statement (i) is obtained by a straightforward estimation of the Hilbert–Schmidt norm.

Statement (ii) follows from [6, Theorem 11.1]. □

Proposition 5.6 (see [5])**.** *Let H_0 be the operator* (5.15), *let $\kappa > 1$, and let V be the operator of multiplication by a function $V(x) \geq 0$, $V \in L_\kappa(\mathbb{R}^d)$. Then V is H_0-form compact and*

$$N_+(\lambda; H_0, \sqrt{V}, 1) \leq C_{5.19}(d,l) \int V^\kappa(x)\, dx, \quad \lambda < 0. \tag{5.19}$$

Estimates of $N_+(\lambda; H_0, \sqrt{V}, 1)$ for $\varkappa \leq 1$ can also be found in [5]; they are slightly more involved than (5.19).

§6. Estimates for the SSF: applications

Below we apply the abstract results of §§2–4 to the differential operators H_0 introduced in §5. In Subsection 6.1 we present conditions sufficient for (2.11), and in Subsections 6.2, 6.3 we obtain estimates for $\mathcal{N}_\pm$. Estimates for the SSF arise as direct combinations of the results of Subsections 6.1 and 6.2, 6.3; we do not record these statements explicitly.

To treat the Schrödinger operator $H_0(g, \mathbf{A}, U)$, we need only properties (5.1) and (5.2). Therefore, keeping in mind that the Schrödinger operator is of primary interest for the applications, we formulate the theorems for an arbitrary operator H_0 satisfying (5.1), (5.2).

6.1. Relationship between $\mathcal{N}_\pm$ and the SSF.

Theorem 6.1. (i) *If H_0 satisfies conditions* (5.1) *and* (5.2), *and the perturbation potential $V = V(x) \geq 0$ in $\mathbb{R}^d$ is $(-\triangle)$-form compact and $V \in L_1(\mathbb{R}^d)$, then Condition* 2.1 *for the pair H_0, $\sqrt{V}$ is satisfied for a.e. $\lambda \in \mathbb{R}$, and $\mathcal{N}_\pm(\lambda; H_0, \sqrt{V}) \in L_{1,\mathrm{loc}}(\mathbb{R})$.*

(ii) *If H_0 is the polyharmonic operator* (5.15), *and the perturbation potential $V = V(x) \geq 0$ in $\mathbb{R}^d$ is H_0-form compact and $V \in L_1(\mathbb{R}^d)$, then Condition* 2.1 *for the pair H_0, $\sqrt{V}$ is satisfied for a.e. $\lambda \in \mathbb{R}$, and $\mathcal{N}_\pm(\lambda; H_0, \sqrt{V}) \in L_{1,\mathrm{loc}}(\mathbb{R})$.*

Proof. We check the assumptions of Corollary 3.8 with $\mathcal{K} = \mathcal{H}$ and $G = \sqrt{V}$.

(i) Condition (1.3) (i.e., the H_0-form compactness) follows from Proposition 5.1. Relation (3.13) is valid for any bounded interval $\delta \subset \mathbb{R}$ by Proposition 5.2(i).

(ii) Relation (3.13) is valid for any bounded interval $\delta \subset \mathbb{R}$ by Proposition 5.5(i). □

In connection with the assumptions of Theorem 6.1(i) we note that the relation $V \in L_1(\mathbb{R}^d)$ implies the $(-\triangle)$-form compactness of V for $d = 1$, but, in general, it does not imply even the $(-\triangle)$-form boundedness of V for $d \geq 2$.

