SPECTRAL THEORY OF PIECEWISE CONTINUOUS FUNCTIONS OF SELF-ADJOINT OPERATORS

ALEXANDER PUSHNITSKI AND DMITRI YAFAEV

Abstract. Let $H_0$, $H$ be a pair of self-adjoint operators for which the standard assumptions of the smooth version of scattering theory hold true. We give an explicit description of the absolutely continuous spectrum of the operator $D_\theta = \theta(H) - \theta(H_0)$ for piecewise continuous functions $\theta$. This description involves the scattering matrix for the pair $H_0$, $H$, evaluated at the discontinuities of $\theta$. We also prove that the singular continuous spectrum of $D_\theta$ is empty and that the eigenvalues of this operator have finite multiplicities and may accumulate only to the “thresholds” of the absolutely continuous spectrum of $D_\theta$. Our approach relies on the construction of “model” operators for each jump of the function $\theta$. These model operators are defined as certain symmetrised Hankel operators which admit explicit spectral analysis. We develop the multichannel scattering theory for the set of model operators and the operator $\theta(H) - \theta(H_0)$. As a by-product of our approach, we also construct the scattering theory for general symmetrised Hankel operators with piecewise continuous symbols.

1. Introduction

1.1. Overview. Let $H_0$ and $H$ be self-adjoint operators and suppose that the difference $V = H - H_0$ is a compact operator. If $\theta$ is a continuous function which tends to zero at infinity then the difference

$$D_\theta = \theta(H) - \theta(H_0)$$

is also compact. On the contrary, if $\theta$ has discontinuities, then the operator $D_\theta$ may acquire the (absolutely) continuous spectrum. This phenomenon was observed in [6] in a concrete example and established in [13] under fairly general assumptions.

Our goal here is to study the structure of the operator $D_\theta$ for piecewise continuous functions under assumptions on $H_0$, $H$ typical for smooth scattering theory (see, e.g., [7, 19]). Roughly speaking, these assumptions mean that the perturbation $V = H - H_0$ is an integral operator with a sufficiently smooth

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kernel in the spectral representation of the “unperturbed” operator $H_0$. Under our assumptions the scattering matrix $S(\lambda)$ for the pair $H_0, H$ is well defined for $\lambda$ in the absolutely continuous (a.c.) spectrum of the operator $H_0$. The scattering matrix is a unitary operator in some auxiliary Hilbert space $\mathcal{N}(\lambda)$ which is the fiber space in the spectral representation of $H_0$. We denote by $N(\lambda) = \dim \mathcal{N}(\lambda) \leq \infty$ the multiplicity of the spectrum of the operator $H_0$ at the point $\lambda$. Moreover, the operator $S(\lambda) - I$ is compact, so that the spectrum of $S(\lambda)$ consists of eigenvalues $\{\sigma_n(\lambda)\}_{n=1}^N$ lying on the unit circle in $\mathbb{C}$. Eigenvalues of $S(\lambda)$ distinct from 1 have finite multiplicities and can accumulate only to the point 1.

We suppose that $\theta(\lambda)$ is a continuous function except at points $\lambda_1, \ldots, \lambda_L$, $L < \infty$, where it has jump discontinuities. That is, at each of these points $\lambda_\ell$, the limits $\theta(\lambda_\ell \pm 0)$ exist and are finite, but $\theta(\lambda_\ell + 0) \neq \theta(\lambda_\ell - 0)$. We denote the jumps of $\theta$ by

$$\tau_\ell = \theta(\lambda_\ell + 0) - \theta(\lambda_\ell - 0).$$

(1.2)

We prove that the a.c. spectrum of the operator $D_\theta$ consists of the union of the intervals:

$$\text{spec}_{ac} D_\theta = \bigcup_{\ell=1}^L \bigcup_{n=1}^{N_\ell} [-a_{n\ell}, a_{n\ell}], \quad a_{n\ell} = \frac{1}{2} |\tau_\ell| |\sigma_n(\lambda_\ell) - 1|, \quad N_\ell = N(\lambda_\ell).$$

(1.3)

Here and in similar formulas below describing the a.c. spectrum, we use two conventions:

(i) the union is taken over all non-degenerate intervals, i.e. if $a_{n\ell} = 0$, then the corresponding degenerate interval is dropped from the union;

(ii) each interval contributes multiplicity one to the a.c. spectrum of $D_\theta$. That is, denoting by $A(\Lambda)$ the operator of multiplication by $\lambda$ in $L^2(\Lambda, d\lambda)$, one can state (1.3) more precisely as follows: the a.c. part of $D_\theta$ is unitarily equivalent to the orthogonal sum

$$D_\theta^{(ac)} \simeq \bigoplus_{\ell=1}^L \bigoplus_{n=1}^{N_\ell} A([-a_{n\ell}, a_{n\ell}]).$$

We also prove that the singular continuous spectrum of $D_\theta$ is empty, the eigenvalues of $D_\theta$ can accumulate only to 0 and to the points $\pm a_{n\ell}$, and all eigenvalues of $D_\theta$ distinct from 0 and $\pm a_{n\ell}$ have finite multiplicity.

It follows from (1.3) that $\text{spec}_{ac} D_\theta = \emptyset$ if and only if $S(\lambda_\ell) = I$ for all $\ell = 1, \ldots, L$. In the latter case the operator $D_\theta$ is compact. We emphasize that only the jumps $\tau_\ell$ of $\theta(\lambda)$ at the points $\lambda_\ell$ of discontinuity and the spectrum of the scattering matrix $S(\lambda_\ell)$ at these points are essential for our construction.

All of our results apply to the case where $H_0$ and $H$ are the free and perturbed Schrödinger operators – see Example 2.3.
1.2. $\mathcal{D}_\theta$ and Hankel operators. In [9], V. V. Peller discovered a link between the operators $\mathcal{D}_\theta$ (for smooth $\theta$) and Hankel operators with the symbol $\theta$. This link was used by Peller and his collaborators to establish estimates of various norms of $\mathcal{D}_\theta$ in terms of the norms of $\theta$ in classes of smooth functions.

Here we take this idea one step further and show that Hankel operators provide a useful model for operators $\mathcal{D}_\theta$ even for discontinuous functions $\theta$. The new feature of our approach is that our model Hankel type operators have operator-valued symbols, and these symbols involve not only the function $\theta(\lambda)$, but also the scattering matrix $S(\lambda)$.

We assume that the multiplicity of the spectrum of $H_0$ is constant (equal to $N_\ell$) in a neighbourhood of each point $\lambda_\ell$, $\ell = 1, \ldots, L$. In order to simplify this preliminary discussion, in this subsection and in the next one we also assume temporarily that $N_1 = \cdots = N_L$. Thus, the corresponding fiber space $N(\lambda) = N$ can be chosen to be constant a neighbourhood of $\{\lambda_1, \ldots, \lambda_L\}$.

Next, we define an operator-valued function $\Xi(\lambda)$, $\lambda \in \mathbb{R}$, with values in the set of compact operators on $N$, by setting

$$\Xi(\lambda) = (2\pi i)^{-1} (S(\lambda) - I)\theta(\lambda),$$

when $\lambda$ is in a neighbourhood of $\{\lambda_1, \ldots, \lambda_L\}$ and by extending $\Xi(\lambda)$ to all $\lambda \in \mathbb{R}$ in a smooth way.

Our key tool is a new formula representation for $\mathcal{D}_\theta$ (see Theorem 6.1), which relates $\mathcal{D}_\theta$ to the operator

$$M_\Xi = P_- \Xi P_+ + P_+ \Xi^* P_- \quad \text{in } L^2(\mathbb{R}; N).$$

Here $P_\pm$ are the orthogonal projections in $L^2(\mathbb{R}; N)$ onto the Hardy classes $H^2_\pm(\mathbb{R}; N)$. For obvious reasons, we call operators of the form (1.5) symmetrised Hankel operators, SHOs for short. It is important that SHOs are automatically self-adjoint and, for a particular choice of $\Xi$, admit an explicit diagonalization.

Our Theorem 6.1 shows that the operators $\mathcal{D}_\theta$ and $M_\Xi$ are, in some sense, close to each other. Thus the SHO $M_\Xi$ with the symbol (1.4) symbol plays the role of a model operator for $\mathcal{D}_\theta$. We analyse the operator $M_\Xi$ and, as a consequence, establish the spectral results for $\mathcal{D}_\theta$ mentioned above.

We note that our representation for $\mathcal{D}_\theta$ is different from the one given by the double operator integral (see the survey [1] by M. Sh. Birman and M. Z. Solomyak and references therein). In particular, the double operator integral approach treats the operators $H_0$ and $H$ in a symmetric way while our representation relies on the spectral representation of $H_0$ only. This is convenient for our purposes.

1.3. Main ideas of the approach. Roughly speaking, our approach relies on the construction of scattering theory for the pair $M_\Xi, \mathcal{D}_\theta$, that is, on the comparison of asymptotic behaviour as $t \to \pm \infty$ of functions $\exp(-iM_\Xi t) f_0$.

and \( \exp(-i\mathcal{D}_\theta t)f \) for elements \( f_0 \) and \( f \) in the a.c. subspaces of the operator \( M_\Xi \) and \( \mathcal{D}_\theta \), respectively. It turns out that the functions \( \exp(-iM_\Xi t)f_0 \) are asymptotically concentrated as \( t \to \pm \infty \) in neighbourhoods of the points \( \lambda_1, \ldots, \lambda_L \). This means that every discontinuity of \( \theta \) yields its own band of the a.c. spectrum.

To handle this situation, we introduce the model operators \( M_{\Xi\ell} \) for all discontinuity points \( \lambda_\ell, \ell = 1, \ldots, L \). The model operators \( M_{\Xi\ell} \) are defined as SHO in the space \( L^2(\mathbb{R}; \mathcal{N}) \) with the symbols \( \Xi_\ell \), each of which contains only one jump. We choose these symbols in such a way that, up to smooth terms, the sum \( \Xi_1 + \cdots + \Xi_L \) equals the function \( \Xi \) defined by (1.4). It is important that each operator \( M_{\Xi\ell} \) can be explicitly diagonalized. Then we develop the scattering theory for the set of model operators \( M_{\Xi_1}, \ldots, M_{\Xi_L} \) and the operator \( \mathcal{D}_\theta \). To be more precise, we prove the existence of wave operators for all pairs \( M_{\Xi\ell}, \mathcal{D}_\theta, \ell = 1, \ldots, L \). The ranges of these wave operators for different \( \ell \) are orthogonal to each other, and their orthogonal sum exhausts the a.c. subspace of the operator \( \mathcal{D}_\theta \). The results of this type are known as the asymptotic completeness of wave operators. Our proofs of these results require a version of multichannel scattering theory constructed in our earlier publication [17]. In the important particular case \( L = 1 \) the multichannel scheme is not necessary, and it suffices to apply the usual results of smooth scattering theory to the pair \( M_\Xi, \mathcal{D}_\theta \).

The model operators \( M_{\Xi\ell} \) are constructed in terms of some bounded Hankel operator with simple a.c. spectrum which can be explicitly diagonalized. The choice of such Hankel operator is not unique. We proceed from the Mehler operator, that is, the Hankel operator in the space \( L^2(\mathbb{R}_+; \mathcal{N}) \) with the integral kernel \( \pi^{-1}(2 + t + s)^{-1} \). Alternatively, we could have used the Hankel operator with integral kernel \( \pi^{-1}(t + s)^{-1}e^{-t-s} \) diagonalized by W. Magnus and M. Rosenblum.

We use the smooth method of scattering theory and work in the spectral representation of the operator \( \mathcal{H}_0 \). Thus we “transplant” the operator \( \mathcal{D}_\theta \) by an isometric (but not necessary unitary) transformation into the space \( L^2(\mathbb{R}; \mathcal{N}) \). An important and technically difficult step is to localize the problem onto a neighbourhood of the set \( \{\lambda_1, \ldots, \lambda_L\} \). It turns out that, after such a localization, the transplanted operator \( \mathcal{D}_\theta \) is close to the SHO \( M_\Xi \) with symbol (1.4).

As a by-product of our approach, we develop the scattering theory for general SHOs \( M_\Xi \) with piecewise continuous symbols \( \Xi \). This theory is one of the key ingredients of our analysis and is perhaps of interest in its own sake.

We note that the SHOs introduced here are very well adapted to the study of discontinuous functions of self-adjoint operators and their spectral theory is simpler than that of standard Hankel operators. We plan to apply the approach
of the present paper to the usual Hankel operators with discontinuous symbols in a separate publication.

1.4. **History.** The analysis of \( D_\theta \) for discontinuous \( \theta \) was initiated in \([6, 13]\). Formula (1.3) first appeared in \([13]\) under relatively stringent assumptions on the perturbation \( V \) and for \( \theta \) being the characteristic function of a half-line. In \([13]\) a mixture of trace class and smooth methods of scattering theory has been used.

To a certain extent, this paper can be considered as a continuation of \([16]\) where the purely smooth approach has been applied to the study of the operator \( D_\theta^2 \). The main difference between \([16]\) and this work is that here we analyse the operator \( D_\theta \) directly whereas in \([16]\) only the spectral properties of the operator \( D_\theta^2 \) were considered. Thus the approach of \([16]\) does not capture information about the structure of the operator \( D_\theta \) and, in particular, about its eigenfunctions. Another important difference is that here we treat arbitrary piecewise continuous functions \( \theta \) with finite limits at \( \pm \infty \) whereas in \([16]\) only the case of \( \theta \) being the characteristic function of a half-line was considered.

Under somewhat less restrictive assumptions than here, it was shown in \([14]\) that the essential spectrum of \( D_\theta \) coincides with the union of the intervals in the r.h.s. of (1.3). This result is of course consistent with formula (1.3) for the a.c. spectrum of \( D_\theta \). The operators \( D_{\theta_\varepsilon} \) for smooth functions \( \theta_\varepsilon \) with supports shrinking as \( \varepsilon \to 0 \) to some point \( \lambda_0 \) were studied in \([15]\).

As far as the spectral theory of Hankel operators with piecewise continuous symbols is concerned, we first note S. Power’s characterisation \([11]\) of the essential spectrum. In the self-adjoint case, the absolutely continuous spectrum was described in \([3]\) by J. Howland who used the trace class method. Moreover, he applied in \([4]\) the Mourre method to perturbations of the Carleman operator and proved the absence of singular continuous spectrum in this case. To a certain extent, the paper \([3]\) can be considered as a precursor of our results on SHOs.

1.5. **The structure of the paper.** The basic objects of scattering theory are introduced in Section 2, where we also state the precise assumptions on the operators \( H_0 \) and \( H \) specific for the smooth scattering theory. In particular, in Subsection 2.6 we summarize the results of \([17]\) concerning the multichannel version of the scattering theory. This theory is used in the study of both \( \text{SHO} (1.5) \) and the operators \( D_\theta \).

In Sections 3 and 4 we collect diverse analytic results which are used in Section 5 for the study of SHOs and in Section 7 for the study of the operators \( D_\theta \). In Section 3 we diagonalize explicitly some special SHO that will be used as a model operator. In Section 4 we prove the compactness of Hankel operators sandwiched by some singular weights.
Spectral and scattering theory of SHOs with piecewise continuous symbols is developed in Section 5. Here the main results are Theorems 5.1 and 5.3.

In Section 6, we obtain convenient representations for operator (1.1) sandwiched by appropriate functions of the operator $H_0$. These representations play an important role in our analysis and are perhaps of an independent interest.

Our main results (Theorems 7.2 and 7.3) concerning the operators $D_\theta$ are stated and proven in Section 7.

1.6. Notation. Let $H$ be a separable Hilbert space. We denote by $B = B(H)$ (resp. by $\mathcal{S}_\infty = \mathcal{S}_\infty(H)$) the class of all bounded (resp. compact) operators on $H$. For a self-adjoint operator $A$, we denote by $E_A(\cdot)$ the projection-valued spectral measure of $A$; $\text{spec} A$ is the spectrum of $A$ and $\text{spec}_p A$ is its point spectrum. We denote by $H_{A}^{\text{ac}}$ the a.c. subspace of $A$, $P_{A}^{\text{ac}}$ is the orthogonal projection onto $H_{A}^{\text{ac}}$, $E_{A}^{\text{ac}}(\cdot) = E_{A}(\cdot)P_{A}^{\text{ac}}$ and $\text{spec}_{ac} A$ is the a.c. spectrum of $A$. For $K \in \mathcal{S}_\infty(H)$ we denote by $s_n(K)$, $1 \leq n \leq \dim H$, the sequence of singular values of $A$ (which may include zeros) enumerated in the decreasing order with multiplicities taken into account. We denote by $C_0(\mathbb{R}; H)$ the space of all continuous functions $f : \mathbb{R} \to H$ such that $\|f(x)\|_H \to 0$ as $|x| \to 0$. We denote by $\chi_A$ the characteristic function of a set $A \subset \mathbb{R}$ and write $\mathbb{R}^\pm = \{ x \in \mathbb{R} : \pm x > 0 \}$. We often use the same notation for a bounded function and the operator of multiplication by this function in the Hilbert space $L^2(\mathbb{R})$. The same convention applies to operator valued functions.

