Representation for the spectral shift function for perturbations of a definite sign

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Abstract

Let H, H_0 be a pair of selfadjoint operators whose difference $V = H - H_0$ is trace-class and has a definite sign. In this case a new (integral) representation for the Kreĭn spectral shift function of the pair H, H_0 in terms of the spectral characteristics of the boundary values of the operator $|V|^{1/2}(H_0 - zI)^{-1}|V|^{1/2}$ is obtained. This representation is extended to the case of perturbations V of a definite sign which satisfy some rather broad conditions of the relatively trace-class type. Applications of this result include pointwise estimates for the spectral shift function and sufficient conditions for its continuity.

0 Introduction

Let H_0 and H be selfadjoint operators in a Hilbert space $\mathcal{H}$ and let their difference, V, be trace-class:

$$V := H - H_0 \in \mathbf{S_1}. \tag{0.1}$$

Then, there exists a *spectral shift function* (SSF) $\xi(\lambda; H, H_0)$ for the pair H, H_0 which plays an important role in the spectral theory. The SSF was first introduced by I. M. Lifshits [1] (on a formal level) and M. G. Kreĭn [2]. For an exposition of the modern stage of the SSF theory, see the review [7] and the book [9]; one can find a detailed bibliography on the subject there. The *Kreĭn's Theorem* (see [2], [3]) provides a representation for the SSF in terms of the boundary values of the perturbation determinant of the pair H_0, H:

$$\xi(\lambda; H, H_0) = \frac{1}{\pi} \lim_{\varepsilon \to +0} \arg \det(I + V(H_0 - (\lambda + i\varepsilon)I)^{-1}), \quad \text{a.e. } \lambda \in \mathbf{R} \tag{0.2}$$

where the branch of the argument is fixed by the condition:

$$\arg \det(I + V(H_0 - zI)^{-1}) \to 0, \quad \operatorname{Im} z \to +\infty.$$

The SSF is related to the scattering matrix $S(\lambda; H, H_0)$ by the *Birman-Krein formula* (see [4] as well as [7],[9]):

$$\det S(\lambda; H, H_0) = e^{-2\pi i \xi(\lambda; H, H_0)}, \quad \text{a.e. } \lambda \in \sigma_{ac}(H_0). \tag{0.3}$$

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This connection allows one to interpret SSF as a scattering phase and stimulates its investigation in quantum-mechanical problems.

The main result of the present paper is Theorem 1.1 which provides a new representation for the SSF (see (1.7) below) in terms of the spectral characteristics of the boundary values of the "sandwiched" resolvent of H_0 in the case of perturbations V of a definite sign. To a certain extent, this representation can be considered as the *Birman-Schwinger principle on the continuous spectrum.* In the Theorem 1.2 the representation (1.7) is extended to the case of relatively trace-class perturbations (see conditions (1.8)-(1.11) below). This result allows us to obtain some pointwise estimates for the SSF (Corollaries 2.3 and 2.4) which are convenient for applications and a sufficient condition for the continuity of the SSF with respect to the spectral parameter (Corollary 2.6). Note that some pointwise estimates for the SSF close to Corollaries 2.3 and 2.4 were obtained earlier in [14] (without the restriction on the sign of a perturbation) on the basis of the formula (0.2) using the invariance principle.

The representation (1.7) is well adapted for the computation of the asymptotics of the SSF in the large coupling constant in concrete problems. Another paper will be devoted to these applications.

The proof of the representation (1.7) is based on some abstract operator formulas which are close to (0.3) (see Lemmas 3.1, 3.2). Similar relations (for another purpose) were used in [15], [16] (see also [8], [9]).

In §1, 2 we formulate the main results of the paper and their applications, while the proofs are presented in §3-5.

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1 Main results

1. Notations Below $\mathcal{H}$, $\mathcal{H}_1$, $\mathcal{H}_2$ are separable Hilbert spaces. By $\rho(A)$, $\sigma(A)$, $\sigma_p(A)$, $\sigma_c(A)$, $\sigma_{ac}(A)$ we denote respectively the resolvent set, the spectrum, the point spectrum, the continuous spectrum and the absolutely continuous spectrum of a linear operator A. For a selfadjoint operator A let $E_A(\delta)$ be the spectral measure of a Borel set $\delta \subset \mathbf{R}$ and $2A_\pm = |A| \pm A$. By $\mathbf{S}_\infty(\mathcal{H}_1, \mathcal{H}_2)$ we denote the Banach space of compact operators from $\mathcal{H}_1$ to $\mathcal{H}_2$; $\mathbf{S}_\infty(\mathcal{H}) := \mathbf{S}_\infty(\mathcal{H}, \mathcal{H})$. For $T = T^* \in \mathbf{S}_\infty(\mathcal{H})$ and $s > 0$ let $n_\pm(s, T) = \dim \operatorname{Ran} E_{T_\pm}(s, +\infty)$, and for $T \in \mathbf{S}_\infty(\mathcal{H}_1, \mathcal{H}_2)$ and $s > 0$ let $n(s, T) = n_+(s^2; T^*T)$. Recall (see, e.g., [17]) the estimates which are equivalent to the Weyl inequalities for selfadjoint operators T_1, T_2:

$$n_\pm(s_1 + s_2, T_1 + T_2) \le n_\pm(s_1, T_1) + n_\pm(s_2, T_2), \quad s_1, s_2 > 0 \qquad (1.1_\pm)$$

and their corollaries

$$n_\pm(s, T_1 + T_2) \ge n_\pm(s + s_2, T_1) - n_\mp(s_2, T_2), \quad s, s_2 > 0. \qquad (1.2_\pm)$$

For $0 < p < \infty$ a class $\mathbf{S_p}(\mathcal{H}_1, \mathcal{H}_2) \subset \mathbf{S}_\infty(\mathcal{H}_1, \mathcal{H}_2)$ is defined as the set of all compact operators T such that the following functional is finite:

$$\|T\|_{\mathbf{S_p}}^p := p \int_0^\infty s^{p-1} n(s, T) ds.$$

Functional $\|\cdot\|_{\mathbf{S_p}}$ is a norm for $p \geq 1$ and a quasinorm for $p < 1$. For $0 < p < \infty$ a class $\mathbf{\Sigma_p}(\mathcal{H}_1, \mathcal{H}_2) \subset \mathbf{S}_\infty(\mathcal{H}_1, \mathcal{H}_2)$ is defined (see [17]) as the set of all compact operators T such that the following functional (which is a quasinorm) is finite:

$$\|T\|_{\mathbf{\Sigma_p}}^p := \sup_{s>0} s^p n(s, T).$$

For a measurable set $\delta \subset \mathbf{R}$ its Lebesgue measure is denoted by mes δ.