Theorem 6.2. (i) *Under the assumptions of Theorem* 6.1(i), *suppose that $d \leq 3$. Let $H_\pm = H_\pm(H_0, \sqrt{V})$. Then, for any $\lambda_0 < \inf(\sigma(H_\pm) \cup \sigma(H_0))$ relation* (2.10) *is fulfilled with $k = 1$; thus, the SSF for the pair $H_\pm$, H_0 is well defined. Identity* (2.11) *is true for a.e. $\lambda \in \mathbb{R}$.*

(ii) *Under the assumptions of Theorem* 6.1(ii), *suppose that $\varkappa < 2$. Let $H_\pm = H_\pm(H_0, \sqrt{V})$. Then for any $\lambda_0 < \inf(\sigma(H_\pm) \cup \sigma(H_0))$ relation* (2.10) *is fulfilled with $k = 1$; thus, the SSF for the pair $H_\pm$, H_0 is well defined. Identity* (2.11) *is true for a.e. $\lambda \in \mathbb{R}$.*

Proof. It suffices to refer to Propositions 2.7 and 2.8(i) with $\mathcal{K} = \mathcal{H}$ and $G = \sqrt{V}$. The relation $\sqrt{V} R_0(-1) \in \mathbf{S}_2$ is a consequence of Proposition 5.2(i) in case (i) and of Proposition 5.5(i) in case (ii) (cf. [1, Theorem 2.11]). □

We have not succeeded in extending Theorem 6.2(i) to the case where $d \geq 4$ and in extending Theorem 6.2(ii) to the case where $\varkappa \geq 2$. However, the quantities $\mathcal{N}_\pm(\lambda; H_0, \sqrt{V})$ are well defined in these cases (see Theorem 6.1).

Theorem 6.3. (i) *Under the assumptions of Theorem* 6.1(i), *suppose that $V \in l_1(\mathbb{Z}^d; L_2(\mathbb{Q}^d))$. Let $H_\pm = H_\pm(H_0, \sqrt{V})$. Then* (2.10) *is fulfilled for all integers $k > (d-1)/2$ and all $\lambda_0 < \inf(\sigma(H_0) \cup \sigma(H_\pm))$ with sufficiently large absolute value; thus, the SSF for the pair H_0, $H_\pm$ is well defined. Identity* (2.11) *is true for a.e. $\lambda \in \mathbb{R}$.*

(ii) *Under the assumptions of Theorem* 6.1(i), *suppose that $V \in l_1(\mathbb{Z}^d; L_2(\mathbb{Q}^d))$. Let $H_\pm = H_\pm(H_0, \sqrt{V})$. Then* (2.10) *is fulfilled for all integers $k > \varkappa - 1/2$ and all $\lambda_0 < \inf(\sigma(H_0) \cup \sigma(H_\pm))$ with sufficiently large absolute value; thus, the SSF for the pair H_0, $H_\pm$ is well defined. Identity* (2.11) *is true for a.e. $\lambda \in \mathbb{R}$.*

Proof. It suffices to refer to Propositions 2.7 and 2.8(ii) with $\mathcal{K} = \mathcal{H}$ and $G = \sqrt{V}$. Relation (2.9) follows from Proposition 5.2(i) in case (i) and from Proposition 5.5(i) in case (ii). The relation

$$(\sqrt{V} R_0^{1/2}(-1))^* (\sqrt{V} R_0^{k+1/2}(-1)) = R_0^{1/2}(-1) V R_0^{k+1/2}(-1) \in \mathbf{S}_1(\mathcal{H})$$

is ensured by Proposition 5.2(ii) in case (i) and by Proposition 5.5(ii) in case (ii) (cf. [1, Corollary 2.13]). □

In connection with the assumptions of Theorem 6.3(i) we note that the relation $V \in l_1(\mathbb{Z}^d; L_2(\mathbb{Q}^d))$ implies the $(-\triangle)$-form compactness of V for $d \leq 4$, but, in general, it does not imply even the $(-\triangle)$-form boundedness of V for $d \geq 5$.