2. Spectral and scattering theory. Generalities

2.1. The strong smoothness. Let $A$ be a self-adjoint operator in a Hilbert space $H$. Suppose that $\delta$ is an open interval where the spectrum of $A$ is purely absolutely continuous with a constant multiplicity $N \leq \infty$. More explicitly, we assume that for some auxiliary Hilbert space $N$, there exists a unitary operator $F$ from $\text{Ran} E_A(\delta)$ onto $L^2(\delta; N)$, $\dim N = N$, such that $F$ diagonalizes $A$: if $f \in \text{Ran} E_A(\delta)$ then

$$ (FAf)(\mu) = \mu (Ff)(\mu), \quad \mu \in \delta. \quad (2.1) $$

Let us formulate an assumption typical for smooth scattering theory. Let $Q$ be a bounded operator in $H$. Suppose that the operators $Z(\mu) : H \to N$ defined by the relation

$$ Z(\mu)f = (FQ^*f)(\mu), \quad \mu \in \delta, \quad (2.2) $$

are compact and satisfy the estimates

$$ \|Z(\mu)\| \leq C, \quad \|Z(\mu) - Z(\mu')\| \leq C|\mu - \mu'|^\gamma, \quad \mu, \mu' \in \delta, \quad \gamma \in (0, 1], \quad (2.3) $$
where the constant $C$ is independent of $\mu$, $\mu'$ in compact subintervals of $\delta$ and $\gamma \in (0, 1]$. Thus we accept the following

**Definition 2.1.** We say that $Q$ is strongly $A$-smooth on $\delta$ with the exponent $\gamma$ if, for some diagonalization $F$ of the operator $E_A(\delta)A$ and for the operator $Z(\lambda)$ defined by (2.2), condition (2.3) is satisfied.

It follows from (2.1), (2.2) that for any bounded function $\varphi$ which is compactly supported on $\delta$ and for all $f \in \mathcal{H}$, we have

$$Q\varphi(A)f = \int_{-\infty}^{\infty} Z^*(\mu)(\mathcal{F}f)(\mu)\varphi(\mu)d\mu.$$  \hspace{1cm} (2.4)

We emphasize that in applications the map $\mathcal{F}$ emerges naturally.

2.2. **Operators $H_0$ and $H$.** Let $H_0$ and $V$ be self-adjoint operators in a Hilbert space $\mathcal{H}$. For simplicity, we assume that the “perturbation” $V$ is a bounded operator so that the sum $H = H_0 + V$ is self-adjoint on the domain of the operator $H_0$. Similarly to [16], all our constructions can easily be extended to a class of unbounded operators $V$, but we will not dwell upon this here.

Suppose that $V$ is factorized as $V = G_* V_0 G$ where

$$V_0 = V_0^* \in \mathcal{B}(\mathcal{H}), \quad G(|H_0| + I)^{-1/2} \in \mathcal{S}_\infty(\mathcal{H}).$$ \hspace{1cm} (2.5)

It is also convenient to assume that $\text{Ker} G = \{0\}$. Of course the resolvents $R_0(z) = (H_0 - zI)^{-1}$ and $R(z) = (H - zI)^{-1}$ of the operators $H_0$ and $H$ are related by the usual identity

$$R(z) - R_0(z) = -(GR_0(\bar{z}))^* V_0 G R(z).$$

Let

$$T(z) = GR_0(z)G^*$$ \hspace{1cm} (2.6)

be the sandwiched resolvent of the operator $H_0$. It follows from the assumption (2.5) that $T(z) \in \mathcal{S}_\infty$, the operator $I + T(z)V_0$ has a bounded inverse for all $z \in \mathbb{C} \setminus \mathbb{R}$ and

$$R(z) = R_0(z) - (GR_0(\bar{z}))^* V_0 (I + T(z)V_0)^{-1} GR_0(z).$$ \hspace{1cm} (2.7)

We set

$$Y(z) = -2\pi i V_0 (I + T(z)V_0)^{-1}, \quad \text{Im} z \neq 0.$$ \hspace{1cm} (2.8)

Then the resolvent identity (2.7) can be rewritten in the more concise form that we use below:

$$R(z) - R_0(z) = \frac{1}{2\pi i} (GR_0(\bar{z}))^* Y(z) GR_0(z), \quad \text{Im} z \neq 0.$$ \hspace{1cm} (2.9)

Note that

$$Y(\bar{z}) = -Y^*(z).$$ \hspace{1cm} (2.10)
Let $\Delta_\ell, \ell = 1, \ldots, L$, be pairwise disjoint open intervals, and let
\[ \Delta = \Delta_1 \cup \cdots \cup \Delta_L. \]
Without loss of generality we will assume that the function $\theta$ has exactly one jump discontinuity on each of the intervals $\Delta_\ell$. We suppose that the spectrum of $H_0$ in $\Delta_\ell, \ell = 1, \ldots, L$, is purely a.c. with a constant multiplicity $N_\ell \leq \infty$. More explicitly, we assume that for some auxiliary Hilbert space $\mathcal{N}_\ell$, $\dim \mathcal{N}_\ell = N_\ell$, there exists a unitary operator $F_\ell : \text{Ran} \ E_{H_0}(\Delta_\ell) \to L^2(\Delta_\ell; \mathcal{N}_\ell)$ (2.11) which diagonalizes $E_{H_0}(\Delta_\ell)H_0$. The corresponding operator (2.2) will be denoted by $Z_\ell(\lambda)$:
\[ Z_\ell(\lambda)f = (F_\ell G^*f)(\lambda), \quad \lambda \in \Delta_\ell. \] (2.12)

Let us summarize our assumptions:

**Assumption 2.2.** (A) $H_0$ has a purely a.c. spectrum with multiplicity $N_\ell$ on each interval $\Delta_\ell, \ell = 1, \ldots, L$.

(B) $V$ admits a factorization $V = G^*V_0G$ satisfying (2.5).

(C) $G$ is strongly $H_0$-smooth on all intervals $\Delta_\ell$ with an exponent $\gamma \in (0, 1]$.

**Example 2.3.** Let $H_0 = -\Delta$ be the Laplace operator in the space $L^2(\mathbb{R}^d), d \geq 1$. Applying the Fourier transform $\Phi$, we see that the operator $\Phi H_0 \Phi^*$ acts as the multiplication by $|\xi|^2$ in $L^2(\mathbb{R}^d; d\xi)$ (in the momentum representation). To diagonalize $H_0$, it remains to pass to the spherical coordinates in $\mathbb{R}^d$ and to make the change of variables $\lambda = |\xi|^2$. Now $L = 1, \Delta_1 = \mathbb{R}_+, \mathcal{N} = L^2(S^{d-1})$ (here $L^2(S^{d-1}) = \mathbb{C}^2$ if $d = 1$) and the operator $F_1 : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}_+; \mathcal{N})$ diagonalizing $H_0$ is defined by the formula
\[ (F_1f)(\lambda; \omega) = 2^{-1/2}(d-2)/4(\Phi f)(\lambda^{1/2}\omega), \quad \lambda > 0, \quad \omega \in S^{d-1}. \]

Of course, the operator $H_0$ has the purely a.c. spectrum $[0, \infty)$. It has infinite multiplicity for $d \geq 2$ and multiplicity 2 for $d = 1$.

Suppose that $V$ acts as the multiplication by a real short-range function $V(x)$, that is,
\[ |V(x)| \leq C(1 + |x|)^{-r}, \quad r > 1. \] (2.13)
Then Assumption 2.2(B) is satisfied with the operator $G$ acting as the multiplication by the function $(1 + |x|)^{-r/2}$ and the operator $V_0$ acting as the multiplication by the bounded function $(1 + |x|)^rV(x)$. For this operator $G$, Assumption 2.2(C) is satisfied with an arbitrary $\gamma < (r - 1)/2$ according to the Sobolev trace theorem.

It would have been too restrictive to assume that all discontinuities of $\theta$ lie in the same interval where the spectrum of $H_0$ has a constant multiplicity. This
would exclude applications at least to two classes of operators $H_0$: (a) operators with spectral gaps (such as periodic operators, or Dirac operators) (b) multichannel systems such as wave guides.

2.3. The limiting absorption principle and spectral results. The following well-known result (see, e.g., [19, Theorems 4.7.2 and 4.7.3]) is called the limiting absorption principle.

**Proposition 2.4.** Let Assumption 2.2 hold true. Then the operator-valued function $T(z)$ defined by (2.6) is Hölder continuous for $\text{Re } z \in \Delta$, $\text{Im } z \geq 0$. In particular, the limits $T(\lambda + i0)$ exist in the operator norm and are Hölder continuous in $\lambda \in \Delta$. Let $\mathcal{R} \subset \Delta$ be the set of $\lambda$ such that the equation

$$f + T(\lambda + i0)V_0f = 0$$

has a non-zero solution $f \in \mathcal{H}$. Then $\mathcal{R}$ is closed and has the Lebesgue measure zero. For all $\lambda \in \Omega := \Delta \setminus \mathcal{R}$, the inverse operator $(I + T(\lambda + i0)V_0)^{-1}$ exists, is bounded and is Hölder continuous in $\lambda \in \Omega$.

**Corollary 2.5.** Let the operator $Y(z)$ be defined by equation (2.8). The limits $Y(\lambda + i0)$ exist in the operator norm and are Hölder continuous in $\lambda \in \Omega$.

The limiting absorption principle can be supplemented by the following spectral results (see, e.g., [19, Theorems 4.7.9 and 4.7.10]).

**Proposition 2.6.** Let Assumption 2.2 hold true. Then the spectrum of $H$ in $\Omega$ is purely absolutely continuous. If, in addition, $\gamma > 1/2$ in (2.3), then the singular continuous spectrum of $H$ in $\Delta$ is absent, $\mathcal{R} = (\text{spec}_p H) \cap \Delta$, and the eigenvalues of $H$ in $\Delta$ have finite multiplicities and can accumulate only to the endpoints of the intervals $\Delta_i$. In this case

$$\Omega = \Delta \setminus \text{spec}_p H. \tag{2.14}$$

Note that, for the Schrödinger operator $H$ defined in Example 2.3, the exponent $\gamma$ in (2.3) can be an arbitrary small number if $r$ in (2.13) is close to 1. Nevertheless, all assertions of Proposition 2.6 remain true in this case. Moreover, according to the Kato theorem the operator $H$ does not have positive eigenvalues so that, under assumption (2.13), $H$ has purely absolutely continuous spectrum on $\mathbb{R}_+$ (see, e.g., [18]).

2.4. Wave operators. For an interval (or the union of pairwise disjoint intervals) $\Delta \subset \mathbb{R}$, the local wave operators are introduced by the relation

$$W_\pm(H, H_0; \Delta) = \text{s-lim}_{t \to \pm \infty} e^{iHt} e^{-iH_0t} E_{H_0}^{(ac)}(\Delta), \tag{2.15}$$

provided these strong limits exist. The wave operators are isometric on $\text{Ran} E_{H_0}^{(ac)}(\Delta)$, enjoy the intertwining property

$$HW_\pm(H, H_0; \Delta) = W_\pm(H, H_0; \Delta)H_0$$
and
\[
\text{Ran } W_\pm(H, H_0; \Delta) \subset \text{Ran } E^{(ac)}_H(\Delta).
\]
The wave operators are called complete if this inclusion reduces to the equality. We need the following well-known result (see, e.g., [19, Theorem 4.6.4]).

**Proposition 2.7.** Under Assumption 2.2 the local wave operators \( W_\pm(H, H_0; \Delta) \) exist and are complete. In particular, the operators \( H_0 E_{H_0}(\Delta) \) and \( H E^{(ac)}_H(\Delta) \) are unitarily equivalent.

Note that if \( E^{(ac)}_{H_0}(\mathbb{R} \setminus \Delta) = 0 \), then the operator \( E_{H_0}(\Delta) \) in definition (2.15) can be dropped and the global wave operators
\[
W_\pm(H, H_0) = \lim_{t \to \pm \infty} e^{iHt} e^{-iH_0 t} P^{(ac)}_{H_0}
\]
exist. This is the case for the pair \( H_0, H \) considered in Example 2.3.

### 2.5. Scattering matrix.

If the wave operators \( W_\pm(H, H_0; \Delta) \) exist, then the (local) scattering operator is defined as
\[
S = S(H, H_0; \Delta) = W_+^*(H, H_0; \Delta) W_-(H, H_0; \Delta).
\]
Moreover, the scattering operator \( S \) is unitary on the subspace \( \text{Ran } E_{H_0}(\Delta) \) if these wave operators are complete. The scattering operator \( S \) commutes with \( H_0 \), and therefore, for almost all \( \lambda \in \Delta_\ell, \ell = 1, \ldots, L \), we have a representation
\[
(F_\ell S F_\ell^*) f(\lambda) = S(\lambda) f(\lambda)
\]
where the operator \( S(\lambda) : \mathcal{N}_\ell \to \mathcal{N}_\ell \) is called the scattering matrix for the pair of operators \( H_0, H \). The scattering matrix \( S(\lambda) \) where \( \lambda \in \Delta_\ell \) is a unitary operator in \( \mathcal{N}_\ell \) if \( S \) is unitary. Moreover, we have the following result (see [19, Theorem 7.4.3]).

**Proposition 2.8.** Let Assumption 2.2 hold. Let the operator \( Y(z) \) be defined by formula (2.8), and let the operators \( Z_\ell(\lambda) \) for \( \lambda \in \Delta_\ell \) be defined by formula (2.12). Then the scattering matrix admits the representation
\[
S(\lambda) = I + Z_\ell(\lambda) Y(\lambda + i0) Z_\ell^*(\lambda), \quad \lambda \in \Omega \cap \Delta_\ell. \quad (2.16)
\]

The representation (2.16) implies that \( S(\lambda) \) is a Hölder continuous function of \( \lambda \in \Omega \). Since the operator \( Y(\lambda + i0) \) is bounded and \( Z_\ell(\lambda) \) is compact, it follows that the operator \( S(\lambda) - I \) is compact. Thus, the spectrum of \( S(\lambda) \) consists of eigenvalues on the unit circle in \( \mathbb{C} \) accumulating possibly only to the point 1. All eigenvalues \( \sigma_n(\lambda), 1 \leq n \leq N_\ell \) of \( S(\lambda) \) distinct from 1 have finite multiplicities; they are enumerated with multiplicities taken into account.
2.6. Multichannel scheme. In this subsection, we recall a key result from [17] which will allow us to put together the contributions from each of the jumps of $\theta$. Let $A_\ell$, $\ell = 1, 2, \ldots, L$, and $A_\infty$ be bounded self-adjoint operators. The abstract results below allow us to construct spectral theory of the operator

$$A = A_1 + \cdots + A_L + A_\infty$$

(2.17)

and a smooth version of scattering theory for the pair of operators

$$A_1 \oplus A_2 \oplus \cdots \oplus A_L \text{ and } A$$

under certain smoothness assumptions on all pair products $A_jA_k$, $j \neq k$, and the operator $A_\infty$.

Spectral results are formulated in the following assertion.

**Proposition 2.9** ([17]). Let $A_\ell$, $\ell = 1, \ldots, L$, and $A_\infty$ be bounded self-adjoint operators in a Hilbert space $H$. Let $\delta \subset \mathbb{R}$ be an open interval such that $0 \not\in \delta$. Assume that the spectra of $A_1, \ldots, A_L$ are purely a.c. on $\delta$ with constant multiplicities. Let $X$ be a bounded self-adjoint operator in $H$ such that $\ker X = \{0\}$. Assume that:

- For all $\ell = 1, \ldots, L$, the operator $X$ is strongly $A_\ell$-smooth on $\delta$ with an exponent $\gamma > 1/2$.
- For all $1 \leq j, k \leq L$, $j \neq k$, the operators $A_jA_k$ can be represented as
  $$A_jA_k = XK_{jk}X \quad \text{where} \quad K_{jk} \in \mathcal{S}_\infty.$$  
  (2.18)
- The operator $A_\infty$ can be represented as
  $$A_\infty = XK_\infty X \quad \text{where} \quad K_\infty \in \mathcal{S}_\infty.$$  
  (2.19)
- The operators $XA_\ell X^{-1}$ are bounded for all $\ell = 1, \ldots, L$.

Let the operator $A$ be defined by equality (2.17). Then:

(i) The a.c. spectrum of the operator $A$ on $\delta$ has a uniform multiplicity which is equal to the sum of the multiplicities of the spectra of $A_1, \ldots, A_L$ on $\delta$.

(ii) The singular continuous spectrum of $A$ on $\delta$ is empty.

(iii) The eigenvalues of $A$ on $\delta$ have finite multiplicities and cannot accumulate at interior points of $\delta$.

(iv) The operator-valued function $X(A - zI)^{-1}X$ is continuous in $z$ for $\pm \text{Im } z \geq 0$, $\text{Re } z \in \delta$, except at the eigenvalues of $A$.

The scattering theory for the set of operators $A_1, \ldots, A_L$ and the operator $A$ is described in the following assertion.