Formulas and statements with double indices ($\pm$ and $\mp$) are understood independently, as pairs of formulas and statements in one of which all indices take their upper values while in another – the lower ones.

2. Trace-class perturbations Let $\mathcal{H}$ be a basic and $\mathcal{K}$ an auxiliary Hilbert space, and H_0 be a selfadjoint operator in $\mathcal{H}$. Let

$$G \in \mathbf{S_2}(\mathcal{H}, \mathcal{K}) \tag{1.3}$$

and

$$V := G^* G (\in \mathbf{S_1}(\mathcal{H})). \tag{1.4}$$

Define the operators

$$H_\pm = H_0 \pm V. \tag{1.5}$$

For $\operatorname{Im} z > 0$ we denote

$$T(z) = G(H_0 - zI)^{-1} G^*, \quad A(z) = \operatorname{Re} T(z), \quad K(z) = \operatorname{Im} T(z). \tag{1.6}$$

It is well-known that under the condition (1.3) the operator-valued function $T(z)$ has limit values as $z \to \lambda + i0$ in $\mathbf{S_2}(\mathcal{K})$ (and even in $\mathbf{S_p}(\mathcal{K})$ for any $p > 1$) for a.e. $\lambda \in \mathbf{R}$; in this case $K(\lambda + i0) \in \mathbf{S_1}(\mathcal{K})$, $K(\lambda + i0) \geq 0$. See [11], [9] and [12] for the case $p > 1$.

Below we formulate the main result of the paper.

Theorem 1.1 *Suppose condition* (1.3) *holds. Then for a.e.* $\lambda \in \mathbf{R}$ *the SSF* $\xi(\lambda; H_\pm, H_0)$ *admits the following representation via the converging integral:*

$$\xi(\lambda; H_\pm, H_0) = \pm \frac{1}{\pi} \int_{-\infty}^\infty \frac{dt}{1 + t^2} n_\mp(1, A(\lambda + i0) + tK(\lambda + i0)). \tag{1.7$_\pm$}$$

Let us discuss the formula (1.7). First suppose $\lambda \in \mathbf{R} \cap \rho(H_0)$. Then (1.7) transforms into the relation from [14]:

$$\xi(\lambda; H_\pm, H_0) = \pm n_\mp(1, G(H_0 - \lambda)^{-1} G^*)$$

which can be considered as a certain variance of the *Birman-Schwinger principle.* The case of λ lying on the continuous spectrum $\sigma_c(H_0)$ is most interesting. Here, the integral in (1.7) performs a "smoothing" of the integer-valued function $n_\mp(1, A(\lambda+i0))$. Note that the integral converges due to the inclusion $K(\lambda+i0) \in \mathbf{S_1}(\mathcal{K})$.

3. Relatively trace-class perturbations Below we extend the formula (1.7) to a broader class of perturbations V. We consider only the case of semibounded from below operators H_0.

Let H_0 be a selfadjoint semibounded from below operator in a Hilbert space $\mathcal{H}$. Fix $\gamma \in \mathbf{R}$ so that

$$H_0 + \gamma I \geq I. \tag{1.8}$$

Let G be a closable operator from $\mathcal{H}$ to an auxiliary Hilbert space $\mathcal{K}$, such that $\operatorname{Dom} G \supset \operatorname{Dom}(H_0+\gamma I)^{1/2}$. Then the operator $G(H_0+\gamma I)^{-1/2}$ is bounded. We suppose that a more restrictive condition holds true:

$$G(H_0+\gamma I)^{-1/2} \in \mathbf{S}_\infty(\mathcal{H}, \mathcal{K}). \tag{1.9}$$

Moreover, suppose for some $m > 0$

$$G(H_0+\gamma I)^{-m} \in \mathbf{S_2}(\mathcal{H}, \mathcal{K}). \tag{1.10}$$

The inclusion (1.9) means that the operator $V = G^*G$ is compact with respect to H_0 in the form sense. This allows one to define the operators (1.5) via the corresponding sesquilinear forms. We suppose that for some $l > 0$, $\lambda_0 < \inf \sigma(H_0) \cup \sigma(H_\pm)$ the following condition holds:

$$(H_\pm - \lambda_0 I)^{-l} - (H_0 - \lambda_0 I)^{-l} \in \mathbf{S_1}(\mathcal{H}). \tag{1.11}$$

This enables one to define the SSF $\xi(\lambda; H_\pm, H_0)$ on the basis of the invariance principle (see, e.g., [7]). As above, we introduce the operators (1.6), which are compact in $\mathcal{K}$ due to (1.9). By (1.9), (1.10), for a.e. $\lambda \in \mathbf{R}$ the operator $T(z)$ has limit values as $z \to \lambda + i0$ in $\mathbf{S}_\infty(\mathcal{K})$ and $K(\lambda+i0) \in \mathbf{S_1}(\mathcal{K})$; see Lemma 4.1 below. Thus, under the above conditions both left and right sides of (1.7) are well-defined.

Theorem 1.2 *Suppose conditions* (1.8)–(1.11) *hold true. Then for a.e.* $\lambda \in \mathbf{R}$ *the SSF* $\xi(\lambda; H_\pm, H_0)$ *admits the representation* (1.7).

2 Applications of the main results

In this section by means of an elementary analysis of the right side of the representation (1.7) we obtain estimates for the SSF and a sufficient condition for the continuity of the SSF.