6.2. Estimates for $\mathcal{N}_-$.

Theorem 6.4. (i) *Under the assumptions of Theorem* 6.1(i), *for any $t > 0$ we have*

$$\begin{aligned}\int_0^\infty & \mathcal{N}_-(\lambda; H_0, \sqrt{V}) e^{-t\lambda}\, d\lambda \\ & \leq (2\pi)^{-d} \omega_d \frac{d}{2} M^2 \beta^{-d/2} \int_0^\infty \lambda^{d/2-1} e^{-t\lambda}\, d\lambda \int V(x)\, dx \\ & = (2\pi)^{-d} \omega_d \frac{d}{2} M^2 \beta^{-d/2} t^{-d/2} \Gamma\left(\frac{d}{2}\right) \int V(x)\, dx.\end{aligned} \tag{6.1}$$

(ii) *Under the assumptions of Theorem* 6.1(ii), *let $f\colon (0,\infty) \to [0,\infty)$ be a monotone nonincreasing function. Then*

$$\int_0^\infty \mathcal{N}_-(\lambda; H_0, \sqrt{V}) f(\lambda)\, d\lambda \leq (2\pi)^{-d} \omega_d \varkappa \int_0^\infty \lambda^{\varkappa-1} f(\lambda)\, d\lambda \int V(x)\, dx. \tag{6.2}$$

Proof. (i) It suffices to apply Lemma 3.3 with $\mathcal{K} = \mathcal{H}$, $G = \sqrt{V}$, $f(\lambda) = e^{-t\lambda}$ (the case of lower signs) and estimate the right-hand side of (3.4−) in accordance with (5.6).

(ii) We apply Lemma 3.3 with $\mathcal{K} = \mathcal{H}$, $G = \sqrt{V}$ and estimate the right-hand side of (3.4−) in accordance with (5.16). □

Remark 6.5. 1) Let $\eta = \eta(\lambda)$ be a function of the form (5.3). Under the assumptions of Theorem 6.4(i), inequality (6.1) implies the estimate

$$\int_0^\infty \mathcal{N}_-(\lambda; H_0, \sqrt{V}) \eta(\lambda)\, d\lambda \leq (2\pi)^{-d} \omega_d \frac{d}{2} M^2 \beta^{-d/2} \int_0^\infty \lambda^{d/2-1} \eta(\lambda)\, d\lambda \int V(x)\, dx. \tag{6.3}$$

2) Suppose that the perturbation potential $V = V(x) \geq 0$ decays rapidly as $|x| \to \infty$. We recall the asymptotic formulas for the SSF as $\lambda \to \infty$:

$$\xi(\lambda; H_0(g, \mathbf{A}, U) + V, H_0(g, \mathbf{A}, U)) \sim (2\pi)^{-d} \omega_d \frac{d}{2} \lambda^{d/2-1} \int V(x) (\det g(x))^{-1/2}\, dx, \tag{6.4}$$

$$\xi(\lambda; (-\triangle)^l + V, (-\triangle)^l) \sim (2\pi)^{-d} \omega_d \varkappa \lambda^{\varkappa-1} \int V(x)\, dx. \tag{6.5}$$

Formally, these relations can be derived by computing the corresponding phase space volume. Under certain conditions on g, $\mathbf{A}$, U, V, formulas (6.4), (6.5) can be justified either as pointwise asymptotics or in the sense of some averaging; we do not dwell on this subject. It is clear that (6.4), (6.5) agree with (6.3), (6.2); moreover, the constant in (6.2) is sharp, and in the case of the flat metric the constant in (6.3) is also sharp.

6.3. Estimates for $\mathcal{N}_+$.

Theorem 6.6. (i) *Under the assumptions of Theorem* 6.1(i), *for any* $t > 0$, $\Lambda_- < 0$, *and* $\theta \in (0,1)$ *we have*

(6.6)
$$\int_{\Lambda_-}^{\infty} (\mathcal{N}_+(\lambda; H_0, \sqrt{V}) - N_+(\Lambda_-; H_0, \sqrt{V}, (1-\theta)^{-2}))_+ e^{-t\lambda}\, d\lambda \\ \leq (2\pi)^{-d}\omega_d \frac{d}{2} M^2 \beta^{-d/2}\theta^{-1-d/2} e^{-(1-\theta)\Lambda_- t} t^{-d/2}\Gamma(d/2) \int V(x)\, dx.$$