**Proposition 2.10** ([17]). Let the hypotheses of Proposition 2.9 hold true. Then:

(i) The local wave operators
  $$W_\pm(A, A_\ell; \delta) = \text{s-lim}_{t \to \pm \infty} e^{iAt} e^{-iA_\ell t} E_{A_\ell}(\delta), \quad \ell = 1, \ldots, L,$$
exist and enjoy the intertwining property

\[ AW_\pm(A, A_\ell; \delta) = W_\pm(A, A_\ell; \delta)A_\ell, \quad \ell = 1, \ldots, L. \]

The wave operators are isometric and their ranges are orthogonal to each other, i.e.

\[ \text{Ran } W_\pm(A, A_j; \delta) \perp \text{Ran } W_\pm(A, A_k; \delta), \quad 1 \leq j, k \leq L, \; j \neq k. \]

(ii) The asymptotic completeness holds:

\[ \text{Ran } W_\pm(A, A_1; \delta) \oplus \cdots \oplus \text{Ran } W_\pm(A, A_L; \delta) = \text{Ran } E_\delta^{(ac)}. \quad (2.20) \]

We note that the statement (i) of Proposition 2.9 is a direct consequence of Proposition 2.10.

We also observe that, for \( L = 1 \), Propositions 2.9 and 2.10 reduce to the results of Subsections 2.3 and 2.4 (see Propositions 2.6 and 2.7) where \( H_0, H, G \) and \( \Delta \) play the roles of \( A_1, A, X \) and \( \delta \), respectively. In this case the assumptions \( 0 \notin \delta \) and \( XA_1X^{-1} \in \mathcal{B} \) (the last hypothesis of Proposition 2.9) are not necessary.

3. Symmetrised Hankel operators. A model operator

Here we introduce the class of symmetrised Hankel operators (SHO) and diagonalize explicitly some special SHO. This SHO will be used as a model operator for the construction in Section 5 of the spectral theory of SHO with piecewise continuous symbols as well as in Section 7 for the proof of our main results concerning the spectral theory of operators (1.1). In view of the applications to the operator \( D_\theta \), we consider SHOs with operator valued symbols. The main result of this section is Theorem 3.3.

3.1. Symmetrised Hankel operators. Recall that the Hardy space \( H^2_{\pm}(\mathbb{R}) \subset L^2(\mathbb{R}) \) is defined as the class of all functions \( f \in L^2(\mathbb{R}) \) that admit the analytic continuation into the half-plane \( \mathbb{C}_\pm = \{ z \in \mathbb{C} : \pm \text{Im } z > 0 \} \) and satisfy the estimate

\[ \sup_{\tau > 0} \int_{-\infty}^{\infty} |f(\lambda \pm i\tau)|^2 d\lambda < \infty. \]

Let \( P_\pm \) be the orthogonal projection in \( L^2(\mathbb{R}) \) onto \( H^2_{\pm}(\mathbb{R}) \). The explicit formula for \( P_\pm \) is

\[ (P_\pm f)(\lambda) = \pm \frac{1}{2\pi i \varepsilon} \lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} \frac{f(x)}{x - \lambda \mp i\varepsilon} dx, \quad (3.1) \]

where the limit exists for almost all \( \lambda \in \mathbb{R} \); it also exists in \( L^2(\mathbb{R}) \). The Hardy spaces of vector valued functions are defined quite similarly. By a slight abuse of notation, we will use the same notation \( P_\pm \) for the Hardy projections in these spaces; they are defined by the same formula (3.1).

Now we are in a position to give the precise
Definition 3.1. Let $\mathfrak{h}$ be a Hilbert space (the case $\dim \mathfrak{h} < \infty$ is not excluded), and let an operator valued function $\Xi \in L^\infty(\mathbb{R}; \mathcal{B}(\mathfrak{h}))$. The symmetrised Hankel operator (SHO) with symbol $\Xi$ is the operator $M_\Xi$ given on the space $L^2(\mathbb{R}; \mathfrak{h}) = L^2(\mathbb{R}) \otimes \mathfrak{h}$ by formula (1.5).

By definition, the operator $M_\Xi$ is bounded and self-adjoint. Since the operator $P_+ - P_-$ is unitary, the simple identity

$$(P_+ - P_-)M_\Xi = -M_\Xi(P_+ - P_-)$$

shows that $M_\Xi$ is unitarily equivalent to $-M_\Xi$. Thus the spectrum of $M_\Xi$ is symmetric with respect to the reflection: $\text{spec } M_\Xi = -\text{spec } M_\Xi$.

Many properties of SHOs are the same as those of the usual Hankel operators; we refer to the book [10] for the theory of Hankel operators. From the definition it follows that $M_\Xi^1 = M_\Xi^2$ if and only if the difference $\Xi^1 - \Xi^2$ belongs to the Hardy class $H^\infty_+(\mathbb{R}; \mathfrak{h})$ of operator valued functions that admit a bounded analytic continuation in the upper half-plane. If $\Xi \in C_0(\mathbb{R}; \mathcal{S}_\infty(\mathfrak{h}))$, then $M_\Xi$ is compact; see [10, § 2.4].

We will consider SHOs with piecewise continuous symbols. Since functions of the class $H^\infty_+(\mathbb{R}; \mathfrak{h})$ cannot have jump discontinuities, different symbols of such SHOs have the same jumps. Curiously, this fact follows from our results, and even more general statements about functions in $H^\infty_+(\mathbb{R}; \mathfrak{h})$ follow from the results of [11] on the essential spectrum of Hankel operators.

3.2. The model operator. Set

$$\zeta(\lambda) = \frac{1}{\pi} \int_0^\infty \frac{\sin(\lambda t)}{2 + t} dt. \quad (3.2)$$

Obviously, $\zeta(\lambda)$ is a real odd function. Since

$$\zeta(\lambda) = \frac{1}{\pi} \text{Im} \left( e^{-2i\lambda} \int_{\lambda}^\infty \frac{e^{2ix}}{x} dx \right), \quad \lambda > 0,$$

the function $\zeta \in C^\infty(\mathbb{R} \setminus \{0\})$ and $\zeta(\lambda) = O(|\lambda|^{-1})$ as $|\lambda| \to \infty$. Moreover, the limits $\zeta(\pm 0)$ exist, $\zeta(\pm 0) = \pm 1/2$ and

$$\zeta'(\lambda) = O(|\ln|\lambda||), \quad \lambda \to 0. \quad (3.3)$$

Let us now define a particular SHO in the space $L^2(\mathbb{R}; \mathfrak{h})$ which can be explicitly diagonalized and plays a key role in our construction. For an arbitrary operator $K \in \mathcal{S}_\infty(\mathfrak{h})$, we consider the SHO $M_\Xi$ defined by formula (1.5) with the symbol $\Xi(\lambda) = \zeta(\lambda)K$.

Next, we define the weight function on $\mathbb{R}$ which vanishes logarithmically at the origin:

$$q_0(\lambda) = \begin{cases} |\log|\lambda||^{-1} & \text{if } |\lambda| < e^{-1}, \\ 1 & \text{if } |\lambda| \geq e^{-1}. \end{cases} \quad (3.4)$$
We will also denote by $q_0$ the operator of multiplication by $q_0(\lambda)$ in the space $L^2(\mathbb{R}; \mathfrak{h})$.

Our goal in this section is to study the spectral properties of the SHO $M_{\zeta K}$. First, we give the result for the scalar case.

**Theorem 3.2.** Let $M_{\zeta}$ be the SHO in $L^2(\mathbb{R})$ with the symbol $\zeta = \zeta K$ where $\zeta$ is the function (3.2) and $K \in \mathbb{C}$, $K \neq 0$. Then:

(i) The operator $M_{\zeta}$ has the purely a.c. spectrum $[-|K|/2, |K|/2]$ of multiplicity one.

(ii) For an arbitrary $\beta > 1/2$, the operator $q_0^\beta$ is strongly $M_{\zeta}$-smooth on intervals $(-|K|/2, 0)$ and $(0, |K|/2)$ with any exponent $\gamma < \beta - 1/2$, $\gamma \in (0, 1]$.

In the general case our result is stated as follows.

**Theorem 3.3.** Let $M_{\zeta}$ be the SHO in $L^2(\mathbb{R}; \mathfrak{h})$ with the symbol $\zeta = \zeta K$ where $\zeta$ is the function (3.2) and $K \in \mathcal{S}_\infty(\mathfrak{h})$. Then:

(i) Apart from the possible eigenvalue $0$, the operator $M_{\zeta}$ has the purely a.c. spectrum:

$$\text{spec}_{\text{ac}} M_{\zeta} = \bigcup_{n=1}^{\dim \mathfrak{h}} [-\frac{1}{2}s_n(K), \frac{1}{2}s_n(K)].$$

The operator $M_{\zeta}$ has eigenvalue $0$ if and only if either $\text{Ker} K \neq \{0\}$ or $\text{Ker} K^* \neq \{0\}$ (or both); if $0$ is an eigenvalue, then its multiplicity in the spectrum of $M_{\zeta}$ is infinite.

(ii) For an arbitrary $\beta > 1/2$, the operator $q_0^\beta$ is strongly $M_{\zeta}$-smooth with any exponent $\gamma < \beta - 1/2$, $\gamma \in (0, 1]$ on all intervals $\delta \subset [-\frac{1}{2}s_1(K), \frac{1}{2}s_1(K)]$ which do not contain the points $0$ and $\pm \frac{1}{2}s_n(K)$.

In (3.5) and in what follows we use the same convention as in formula (1.3), i.e. the union is taken only over the intervals of positive length and each interval contributes multiplicity one to the a.c. spectrum of $M_{\zeta}$.

The proofs of these results will be given in the rest of this section where a concrete diagonalization of the operator $M_{\zeta}$ will also be given.

### 3.3. Hankel operators.

Let $\mathcal{H}$ be an auxiliary Hilbert space, and let $\Sigma$ be the operator of multiplication by a bounded operator valued function $\Sigma(\lambda) : \mathcal{H} \to \mathcal{H}$ in the space $L^2(\mathbb{R}; \mathfrak{h}) = L^2(\mathbb{R}) \otimes \mathfrak{h}$. We set $(\mathcal{J} f)(\lambda) = f(-\lambda)$. Then the Hankel operator $\Gamma_{\Sigma}$ with symbol $\Sigma$ is defined in the space $H^2_{\mathcal{H}}(\mathbb{R}; \mathfrak{h})$ by the formula

$$\Gamma_{\Sigma} f = P_+ \Sigma \mathcal{J} f.$$

Let $\Phi$ be the unitary Fourier transform in $L^2(\mathbb{R})$ (or, more generally, in $L^2(\mathbb{R}; \mathfrak{h})$),

$$(\Phi f)(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\lambda t} f(\lambda) d\lambda.$$
Then $\Phi P_\pm \Phi^* = \chi_\pm$ where $\chi_\pm$ is the operator of multiplication by the characteristic function of $\mathbb{R}_\pm$ so that $\Phi : H^2_\pm(\mathbb{R}) \to L^2(\mathbb{R}_\pm)$. Further, $\Phi \Sigma \Phi^*$ is the operator of convolution with the function $(2\pi)^{-1/2} \widehat{\Sigma}(t)$ where $\widehat{\Sigma} = \Phi(\Sigma)$. It follows that the operator $\widehat{\Gamma}_\Sigma = \Phi \Gamma_\Sigma \Phi^* : L^2(\mathbb{R}_+; \mathbb{H}) \to L^2(\mathbb{R}_+; \mathbb{H})$ acts by the formula

$$\widehat{\Gamma}_\Sigma f(t) = (2\pi)^{-1/2} \int_0^\infty \widehat{\Sigma}(t+s) f(s) ds. \quad (3.7)$$

A SHO $M_\Xi$ in the space $H^2_+ (\mathbb{R}; \mathfrak{h})$ is canonically unitarily equivalent to a special Hankel operator $\Gamma_\Sigma$ acting in the space $H^2_+ (\mathbb{R}; \mathbb{H})$ where $\mathbb{H} = \mathfrak{h} \oplus \mathfrak{h}$. Indeed, let the unitary operator $J : L^2(\mathbb{R}; \mathfrak{h}) \to H^2_+ (\mathbb{R}; \mathfrak{h}) \otimes \mathbb{H}$ be defined by the formula

$$Jf = \left( P_+ f, P_+ \mathcal{J} f \right)^\top. \quad (3.8)$$

This can be checked by the following direct calculation. Since $P_+ \mathcal{J} = \mathcal{J} P_-$, it follows from (1.5) that

$$JM_\Xi f = (P_+ (P_- \Xi P_+ + P_- \Xi^* P_-) f, \mathcal{J} P_- (P_- \Xi P_+ + P_- \Xi^* P_-) f)^\top = (P_+ \Xi^* P_- f, \mathcal{J} P_- \Xi P_+ f)^\top. \quad (3.9)$$

Similarly, it follows from (3.6) that

$$\Gamma_\Sigma f = P_+ \Sigma (\mathcal{J} P_+ f, P_- f)^\top = (P_+ \Xi^* P_- f, P_+ (\mathcal{J} \Xi) \mathcal{J} P_+ f)^\top. \quad (3.10)$$

The r.h.s. in (3.9) and (3.10) coincide because $P_+ (\mathcal{J} \Xi) \mathcal{J} P_+ = P_+ \mathcal{J} \Xi P_+ = \mathcal{J} P_- \Xi P_+$. In particular, a SHO $M_\Xi$ in $L^2(\mathbb{R})$ satisfies (3.8) with $\Sigma(\lambda) : \mathbb{C}^2 \to \mathbb{C}^2$ and $\Gamma_\Sigma$ acting in $H^2_+ (\mathbb{R}; \mathbb{C}^2)$.

3.4. Mehler’s formula. Let us consider an integral operator $\mathcal{M}$ acting in the space $L^2(\mathbb{R}_+)$ by the formula

$$(\mathcal{M} f)(t) = \frac{1}{\pi} \int_0^\infty \frac{f(s)}{2 + t + s} ds.$$

Calculating the Fourier transform of the function (3.2), we find that

$$\widehat{\zeta}(t) = -\frac{i}{\sqrt{2\pi}} \frac{\text{sign } t}{2 + |t|}, \quad t \in \mathbb{R},$$

and hence (see (3.7))

$$\mathcal{M} = \Phi \Gamma_{2\kappa} \Phi^* \quad (3.11)$$

where $\Gamma_{2\kappa}$ is the Hankel operator in $H^2_+ (\mathbb{R})$ defined by formula (3.6).
We use the fact that the operator $\mathcal{M}$ can be explicitly diagonalised. Its diagonalisation is based on Mehler’s formula (see [2, Section 3.14]):

$$\frac{1}{\pi} \int_0^\infty \frac{P_{\frac{1}{2}+i\tau}(1+s)}{2+t+s} ds = \frac{1}{\cosh(\pi \tau)} P_{\frac{1}{2}+i\tau}(1+t), \quad t, \tau \in \mathbb{R}_+, \quad (3.12)$$

where $P_\nu$ is the Legendre function, see formulas (A.2), (A.3) in the Appendix. A systematic approach to the proof of formulas of this type was suggested in [20]. Recall (see [2, §3.14]) that the Mehler-Fock transform $\Psi$ is defined by the formula

$$(\Psi f)(\tau) = \sqrt{\tau \tanh(\pi \tau)} \int_0^\infty P_{\frac{1}{2}+i\tau}(t+1)f(t)dt, \quad \tau > 0. \quad (3.13)$$

**Lemma 3.4.** The Mehler-Fock transform $\Psi$ maps $L^2(\mathbb{R}_+; dt)$ onto $L^2(\mathbb{R}_+; d\tau)$ and is unitary. It diagonalizes the operator $\mathcal{M}$:

$$(\Psi \mathcal{M} f)(\tau) = \frac{1}{\cosh(\pi \tau)}(\Psi f)(\tau). \quad (3.14)$$

The unitarity of $\Psi$ is discussed in [20], and (3.14) is a consequence of (3.12).

Thus, the map $\Psi$ reduces the operator $\mathcal{M}$ to the operator of multiplication by the function $1/\cosh(\pi \tau)$ in the space $L^2(\mathbb{R}_+; d\tau)$. In particular, we see that $\mathcal{M}$ has the simple purely a.c. spectrum which coincides with the interval $[0, 1]$.

Let us now define the operator $W = \Psi \chi \Phi : L^2(\mathbb{R}) \to L^2(\mathbb{R}_+)$. It is a unitary mapping of $H^2_+(\mathbb{R})$ onto $L^2(\mathbb{R}_+)$ and $W f = 0$ if $f \in H^2_-(\mathbb{R})$. It follows from definition (3.13) that, formally, the operator $W$ is given by the equality

$$(W f)(\tau) = \sqrt{\tau \tanh(\pi \tau)} \int_{-\infty}^\infty w_\tau(\lambda)f(\lambda)d\lambda \quad (3.15)$$

where

$$w_\tau(\lambda) = \frac{1}{\sqrt{2\pi}} \int_0^\infty P_{\frac{1}{2}+i\tau}(t+1)e^{-i\lambda t}dt. \quad (3.16)$$

Putting together formulas (3.11) and (3.14), we obtain the following result.