1. Estimates for the SSF

For the operators

$$A = A^* \in \mathbf{S}_\infty(\mathcal{K}), \quad K = K^* \in \mathbf{S_1}(\mathcal{K}) \tag{2.1}$$

let

$$\xi_\pm = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{dt}{1+t^2} n_\mp(1, A + tK). \tag{2.2}$$

One easily checks that the conditions (2.1) are sufficient for the convergence of the integral (2.2).

Lemma 2.1 *Suppose conditions* (2.1) *hold true and* $K \in \mathbf{S_p}(\mathcal{K})$ *for some* $p \le 1$. *Then for the quantities* $\xi_\pm$ *defined by* (2.2)*, the following estimates hold true:*

$$\xi_\pm \le \inf_{0<\varepsilon<1} (n_\mp(1-\varepsilon, A) + \varepsilon^{-p}\mu_p(\varepsilon, K)\|K\|^p_{\mathbf{S_p}}) \tag{2.3$_\pm$}$$

$$\xi_\pm \ge \sup_{\varepsilon>0} (n_\mp(1+\varepsilon, A) - \varepsilon^{-p}\mu_p(\varepsilon, K)\|K\|^p_{\mathbf{S_p}}) \tag{2.4$_\pm$}$$

where $\mu_p(\varepsilon, K)$ *obeys*

$$0 \le 2\pi p \mu_p(\varepsilon, K) \le (1-p)^{\frac{1-p}{2}}(1+p)^{\frac{1+p}{2}}, \quad \mu_p(\varepsilon, K) \to 0 \text{ as } \varepsilon \to +0. \tag{2.5}$$

Similar estimates hold true for the class $\mathbf{\Sigma_p}$ replacing $\mathbf{S_p}$.

Lemma 2.2 *Suppose conditions* (2.1) *hold true and* $K \in \mathbf{\Sigma_p}(\mathcal{K})$ *for some* $p < 1$. *Then for the quantities* $\xi_\pm$ *defined by* (2.2)*, the following estimates hold true:*

$$\xi_\pm \le \inf_{0<\varepsilon<1} (n_\mp(1-\varepsilon, A) + \varepsilon^{-p}(2\cos(\pi p/2))^{-1}\|K\|^p_{\mathbf{\Sigma_p}}) \tag{2.6$_\pm$}$$

$$\xi_\pm \ge \sup_{\varepsilon>0} (n_\mp(1+\varepsilon, A) - \varepsilon^{-p}(2\cos(\pi p/2))^{-1}\|K\|^p_{\mathbf{\Sigma_p}}). \tag{2.7$_\pm$}$$

Lemmas 2.1, 2.2 immediately imply the following corollaries.

Corollary 2.3 *Suppose that under the hypothesis of Theorem 1.1 or 1.2 the representation* (1.7) *holds true at some point* $\lambda \in \mathbf{R}$ *and* $K(\lambda + i0) \in \mathbf{S_p}(\mathcal{K})$ *for some* $p \le 1$. *Then the estimates* (2.3)–(2.5) *are valid with* $\pm\xi_\pm = \xi(\lambda; H_\pm, H_0)$, $A = A(\lambda + i0)$, $K = K(\lambda + i0)$.

Corollary 2.4 *Suppose that under the hypothesis of Theorem 1.1 or 1.2 at some point* $\lambda \in \mathbf{R}$ *the representation* (1.6) *holds true and* $K(\lambda + i0) \in \mathbf{\Sigma_p}(\mathcal{K})$ *for some* $p < 1$. *Then the estimates* (2.6), (2.7)*, are valid with* $\pm\xi_\pm = \xi(\lambda; H_\pm, H_0)$, $A = A(\lambda + i0)$, $K = K(\lambda + i0)$.

Remark Using the monotonicity of the SSF, that is, the inequalities

$$\xi(\lambda; H_0 - V_-, H_0) \le \xi(\lambda; H_0 + V, H_0) \le \xi(\lambda; H_0 + V_+, H_0)$$

it is easy to obtain estimates analogous to (2.3)–(2.7) for the perturbations V without the restriction on the sign.

For comparison, below we present the estimates from [14]. For simplicity, consider the case of the trace-class perturbation V of the type (1.4). In [14], provided that the operators (1.6) have boundary values in $\mathbf{S_p}(\mathcal{K})$, $0 < p < \infty$, the following estimates have been proved:

$$|\xi(\lambda; H_\pm, H_0)| \le C_p(\|K(\lambda + i0)\|_{\mathbf{S_1}} + \|T(\lambda + i0)\|^p_{\mathbf{S_p}}), \quad p \ge 1$$

and

$$|\xi(\lambda; H_\pm, H_0)| \le C_p\|T(\lambda + i0)\|^p_{\mathbf{S_p}}, \quad p < 1.$$

Similarly, provided that the operators (1.6) have boundary values in $\mathbf{\Sigma_p}(\mathcal{K})$, $0 < p < \infty$, the following estimates have been proved:

$$|\xi(\lambda; H_\pm, H_0)| \le C_p(\|K(\lambda + i0)\|_{\mathbf{S_1}} + \|T(\lambda + i0)\|^p_{\mathbf{\Sigma_p}}), \quad p \ge 1$$

and

$$|\xi(\lambda; H_\pm, H_0)| \le C_p \|T(\lambda + i0)\|^p_{\mathbf{\Sigma_p}}, \quad p < 1.$$

Note that the scope of applications of the estimates (2.3)–(2.7) is broader, for they do not require $A(\lambda + i0)$ to belong to the classes $\mathbf{S_p}$, $\mathbf{\Sigma_p}$.

2. Continuity of the SSF

Lemma 2.5 *Let operators $A = A^* \in \mathbf{S}_\infty(\mathcal{K})$, $K = K^* \in \mathbf{S_1}(\mathcal{K})$, $A_j = A_j^* \in \mathbf{S}_\infty(\mathcal{K})$, $K_j = K_j^* \in \mathbf{S_1}(\mathcal{K})$, $j \in \mathbf{N}$ be such that $\|A_j - A\| \to 0$, $\|K_j - K\|_{\mathbf{S_1}} \to 0$ as $j \to \infty$. Suppose that for some $z_0 \in \mathbf{C}$: $\mp 1 \in \sigma(A + z_0 K)$. Then, for $j \to \infty$*

$$\int_{-\infty}^{\infty} \frac{dt}{1+t^2} n_\mp(1, A_j + tK_j) \to \int_{-\infty}^{\infty} \frac{dt}{1+t^2} n_\mp(1, A + tK). \qquad (2.8_\pm)$$

The above lemma implies the following sufficient condition for the continuity of the SSF.