(ii) *Under the assumptions of Theorem* 6.1(ii), *let* $\Lambda_- < 0$, *and let* $f\colon (\Lambda_-, \infty) \to [0, \infty)$ *be a monotone nonincreasing function. Then for any* $\theta \in (0,1)$ *we have*

(6.7)
$$\int_{\Lambda_-}^{\infty} (\mathcal{N}_+(\lambda; H_0, \sqrt{V}) - N_+(\Lambda_-; H_0, \sqrt{V}, (1-\theta)^{-2}))_+ f(\lambda)\, d\lambda \\ \leq (2\pi)^{-d}\omega_d \varkappa \theta^{-1-\varkappa} \int_{(1-\theta)\Lambda_-}^{\infty} (\lambda - (1-\theta)\Lambda_-)^{\varkappa - 1} f(\lambda)\, d\lambda \int V(x)\, dx.$$

Proof. (i) It suffices to apply Lemma 4.3 with $\mathcal{K} = \mathcal{H}$, $G = \sqrt{V}$, $\theta_1 = \theta_2 = \theta$, $f(\lambda) = e^{-t\lambda}$ and to estimate the right-hand side of (4.8) in accordance with (5.6).

(ii) It suffices to apply Lemma 4.3 with $\mathcal{K} = \mathcal{H}$, $G = \sqrt{V}$, $\theta_1 = \theta_2 = \theta$ and to estimate the right-hand side of (4.8) in accordance with (5.16). □

Remark 6.7. Let $\eta = \eta(\lambda)$ be a function of the form (5.3) where the measure μ has the property that the integral in (5.3) is convergent for any $\lambda > \Lambda_-$; for example, we can take $\eta(\lambda) = (\lambda - \lambda_0)^{-\gamma}$, $\lambda_0 < \Lambda_-$, $\gamma > 0$. Then, under the assumptions of Theorem 6.6, inequality (6.6) implies the estimate

(6.8)
$$\int_{\Lambda_-}^{\infty} (\mathcal{N}_+(\lambda; H_0, \sqrt{V}) - N_+(\Lambda_-; H_0, \sqrt{V}, (1-\theta)^{-2}))_+ \eta(\lambda)\, d\lambda \\ \leq (2\pi)^{-d}\omega_d \frac{d}{2} M^2 \beta^{-d/2}\theta^{-1-d/2} \int_{(1-\theta)\Lambda_-}^{\infty} (\lambda - (1-\theta)\Lambda_-)^{d/2-1} \eta(\lambda)\, d\lambda \int V(x)\, dx.$$

Corollary 6.8. (i) *Under the assumptions of Theorem* 6.1(i), *let* $H_0 = H_0(g, \mathbf{A}, U)$, *and let* $d \geq 3$. *Then for any function* η *of the form* (5.3) *and any* $\theta \in (0,1)$ *we have*

(6.9)
$$\int_0^{\infty} \mathcal{N}_+(\lambda; H_0, \sqrt{V})\eta(\lambda)\, d\lambda \\ \leq C_{5.14}(d) g_-^{-d/2}(1-\theta)^{-d} \int_0^{\infty} \eta(\lambda)\, d\lambda \int V^{d/2}(x)\, dx \\ + (2\pi)^{-d}\omega_d \frac{d}{2} M^2 \beta^{-d/2}\theta^{-1-d/2} \int_0^{\infty} \lambda^{(d/2)-1}\eta(\lambda)\, d\lambda \int V(x)\, dx.$$

(ii) *Under the assumptions of Theorem* 6.1(ii), *let* $\kappa > 1$. *Then for any* $\theta \in (0,1)$ *and any monotone nonincreasing function* $f\colon (0,\infty) \to [0,\infty)$ *we have*

(6.10)
$$\begin{aligned}\int_0^\infty &\mathcal{N}_+(\lambda; H_0, \sqrt{V}) f(\lambda)\, d\lambda \\ &\le C_{5.19}(d,l)(1-\theta)^{-2\varkappa} \int_0^\infty f(\lambda)\, d\lambda \int V^{\varkappa}(x)\, dx \\ &\quad + (2\pi)^{-d}\omega_d \kappa \theta^{-1-\lambda} \int_0^\infty \lambda^{\varkappa-1} f(\lambda)\, d\lambda \int V(x)\, dx.\end{aligned}$$