**Lemma 3.5.** Let $\zeta$ be the function (3.2) and let the Hankel operator $\Gamma_{2i\zeta}$ in $H^2_+(\mathbb{R})$ be defined by formula (3.6). Then

$$(W \Gamma_{2i\zeta} f)(\tau) = \frac{1}{\cosh(\pi \tau)}(W f)(\tau), \quad \tau > 0. \quad (3.17)$$

3.5. The diagonalization of $M_\Xi$. Let us return to the SHO $M_\Xi$ with the symbol $\Xi(\lambda) = \zeta(\lambda)K$. For an arbitrary operator $K \in \mathcal{B}(\mathfrak{h})$, a real odd function $\zeta$ and $\Xi(\lambda) = \zeta(\lambda)K$, relation (3.8) holds with

$$\Sigma(\lambda) = i\zeta(\lambda)K \quad \text{where} \quad K = \begin{pmatrix} 0 & -iK^* \\ iK & 0 \end{pmatrix}. \quad (3.18)$$
Let us define the unitary operator \( W : L^2(\mathbb{R}) \otimes \mathfrak{h} \to L^2(\mathbb{R}_+) \otimes H \) by the formula
\[
Wf = ((W \otimes I)f, (WJ \otimes I)f)^\top.
\] (3.19)

Putting together formulas (3.8) and (3.17), we obtain the following result.

**Lemma 3.6.** Let \( K \) be a bounded operator in a Hilbert space \( \mathfrak{h} \), let the function \( \zeta \) be defined by formula (3.2), and let \( \Xi(\lambda) = \zeta(\lambda)K \). Then
\[
WM_\Xi W^* = \frac{1}{2 \cosh(\pi \tau)} \otimes K
\] (3.20)
where the operator \( K \) is given by formula (3.18).

The operator \((2 \cosh(\pi \tau))^{-1} \otimes K\) acting in the space \( L^2(\mathbb{R}_+; d\tau) \otimes H \) is obviously unitarily equivalent to the operator \( \mu \otimes K \) acting in the space \( L^2((0,1/2); d\mu) \otimes H \). Thus the diagonalization of the operator \( M_\Xi \) reduces to that of the operator \( K \).

Suppose now that \( K \) is compact in \( \mathfrak{h} \). Then the operator \( K \) is compact and self-adjoint in the space \( H \). Let \( s_n = s_n(K) \), \( 1 \leq n \leq \dim \mathfrak{h} \), be the singular values of \( K \). It is easy to check the following result.

**Lemma 3.7.** The nonzero spectrum of \( K \) consists of the eigenvalues
\[
\{ \pm s_n(K) : 1 \leq n \leq \dim \mathfrak{h}, \ s_n(K) > 0 \}.
\]
Moreover, \( \ker K = \ker K \oplus \ker K^* \).

Thus, diagonalising the operator \( K \), we get the following result:

**Lemma 3.8.** Let \( A = (2 \cosh(\pi \tau))^{-1} \otimes K \) in \( L^2(\mathbb{R}_+) \otimes H \). Then, apart from the possible eigenvalue at zero, the operator \( A \) has the purely a.c. spectrum:
\[
\text{spec}_{ac} A = \bigcup_{n=1}^{\dim \mathfrak{h}} [-\frac{1}{2}s_n(K), \frac{1}{2}s_n(K)].
\] (3.21)

The operator \( A \) has eigenvalue zero if and only if \( \ker K \neq \{0\} \); if zero is the eigenvalue of \( A \), then it has infinite multiplicity.

Putting together Lemmas 3.6 and 3.8, we conclude the proof of Theorem 3.3(i).

In the scalar case \( \mathfrak{h} = \mathbb{C} \), we have \( K \in \mathbb{C} \), \( H = \mathbb{C}^2 \) and the operator \( K \) in (3.18) is the \( 2 \times 2 \) matrix. If \( K \neq 0 \), the spectrum of \( K \) consists of the two eigenvalues \( \pm |K| \) and formula (3.21) for \( \dim \mathfrak{h} = 1 \) means that
\[
\text{spec}_{ac} A = [-\frac{1}{2}|K|, \frac{1}{2}|K|].
\]
In particular, the spectrum of \( A \) is a.c. and simple.
Remark 3.9. It is easy to calculate eigenvectors of the operator $K$ in terms of eigenvectors of the operator $\sqrt{K^*K}$. Indeed, if $\sqrt{K^*K}a_n = s_n a_n$, then $b_n^{(\pm)} = (s_n a_n, \pm iKa_n)^T$ satisfy the equation $Kb_n^{(\pm)} = \pm s_n b_n^{(\pm)}$. In particular, in the case $\dim \mathfrak{h} = 1$ we have $b^{(\pm)} = (|K|, \pm iK)^T$.

3.6. Smoothness with respect to the model operator. It remains to check Theorems 3.2(ii) and 3.3(ii). We use the diagonalization $\mathcal{F}$ of the operator $M_\Xi$ defined by formulas (3.15), (3.19) and (3.25). By Definition 2.1, the proof of the strong $M_\Xi$-smoothness of the operator $q^\beta$ requires estimates on the function $w_\tau(\lambda)$. They are collected in the following assertion.

Lemma 3.10. For $\tau > 0$ and $\lambda \in \mathbb{R}$, $\lambda \neq 0$, the function $w_\tau(\lambda)$ is differentiable in $\tau$. If $\delta \subset \mathbb{R}_+$ is a compact interval and $\tau \in \delta$, then for some constant $C = C(\delta)$ we have the estimates:

$$|w_\tau(\lambda)| \leq C|\lambda|^{-1}, \quad \left|\partial w_\tau(\lambda)/\partial \tau\right| \leq C|\lambda|^{-1}, \quad |\lambda| \geq 1/2, \quad (3.22)$$

and

$$|w_\tau(\lambda)| \leq C|\lambda|^{-1/2}, \quad \left|\partial w_\tau(\lambda)/\partial \tau\right| \leq C|\lambda|^{-1/2} \ln|\lambda|, \quad |\lambda| \leq 1/2, \quad \lambda \neq 0. \quad (3.23)$$

The proof is quite elementary and is given in the Appendix. The following assertion is an easy consequence of Lemma 3.10.

Lemma 3.11. Let the operator $W$ be defined by formula (3.15). Set $W_1 = W$, $W_2 = W\mathcal{F}$. Suppose that $\beta > 1/2$ and $\gamma < \beta - \frac{1}{2}$, $\gamma \leq 1$. Then, for $\tau$ and $\tau'$ in compact subintervals of $\mathbb{R}_+$ and for all $f \in L^2(\mathbb{R})$, we have the estimates

$$|(W_{1,2}q_0^\beta f)(\tau)| \leq C \|f\|, \quad (3.24)$$

$$|(W_{1,2}q_0^\beta f)(\tau) - (W_{1,2}q_0^\beta f)(\tau')| \leq C|\tau - \tau'|^{\gamma} \|f\|.$$

Proof. It follows directly from Lemma 3.10 that

$$\int_{-\infty}^{\infty} q_0^{2\beta}(\lambda)|w_\tau(\pm \lambda)|^2d\lambda \leq C,$$

$$\int_{-\infty}^{\infty} q_0^{2\beta}(\lambda)|w_\tau(\pm \lambda) - w_{\tau'}(\pm \lambda)|^2d\lambda \leq C|\tau - \tau'|^{2\gamma}.$$

These estimates imply estimates (3.24). □

Of course Lemma 3.11 implies a similar statement about the operator $W$ defined by formula (3.19).

Let us denote by $Y$ the unitary operator in $H$ that diagonalises $K$:

$$YKY^* = \text{diag}\{s_1, -s_1, s_2, -s_2, \ldots\}, \quad s_n = s_n(K); \quad (3.25)$$

the sequence in the r.h.s. of (3.25) has the same number of zeros as $\dim \text{Ker} K$. 

Proof of Theorem 3.2(ii). It follows from formulas (3.20) and (3.25) that the operator
\[(I \otimes Y)WM_{\Xi}W^*(I \otimes Y)^*\] acts in the space \(L^2(\mathbb{R}_+; \mathbb{C}^2)\) as the multiplication by the matrix valued function
\[
\frac{|K|}{2 \cosh(\pi \tau)} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]
Making the changes of variables \(\mu = \pm 2^{-1}|K|(\cosh(\pi \tau))^{-1}\), we see that it reduces to the multiplication by \(\mu\) in the space \(L^2(\delta_{s_n}/2, \delta_{s_n}/2)\). Therefore estimates (2.3) for the diagonalization \(F = (I \otimes Y)W\) of the operator \(M_{\Xi}\), \(Q = q_0^\beta\) and the operator \(Z(\mu)\) defined by (2.2) are equivalent to estimates (3.24). This concludes the proof. □

Proof of Theorem 3.3(ii). In view of (3.20) and (3.25), the operator (3.26) acts in the space \(L^2(\mathbb{R}_+) \otimes H\) as the multiplication by the matrix valued function
\[
(2 \cosh(\pi \tau))^{-1} \otimes \text{diag}\{s_1, -s_1, s_2, -s_2, \ldots\}, \quad s_n = s_n(K).
\] After the changes of variables \(\mu = \pm 2^{-1}s_n(\cosh(\pi \tau))^{-1}\), the part of the operator (3.27) in the orthogonal complement to its kernel reduces to the multiplication by \(\mu\) in the space \(\bigoplus_{n=1}^{\dim h} L^2(-s_n/2, s_n/2)\) (the sum is taken over \(s_n > 0\)).

We have to verify the \(M_{\Xi}\)-smoothness of the operator \(Q = q_0^\beta\) on all intervals \(\delta_{m}^{(\pm)} = (s_{m+1}/2, s_m/2)\) and \(\delta_{m}^{(-)} = (-s_m/2, -s_{m+1}/2)\) where \(s_{m+1} < s_m\). The operator \(M_{\Xi}|_{\text{Ran} E_{M_{\Xi}}(\delta_{m}^{(\pm)})}\) is unitarily equivalent to the operator of multiplication by \(\mu\) in the space \(L^2(\delta_{m}^{(\pm)}) \otimes \mathbb{C}^m\). Thus, estimates (2.3) for this diagonalization of the operator \(M_{\Xi}|_{\text{Ran} E_{M_{\Xi}}(\delta_{m}^{(\pm)})}\) are again equivalent to estimates (3.24). □

Remark 3.12. In Sections 5 and 7 we need the results which are formally more general than Theorem 3.3 but are its direct consequences. Let
\[\Xi(\lambda) = \zeta(\lambda - \lambda_0)K\]
for some \(\lambda_0 \in \mathbb{R}\). Then the first assertion of Theorem 3.3 is true for the SHO \(M_{\Xi}\) with the symbol
\[\Xi(\lambda) = \zeta(\lambda - \lambda_0)K, \quad K \in \mathfrak{S}_\infty(h)\].
Set
\[q(\lambda) = \prod_{\ell=1}^{L} q_0(\lambda - \lambda_\ell)\]
(3.28)
where the function \(q_0(\lambda)\) is defined by formula (3.4). Then the assertion of Theorem 3.3 about strong \(M_{\Xi}\)-smoothness remains true for the operator of multiplication by the function \(q^\beta(\lambda)\) if \(\lambda_\ell = \lambda_0\) for one of \(\ell\).
4. COMPACTNESS OF SANDWICHED HANKEL OPERATORS

In this section we prepare auxiliary statements about the compactness of some Hankel type operators appearing in our construction.

4.1. Muckenhoupt weights. Recall that a function \( v \in L^1_{\text{loc}}(\mathbb{R}) \), \( v \geq 0 \), such that \( v^{-1} \in L^1_{\text{loc}}(\mathbb{R}) \), is called a Muckenhoupt weight if

\[
\sup_{\Lambda} \frac{1}{|\Lambda|} \int_{\Lambda} v(\lambda) d\lambda \cdot \frac{1}{|\Lambda|} \int_{\Lambda} v(\lambda)^{-1} d\lambda < \infty,
\]

where the supremum is taken over all bounded intervals \( \Lambda \subset \mathbb{R} \), and \( |\Lambda| \) denotes the length of \( \Lambda \). It is a classical result of [8] that the operators \( P_{\pm} \) are bounded in \( L^2(\mathbb{R}; v(\lambda) d\lambda) \) iff \( v \) is a Muckenhoupt weight. Thus, for a Muckenhoupt weight \( v \), the operators \( v^{1/2} P_{\pm} v^{-1/2} \) are bounded. Of course, the same result is true for the operators \( P_{\pm} \) in the vector-valued space \( L^2(\mathbb{R}, h) \), if \( v \) is a scalar function satisfying (4.1).

We consider operators acting in the space \( L^2(\mathbb{R}; h) \) where \( h \) is an auxiliary Hilbert space. Let the function \( q(\lambda) \) be defined by formulas (3.4) and (3.28). We use the same notation \( q \) for the operator of multiplication by the function \( q(\lambda) \) acting in different spaces (for example, in \( L^2(\mathbb{R}) \) and \( L^2(\mathbb{R}; h) \)). It is easy to check that, for any \( \beta \in \mathbb{R} \), the function \( q^2 \beta(\lambda) \) is a Muckenhoupt weight. This yields the following result.

**Proposition 4.1.** For any \( \beta \in \mathbb{R} \), the operators \( q^\beta P_{\pm} q^{-\beta} \) and hence \( q^\beta M_\Xi q^{-\beta} \) are bounded in \( L^2(\mathbb{R}; h) \) for all \( \Xi \in L^\infty(\mathbb{R}; \mathcal{B}(h)) \).

4.2. Singular weight functions. Recall (see [10]) that for \( \Xi \in C_0(\mathbb{R}; \mathcal{S}_\infty(h)) \), we have

\[
\Xi P_+ - P_+ \Xi = P_- \Xi P_+ - P_+ \Xi P_- \in \mathcal{S}_\infty.
\]

We shall see that the operator (4.2) remains compact even after being sandwiched between singular weights \( q^{-\beta} \) provided the symbol \( \Xi \) is logarithmically Hölder continuous at the singular points of \( q^{-\beta} \). Let us first consider sufficiently smooth symbols with compact supports.

**Lemma 4.2.** Let \( \Xi \in C(\mathbb{R}; \mathcal{S}_\infty(h)) \) and \( \Xi \in C^1 \) in some neighbourhoods of the singular points \( \lambda_1, \ldots, \lambda_L \) of the weight \( q^{-\beta} \). Suppose also that \( \Xi \) has compact support. Then the operator

\[
G = q^{-\beta}(\Xi P_+ - P_+ \Xi)q^{-\beta}
\]

is compact in the space \( L^2(\mathbb{R}; h) \) for all \( \beta \in \mathbb{R} \).

**Proof.** Let \( A \) be an operator in \( L^2(\mathbb{R}; h) \) with the integral kernel \( a(x, y) \). We proceed from the obvious estimate

\[
\|A\|^2 \leq \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \|a(x, y)\|_{\mathcal{B}(h)}^2 dx dy.
\]
According to formula (3.1) the integral kernel of the operator $K = \Xi P_+ - P_+ \Xi$ equals
\[
k(x, y) = -\frac{1}{2\pi i} \frac{\Xi(x) - \Xi(y)}{x - y}.
\]
It is a continuous function with values in $\mathcal{G}_\infty(\mathfrak{h})$, and it satisfies the bound
\[
\|k(x, y)\|_{\mathcal{B}(\mathfrak{h})} \leq C(1 + |x|)^{-1}(1 + |y|)^{-1}.
\] (4.5)

Let $\chi_\varepsilon(x)$ be the characteristic function of the set
\[
Q_\varepsilon = \{x \in \mathbb{R} : |x - \lambda_\ell| > \varepsilon, \ \forall \ell = 1, \ldots, L \quad \text{and} \quad |x| < \varepsilon^{-1}\}. \tag{4.6}
\]
Set $G_\varepsilon = \chi_\varepsilon G \chi_\varepsilon$. It follows from estimates (4.4) and (4.5) that
\[
\|G - G_\varepsilon\|^2 \leq C \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |1 - \chi_\varepsilon(x)|^2 q(x)^{-2\beta} (1 + |x|)^{-2} (1 + |y|)^{-2} q(y)^{-2\beta} dy dx
\]
tends to zero as $\varepsilon \to 0$.

It remains to show that $G_\varepsilon$ is compact in $L^2(\mathbb{R}; \mathfrak{h})$ for all $\varepsilon > 0$. Observe that the integral kernel $g_\varepsilon(x, y)$ of this operator is a continuous operator valued function on $Q_\varepsilon \times Q_\varepsilon$ with values in $\mathcal{G}_\infty(\mathfrak{h})$. It is almost obvious that such operators are compact. Indeed, let us split the interval $(-\varepsilon^{-1}, \varepsilon^{-1})$ in $N$ equal intervals $(x_1, x_2), \ldots, (x_N, x_{N+1})$, and let $\tilde{\chi}_n(x)$ be the characteristic function of the interval $(x_n, x_{n+1})$. Set
\[
g_{\varepsilon, N}(x, y) = \sum_{n,m=1}^{N} g_\varepsilon(x_n, y_m) \tilde{\chi}_n(x) \tilde{\chi}_m(y).
\]
Clearly, the operator $G_{\varepsilon, N}$ with such kernel is compact in $L^2(\mathbb{R}; \mathfrak{h})$. Since $g_\varepsilon(x, y)$ is a continuous function, we have
\[
\lim_{N \to \infty} \sup_{x \in Q_\varepsilon, y \in Q_\varepsilon} \|g_\varepsilon(x, y) - g_{\varepsilon, N}(x, y)\|_{\mathcal{B}(\mathfrak{h})} = 0.
\]
Therefore $\|G_\varepsilon - G_{\varepsilon, N}\| = 0$ as $N \to \infty$ according again to (4.4).