Corollary 2.6 *Suppose that the hypothesis of Theorem 1.1 or 1.2 holds true, and for some interval $\delta \subset \mathbf{R}$ the operator-valued function $A(\lambda + i0)$ is continuous in $\lambda \in \delta$ in $\mathbf{S}_\infty(\mathcal{K})$, and $K(\lambda + i0)$ is continuous in $\lambda \in \delta$ in $\mathbf{S_1}(\mathcal{K})$. In addition, let $\mp 1 \notin \sigma(A(\lambda + i0) + iK(\lambda + i0))$ for all $\lambda \in \delta$. Then for a.e. $\lambda \in \delta$ the SSF $\xi(\lambda; H_\pm, H_0)$ coincides with a continuous in $\lambda \in \delta$ function.*

The proofs of Lemmas 2.1, 2.2, 2.5 are given in §5.

3 Proof of Theorem 1.1

1. In the course of the preliminary considerations (see Lemmas 3.1 and 3.2 below) it will be convenient for us to impose on the operator G a less restrictive condition than (1.3) in Theorem 1.1:

$$G(|H_0| + I)^{-1/2} \in \mathbf{S_2}(\mathcal{H}, \mathcal{K}). \qquad (3.1)$$

The proof of Theorem 1.1 depends on some abstract relations close to (0.3). The part of a scattering matrix in our considerations is taken by the following operator in $\mathcal{K}$:

$$S_\pm(z) = I \mp 2iK^{1/2}(z)(I \pm T(z))^{-1} K^{1/2}(z) \qquad (3.2)$$

where $\operatorname{Im} z > 0$.

Remark Due to the identity

$$(I \pm G(H_0 - zI)^{-1}G^*)(I \mp G(H_\pm - zI)^{-1}G^*) = I \qquad (3.3)$$

which holds true under the condition $G(|H_0| + I)^{-1/2} \in \mathbf{S}_\infty(\mathcal{H}, \mathcal{K})$), the inverse operator in (3.2) exists for all $z \in \mathbf{C} \setminus \mathbf{R}$ (see [9], §1.9, 1.10).

Operators of the form (3.2) have been introduced and studied in [15], [16] (see also [8], [9]). Below for the sake of brevity we omit variable z in the notations A, K, $S_\pm$ if it does not lead to a confusion. In the next lemma we collect the properties of the operators $S_\pm$ that we shall need later on.

Lemma 3.1 *Let G satisfy the condition (3.1). Then for the operators $S_\pm$ the following properties hold true:*

(i) for any z, $\operatorname{Im} z > 0$, the operators $S_\pm(z)$ are unitary in $\mathcal{K}$ and $S_\pm(z) - I \in \mathbf{S_1}(\mathcal{K})$;

(ii) $\|S_\pm(z) - I\|_{\mathbf{S_1}} \to 0$ for $\operatorname{Im} z \to +\infty$;

(iii) for any z, $\operatorname{Im} z > 0$ and any $\varphi \in (0, 2\pi)$ the following relation holds:

$$e^{i\varphi} \in \sigma_p(S_\pm(z)) \iff \mp 1 \in \sigma_p(A(z) + \cot(\varphi/2)K(z)) \tag{3.4$_\pm$}$$

where the dimensions of the corresponding eigenspaces coincide;

(iv) for any compact set $\Omega \subset \{z \in \mathbf{C} \mid \operatorname{Im} z > 0\}$ there exist the numbers $\varphi_0^{(\pm)} \in (0, 2\pi)$ such that:

$$\forall z \in \Omega : \quad \sigma(S_+(z)) \subset \{e^{i\varphi} \mid \varphi \in [\varphi_0^{(+)}, 2\pi]\} \tag{3.5$_+$}$$

$$\forall z \in \Omega : \quad \sigma(S_-(z)) \subset \{e^{i\varphi} \mid \varphi \in [0, \varphi_0^{(-)}]\}. \tag{3.5$_-$}$$

Proof (i) was proved in [16] (see also [9]).

(ii) Let us write the operator $T(z)$ as

$$T(z) = G(|H_0| + I)^{-1/2} \frac{|H_0| + I}{H_0 - zI} (G(|H_0| + I)^{-1/2})^*.$$

Since $(|H_0|+I)(H_0 - zI)^{-1} \xrightarrow{s} 0$ for $\operatorname{Im} z \to +\infty$, by (3.1) we obtain $\|T(z)\|_{\mathbf{S_1}} \to 0$ as $\operatorname{Im} z \to \infty$. This immediately implies (ii).

(iii) To be definite, let us check (3.4$_+$):

$$e^{i\varphi} \in \sigma_p(S_+) \iff e^{i\varphi} \in \sigma_p(I - 2iK(I + A + iK)^{-1})$$

$$\iff e^{i\varphi} \in \sigma_p((I + A - iK)(I + A + iK)^{-1})$$

$$\iff 0 \in \sigma_p((I + A - iK - e^{i\varphi}(I + A + iK))$$

$$\iff 0 \in \sigma_p((I + A + \cot(\varphi/2)K) \iff (-1) \in \sigma_p(A + \cot(\varphi/2)K)$$

and the dimensions of the corresponding eigenspaces coincide.