Proof. It suffices to relax inequalities (6.7), (6.8) by transferring the terms that involve N_+ to the right-hand side, then applying estimates (5.14), (5.19), and putting $\Lambda_- = 0$. □

Theorem 6.9. (i) *Under the assumptions of Theorem* 6.1(i), *for any* $\Lambda_- < 0$, *any* $p > \max\{1, \varkappa\}$, *and any* $\theta \in (0,1)$ *we have*

(6.11)
$$\begin{aligned}\int_0^\infty &(\mathcal{N}_+(\lambda; H_0, \sqrt{V}) - N_+(\Lambda_-; H_0, \sqrt{V}, (1+\theta)^{-1}))_- (\lambda - \Lambda_-)^{-p}\, d\lambda \\ &\le (2\pi)^{-d}\omega_d \frac{d}{2} M^2 \beta^{-d/2} \theta^{-1} |\Lambda_-|^{\varkappa - p} B(d/2, p - d/2) \frac{4p-2}{p-1} \int V(x)\, dx.\end{aligned}$$

(ii) *Under the assumptions of Theorem* 6.1(ii), *let* $\Lambda_- < 0$, *and let* $f\colon (0,\infty) \to [0,\infty)$ *be a monotone nonincreasing integrable function. Then for any* $\theta \in (0,1)$ *we have*

(6.12)
$$\begin{aligned}\int_0^\infty &(\mathcal{N}_+(\lambda; H_0, \sqrt{V}) - N_+(\Lambda_-; H_0, \sqrt{V}, (1+\theta)^{-1}))_- f(\lambda)\, d\lambda \\ &\le (2\pi)^{-d}\omega_d \varkappa \theta^{-1} \int_0^\infty \Big(4\lambda^{\varkappa-1} + 2\int_0^\lambda \frac{t^{\varkappa-1}}{t - \Lambda_-} dt\Big) f(\lambda)\, d\lambda \int V(x)\, dx.\end{aligned}$$

Proof. (i) It suffices to apply Lemma 4.4 with $\mathcal{K} = \mathcal{H}$, $G = \sqrt{V}$, $\theta_1 = \theta_2 = \theta/2$, $f(\lambda) = (\lambda - \Lambda_-)^{-p}$ and to estimate the right-hand side of (4.10) in accordance with (5.6).

(ii) We apply Lemma 4.4 with $\mathcal{K} = \mathcal{H}$, $G = \sqrt{V}$, $\theta_1 = \theta_2 = \theta/2$ and estimate the right-hand side of (4.10) in accordance with (5.16). □

Remark 6.10. For $l \le 1$, the semigroup $e^{-t(-\triangle)^l}$ in $L_2(\mathbb{R}^d)$ is positivity preserving. Using this, we can apply the construction of §§5–6 to the differential operators H_0 of order $l < 1$ that satisfy the condition

$$|e^{-tH_0}\psi| \le M e^{-t\beta(-\triangle)^l} |\psi|, \quad \psi \in L_2(\mathbb{R}^d).$$

As a typical example we mention the pseudorelativistic magnetic Schrödinger operator

$$(H_0(\mathbf{1}, \mathbf{A}, 0) + I)^{1/2} + U(x);$$

see, e.g., [26].

§7. Appendix. Remarks on the proof of Proposition 5.3

7.1. In [11], Proposition 5.3 was proved under the additional assumption $U \in L_\infty(\mathbb{R}^d)$. It is easy to carry the desired statement over to the general case. Indeed, first we observe that

$$g_- h^{(0)}_{\mathbf{1},\mathbf{A},U} \le h^{(0)}_{g,\mathbf{A},U} \le g_+ h^{(0)}_{\mathbf{1},\mathbf{A},U}. \tag{7.1}$$

Under conditions (5.10), (5.11), the form $h^{(0)}_{\mathbf{1},\mathbf{A},U}$ is closable, and its domain is given by the expression on the right-hand side of (5.13); see [34]. Now, the first statement of Proposition 5.3 follows from (7.1).