Next we pass to the general case.

**Lemma 4.3.** Let $\Xi \in C_0(\mathbb{R}; \mathcal{G}_\infty(\mathfrak{h}))$ be such that
\[
\lim_{\lambda \to \lambda_\ell} \|\Xi(\lambda) - \Xi(\lambda_\ell)\|_{\mathcal{B}(\mathfrak{h})} q(\lambda)^{-2\beta} = 0 \tag{4.7}
\]
for all $\ell = 1, \ldots, L$ and some $\beta > 0$. Then the operator (4.3) is compact in $L^2(\mathbb{R}; \mathfrak{h})$.

**Proof.** Let us reduce the question to the case
\[
\Xi(\lambda_1) = \cdots = \Xi(\lambda_L) = 0. \tag{4.8}
\]
For \( \ell = 1, \ldots, L \), let \( \rho_\ell \in C^\infty_0(\mathbb{R}) \) be such that \( \rho_\ell(\lambda_k) = \delta_{\ell k} \). Consider the function
\[
\Xi_0(\lambda) = \sum_{\ell=1}^L \Xi(\lambda_\ell) \rho_\ell(\lambda).
\]
The operator \( q^{-\beta}(P_+\Xi_0 - \Xi_0 P_+)q^{-\beta} \) is compact according to Lemma 4.2. Thus, we may check the compactness of the operator (4.3) with \( \Xi - \Xi_0 \) in place of \( \Xi \). The function \( \Xi - \Xi_0 \) satisfies the condition \( \Xi(\lambda_\ell) - \Xi_0(\lambda_\ell) = 0 \) for all \( \ell \). To simplify our notation, we will assume that \( \Xi \) already satisfies (4.8) and hence
\[
\lim_{\lambda \to \lambda_\ell} \left( \|\Xi(\lambda)\|_{B(h)} q(\lambda)^{-2\beta} \right) = 0. \quad (4.9)
\]
Let \( \sigma_\varepsilon \in C^\infty_0(\mathbb{R}) \), \( \sigma_\varepsilon(\lambda) = 0 \) in neighbourhoods of all points \( \lambda_1, \ldots, \lambda_L \), \( 0 \leq \sigma_\varepsilon(\lambda) \leq 1 \) and \( \sigma_\varepsilon(\lambda) = 1 \) for \( \lambda \in Q_\varepsilon \) (see (4.6)). According to Lemma 4.2 applied to \( \Xi \sigma_\varepsilon \) in place of \( \Xi \), the operators
\[
 q^{-\beta}(\Xi \sigma_\varepsilon P_+ - P_+ \Xi \sigma_\varepsilon)q^{-\beta}
\]
are compact. Observe that
\[
\|q^{-\beta}(\Xi_0 P_+ - P_+ \Xi_0)q^{-\beta}\|
\leq \|q^{-\beta}(\sigma_\varepsilon - 1)P_+ q^{-\beta}\| \leq \sup_{\lambda \in \mathbb{R}} \left( \|\Xi(\lambda)\|_{B(h)} q(\lambda)^{-2\beta} |1 - \sigma_\varepsilon(\lambda)| \right) \|q^\beta P_+ q^{-\beta}\|.
\]
Since \( q^\beta P_+ q^{-\beta} \in B \), it follows from (4.9) and the condition \( \|\Xi(\lambda)\| \to 0 \) as \( |\lambda| \to \infty \) that the r.h.s. here tends to 0 as \( \varepsilon \to 0 \). Thus the operators (4.10) approximate the operator (4.3) in the operator norm. This proves the compactness of (4.3).

**Corollary 4.4.** Under the hypothesis of Lemma 4.3, the operators \( q^{-\beta}P_+ \Xi P_+ q^{-\beta} \) and therefore \( q^{-\beta}Mq^{-\beta} \) are compact in \( L^2(\mathbb{R}; h) \).

**Proof.** Consider, for example, the operator
\[
 q^{-\beta}P_+ \Xi P_+ q^{-\beta} = q^{-\beta}P_- (\Xi P_+ - P_+ \Xi)q^{-\beta} = (q^{-\beta}P_- q^\beta)(\Xi P_+ - P_+ \Xi)q^{-\beta}.
\]
It suffices to use the fact that the operator \( q^{-\beta}P_- q^\beta \) is bounded and the operator (4.3) is compact.

**Remark 4.5.** Lemma 4.3 and Corollary 4.4 can be extended in an obvious way to operators acting from a space \( L^2(\mathbb{R}; h_1) \) to a space \( L^2(\mathbb{R}; h_2) \) where \( h_1 \neq h_2 \).
4.3. Disjoint singularities. The following assertion will allow us to separate contributions of different jumps of the function Ξ. It suffices to consider scalar valued symbols, that is, SHOs in the space $L^2(\mathbb{R})$.

**Lemma 4.6.** Let $\zeta$ be the function defined by (3.2) and $\zeta_j(\lambda) = \zeta(\lambda - \lambda_j)$, $j = 1, 2$. Suppose that $\lambda_1 \neq \lambda_2$. Then, for all $\beta \in \mathbb{R}$, the operators

$$q^{-\beta}P_\pm \zeta_1 P_\pm \zeta_2 P_\pm q^{-\beta}$$

(4.11)

are compact and therefore the operators $q^{-\beta}M_{\zeta_1}M_{\zeta_2}q^{-\beta}$ are also compact in $L^2(\mathbb{R})$.

**Proof.** Choose functions $\rho_j \in C_0^\infty(\mathbb{R})$, $j = 1, 2$, such that $\rho_j(\lambda) = 1$ in a neighbourhood of the point $\lambda_j$ and

$$\text{dist}(\text{supp } \rho_1, \text{supp } \rho_2) > 0.$$  

(4.12)

Set $\tilde{\rho}_j = 1 - \rho_j$. Since $\tilde{\rho}_j \zeta_j \in C^\infty(\mathbb{R})$ and $\tilde{\rho}_j(\lambda)\zeta_j(\lambda) \to 0$ as $|\lambda| \to \infty$, it follows from Corollary 4.4 that

$$q^{-\beta}P_\pm \zeta_j \tilde{\rho}_j P_\pm q^{-\beta} \in \mathcal{S}_\infty, \quad j = 1, 2.$$  

(4.13)

Let us consider the operator (4.11), for example, for the upper signs. Writing $P_-$ as

$$P_- = \rho_1 P_- \rho_2 + \tilde{\rho}_1 P_- \rho_1 P_- \tilde{\rho}_2,$$

we see that

$$q^{-\beta}P_+ \zeta_1 P_- \zeta_2 P_+ q^{-\beta} = (q^{-\beta}P_+ q^{-\beta})(q^{-\beta} \zeta_1 \rho_1 P_- \zeta_2 \rho_2 q^{-\beta})(q^{-\beta}P_+ q^{-\beta})$$

$$+ (q^{-\beta}P_+ \zeta_1 \tilde{\rho}_1 P_+ q^{-\beta})\zeta_2 (q^{-\beta}P_+ q^{-\beta}) + (q^{-\beta}P_+ q^{-\beta})\zeta_1 \rho_1 (q^{-\beta}P_- \zeta_2 \tilde{\rho}_2 P_+ q^{-\beta}).$$

(4.14)

Recall that, by Proposition 4.1, the operator $q^{-\beta}P_+ q^{-\beta}$ is bounded. Therefore to show that the first term in the r.h.s. of (4.14) is compact, it suffices to check that

$$\psi_1 P_- \psi_2 \in \mathcal{S}_\infty$$

where $\psi_j = q^{-\beta} \zeta_j \rho_j \in L^2(\mathbb{R})$. Using formula (3.1) for the integral kernel of $P_-$ and condition (4.12), we find that the integral kernel of the operator $\psi_1 P_- \psi_2$ equals

$$(2\pi i)^{-1}\psi_1(x)(x - y)^{-1}\psi_2(y).$$

This function belongs to $L^2(\mathbb{R}^2; dx dy)$ so that the operator $\psi_1 P_- \psi_2$ is Hilbert-Schmidt.

The second and third terms in the r.h.s. of (4.14) are compact because, by (4.13), the operators $q^{-\beta}P_+ \zeta_1 \tilde{\rho}_1 P_- q^{-\beta}$ and $q^{-\beta}P_- \zeta_2 \tilde{\rho}_2 P_+ q^{-\beta}$ are compact.

In view of definition (1.5) the compactness of operators (4.11) implies the compactness of $q^{-\beta}M_{\zeta_1}M_{\zeta_2}q^{-\beta}$. \qed
5. SPECTRAL AND SCATTERING THEORY OF SYMMETRIZED HANKEL OPERATORS

Here we construct the spectral theory of SHOs with piecewise continuous symbols. This theory may be interesting in its own right but, most importantly for us, it serves as a model for the spectral theory of the operators $D_{\theta}$.

5.1. Main results for SHOs. Let $\Xi \in L^\infty(\mathbb{R}; \mathfrak{S}_\infty(\mathfrak{h}))$ and let $M_\Xi$ be the SHO in $\mathcal{H} = L^2(\mathbb{R}; \mathfrak{h})$ introduced in Definition 3.1. We consider the case of piecewise continuous operator valued symbols $\Xi$ with jump discontinuities at the points $\lambda_1, \ldots, \lambda_L$. We use the notation

$$K_\ell = \Xi(\lambda_\ell^+ - 0) - \Xi(\lambda_\ell^- - 0)$$

for the jumps of $\Xi$. These jumps are compact operators in $\mathfrak{h}$. We denote by $s_n(K_\ell), 1 \leq n \leq \dim \mathfrak{h}$, the sequence of singular values of the operators $K_\ell$. Recall that the function $q(\lambda)$ was defined by formulas (3.4) and (3.28).

**Theorem 5.1.** Let a symbol $\Xi(\lambda)$ with values in $\mathfrak{S}_\infty(\mathfrak{h})$ be a norm-continuous function of $\lambda$ apart from some jump discontinuities at finitely many points $\lambda_1, \ldots, \lambda_L$. Assume that for each $\ell = 1, \ldots, L$, the symbol $\Xi$ satisfies the logarithmic regularity condition

$$\|\Xi(\lambda_\ell^+ \pm \varepsilon) - \Xi(\lambda_\ell^- + 0)\| = O(|\log \varepsilon|^{-\beta_0}), \quad \varepsilon \to +0, \quad (5.2)$$

with an exponent $\beta_0 > 2$, and assume

$$\lim_{|\lambda| \to \infty} \|\Xi(\lambda)\| = 0. \quad (5.3)$$

Then:

(i) The a.c. spectrum of the operator $M_\Xi$ consists of the union of the intervals:

$$\text{spec}_{ac} M_\Xi = \bigcup_{\ell=1}^L \bigcup_{n=1}^{\dim \mathfrak{h}} \left[-\frac{1}{2}s_n(K_\ell), \frac{1}{2}s_n(K_\ell)\right]. \quad (5.4)$$

(ii) The singular continuous spectrum of $M_\Xi$ is empty.

(iii) The eigenvalues of $M_\Xi$ can accumulate only to $0$ and to the points $\pm \frac{1}{2}s_n(K_\ell)$. All eigenvalues of $M_\Xi$, distinct from $0$ and from $\pm \frac{1}{2}s_n(K_\ell)$, have finite multiplicities.

(iv) The operator valued function $q^\beta(M_\Xi - zI)^{-1}q^\beta$, $\beta > 1$, is continuous in $z$ for $\pm \text{Im} z \geq 0$ if $z$ is separated away from the thresholds $0, \pm \frac{1}{2}s_n(K_\ell)$ and from the eigenvalues of $M_\Xi$.

Note that already the case $\dim \mathfrak{h} = 1$ is non-trivial. In fact, this case contains most of the essential difficulties and the generalisation to the case $\dim \mathfrak{h} > 1$ and even to $\dim \mathfrak{h} = \infty$ is almost automatic.
Our next goal is to describe the structure of the a.c. subspace of the SHO $M_\Xi$. We use tools of the scattering theory which also give information on the behaviour of the unitary group $\exp(-iM_\Xi t)f$ as $t \to \pm \infty$ for $f$ in the a.c. subspace of the operator $M_\Xi$. The following assertion shows that, for large $|t|$, the function $\exp(-iM_\Xi t)f$ “lives” only in neighbourhoods of the singular points $\lambda_1, \ldots, \lambda_L$.

**Lemma 5.2.** Let $\Xi$ satisfy the assumptions of Theorem 5.1. Let $Q$ be a closed set such that $Q \cap \{\lambda_1, \ldots, \lambda_L\} = \emptyset$ and let $\chi_Q$ be the characteristic function of $Q$. Then the operator $\chi_Q M_\Xi$ is compact and

$$\lim_{|t| \to \infty} \chi_Q \exp(-iM_\Xi t) P_{ac}^{(ac)} = 0. \quad (5.5)$$

**Proof.** Choose a function $\omega \in C^\infty(\mathbb{R})$ such that $\omega(\lambda) = 0$ in a neighbourhood of the set $\{\lambda_1, \ldots, \lambda_L\}$, $\omega(\lambda) = 1$ away from some larger neighbourhood of this set and $\chi_Q = \chi_Q \omega$. We have

$$\omega P_\pm = (\omega P_\pm - P_\pm \omega) \Xi P_\pm + P_\pm \omega \Xi P_\pm. \quad (5.6)$$

The first term in r.h.s. is compact because $\omega P_\pm - P_\pm \omega = (\omega - 1)P_\pm - P_\pm (\omega - 1) \in \mathcal{S}_\infty$ according to (4.2). The second term in r.h.s. of (5.6) is compact because $\omega \Xi \in C_0(\mathbb{R}; \mathcal{S}_\infty(h))$. Thus $\omega M_\Xi \in \mathcal{S}_\infty$ and hence $\chi_Q M_\Xi \in \mathcal{S}_\infty$. It follows that (5.5) holds on all elements of the form $f = M_\Xi g$ and therefore it holds for all $f \in L^2(\mathbb{R}, h)$.

Note that condition (5.2) and even the existence of the limits $\Xi(\lambda_\ell \pm 0)$ were not used in the proof.

Lemma 5.2 shows that it is natural to construct model operators for each jump of $\Xi$. Set

$$\Xi_\ell(\lambda) = \zeta(\lambda - \lambda_\ell)K_\ell, \quad \ell = 1, \ldots, L, \quad (5.7)$$

where $\zeta$ is given by (3.2) and $K_\ell$ is given by (5.1). As the model operator for the point $\lambda_\ell$ we choose the SHO $M_{\Xi_\ell}$. Note that each of the symbols $\Xi_\ell$, $\ell \geq 1$, has only one jump at the point $\lambda_\ell$ and the spectral structure of $M_{\Xi_\ell}$ is described in Section 3.

**Theorem 5.3.** Let the assumptions of Theorem 5.1 hold true, let $\Xi_\ell$ be as defined in (5.7), and let $M_{\Xi_\ell}$ be as defined in (1.5). Then:

(i) The wave operators

$$W_\pm(M_\Xi, M_{\Xi_\ell}) = \lim_{t \to \pm \infty} e^{iM_{\Xi_\ell}t} e^{-iM_\Xi t} P_{ac}^{(ac)}(M_{\Xi_\ell}), \quad \ell = 1, \ldots, L,$$

exist and enjoy the intertwining property

$$M_\Xi W_\pm(M_\Xi, M_{\Xi_\ell}) = W_\pm(M_\Xi, M_{\Xi_\ell}) M_{\Xi_\ell}.$$
These operators are isometric and their ranges are orthogonal to each other, i.e.

\[ \text{Ran} W_{\pm}(M_{\mathbb{E}}, M_{\mathbb{E}_j}) \perp \text{Ran} W_{\pm}(M_{\mathbb{E}}, M_{\mathbb{E}_k}), \quad j \neq k. \]

(ii) The asymptotic completeness holds:

\[ \text{Ran} W_{\pm}(M_{\mathbb{E}}, M_{\mathbb{E}_1}) \oplus \cdots \oplus \text{Ran} W_{\pm}(M_{\mathbb{E}}, M_{\mathbb{E}_L}) = \mathcal{H}^{(ac)}_{M_{\mathbb{E}}}. \]

Instead of (5.3), it suffices to assume that \( \Xi(\lambda) \) has a finite limit \( \xi_\infty \) as \( |\lambda| \to \infty \), that is,

\[ \lim_{|\lambda| \to \infty} \| \Xi(\lambda) - \xi_\infty \| = 0. \tag{5.8} \]

Indeed, set \( \tilde{\Xi}(\lambda) = \Xi(\lambda) - \xi_\infty \). Then Theorems 5.1 and 5.3 can be applied to the operator \( M_{\tilde{\mathbb{E}}} \). Since \( M_{\mathbb{E}} = M_{\tilde{\mathbb{E}}} \), this yields all required results about the operator \( M_{\mathbb{E}} \). However, assumption (5.8) can also be relaxed — see Remark 5.6.