(iv) To be definite, let us prove (3.5$_-$). By (3.4$_-$), it is sufficient to prove existence of such $t_0 > 0$ that:

$$\forall z \in \Omega, \quad \forall t \geq t_0 : \quad \sigma(A(z) - tK(z)) \subset (-\infty, 1). \tag{3.6}$$

Then for $t_0 = -\cot(\varphi_0^{(-)}/2)$ we will have (3.5$_-$). To prove (3.6), let us take arbitrary $z = \lambda + i\varepsilon \in \Omega$ and note that

$$A(z) - tK(z) = G \frac{H_0 - (\lambda + t\varepsilon)I}{(H_0 - \lambda I)^2 + \varepsilon^2} G^*$$

$$\leq G(|H_0| + I)^{-1/2} \frac{(|H_0| + I)}{(H_0 - \lambda I)^2 + \varepsilon^2} (H_0 - (\lambda + t\varepsilon)I)_+ (G(|H_0| + I)^{-1/2})^*. \tag{3.7}$$

Consider the operator in the right-hand side of (3.7). Since Ω is a compact in $\{z \in \mathbf{C} \mid \operatorname{Im} z > 0\}$, there exist $\varepsilon_0 > 0$ and $\lambda_1, \lambda_2 \in \mathbf{R}$, $\lambda_1 < \lambda_2$ such that $\Omega \subset \{z \in \mathbf{C} \mid \operatorname{Im} z > \varepsilon_0,\ \lambda_1 < \operatorname{Re} z < \lambda_2\}$. From here one gets:

$$\frac{|H_0| + I}{(H_0 - \lambda I)^2 + \varepsilon^2}(H_0 - (\lambda + t\varepsilon)I)_+ \le \frac{|H_0| + I}{(H_0 - \lambda I)^2 + \varepsilon_0^2}(H_0 - (\lambda_1 + t\varepsilon_0)I)_+$$

$$\le \frac{(|H_0| + I)(H_0 - \lambda_1 I)}{(H_0 - \lambda I)^2 + \varepsilon_0^2}\theta(H_0 - (\lambda_1 + t\varepsilon_0)I) \le C(\Omega)\theta(H_0 - (\lambda_1 + t\varepsilon_0)I)$$

where θ is the characteristic function of $(0, \infty)$ and $C(\Omega)$ is a positive constant. Substituting the above estimate into (3.7), we obtain:

$$A(z) - tK(z) \le C(\Omega)G(|H_0| + I)^{-1/2}\theta(H_0 - (\lambda_1 + t\varepsilon_0)I)(G(|H_0| + I)^{-1/2})^*.$$

Taking into account (3.1) and the relation $\theta(H_0 - (\lambda_1 + t\varepsilon_0)I) \xrightarrow{s} 0$ as $t \to \infty$, we get that the norm (and even the trace norm) of the operator in the right-hand side of the last inequality tends to zero as $t \to \infty$. This gives (3.6). □

2. Below we prove the lemma which contains the main idea of the proof of Theorem 1.1.

Lemma 3.2 *Let G satisfy the condition* (3.1), *and let for* $\operatorname{Im} z > 0$ *a function $D_{H_\pm/H_0}(z)$ be defined by*

$$D_{H_\pm/H_0}(z) = \det(I \pm T(z)). \tag{3.8$_\pm$}$$

Then the following formula holds true:

$$\arg D_{H_\pm/H_0}(z) = \pm \int_{-\infty}^{\infty} \frac{dt}{1+t^2} n_\mp(1, A(z) + tK(z)) \tag{3.9$_\pm$}$$

where the branch of the argument is fixed by the condition

$$\arg D_{H_\pm/H_0}(z) \to 0, \quad \operatorname{Im} z \to +\infty. \tag{3.10$_\pm$}$$

Proof To be definite, consider the case of the upper signs. First rewrite the left-hand side of (3.9_+) as follows:

$$\arg D_{H_+/H_0} = \arg\det(I + A + iK) = -\frac{1}{2}(\arg\det(I + A - iK)$$

$$-\arg\det(I + A + iK)) = -\frac{1}{2}\arg\det(I - 2iK(I + A + iK)^{-1}) = -\frac{1}{2}\arg\det S_+.$$

Thus, it suffices to calculate $\arg\det S_+(z)$, where the branch of the argument is fixed by the condition

$$\arg\det S_+(z) \to 0, \quad \operatorname{Im} z \to \infty. \tag{3.11}$$

Lemma 3.1 (iv) together with the condition (3.11) allows one to write down the following formula for $\arg\det S_+$

$$\arg\det S_+ = -\sum_{\substack{e^{i\varphi_n} \in \sigma_p(S_+) \\ \varphi_n \in (0, 2\pi)}} (2\pi - \varphi_n).$$

By (3.4$_+$), one has (counting multiplicity):

$$\sum_{\substack{e^{i\varphi_n}\in\sigma_p(S_+)\\ \varphi_n\in(0,2\pi)}} (2\pi-\varphi_n) = \int_0^{2\pi} \operatorname{card}\{n \mid e^{i\varphi_n}\in\sigma_p(S_+), \varphi_n<\varphi\}d\varphi$$

$$= \int_0^{2\pi} \operatorname{card}\{n \mid (-1)\in\sigma_p(A+\cot(\varphi_n/2)K), \varphi_n<\varphi\}d\varphi$$

$$= \int_0^{2\pi} n_-(1, A+\cot(\varphi/2)K)d\varphi = 2\int_{-\infty}^{\infty}\frac{dt}{1+t^2}n_-(1, A+tK),$$

which gives (3.9$_+$). □

3. Proof of Theorem 1.1 Let us write down (0.2) in the following form:

$$\xi(\lambda; H_\pm, H_0) = \frac{1}{\pi}\lim_{\varepsilon\to+0}\arg D_{H_\pm/H_0}(\lambda+i\varepsilon)$$

where the prelimit expression in the right-hand side is defined by (3.8), (3.10). By Lemma 3.2, for a.e. $\lambda\in\mathbf{R}$ one has

$$\xi(\lambda; H_\pm, H_0) = \pm\frac{1}{\pi}\lim_{\varepsilon\to+0}\int_{-\infty}^{\infty}\frac{dt}{1+t^2}n_\mp(1, A(\lambda+i\varepsilon)+tK(\lambda+i\varepsilon)). \tag{3.12}$$

Note that $\{-1,1\}\cap\sigma_p(A(\lambda+i0)+iK(\lambda+i0))=\emptyset$ for a.e. $\lambda\in\mathbf{R}$. This can be deduced, say, from the identity (3.3) and the fact that, by (1.3), the operator-valued functions $G(H_\pm-zI)^{-1}G^*$ have limit values as $z\to\lambda+i0$ for a.e. $\lambda\in\mathbf{R}$. This remark allows one to apply Lemma 2.5. Passing to the limit in the right-hand side of (3.12) for a.e. $\lambda\in\mathbf{R}$, we obtain (1.7). □

4 Proof of Theorem 1.2

1. First let us discuss existence of the boundary values of the operators $A(z)$ and $K(z)$. The next statement follows directly from the basic facts of the stationary trace-class scattering theory.