Next, the results of [11] imply that if conditions (5.9), (5.10) are fulfilled and

$$\widetilde{U} \ge 0, \qquad \widetilde{U} \in L_\infty(\mathbb{R}^d), \tag{7.2}$$

then

$$|e^{-tH_0(g,\mathbf{A},\widetilde{U})}\psi| \le e^{-tH_0(g,0,0)}|\psi|, \quad t>0, \ \psi \in L_2(\mathbb{R}^d). \tag{7.3}$$

We put $U_n(x) = \min\{U(x), n\}$; arguing as in [34, Theorem 2.1], we can check that $H_0(g,\mathbf{A},U_n) \to H_0(g,\mathbf{A},U)$ in the strong resolvent sense. Then, writing (7.3) with $\widetilde{U} = U_n$ and passing to the limit as $n \to \infty$ over a subsequence that converges almost everywhere in $\mathbb{R}^d$, we get

$$|e^{-tH_0(g,\mathbf{A},U)}\psi| \le e^{-tH_0(g,0,0)}|\psi|, \quad t>0, \ \psi \in L_2(\mathbb{R}^d) \tag{7.4}$$

(cf. the proof of Theorem 3.1 in [16]). It is well known (see, e.g., [10]), that the semigroup $e^{-tH_0(g,0,0)}$ is positivity preserving and that for some constants $M>0$, $\beta>0$ we have

$$e^{-tH_0(g,0,0)}|\psi| \le M e^{-t\beta(-\triangle)}|\psi|, \quad t>0, \ \psi \in L_2(\mathbb{R}^d). \tag{7.5}$$

Combining (7.4) and (7.5), we arrive at (5.2).

7.2. In [11], the proof of (7.3) was based on a certain new domination criterion for semigroups (see [24]). We are going to show that (7.3) can be proved by using the "classical" semigroup theory technique: the Kato inequality [15], the domination criterion [33, 13], and an approximation argument [16, 34].

Let $g \in C^1(\mathbb{R}^d)$ be of the form (5.9), let $\mathbf{A} \in C^1(\mathbb{R}^d)$, and let $\widetilde{U}$ be of the form (7.2). Lemma A in [15] gives the pointwise estimate

$$\operatorname{Re}(\overline{\operatorname{sgn} u} H_0(g,\mathbf{A},\widetilde{U})u) \ge H_0(g,0,0)|u|,$$

in the sense of distributions. Using the domination criterion [33, 13], from this estimate we deduce (7.3) for the class of functions g, $\mathbf{A}$, $\widetilde{U}$ specified above. Now, let g, $\mathbf{A}$ be of the form (5.9), (5.10). We take a sequence $\mathbf{A}_n \in C^1(\mathbb{R}^d)$ such that $\mathbf{A}_n \to \mathbf{A}$ in $L_{2,\mathrm{loc}}$, and a sequence of matrix-valued functions $g_n(x)$ satisfying $g_-\mathbf{1} \le g_n(x) \le g_+\mathbf{1}$ (with the same g_-, g_+ as in (5.9)) and such that $g_n \to g$ almost everywhere in $\mathbb{R}^d$. Arguing as in the proof of [34, Theorem 4.1] (see also [23, Proposition 2.5]), we check that $H(g_n,\mathbf{A}_n,\widetilde{U}) \to H(g,\mathbf{A},\widetilde{U})$ in the strong resolvent sense. Here the key point of the proof is the fact that if $\varphi \in C_0^\infty(\mathbb{R}^d)$, then $g_n(\nabla - i\mathbf{A}_n)\varphi \to g(\nabla - i\mathbf{A})\varphi$ strongly in L_2. Next, in the inequality

$$|e^{-tH_0(g_n,\mathbf{A}_n,\widetilde{U})}\psi| \le e^{-tH_0(g_n,0,0)}|\psi|$$

we pass to the limit as $n \to \infty$ along a subsequence converging almost everywhere in $\mathbb{R}^d$. This yields (7.3).