5.2. Proofs of Theorems 5.1 and 5.3. Theorems 5.1 and 5.3 will be deduced from Propositions 2.9 and 2.10, respectively. We will check the hypotheses of Proposition 2.9 for \( A = M_{\mathbb{E}}, A_\ell = M_{\mathbb{E}_\ell}, \ell = 1, \ldots, L, \) and \( A_\infty = M_{\mathbb{E}_\infty} \) where the symbol \( \Xi_\infty \) is given by

\[ \Xi_\infty(\lambda) = \Xi(\lambda) - \sum_{\ell=1}^L \Xi_\ell(\lambda). \]

Then equality (2.17) is satisfied. As \( \delta \), we choose an arbitrary open interval \( \delta \subset \mathbb{R} \) which does not contain the points \( 0, \pm \frac{1}{2} s_n(K_\ell), 1 \leq n \leq \dim \mathfrak{h}, \ell = 1, \ldots, L. \)

Set \( X = q^\beta \). The parameter \( \beta > 0 \) is chosen sufficiently large to guarantee strong \( A_\ell \)-smoothness of \( X \). On the other hand, it should be sufficiently small to ensure conditions (2.18) and (2.19). To be more precise, we suppose that \( 1 < \beta < \beta_0/2, \) where \( \beta_0 \) is the exponent from (5.2).

By Theorem 3.3, the operator \( q^\beta \) is strongly \( A_\ell \)-smooth on \( \delta \) with any exponent \( \gamma < \beta - 1/2 \) (and of course \( \gamma \leq 1 \)) which allows us to choose \( \gamma > 1/2 \).

The operators \( q^{-\beta} M_{\mathbb{E}_j} M_{\mathbb{E}_k} q^{-\beta}, 1 \leq j, k \leq L, j \neq k, \) are compact by Lemma 4.6. This yields condition (2.18).

Next, \( \Xi_\infty \) is a continuous function because the functions \( \Xi(\lambda) \) and \( \Xi_\ell(\lambda) \) have the same jumps at all points \( \lambda = \lambda_\ell \) and the functions \( \Xi_\ell(\lambda) \) are continuous at \( \lambda = \lambda_\ell \) if \( j \neq \ell \). Moreover, according to (5.2) the function \( \Xi_\infty \) satisfies assumption (4.7) with any \( \beta < \beta_0/2 \). By Corollary 4.4, it follows that the operator \( q^{-\beta} M_{\mathbb{E}_\infty} q^{-\beta} \) is compact.

Finally, the operators \( q^\beta M_{\mathbb{E}_j} q^{-\beta} \) are bounded according to Proposition 4.1. Thus, all the assumptions of Proposition 2.9 are satisfied.
On each interval \( \delta \), every statement of Proposition 2.9 yields the corresponding statement of Theorem 5.1 about the operator \( M_\Sigma \). Using that \( \delta \) is arbitrary, we obtain the same statements on the whole line \( \mathbb{R} \) with the points 0 and \( \pm \frac{1}{2} s_n(K_\ell) \) removed. This concludes the proof of Theorem 5.1.

Similarly, all conclusions of Proposition 2.10 are true for the wave operators \( W_\pm(M_\Sigma, M_\Sigma; \delta) \). Let us now use the fact that linear combinations of all elements \( f \) such that \( f \in \text{Ran} \, E_{M_\Sigma}(\delta) \) for some admissible \( \delta \) are dense in \( H^{ac}_{M_\Sigma} \). Therefore all statements of Proposition 2.10 about the wave operators \( W_\pm(M_\Sigma, M_\Sigma; \delta) \) yield the corresponding statements of Theorem 5.3 about the wave operators \( W_\pm(M_\Sigma, M_\Sigma) \).

\[ \square \]

**Remark 5.4.** If \( \Xi \) has only one jump (i.e. \( L = 1 \)), the proof simplifies considerably. In this case it suffices to use the usual smooth scheme of scattering theory for the pair \( M_\Xi, M_\Xi \). Lemma 4.6 is not required either.

5.3. **SHOs on the circle.** Let us briefly discuss the analogues of Theorems 5.1 and 5.3 for SHOs on the unit circle. Let \( H^2_T(\mathbb{R}; \mathfrak{h}) \subset L^2_T(\mathbb{R}; \mathfrak{h}) \) be the Hardy space of \( \mathfrak{h} \)-valued functions analytic in the unit disc, let \( H^2_{\pm}(T; \mathfrak{h}) \) be the orthogonal complement of \( H^2_T(T; \mathfrak{h}) \) in \( L^2_T(\mathbb{R}; \mathfrak{h}) \), and let \( P_{\pm} \) be the orthogonal projection in \( L^2_T(\mathbb{R}; \mathfrak{h}) \) onto \( H^2_{\pm}(T; \mathfrak{h}) \). For \( \Psi \in L^\infty(T; \mathcal{B}(\mathfrak{h})) \) we define, similarly to (1.5), the SHO

\[ M_\Psi = P_- \Psi P_+ + P_+ \Psi^* P_- \quad \text{in} \quad L^2_T(T; \mathfrak{h}). \]

The spectral analysis of \( M_\Psi \) can be obtained through a unitary map from \( L^2_T(T; \mathfrak{h}) \) onto \( L^2_T(\mathbb{R}; \mathfrak{h}) \). Indeed, the map

\[ \mathbb{R} \ni \lambda \mapsto \frac{\lambda - i}{\lambda + i} = \mu \in \mathbb{T} \]

generates the unitary operator

\[ U : L^2(\mathbb{R}; \mathfrak{h}) \to L^2_T(\mathbb{R}; \mathfrak{h}), \]

\[ (Uf)(\mu) = 2 i \sqrt{\pi} (1 - \mu)^{-1} f(i \frac{1 + \mu}{1 - \mu}). \]  

(5.9)

This operator transforms the SHO \( M_\Xi \) in \( L^2_T(\mathbb{R}; \mathfrak{h}) \) into the SHO \( M_\Psi \) in \( L^2_T(\mathbb{R}; \mathfrak{h}) \):

\[ UM_\Xi U^* = M_\Psi, \quad \Psi(\mu) = \Xi(i \frac{1 + \mu}{1 - \mu}), \quad \mu \in \mathbb{T}. \]

Thus, Theorems 5.1 and 5.3 extend to the SHOs on the circle with piecewise continuous symbols.

Assumption (5.3) means that the symbol \( \Psi \) must be continuous at \( \mu = 1 \). But of course this assumption can be lifted by means of a rotation. Indeed, suppose that \( \Psi \) has a jump at \( \mu = 1 \). Choose \( \alpha \) such that \( \Psi \) is continuous at \( \mu = e^{i \alpha} \). Then all above mentioned spectral results are true for the SHO \( M_{\tilde{\Psi}} \) with the symbol \( \tilde{\Psi}(\mu) = \Psi(\mu e^{i \alpha}) \). Since the operator \( M_{\tilde{\Psi}} \) is unitarily
equivalent to $M_\Psi$ through the unitary transformation $f(\mu) \mapsto f(\mu e^{i\alpha})$, we can reformulate these results in terms of the operator $M_\Psi$. This reasoning yields the following spectral results.

**Theorem 5.5.** Let a symbol $\Psi(\mu) \in \mathcal{S}_\infty(\mathfrak{h})$ be a norm-continuous function apart from some jump discontinuities at finitely many points $\mu_1, \ldots, \mu_L$. Set

$$K_\ell = \lim_{\varepsilon \to +0} (\Psi(\mu_\ell e^{i\varepsilon}) - \Psi(\mu_\ell e^{-i\varepsilon}))$$

and assume that

$$\|\Psi(\mu_\ell e^{\pm i\varepsilon}) - \Psi(\mu_\ell e^{\pm i0})\| = O(\log |\varepsilon|^{-\beta_0}), \quad \varepsilon \to +0,$$

with some exponent $\beta_0 > 2$ for each $\ell = 1, \ldots, L$. Then the a.c. spectrum of the operator $M_\Psi$ in $L^2(\mathbb{T}; \mathfrak{h})$ consists of the union of the intervals in the r.h.s. of (5.4). The singular continuous spectrum of $M_\Psi$ is empty. The eigenvalues of $M_\Psi$ can accumulate only to 0 and to the points $\pm \frac{1}{2}s_n(K_\ell)$. All eigenvalues of $M_\Psi$, distinct from 0 and from $\pm \frac{1}{2}s_n(K_\ell)$, have finite multiplicities.

The results of Theorem 5.3 concerning the wave operators can also be extended to SHOs on the unit circle if model operators $M_\Xi_\ell$ are transplanted into the space $L^2(\mathbb{T}; \mathfrak{h})$ via the unitary transform (5.9).

**Remark 5.6.** Using the unitary operator $U$ (see (5.9)) to transform $M_\Psi$ back to $M_\Xi$, we see that the assumption (5.3) in Theorem 5.1 can be relaxed. Instead, one can assume that the limits $\Xi(\pm \infty)$ exist and

$$\|\Xi(\pm \lambda) - \Xi(\pm \infty)\| = O(|\log \lambda|^{-\beta_0}), \quad \lambda \to +\infty,$$

with $\beta_0 > 2$. Then the jump $K_\infty = \Xi(-\infty) - \Xi(+\infty)$ at infinity will also contribute to the orthogonal sum (5.4).

### 6. Representations for $D_\theta$

Our goal here is to derive a formula (see Theorem 6.1) for the operator $D_\theta$ sandwiched between appropriate functions of $H_0$ in the spectral representation of $H_0$. This formula motivates and explains much of our construction. Eventually, this formula will allow us to relate $D_\theta$ to the SHO (1.5) with the symbol (1.4). Technically, we need also another representation (see Theorem 6.2) for $D_\theta$ sandwiched between functions of $H_0$ with disjoint supports. In fact, our representations below are valid for $D_\phi$ with arbitrary bounded functions $\phi$ of compact support.

#### 6.1. Two formulas.

We work under Assumption 2.2. First we need to deal with the following technical difficulty of a formal nature. The multiplicity $N_\ell$ of the spectrum of $H_0$ on $\Delta_\ell$ may vary with $\ell$, whilst we would like to relate $D_\phi$ to a SHO acting in $L^2(\mathbb{R}; \mathcal{N})$ with a fixed space $\mathcal{N}$. To achieve this, we set

$$\mathcal{N} = \mathcal{N}_1 \oplus \cdots \oplus \mathcal{N}_L$$

\hspace{1cm} (6.1)
and extend the scattering matrix to a unitary operator on \( \mathcal{N} \). More precisely, for \( \lambda \in \Delta_\ell \) the scattering matrix \( S(\lambda) \) is initially defined as an operator on \( \mathcal{N}_\ell \); we extend it to an operator on \( \mathcal{N} \) by setting \( S(\lambda)\psi = \psi \) for \( \psi \in \mathcal{N}_\ell^- \). We also define similar extensions for other relevant objects. We extend the unitary operator \( F_\ell \) (see (2.11)) by zero onto \( \text{Ran} \ E_{H_0}(\mathbb{R} \setminus \Delta_\ell) \), which yields a partially isometric operator
\[
F_\ell : \mathcal{H} \to L^2(\mathbb{R}; \mathcal{N}). \tag{6.2}
\]
In a similar way, the operators \( Z_\ell(\lambda), \lambda \in \Delta_\ell \) (see (2.12)), will be considered as operators from \( \mathcal{H} \) to \( \mathcal{N} \). Further, we combine the operators \( F_\ell, \ell = 1, \ldots, L \) together as follows: we set \( \Delta = \Delta_1 \cup \cdots \cup \Delta_L \) and define the unitary map
\[
F_\Delta : \text{Ran} \ E_{H_0}(\Delta) \to L^2(\Delta; \mathcal{N})
\]
by setting \( F_\Delta f = F_\ell f \) if \( f \in \text{Ran} \ E_{H_0}(\Delta_\ell) \). Finally, we extend the unitary operator \( F_\Delta \) by zero onto \( \text{Ran} \ E_{H_0}(\mathbb{R} \setminus \Delta) \), which yields a partial isometry
\[
F_\Delta : \mathcal{H} \to L^2(\mathbb{R}; \mathcal{N}). \tag{6.3}
\]
Note that the adjoint operator \( F_\Delta^* : L^2(\mathbb{R}; \mathcal{N}) \to \mathcal{H} \) is a partial isometry which sends the subspace
\[
\mathcal{K} := L^2(\Delta_1; \mathcal{N}_1) \oplus \cdots \oplus L^2(\Delta_L; \mathcal{N}_L)
\]
unitarily onto \( \text{Ran} \ E_{H_0}(\Delta) \) and is zero on \( \mathcal{K}^- \). We also combine the operators \( Z_\ell(\lambda) \) together by setting \( Z(\lambda) = Z_\ell(\lambda) \) for \( \lambda \in \Delta_\ell \).

Recall that \( \Omega \) is defined by (2.14) and \( Y(z) \) is defined by (2.6), (2.8). Let \( \omega \) and \( \varphi \) be bounded functions with compact supports \( \text{supp} \ \omega \subset \Delta \) and \( \text{supp} \ \varphi \subset \Omega \). Define the operators
\[
Z_\omega : L^2(\mathbb{R}; \mathcal{H}) \to L^2(\mathbb{R}; \mathcal{N}) \quad \text{and} \quad Y_\varphi : L^2(\mathbb{R}; \mathcal{H}) \to L^2(\mathbb{R}; \mathcal{H})
\]
by formulas
\[
(Z_\omega u)(\lambda) = \omega(\lambda)Z(\lambda)u(\lambda), \quad (Y_\varphi u)(\lambda) = \varphi(\lambda)Y(\lambda + i0)u(\lambda). \tag{6.5}
\]
Recall that \( Z(\lambda) \) and \( Y(\lambda + i0) \) are Hölder continuous in \( \lambda \) on \( \Delta \) and \( \Omega \), respectively. In particular, this implies that the operators \( Z_\omega \) and \( Y_\varphi \) are bounded.

The following representation of the sandwiched operator \( D_\varphi \) is central to our construction.

**Theorem 6.1.** Let Assumption 2.2 be satisfied. Let \( \omega \) and \( \varphi \) be bounded functions with compact supports \( \text{supp} \ \omega \subset \Delta \) and \( \text{supp} \ \varphi \subset \Omega \), and let the operators \( Z_\omega \) and \( Y_\varphi \) be defined by formulas (6.5). Then the representation
\[
F_\Delta \omega(H_0)D_\varphi \omega(H_0)F_\Delta^* = Z_\omega(P_-Y_\varphi P_+ + P_+Y_\varphi^* P_-)Z_\omega^* \tag{6.6}
\]
holds.
The proof is given in Subsection 6.3.

Now we consider $D_\varphi$ as an operator acting from $H$ to the spectral representation of $H_0$. Let $\varphi$ and $v$ be bounded functions such that $\text{supp} \varphi \subset \Omega$ and
\[
\text{dist}(\text{supp} \varphi, \text{supp} v) > 0.
\]
(6.7)
Then we can define the operators $Y^{(\pm)}_{\varphi,v} : H \to L^2(\mathbb{R}; H)$ by the formula
\[
(Y^{(\pm)}_{\varphi,v} f)(\lambda) = \varphi(\lambda) Y(\lambda \pm i0) GR_0(\lambda) v(H_0) f.
\]
(6.8)
In view of condition (6.7) the operators $GR_0(\lambda) v(H_0)$ are compact in $H$ and depend Hölder continuously on $\lambda \in \text{supp} \varphi$.

**Theorem 6.2.** Let functions $\omega$ and $\varphi$ be the same as in Theorem 6.1. Let $v$ be a bounded function satisfying condition (6.7). Then under Assumption 2.2 the representation
\[
2\pi i F_{\Delta} \omega(H_0) D_\varphi v(H_0) = Z_\omega(P_- Y^{(+)}_{\varphi,v} + P_+ Y^{(-)}_{\varphi,v})
\]
(6.9)
holds.

The proof is given in Subsection 6.4.

6.2. **Auxiliary results.** We start with a simple identity.

**Lemma 6.3.** Suppose that $\varphi$ is a bounded function and that both the spectra of $H_0$ and $H$ are purely a.c. on $\text{supp} \varphi$. Then for all $f, g \in H$, we have the identity
\[
(2\pi i)^2 (D_\varphi f, g) = \int_{-\infty}^{\infty} \lim_{\varepsilon \to +0} (Y(\lambda + i\varepsilon) GR_0(\lambda + i\varepsilon) f, GR_0(\lambda - i\varepsilon) g)\varphi(\lambda) d\lambda
\]
\[
- \int_{-\infty}^{\infty} \lim_{\varepsilon \to +0} (Y(\lambda - i\varepsilon) GR_0(\lambda - i\varepsilon) f, GR_0(\lambda + i\varepsilon) g)\varphi(\lambda) d\lambda
\]
(6.10)
where the limits in the r.h.s. exist for almost all $\lambda \in \mathbb{R}$.