Lemma 4.1 *Let the conditions* (1.8)–(1.10) *hold true. Then for a.e.* $\lambda\in\mathbf{R}$*:*
(i) *the operator* $T(z)$ *has limit values in* $\mathbf{S}_\infty(\mathcal{K})$ *as* $z\to\lambda+i0$*;*
(ii) $K(\lambda+i0)\in\mathbf{S_1}(\mathcal{K})$;
(iii) $\{-1,1\}\cap\sigma_p(T(\lambda+i0))=\emptyset$.

Proof

1. Let $\delta\subset\mathbf{R}$ be an arbitrary bounded interval. Let us represent $T(z)$ as follows:

$$T(z) = GE_{H_0}(\delta)(H_0-zI)^{-1}(GE_{H_0}(\delta))^*$$

$$+GE_{H_0}(\mathbf{R}\setminus\delta)(H_0-zI)^{-1}(GE_{H_0}(\mathbf{R}\setminus\delta))^* =: T_1(z)+T_2(z).$$

By (1.10), the operator $T_1(\lambda+i\varepsilon)$ has limit values in $\mathbf{S_2}(\mathcal{K})$ for a.e. $\lambda\in\delta$ and $\operatorname{Im}T_1(\lambda+i0)\in\mathbf{S_1}$ (see [11]). On the other hand, the operator $T_2(\lambda+i\varepsilon)$ has limit values as $\varepsilon\to+0$ in $\mathbf{S}_\infty(\mathcal{K})$ for all $\lambda\in\delta$, and $\operatorname{Im}T_2(\lambda+i0)=0$. This proves (i), (ii).

2. To prove (iii), let us first show that (1.8)–(1.10) imply the inclusions

$$G(H_\pm + aI)^{-1/2} \in \mathbf{S}_\infty(\mathcal{H}, \mathcal{K}) \tag{4.1}$$

$$G(H_\pm + aI)^{-m} \in \mathbf{S_2}(\mathcal{H}, \mathcal{K}) \tag{4.2}$$

for some $-a < \inf \sigma(H_-)$.

Obviously, (4.1) follows from (1.9) and the boundedness of $(H_0 + \gamma I)^{1/2}(H_\pm + aI)^{-1/2}$. The relation (4.2) is equivalent to the inclusion

$$G(H_\pm + aI)^{-2m} G^* \in \mathbf{S_1}(\mathcal{K}) \tag{4.3}$$

which we check below.

3. Take $-a < \inf \sigma(H_-)$ so that

$$\|G(H_0 + aI)^{-1/2}\|_{\mathbf{S}_\infty} < 1 \tag{4.4}$$

$$\|G(H_0 + aI)^{-m}\|_{\mathbf{S_2}} < 1. \tag{4.5}$$

It is easy to see that such a exists by (1.8)–(1.10). Interpolating between (4.4) and (4.5), one gets

$$\|G(H_0 + aI)^{-1/2-k}\|_{\mathbf{S_{(2m-1)/k}}} < 1, \quad 0 \le k \le m - 1/2. \tag{4.6}$$

Writing the perturbation power series for the operator (4.3) and taking into account (4.6), one easily verifies its convergence in $\mathbf{S_1}(\mathcal{K})$, which gives (4.3). A similar argument see, e.g., in [10], Theorem XI.12.

4. From (4.1), (4.2), as in part (i), we obtain that the operator $G(H_\pm - zI)^{-1}G^*$ has limit values in $\mathbf{S}_\infty(\mathcal{K})$ for a.e. $\lambda \in \mathbf{R}$ as $z \to \lambda + i0$. Applying (3.3), we get the statement (iii). □

2. The plan for the proof of Theorem 1.2 is to approximate V by a sequence of trace-class operators V_j, apply Theorem 1.1 for a fixed j and pass to the limit as $j \to \infty$ in (1.7). Lemma 2.5 enables one to pass to the limit in the right side of (1.7), while the next lemma ensures the passage to the limit in the left side.

Lemma 4.2 *Suppose that for a sequence of operators, M_j, $j = 0, 1, 2, \dots$ in $\mathcal{H}$ the following conditions hold true:*

$$0 \le M_j \le aI, \quad a > 0 \tag{4.7}$$

$$\pm(M_j - M_{j+1}) \ge 0 \tag{4.8$_\pm$}$$

$$\operatorname{rank}(M_j - M_0) < \infty. \tag{4.9}$$

In addition, suppose that for

$$M := \operatorname*{s-lim}_{j \to \infty} M_j \tag{4.10}$$

(this limit exists by (4.8)*) one has*

$$M_0^l - M^l \in \mathbf{S_1} \text{ for some } l > 1. \tag{4.11}$$

Then for a.e. $\lambda \in (0, a)$
$$\xi(\lambda; M_j, M) \to 0, \quad j \to \infty. \tag{4.12}$$

Proof

1. First note that by (4.11) and (4.9) the SSF $\xi(\lambda; M_j, M)$ is correctly defined. Next, (4.8$_\pm$) implies the monotonicity of the sequence $\xi(\lambda; M_j, M)$. Thus, it suffices to show that the sequence $\xi(\lambda; M_j, M)$ tends to zero in $L_1(a_0, a)$ for any $a_0 \in (0, a)$.