7.3. We note that in [11] the condition on the matrix $g(x)$ was much weaker than (5.9); this allowed the authors of [11] to cover the case where $H_0(g, \mathbf{A}, U)$ is non-selfadjoint. Of course, the alternative proof sketched above does not include this case.

References

1. J. Avron, I. Herbst, and B. Simon, *Schrödinger operators with magnetic fields.* I. *General interactions*, Duke Math. J. **45** (1978), 847–883.
2. M. Sh. Birman and S. B. Entina, *Stationary approach in abstract scattering theory*, Izv. Akad. Nauk SSSR Ser. Mat. **31** (1967), no. 2, 401–430; English transl., Math. USSR-Izv. **1** (1967), no. 1, 391–420.
3. M. Sh. Birman and M. Z. Solomyak, *Spectral theory of selfadjoint operators in Hilbert space*, Leningrad. Gos. Univ, Leningrad, 1980; English transl., Math. Appl. (Soviet Ser.), D. Reidel Publishing Co., Dordrecht, 1987.
4. ______, *Remarks on the spectral shift function*, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **27** (1972), 33–46; English transl., J. Soviet Math. **3** (1975), no. 4, 408–419.
5. ______, *Schrödinger operator. Estimates for the number of bound states as function-theoretical problem*, Spectral Theory of Operators (Novgorod, 1989), Amer. Math. Soc. Transl. (2), vol. 150, Amer. Math. Soc., Providence, RI, 1992, pp. 1–54.
6. ______, *Estimates for the singular numbers of integral operators*, Uspekhi Mat. Nauk **32** (1977), no. 1, 17–84; English transl., Russian Math. Surveys **32** (1977), no. 1, 15–89.
7. ______, *Estimates for the number of negative eigenvalues of the Schrödinger operator and its generalizations*, Estimates and Asymptotics for Discrete Spectra of Integral and Differential Equations (Leningrad, 1989–90), Adv. Soviet Math., vol. 7, Amer. Math. Soc., Providence, RI, 1991, pp. 1–55.
8. M. Sh. Birman and D. R. Yafaev, *The spectral shift function. The papers of M. G. Kreĭn and their further development*, Algebra i Analiz **4** (1992), no. 5, 1–44; English transl., St. Petersburg Math. J. **4** (1993), no. 5, 833–870.
9. V. S. Buslaev, *The trace formulas and certain asymptotic estimates of the kernel of the resolvent for the Schrödinger operator in three-dimensional space*, Spectral Theory and Wave Processes, Probl. Mat. Fiz., No. 1, Leningrad. Univ., Leningrad, 1966, pp. 82–101. (Russian)
10. E. B. Davies, *Heat kernels and spectral theory*, Cambridge Tracts in Math., vol. 92, Cambridge Univ. Press, Cambridge–New York, 1989.
11. M. Demuth and E. M. Ouhabaz, *Scattering theory for Schrödinger operators with magnetic fields*, Math. Nachr. **185** (1997), 49–58.
12. R. Geisler, V. Kostrykin, and R. Schrader, *Concavity properties of Kreĭn's spectral shift function*, Rev. Math. Phys. **7** (1995), no. 2, 161–181.
13. H. Hess, R. Schrader, and D. A. Uhlenbrock, *Domination of semigroups and generalization of Kato's inequality*, Duke Math. J. **44** (1977), 893–904.
14. T. Kato, *Wave operators and similarity for some non-selfadjoint operators*, Math. Ann. **162** (1965), 258–279.
15. ______, *Schrödinger operators with singular potentials*, Israel J. Math. **13** (1972), 135–148.
16. ______, *Remarks on Schrödinger operators with vector potentials*, Integral Equations Operator Theory **1** (1978), no. 1, 103–113.
17. R. Konno and S. T. Kuroda, *On the finiteness of perturbed eigenvalues*, J. Fac. Sci. Univ. Tokyo Sect. I **13** (1966), 55–63.
18. L. S. Koplienko, *On the theory of the spectral shift function*, Spectral Theory, Probl. Mat. Fiz., No. 5, Leningrad. Gos. Univ., Leningrad, 1971, pp. 62–72; English transl., Topics in Math. Phys., vol. 5, Consultants Bureau, New York–London, 1972, pp. 51–59.
19. M. G. Kreĭn, *On the trace formula in perturbation theory*, Mat. Sb. **33 (75)** (1953), no. 3, 597–626. (Russian)
20. E. Lieb, *The number of bound states of one-body Schrödinger operators and the Weyl problem*, Geometry of the Laplace Operator (Proc. Sympos. Pure Math., Univ. Hawaii, Honolulu, Hawaii, 1979), Proc. Sympos. Pure Math., vol. 36, Amer. Math. Soc., Providence, RI, 1980, pp. 241–252.