**Proof.** Since the measure $(E_H(\lambda)f, g)$ is absolutely continuous on $\text{supp} \varphi$, it follows from the spectral theorem that
\[
(\varphi(H)f, g) = \int_{-\infty}^{\infty} \varphi(\lambda) d(E_H(\lambda)f, g) = \int_{-\infty}^{\infty} \varphi(\lambda) \frac{d(E_H(\lambda)f, g)}{d\lambda} d\lambda.
\]
Recall that for an arbitrary self-adjoint operator $H$ the relation holds:
\[
2\pi i \frac{d(E_H(\lambda)f, g)}{d\lambda} = \lim_{\varepsilon \to +0} (R(\lambda + i\varepsilon) f, g) - \lim_{\varepsilon \to +0} (R(\lambda - i\varepsilon) f, g)
\]
where the derivative in the l.h.s. and the limits in the r.h.s. exist for almost all $\lambda \in \mathbb{R}$. Similar relations are of course also true for the operator $H_0$. Putting
these relations for $H$ and $H_0$ together and collecting terms corresponding to $\lambda + i\varepsilon$ and to $\lambda - i\varepsilon$, we obtain the formula

$$2\pi i(D_\varphi f, g) = \int_{-\infty}^{\infty} \lim_{\varepsilon \to +0} \left( (R(\lambda + i\varepsilon) - R_0(\lambda + i\varepsilon))f, g \right) \varphi(\lambda) d\lambda$$

$$- \int_{-\infty}^{\infty} \lim_{\varepsilon \to -0} \left( (R(\lambda - i\varepsilon) - R_0(\lambda - i\varepsilon))f, g \right) \varphi(\lambda) d\lambda.$$  

Substituting here expression (2.9) for $z = \lambda \pm i\varepsilon$, we conclude the proof of (6.10). \hfill \square

Our next goal is to pass to the limit $\varepsilon \to 0$ in (6.10). This is possible if $D_\varphi$ is sandwiched between appropriate functions of $H_0$. Passing to the limit relies on the following assertion.

**Lemma 6.4.** For all $f \in \mathcal{H}$, all bounded functions $\omega$ with compact support $\text{supp} \omega \subset \Delta$ and almost all $\lambda \in \mathbb{R}$, we have

$$GR_0(\lambda \pm i\varepsilon)\omega(H_0)f \to \pm 2\pi i(P_\pm Z_\omega^* \mathcal{F}_\Delta f)(\lambda)$$

as $\varepsilon \to +0$.

**Proof.** It follows from (2.4) that

$$GR_0(\lambda \pm i\varepsilon)\omega(H_0)f = \int_{-\infty}^{\infty} \omega(x)Z^*(x)(\mathcal{F}_\Delta E_{H_0}(\Delta)f)(x) \frac{dx}{x - \lambda \mp i\varepsilon}.$$  

Observe that $\mathcal{F}_\Delta E_{H_0}(\Delta)f \in L^2(\mathbb{R}; \mathcal{N})$ and the operator valued function $\omega(x)Z^*(x)$ is bounded. Therefore (6.11) follows from formula (3.1). \hfill \square

**6.3. Proof of Theorem 6.1.** Recall (see Proposition 2.6) that the spectrum of $H$ is a.c. on $\Omega$. Let us apply identity (6.10) to the elements $\omega(H_0)f$ and $\omega(H_0)g$ instead of $f$ and $g$ and then pass to the limit $\varepsilon \to +0$. The operator valued function $Y(\lambda + i\varepsilon)$ converges to $Y(\lambda + i0)$ uniformly on $\text{supp} \varphi$. Therefore it follows from (6.10) and (6.11) that

$$(D_\varphi \omega(H_0)f, \omega(H_0)g)$$

$$= \int_{-\infty}^{\infty} (Y(\lambda + i0)(P_+ Z_\omega^* \mathcal{F}_\Delta f)(\lambda), (P_- Z_\omega^* \mathcal{F}_\Delta g)(\lambda)) \varphi(\lambda) d\lambda$$

$$- \int_{-\infty}^{\infty} (Y(\lambda - i0)(P_- Z_\omega^* \mathcal{F}_\Delta f)(\lambda), (P_+ Z_\omega^* \mathcal{F}_\Delta g)(\lambda)) \varphi(\lambda) d\lambda.$$  

Taking into account (2.10) and using notation (6.5), we get

$$(D_\varphi \omega(H_0)f, \omega(H_0)g) = \left( Y_\varphi P_+ Z_\omega^* \mathcal{F}_\Delta f, P_- Z_\omega^* \mathcal{F}_\Delta g \right)_{L^2(\mathbb{R}; \mathcal{H})}$$

$$+ \left( Y_\varphi P_- Z_\omega^* \mathcal{F}_\Delta f, P_+ Z_\omega^* \mathcal{F}_\Delta g \right)_{L^2(\mathbb{R}; \mathcal{H})}. \quad (6.12)$$

Set here $f = \mathcal{F}_\Delta \tilde{f}$, $g = \mathcal{F}_\Delta \tilde{g}$ where $\tilde{f}$, $\tilde{g}$ are arbitrary elements in $L^2(\mathbb{R}; \mathcal{H})$ and observe that $Z_\omega = \mathcal{F}_\Delta \mathcal{F}_\Delta^* Z_\omega$. Thus (6.12) implies (6.6). \hfill \square
6.4. Proof of Theorem 6.2. Let us apply identity (6.10) to the elements \( v(H_0)f \) and \( \omega(H_0)g \) instead of \( f \) and \( g \). We have to pass to the limit \( \varepsilon \to +0 \) in the expression

\[
(Y(\lambda \pm i\varepsilon)GR_0(\lambda \pm i\varepsilon)v(H_0)f, GR_0(\lambda \mp i\varepsilon)\omega(H_0)g)\varphi(\lambda). \tag{6.13}
\]

Similarly to the proof of Theorem 6.1, we use the uniform convergence of \( Y(\lambda + i\varepsilon) \) on supp \( \varphi \) and apply Lemma 6.4 to \( GR_0(\lambda \mp i\varepsilon)\omega(H_0)g \). Moreover, we use that \( GR_0(\lambda \pm i\varepsilon)v(H_0)f \to GR_0(\lambda)v(H_0)f \) as \( \varepsilon \to 0 \) uniformly on supp \( \varphi \). Therefore the limit of (6.13) equals

\[
\pm 2\pi i(Y(\lambda \pm i0)GR_0(\lambda)v(H_0)f, (P_\mp Z_\omega^*F\Delta g)(\lambda))\varphi(\lambda)
\]

where notation (6.8) has been used. It now follows from (6.10) that

\[
2\pi i(D_\varphi v(H_0)f, \omega(H_0)g)
\]

\[
= \int_{-\infty}^{\infty} ((Y(\varphi^+_{\varphi,v})f)(\lambda), (P_- Z_\omega^*F\Delta g)(\lambda)) d\lambda + \int_{-\infty}^{\infty} ((Y(\varphi^-_{\varphi,v})f)(\lambda), (P_+ Z_\omega^*F\Delta g)(\lambda)) d\lambda
\]

\[
= (Y(\varphi^+_{\varphi,v})f, P_- Z_\omega^*F\Delta g)_{L^2(\mathbb{R};H)} + (Y(\varphi^-_{\varphi,v})f, P_+ Z_\omega^*F\Delta g)_{L^2(\mathbb{R};H)}. \tag{6.14}
\]

Set here \( g = F_\Delta \tilde{g} \) where \( \tilde{g} \in L^2(\mathbb{R};H) \) is arbitrary. Thus (6.14) implies (6.9).

7. Spectral and Scattering Theory of \( D_\theta \)

This section is organized as follows. We formulate our main results in Subsection 7.1 and give their proofs in Subsection 7.2 modulo an essential analytic result (Theorem 7.5). The rest of the section is devoted to the proof of Theorem 7.5.

7.1. Main results. Let Assumption 2.2 hold true, and let the operator \( D_\theta \) be defined by formula (1.1). Our aim is to describe the spectral structure of this operator for piecewise continuous functions \( \theta \) with discontinuities on the set \( \Omega \) defined by relation (2.14).

Assumption 7.1. Let \( \theta(\lambda) \) be a real function such that:

- \( \theta \) is continuous apart from jump discontinuities at the points \( \lambda_\ell \in \Omega_\ell = \Omega \cap \Delta_\ell, \ell = 1, \ldots, L \);
- at each point of discontinuity \( \lambda_\ell \), the function \( \theta \) satisfies the logarithmic regularity condition
  \[
  \theta(\lambda_\ell \pm \varepsilon) - \theta(\lambda_\ell \pm 0) = O(|\log \varepsilon|^{-\beta_0}), \quad \varepsilon \to +0, \tag{7.1}
  \]
  with an exponent \( \beta_0 > 2 \);
- the limits \( \lim_{\lambda \to \pm \infty} \theta(\lambda) \) exist and are finite.
Thus we suppose that there is exactly one discontinuity point on every set $\Delta_\ell$. To put it differently, given discontinuity points $\lambda_1, \ldots, \lambda_L$, we suppose that the Assumption 2.2(C) is satisfied only in some neighbourhoods of these points. We also suppose that discontinuity points do not coincide with the eigenvalues of the operator $H$.

As in the previous section, we consider the operators $F_\ell$ and $F_\Delta$, see (6.2) and (6.3), as partially isometric operators acting from $H$ to $L^2(\mathbb{R}; \mathcal{N})$ where the space $\mathcal{N}$ is defined by equality (6.1). We denote the jump of $\theta$ at $\lambda_\ell$ by $\kappa_\ell$, see (1.2); $\sigma_n(\lambda_\ell), 1 \leq n \leq N_\ell$, are the eigenvalues of the scattering matrix $S(\lambda_\ell)$ for the pair of operators $H_0, H$ and the numbers $a_{n\ell}$ are defined in (1.3).

As before, the function $q(\lambda)$ is defined by equalities (3.4) and (3.28), and we denote by the same letter $q$ the operator of multiplication by this function in the space $L^2(\mathbb{R}; \mathcal{N})$.

The spectral properties of the operator $D_\theta$ are described in the following assertion.

**Theorem 7.2.** Let Assumptions 2.2 and 7.1 hold true. Then:

(i) The a.c. spectrum of $D_\theta$ consists of the union of the intervals $[-a_{n\ell}, a_{n\ell}]$, that is, relation (1.3) holds.

(ii) The singular continuous spectrum of $D_\theta$ is empty.

(iii) The eigenvalues of $D_\theta$, distinct from 0 and from $\pm a_{n\ell}$, have finite multiplicities and can accumulate only to 0 and to the points $\pm a_{n\ell}$.

(iv) For any $\beta > 1$, the operator-valued function $q^\beta F_\Delta (D_\theta - zI)^{-1} F_\Delta^* q^\beta$ is continuous in $z$ for $\pm \text{Im} z \geq 0$ away from 0, all points $\pm a_{n\ell}$ and the eigenvalues of $D_\theta$.

To construct scattering theory for the operator $D_\theta$, we have to introduce a model operator for each discontinuity $\lambda_\ell$ of the function $\theta$. Let $\zeta$ be as in (3.2). Recall that (see Subsection 6.1) we have agreed to consider $S(\lambda_\ell)$ as a unitary operator in $\mathcal{N}$, see (6.1). For $\ell = 1, \ldots, L$, we define the operator $K_\ell : \mathcal{N} \to \mathcal{N}$ by

$$K_\ell = \zeta_\ell (S(\lambda_\ell) - I)$$

and then define the symbol $\Xi_\ell(\lambda)$ by formula (5.7). We note that the function $\Xi_\ell$ is bounded and has a single point of discontinuity at $\lambda = \lambda_\ell$. It is clear that the singular values of $S(\lambda_\ell) - I$ are $|\sigma_n(\lambda_\ell) - 1|$. According to Lemma 3.6, each operator $M_{\Xi_\ell}$ can be explicitly diagonalized. Except for a possible zero eigenvalue of infinite multiplicity, its spectrum is absolutely continuous and consists of the intervals $[-a_{n\ell}, a_{n\ell}], 1 \leq n \leq N_\ell$. Note that $\text{Ker} K_\ell \neq \{0\}$ and hence $\text{Ker} M_{\Xi_\ell} \neq \{0\}$ if and only if $a_{n\ell} = 0$ for some $n$. Even for $L = 1$, this happens if the scattering matrix $S(\lambda_1)$ has the eigenvalue 1. We emphasize that the zero eigenvalue of the operators $M_{\Xi_\ell}$ is irrelevant for our construction.

**Theorem 7.3.** Let Assumptions 2.2 and 7.1 hold true. Then:
The wave operators
\[ W_\pm(D_\theta, M_{\Xi\ell}) := \text{s-lim}_{t \to \pm \infty} e^{iD_\theta t} F_\ell e^{-iM_{\Xi\ell} t} P^{(ac)}_{M_{\Xi\ell}}, \quad \ell = 1, \ldots, L, \quad (7.3) \]
exist and enjoy the intertwining property
\[ D_\theta W_\pm(D_\theta, M_{\Xi\ell}) = W_\pm(D_\theta, M_{\Xi\ell}) M_{\Xi\ell}. \]
The wave operators are isometric on the a.c. subspaces of \( M_{\Xi\ell} \) and their ranges are orthogonal to each other, i.e.
\[ \text{Ran} W_\pm(D_\theta, M_{\Xi j}) \perp \text{Ran} W_\pm(D_\theta, M_{\Xi k}), \quad j \neq k. \quad (7.4) \]

(ii) The asymptotic completeness holds:
\[ \text{Ran} W_\pm(D_\theta, M_{\Xi 1}) \oplus \cdots \oplus \text{Ran} W_\pm(D_\theta, M_{\Xi L}) = \mathcal{H}^{(ac)}_{D_\theta}. \quad (7.5) \]

Corollary 7.4. For an arbitrary \( f \in \mathcal{H}^{(ac)}_{D_\theta} \), we have the relation
\[ \lim_{t \to \pm \infty} \| e^{-iD_\theta t} f - \sum_{\ell=1}^{L} e^{-iM_{\Xi\ell} t} f_\ell \| = 0, \quad f_\ell = W^*_{\pm}(D_\theta, M_{\Xi\ell}) f. \]

The interval \( \Delta_\ell \) in the definition of the operator \( F_\ell \) can be replaced by a smaller one. In view of Lemma 5.2 this does not change the wave operator \( W_\pm(D_\theta, M_{\Xi\ell}) \). We note also that this operator is not changed if the model operator \( M_{\Xi\ell} \) is considered in the space \( L^2(\mathbb{R}; N_\ell) \) and \( F_\ell \) is considered as the mapping \( F_\ell : \mathcal{H} \to L^2(\mathbb{R}; N_\ell) \).

7.2. Proofs of Theorems 7.2 and 7.3. Our proofs of Theorems 7.2 and 7.3 rely on Propositions 2.9 and 2.10, respectively. They bear a certain resemblance to the proofs of Theorems 5.1 and 5.3. Indeed, the symbols of the model operators are defined by the same formula (5.7). Now the auxiliary space \( \mathfrak{h} = \mathcal{N} \) and the operator \( K_\ell \) is given by (7.2). On the contrary, the role of the operator \( A \) in Proposition 2.9 will now be played by the operator \( D_\theta \) transplanted by an isometric transformation into the space \( L^2(\mathbb{R}; N_\ell) \) whereas in Section 5 the operator \( A \) was itself a SHO. This difference leads to rather serious technical difficulties.

Let us consider an arbitrary isometric transformation
\[ F_\perp : \text{Ran} E_{H_0}(\mathbb{R} \setminus \Delta) \to L^2(\mathbb{R} \setminus \Delta; \mathcal{N}). \quad (7.6) \]
Without loss of generality we may assume that \(|\mathbb{R} \setminus \Delta| > 0\) so that such a mapping exists. Then the operator \( F = F_\Delta \oplus F_\perp : \mathcal{H} \to L^2(\mathbb{R}; \mathcal{N}) \) is also isometric. Of course, the construction of the operators \( F_\perp \) and hence of \( F \) is not unique and has a slightly artificial flavour. However, it is very convenient because it allows us to develop the scattering theory in only one space. Note that if \( E_{H_0}(\Delta) = I \) (this is the case, for example, for the Schrödinger operator – Example 2.3), then \( F = F_\Delta \).
The following result shows that the operator $D_\theta$ transplanted by our isometric (but, in general, not unitary) transformation $\mathcal{F}$ into the space $L^2(\mathbb{R}; \mathcal{N})$ is well approximated by the sum of model operators.

**Theorem 7.5.** Let Assumptions 2.2 and 7.1 hold true, and let the function $q$ be defined by formulas (3.4) and (3.28). Set

$$A_\infty = \mathcal{FD}_\theta \mathcal{F}^* - \sum_{\ell=1}^L M_{\Xi_\ell}. \quad (7.7)$$

Then the operator $q^{-\beta} A_\infty q^{-\beta}$ is compact in $L^2(\mathbb{R}; \mathcal{N})$ for all $\beta < \beta_0/2$.