2. Let $V_j = \pm(M_j - M) \geq 0$ (the signs are taken in accordance with those in (4.8)). It is easy to show that the condition (4.11) implies the compactness of V_0. From here and (4.8), (4.10) one gets
$$\|V_j\| \to 0, \quad j \to \infty. \tag{4.13}$$

3. Let us present a compulsory result from [13] (see also [9], Theorem 8.10.4). Arrange the eigenvalues λ_k of V_0 in order of decreasing, and write the spectral representation of V_0 as
$$V_0 = \sum_{k=1}^{\infty} \lambda_k (\cdot, \psi_k)\psi_k.$$

For $k \geq 1$ denote: $\tilde{P}_k = \sum\limits_{i=1}^{k} (\cdot, \psi_i)\psi_i$, $P_k = I - \tilde{P}_k$, $V_0^{(k)} = P_k V_0$, $\tilde{V}_0^{(k)} = \tilde{P}_k V_0$. It has been proven in [13], [9] that (4.11) implies
$$\|(M \pm V_0^{(k)})^n - M^n\|_{\mathbf{S_1}} \to 0, \quad k \to \infty \tag{4.14}$$
where n is the first odd integer greater than l.

4. Let us fix arbitrary $a_0 \in (0, a)$ and check that
$$\int_{a_0}^{a} |\xi(\lambda; M_j, M)| d\lambda \to 0, \quad j \to \infty. \tag{4.15}$$

For all $j, k = 1, 2, 3, \ldots$ write V_j as $V_j = V_j^{(k)} + \tilde{V}_j^{(k)}$, where $V_j^{(k)} = P_k V_j P_k$ and the projection P_k is defined in the previous paragraph. It is clear that $V_j^{(k)} \leq V_0^{(k)}$ (by (4.8$_\pm$)). Thus, for all k, j one has:
$$|\xi(\lambda; M_j, M)| = |\xi(\lambda; M \pm V_j, M)| \leq |\xi(\lambda; M \pm V_j, M \pm V_j^{(k)})|$$
$$+|\xi(\lambda; M \pm V_j^{(k)}, M)| \leq |\xi(\lambda; M \pm V_j, M \pm V_j^{(k)})| + |\xi(\lambda; M \pm V_0^{(k)}, M)|.$$

From here taking into account the estimate $\|\tilde{V}_j^{(k)}\|_{\mathbf{S_1}} \leq \operatorname{rank} \tilde{V}_j^{(k)} \|V_j\| \leq 2k\|V_j\|$ one gets
$$\int_{a_0}^{a} |\xi(\lambda; M_j, M)| d\lambda \leq \int_{a_0}^{a} |\xi(\lambda; M \pm V_j, M \pm V_j^{(k)})| d\lambda$$
$$+ \int_{a_0}^{a} |\xi(\lambda; M \pm V_0^{(k)}, M)| d\lambda \leq 2k\|V_j\|$$

$$+n^{-1}{a_0}^{1-n}\int_{{a_0}^n}^{a^n}|\xi(\lambda^n;(M\pm V_0^{(k)})^n,M^n)|d\lambda^n\le 2k\|V_j\|$$

$$+n^{-1}{a_0}^{1-n}\|(M\pm V_0^{(k)})^n-M^n\|_{\mathbf{S_1}}.$$

Since k is arbitrary here, (4.13) and (4.14) imply (4.15). □

3. Proof of Theorem 1.2

1. Let P_j be some sequence of finite-dimensional orthogonal projections in $\mathcal{K}$ so that $P_0=0$, $P_{j+1}\ge P_j$, and $P_j\xrightarrow{s} I$ as $j\to\infty$. Denote $G_j=P_jG$ and define the operators $H_\pm^{(j)}=H_0\pm G_j^*G_j$ via the corresponding sesquilinear forms.

2. Let us fix $j\in\mathbf{N}$ and prove the representation:

$$\xi(\lambda;H_\pm^{(j)},H_0)=\pm\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{dt}{1+t^2}n_\mp(1,P_jA(\lambda+i0)P_j+tP_jK(\lambda+i0)P_j). \tag{4.16}$$

The condition (1.10) implies

$$G_j(H_0+\gamma I)^{-1/2}\in\mathbf{S_2}(\mathcal{H},\mathcal{K}). \tag{4.17}$$

It was proved in [5], [6] that (1.8), (4.17) imply the following representation for a.e. $\lambda\in\mathbf{R}$:

$$\xi(\lambda;H_\pm^{(j)},H_0)=\frac{1}{\pi}\lim_{\varepsilon\to+0}\arg D_{H_\pm^{(j)}/H_0}(\lambda+i\varepsilon) \tag{4.18}$$

where the function $D_{H_\pm^{(j)}/H_0}(z)$ is defined by (1.6), (3.8), (3.10) with $G\mapsto G_j$. By (4.17), one can apply Lemma 3.2, which gives the representation

$$\xi(\lambda;H_\pm^{(j)},H_0)=\pm\frac{1}{\pi}\lim_{\varepsilon\to+0}\int_{-\infty}^{\infty}\frac{dt}{1+t^2}n_\mp(1,P_jA(\lambda+i\varepsilon)P_j+tP_jK(\lambda+i\varepsilon)P_j). \tag{4.19}$$

Next, as in Lemma 4.1(iii), one checks that $\{-1,1\}\cap\sigma(P_jT(\lambda+i0)P_j)=\emptyset$ for a.e. $\lambda\in\mathbf{R}$. This allows one to apply Lemma 2.5 and pass to the limit as $\varepsilon\to+0$ in the right-hand side of (4.19), which gives (4.16).

3. Finally, let us prove that one can pass to the limit in (4.16) as $j\to\infty$ and thus obtain (1.7). Let $M_j^\pm=(H_\pm^{(j)}-\lambda_0I)^{-1}$, $M^\pm=(H_0-\lambda_0I)^{-1}$, where λ_0 is the same as in (1.11). Operators $M_j^\pm$, $M^\pm$ satisfy the hypothesis of Lemma 4.2 with $a^{-1}=\min\{\inf\sigma(H_0)-\lambda_0,\inf\sigma(H_\pm)-\lambda_0\}$. Hence, for a.e. $\lambda\in\mathbf{R}$:

$$\xi(\lambda;H_\pm,H_0)-\xi(\lambda;H_\pm^{(j)},H_0)=\xi(\lambda;H_\pm,H_\pm^{(j)})\to 0,\quad j\to\infty.$$