21. I. M. Lifshits, *On a problem of the perturbation theory connected with quantum statistics*, Uspekhi Mat. Nauk **7** (1952), no. 1, 171–180. (Russian)
22. V. G. Maz'ya and T. O. Shaposhnikova, *Theory of multipliers in spaces of differentiable functions*, Monogr. Stud. Math., vol. 23, Pitman, Boston–London, 1985.
23. M. Melgaard and G. Rozenblum, *Spectral estimates for magnetic operators*, Math. Scand. **79** (1996), 237–254.
24. E.-M. Ouhabaz, *Invariance of closed convex sets and domination criteria for semigroups*, Potential Anal. **5** (1996), 611–625.
25. A. B. Pushnitskiĭ, *Representation for the spectral shift function for perturbations of a definite sign*, Algebra i Analiz **9** (1997), no. 6, 197–213; English transl. in St. Petersburg Math. J. **9** (1998), no. 6.
26. G. V. Rozenblum and M. Z. Solomyak, *CLR-estimate for the generators of positivity preserving and positively dominated semigroups*, Algebra i Analiz **9** (1997), no. 6, 214–236; English transl. in St. Petersburg Math. J. **9** (1998), no. 6.
27. ———, *On the number of negative eigenvalues for the two-dimensional magnetic Schrödinger operator*, Differential Operators and Spectral Theory. Collection of papers, dedicated to the 70-th birthday of M. Sh. Birman (to appear).
28. M. Reed and B. Simon, *Methods of modern mathematical physics.* Vol. 3. *Scattering theory*, Academic Press, New York–London, 1979.
29. ———, *Methods of modern mathematical physics.* Vol. 2. *Fourier analysis, selfadjointness*, Academic Press, New York–London, 1975.
30. D. Robert, *Semi-classical asymptotics of the spectral shift function*, Differential Operators and Spectral Theory. Collection of papers, dedicated to the 70-th birthday of M. Sh. Birman (to appear).
31. B. Simon, *Functional integration and quantum physics*, Academic Press, New York, 1979.
32. ———, *Schrödinger semigroups*, Bull. Amer. Math. Soc. (N. S.) **7** (1982), 447–526.
33. ———, *Kato's inequality and the comparison of semigroups*, J. Funct. Anal. **32** (1979), 97–101.
34. ———, *Maximal and minimal Schrödinger forms*, J. Operator Theory **1** (1979), 37–47.
35. A. V. Sobolev, *Efficient bounds for the spectral shift function*, Ann. Inst. H. Poincaré Phys. Théor. **58** (1993), no. 1, 55–83.
36. D. R. Yafaev, *Mathematical scattering theory. General theory*, Transl. Math. Monogr., vol. 105, Amer. Math. Soc., Providence, RI, 1992.

Translated by THE AUTHOR

Division of Mathematical Physics, Department of Physics, St. Petersburg State University, 198904, St. Petersburg, Ul'yanovskaya, 1, RUSSIA
E-mail address: pushnit@mph.phys.spbu.ru