On the other hand, by Lemmas 2.5 and 4.1(iii), one can pass to the limit in the integral in the right-hand side of (4.16) for a.e. $\lambda\in\mathbf{R}$, which gives the desired representation (1.7). □

5 Proof of Lemmas 2.1, 2.2, 2.5

1. Proof of Lemma 2.1 We will prove only the estimate (2.3_-); the other estimates are treated analogously. By (1.1_+), we get for all $\varepsilon \in (0,1)$:

$$n_+(1, A + tK) \le n_+(1 - \varepsilon, A) + n_+(\varepsilon, tK).$$

Integrating by t with the weight $\pi^{-1}(1+t^2)^{-1}$, one obtains:

$$\xi_- \le n_+(1-\varepsilon, A) + \frac{1}{\pi}\int_0^\infty \frac{dt}{1+t^2} n(\varepsilon t^{-1}, K). \tag{5.1}$$

It remains to estimate the integral in the right-hand side of (5.1). One has:

$$\frac{1}{\pi}\int_0^\infty \frac{dt}{1+t^2} n(\varepsilon t^{-1}, K) = \frac{1}{\pi}\int_0^\infty \frac{\varepsilon}{\varepsilon^2+\tau^2} n(\tau, K) d\tau$$
$$= \varepsilon^{-p}\int_0^\infty g_p(\tau/\varepsilon) p\tau^{p-1} n(\tau, K) d\tau$$

where $p\pi g_p(x) = x^{1-p}(1+x^2)^{-1}$. Since $\int_0^\infty p\tau^{p-1}n(\tau,K)d\tau = \|K\|^p_{\mathbf{S_p}}$, $2\pi p g_p(x) \le (1-p)^{\frac{1-p}{2}}(1+p)^{\frac{1+p}{2}}$, and $g_p(x) \to 0$ as $x \to \infty$, one gets

$$\int_0^\infty g_p(\tau/\varepsilon)p\tau^{p-1}n(\tau,K)d\tau = \mu_p(\varepsilon, K)\|K\|^p_{\mathbf{S_p}}$$

where $\mu_p(\varepsilon, K)$ has the required properties (2.5). □

2. Proof of Lemma 2.2 As in the proof of the previous lemma, first we obtain (5.1). Then we estimate the integral in the right-hand side of (5.1) as follows:

$$\frac{1}{\pi}\int_0^\infty \frac{dt}{1+t^2} n(\varepsilon t^{-1}, K) = \frac{\varepsilon^{-p}}{\pi}\int_0^\infty \frac{\tau^{-p}}{1+\tau^2}\tau^p n(\tau, K) d\tau$$
$$\le \frac{\varepsilon^{-p}}{\pi}\left(\sup_{\tau\ge 0}\tau^p n(\tau,K)\right)\int_0^\infty \frac{\tau^{-p}}{1+\tau^2}d\tau = \varepsilon^{-p}(2\cos\frac{\pi p}{2})^{-1}\|K\|^p_{\mathbf{\Sigma_p}},$$

which gives (2.6_-). The rest of the estimates are treated analogously. □

3. Proof of Lemma 2.5 To be definite, let us prove (2.8_-).

1. Let us first estimate the integral in the left-hand side of (2.8_-). By (1.1_+), for a fixed $\varepsilon \in (0,1)$ one has:

$$n_+(1, A_j + tK_j) = n_+(1, A + tK + (A_j - A) + t(K_j - K))$$
$$\le n_+(1-2\varepsilon, A+tK) + n_+(\varepsilon, A_j - A) + n_+(\varepsilon, t(K_j - K)).$$

Integrate the above estimate with the weight $(1+t^2)^{-1}$:

$$\int_{-\infty}^\infty \frac{dt}{1+t^2} n_+(1, A_j + tK_j) \le \int_{-\infty}^\infty \frac{dt}{1+t^2} n_+(1-2\varepsilon, A+tK)$$

$$+\pi n_+(\varepsilon, A_j - A) + \int_{-\infty}^{\infty} \frac{dt}{1+t^2} n_+(\varepsilon, t(K_j - K)). \tag{5.2}$$

2. Let us estimate the last integral in the right-hand side of (5.2):

$$\int_{-\infty}^{\infty} \frac{dt}{1+t^2} n_+(\varepsilon, t(K_j - K)) = \int_{0}^{\infty} \frac{\varepsilon}{\tau^2+\varepsilon^2} n_+(\tau, |K_j - K|) d\tau$$

$$\leq \varepsilon^{-1} \|K_j - K\|_{\mathbf{S_1}}. \tag{5.3}$$

3. Let us pass to the upper limit in j in the estimate (5.2). By (5.3) one gets:

$$\limsup_{j\to\infty} \int_{-\infty}^{\infty} \frac{dt}{1+t^2} n_+(1, A_j + tK_j) \leq \int_{-\infty}^{\infty} \frac{dt}{1+t^2} n_+(1 - 2\varepsilon, A + tK). \tag{5.4}$$

4. By the analytic Fredholm alternative (see, e.g., [9]), the set $\{t \in \mathbf{C} \mid 1 \in \sigma(A + tK)\}$ is either discrete or coincides with $\mathbf{C}$. But the last variance cannot realize because of the condition $1 \notin \sigma(A + z_0K)$. Thus, for all $t \in \mathbf{R}$ but for a discrete set one has:

$$n_+(1 - 2\varepsilon, A + tK) \to n_+(1, A + tK) \text{ as } \varepsilon \to 0.$$

Thus, by the monotone convergence theorem,

$$\int_{-\infty}^{\infty} \frac{dt}{1+t^2} n_+(1 - 2\varepsilon, A + tK) \to \int_{-\infty}^{\infty} \frac{dt}{1+t^2} n_+(1, A + tK) \text{ as } \varepsilon \to 0.$$

From here and (5.4) one gets:

$$\limsup_{j\to\infty} \int_{-\infty}^{\infty} \frac{dt}{1+t^2} n_+(1, A_j + tK_j) \leq \int_{-\infty}^{\infty} \frac{dt}{1+t^2} n_+(1, A + tK).$$

Similarly (using (1.2_+) instead of (1.1_+)) one estimates from below the lower limit in j. These two estimates imply (2.8_-). □

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