The proof of Theorem 7.5 is lengthy and will be given in the following subsections.

Given Theorem 7.5, the proofs of Theorems 7.2 and 7.3 are almost identical to those of Theorems 5.1 and 5.3. We check that the assumptions of Proposition 2.9 are satisfied for the operators $A = \mathcal{FD}_\theta \mathcal{F}^*$ and $A_\ell = M_{\Xi_\ell}$, $\ell = 1, \ldots, L$, acting in the space $L^2(\mathbb{R}; \mathcal{N})$. Equality (2.17) is now true with the operator $A_\infty$ defined by (7.7). We take $X = q^\beta$ where $\beta$ satisfies $1 < \beta < \beta_0/2$, $\beta < 3/2$. Let $\delta$ be any bounded open interval which does not contain the points $0$, $\pm a_n$, $\ell = 1, \ldots, L$, $1 \leq n \leq N_\ell$. The symbol of the SHO $M_{\Xi_\ell}$ is a particular case of (5.7) corresponding to the operator $K_\ell$ defined by formula (7.2). Since $a_n = 2^{-1}s_n(K_\ell)$, the choice of $\delta$ is also the same as in Subsection 5.2. Thus all the assumptions of Proposition 2.9, except (2.19), have been already verified there. Finally, condition (2.19) follows from Theorem 7.5. Thus all assertions of Propositions 2.9 and 2.10 are true for the operators $\mathcal{FD}_\theta \mathcal{F}^*$ and $M_{\Xi_1}, \ldots, M_{\Xi_L}$.

It remains to reformulate the results in terms of the operator $D_\theta$. Since $\mathcal{F}^* \mathcal{F} = I$, the restriction of $\mathcal{FD}_\theta \mathcal{F}^*$ onto $\text{Ran} \mathcal{F}$ is unitarily equivalent to the operator $D_\theta$ and $\mathcal{FD}_\theta \mathcal{F}^* f = 0$ for $f$ in the orthogonal complement to $\text{Ran} \mathcal{F}$. It follows that

$$\mathcal{F}^* \varphi(\mathcal{FD}_\theta \mathcal{F}^*) = \varphi(D_\theta) \mathcal{F}^* \quad (7.8)$$

for, say, continuous functions $\varphi$. In particular, the spectra of $D_\theta$ and $\mathcal{FD}_\theta \mathcal{F}^*$ coincide up to a possible zero eigenvalue. This yields the first three statements of Theorem 7.2. According to (7.8) for $\varphi(\lambda) = (\lambda - z)^{-1}$, we have

$$q^\beta \mathcal{F}_\Delta(D_\theta - zI)^{-1} \mathcal{F}_\Delta q^\beta = q^\beta \mathcal{F}_\Delta(\mathcal{FD}_\theta \mathcal{F}^* - zI)^{-1} \mathcal{F}^* \mathcal{F}_\Delta q^\beta = (\mathcal{F}_\Delta \mathcal{F}^*)(q^\beta(\mathcal{FD}_\theta \mathcal{F}^* - zI)^{-1} q^\beta)(\mathcal{F} \mathcal{F}_\Delta^*).$$

At the last step we have used that the operator $q^\beta$ commutes with $\mathcal{F}_\Delta \mathcal{F}^*$ which is the orthogonal projection in $L^2(\mathbb{R}; \mathcal{N})$ onto the subspace $\mathcal{K}$ defined in (6.4). Therefore the last statement of Theorem 7.2 follows from the continuity of the operator valued function $q^\beta(\mathcal{FD}_\theta \mathcal{F}^* - zI)^{-1} q^\beta$. 
Proposition 2.10 gives the existence of the wave operators $W_{\pm}(\mathcal{F}D_{\theta}\mathcal{F}^*, M_{\Xi_\ell})$ so that according to (7.8) for $\varphi(\lambda) = e^{i\lambda t}$ there exist
\[
\lim_{t \to \pm \infty} e^{iF_{\ell}^* e^{-iM_{\Xi_\ell} t}P_{M_{\Xi_\ell}}} = \lim_{t \to \pm \infty} e^{iF_{\ell} D_{\theta} F^* e^{-iM_{\Xi_\ell} t}P_{M_{\Xi_\ell}}} = \mathcal{F}^* W_{\pm}(\mathcal{F}D_{\theta}\mathcal{F}^*, M_{\Xi_\ell})
\] (7.9)
for all $\ell = 1, \ldots, L$. Since $\chi_{R\setminus \Delta F} = \mathcal{F}_{\bot}$ and $\chi_{\Delta_kF_k} = \mathcal{F}_k$, it follows from Lemma 5.2 that
\[
\lim_{t \to \pm \infty} e^{-iM_{\Xi_\ell} t}P_{M_{\Xi_\ell}}(ac) = 0 \quad \text{and} \quad \lim_{t \to \pm \infty} e^{-iM_{\Xi_k} t}P_{M_{\Xi_k}}(ac) = 0, \quad k \neq \ell.
\]
Therefore relation (7.9) implies that the limits (7.3) exist and
\[
W_{\pm}(D_{\theta}, M_{\Xi_\ell}) = \mathcal{F}^* W_{\pm}(\mathcal{F}D_{\theta}\mathcal{F}^*, M_{\Xi_\ell}), \quad \ell = 1, \ldots, L.
\] (7.10)
The intertwining property of wave operators is a direct consequence of their existence. Since $F_{\ell}F_{\ell}^* P_{M_{\Xi_\ell}}(ac) = \chi_{\Delta_{\ell}} P_{M_{\Xi_\ell}}(ac)$, Lemma 5.2 implies that
\[
\lim_{t \to \pm \infty} (F_{\ell}F_{\ell}^* - I)e^{-iM_{\Xi_\ell} t}P_{M_{\Xi_\ell}}(ac) = 0,
\]
and hence the wave operators $W_{\pm}(\mathcal{F}D_{\theta}\mathcal{F}^*, M_{\Xi_\ell})$ are isometric. Relations (7.4) are trivial because $F_{\ell} = F_{\ell} E_{H_0}(\Delta_{\ell})$ and hence
\[
F_{k}F_{j}^* = F_{k} E_{H_0}(\Delta_k \cap \Delta_j) F_{j}^* = 0, \quad j \neq k.
\]
Finally, according to (2.20) we have
\[
\text{Ran} W_{\pm}(\mathcal{F}D_{\theta}\mathcal{F}^*, M_{\Xi_1}) \oplus \cdots \oplus \text{Ran} W_{\pm}(\mathcal{F}D_{\theta}\mathcal{F}^*, M_{\Xi_L}) = \mathcal{H}_{\mathcal{F}D_{\theta}\mathcal{F}^*}(ac)
\] (7.11)
This relation implies (7.5). Indeed, observe that $\mathcal{H}_{\mathcal{F}D_{\theta}}(ac) = \mathcal{F}^* \mathcal{H}_{\mathcal{F}D_{\theta}\mathcal{F}^*}(ac)$. Hence if $f \in \mathcal{H}_{\mathcal{F}D_{\theta}}(ac)$, then $f = \mathcal{F}^* \tilde{f}$ where $\tilde{f} \in \mathcal{H}_{\mathcal{F}D_{\theta}\mathcal{F}^*}(ac)$. It follows from (7.11) that
\[
\tilde{f} = \sum_{\ell=1}^{L} W_{\pm}(\mathcal{F}D_{\theta}\mathcal{F}^*, M_{\Xi_\ell}) \tilde{f}_{\ell}
\]
where $\tilde{f}_{\ell} = W_{\pm}(\mathcal{F}D_{\theta}\mathcal{F}^*, M_{\Xi_\ell}) \tilde{f}$. Therefore
\[
f = \sum_{\ell=1}^{L} F^* W_{\pm}(\mathcal{F}D_{\theta}\mathcal{F}^*, M_{\Xi_\ell}) \tilde{f}_{\ell} = \sum_{\ell=1}^{L} W_{\pm}(D_{\theta}, M_{\Xi_\ell}) \tilde{f}_{\ell}
\]
according to equality (7.10). This proves (7.5) and hence concludes the proof of Theorem 7.3.

Just as with the analysis of SHOs (see Remark 5.4) in the case $L = 1$ many of the above steps simplify considerably.
7.3. Proof of Theorem 7.5. Here we briefly describe the main steps of the proof of Theorem 7.5. We consecutively replace the operator $\mathcal{F} \mathcal{D}_\theta \mathcal{F}^*$ by simpler operators neglecting all terms that admit the representation $q^\beta K q^\beta$ with an operator $K$ compact in the space $L^2(\mathbb{R};\mathcal{N})$. We call such terms negligible. Our goal is to reduce the operator $\mathcal{F} \mathcal{D}_\theta \mathcal{F}^*$ to the SHO with the explicit symbol

$$\Xi_0(\lambda) = \sum_{\ell=1}^L \Xi_\ell(\lambda).$$

(7.12)

The first step is to replace the function $\theta$ by a function $\varphi$ supported in $\Omega$. To be more precise, we choose a function $\rho \in C_0^\infty(\Omega)$ such that $\rho(\lambda) = 1$ in an open neighbourhood of $\{\lambda_1, \ldots, \lambda_L\}$ and set $\varphi(\lambda) = \rho(\lambda) \theta(\lambda)$.

The operator $\mathcal{F}(\mathcal{D}_\theta - \mathcal{D}_\varphi)\mathcal{F}^*$ is negligible in view of the following

Lemma 7.6. Let $\xi \in C(\mathbb{R})$ be such that the limits $\lim_{\lambda \to \pm \infty} \xi(\lambda)$ exist and are finite and assume that $\xi(\lambda) = 0$ in a neighbourhood of $\{\lambda_1, \ldots, \lambda_L\}$. Then for any $\beta > 0$,

$$q^{-\beta} \mathcal{F} \mathcal{D}\xi \mathcal{F}^* q^{-\beta} \in \mathcal{S}_\infty.$$ 

The second step is to sandwich $\mathcal{D}_\varphi$ between the operators $\omega(H_0)$. We suppose that $\omega \in C_0^\infty(\Delta)$ and $\omega(\lambda) = 1$ for $\lambda \in \text{supp} \varphi$.

Lemma 7.7. For any $\beta > 0$, the difference

$$q^{-\beta} \mathcal{F}(\omega(H_0) D \omega(H_0) - D \omega) \mathcal{F}^* q^{-\beta}$$

is compact.

Observe that $\mathcal{F} \omega(H_0) = \mathcal{F}_\Delta \omega(H_0)$. Thus, up to negligible terms, we get the operator $\mathcal{F}_\Delta \omega(H_0) D \omega(H_0) \mathcal{F}_\Delta^*$ which localizes the problem onto neighbourhoods of singular points. It is important that the operator $\mathcal{F}_\Delta \omega(H_0) D \omega(H_0) \mathcal{F}_\Delta^*$ admits representation (6.6).

Next, we reduce the problem to the study of SHO. To that end, we swap the operators $Z_\omega$ and $P_\pm$ in the r.h.s. of (6.6). In fact, we have

Lemma 7.8. Let

$$\Xi(\lambda) = \omega^2(\lambda) \varphi(\lambda) Z(\lambda) Y(\lambda + i0) Z^*(\lambda) : \mathcal{N} \to \mathcal{N}. $$

(7.13)

Then, for any $\beta > 0$, the difference

$$q^{-\beta} (\mathcal{F}_\Delta \omega(H_0) D \omega(H_0) \mathcal{F}_\Delta^* - M \Xi) q^{-\beta}$$

is compact.
Thus the problem reduces to the analysis of the SHO $M_\Xi$. Up to negligible terms, it is determined by the values $\Xi(\lambda_\ell)$ at the points of discontinuity of $\theta(\lambda)$. Putting together the stationary representation (2.16) of the scattering matrix and the definition (5.7) of $\Xi$, we will prove the following result.

**Lemma 7.9.** Let $\Xi_0$ be given by formula (7.12). Then the symbol $\Xi - \Xi_0$ satisfies the hypothesis of Lemma 4.3.

Combining this result with Corollary 4.4, we directly obtain

**Lemma 7.10.** For any $\beta > 0$, the difference $q^{-\beta}(M_\Xi - M_{\Xi_0})q^{-\beta}$ is compact.

Theorem 7.5 follows from Lemmas 7.6, 7.7, 7.8 and 7.10. Thus for the proof of Theorem 7.5 it remains to establish Lemmas 7.6, 7.7, 7.8 and 7.9. This requires several analytic assertions which are collected in the next subsection.

### 7.4. Compactness of the sandwiched operators $D_{\phi'}$. For the proof of the first assertion, see [12, Theorem 7.3] and [16, Lemma 5.4].

**Lemma 7.11.** Under Assumption 2.2(B) the operator $D_k$ is compact for any function $\xi \in C(\mathbb{R})$ such that the limits $\lim_{\lambda \to \pm \infty} \xi(\lambda)$ exist and are finite.

The next one is a direct consequence of our construction of the operator $F$.

**Lemma 7.12.** Suppose that $v \in L^\infty(\mathbb{R})$ and $vq^{-\beta} \in L^\infty(\mathbb{R})$. Then the operator $q^{-\beta}Fv(H_0) : \mathcal{H} \to L^2(\mathbb{R};\mathcal{N})$ is bounded.

**Proof.** The operator $q^{-\beta}F_\Delta v(H_0) = q^{-\beta}vF_\Delta$ is bounded because $vq^{-\beta} \in L^\infty(\mathbb{R})$. The operator $q^{-\beta}F_\perp v(H_0)$ is bounded because, by (7.6), $F_\perp = \chi_{\mathbb{R}\setminus\Delta}F_\perp$ and $q^{-\beta}\chi_{\mathbb{R}\setminus\Delta} \in L^\infty(\mathbb{R})$. $\square$

The following assertion relies on Theorem 6.2.

**Lemma 7.13.** Under the assumptions of Theorem 6.2 for any $\beta > 0$, the operator $q^{-\beta}F\omega(H_0)D_{\phi'}v(H_0) : \mathcal{H} \to L^2(\mathbb{R};\mathcal{N})$ is compact.

**Proof.** Since $F\omega(H_0) = F_\Delta\omega(H_0)$, we can use representation (6.9) where $Y_{\phi',v}^{(\pm)}$ is the operator (6.8). Thus we have to check that the operators $q^{-\beta}Z_\omega P_{\pm}Y_{\phi',v}^{(\mp)} : \mathcal{H} \to L^2(\mathbb{R};\mathcal{N})$ are compact. Observe that $q^{-\beta}Z_\omega = Z_\omega q^{-\beta}$, the operator $Z_\omega : L^2(\mathbb{R};\mathcal{H}) \to L^2(\mathbb{R};\mathcal{N})$ is bounded and, by Proposition 4.1, the operator $q^{-\beta}P_{\pm}q^{\beta}$ is bounded in $L^2(\mathbb{R};\mathcal{H})$. Therefore it suffices to verify the compactness of the operator $q^{-\beta}Y_{\phi',v}^{(\mp)} : \mathcal{H} \to L^2(\mathbb{R};\mathcal{H})$. 

Set

\[ Y_v^{(\pm)}(\lambda) = Y(\lambda \pm i0)GR_0(\lambda)v(H_0), \quad \lambda \in \text{supp} \varphi. \quad (7.14) \]

It follows from definition (6.8) that

\[ (q^{-\beta}Y_v^{(\mp)} f)(\lambda) = q^{-\beta}(\lambda)\varphi(\lambda)Y_v^{(\mp)}(\lambda)f. \]

Suppose that \( f_n \to 0 \) weakly in \( \mathcal{H} \) as \( n \to \infty \). By definition (7.14), the operators \( Y_v^{(\mp)}(\lambda) \) are compact, and hence \( \|Y_v^{(\mp)}(\lambda)f_n\| \to 0 \) as \( n \to \infty \) for \( \lambda \in \text{supp} \varphi \). Note that the integrand in

\[ \|q^{-\beta}Y_v^{(\mp)} f_n\|^2 = \int_{-\infty}^{\infty} q^{-2\beta}(\lambda)\varphi(\lambda)^2||Y_v^{(\mp)}(\lambda)f_n||^2d\lambda \quad (7.15) \]

is bounded by \( Cq^{-2\beta}(\lambda)\varphi^2(\lambda)||Y_v^{(\mp)}(\lambda)||^2 \) which belongs to \( L^1(\mathbb{R}) \) because the operators \( Y_v^{(\mp)}(\lambda) \) are uniformly bounded on \( \text{supp} \varphi \). Thus expression (7.15) tends to zero as \( n \to \infty \) by the Lebesgue theorem. \( \square \)

Finally, we use Theorem 6.1.

**Lemma 7.14.** Let \( \rho \in C^\infty_0(\Omega) \); then for any \( \beta > 0 \), we have

\[ q^{-\beta}F\mathcal{D}_\rho \in \mathcal{S}_\infty. \]

**Proof.** Let \( \omega \) be a bounded function with \( \text{supp} \omega \subset \Delta \) such that \( \omega(\lambda) = 1 \) for \( \lambda \in \text{supp} \rho \); set \( \tilde{\omega} = 1 - \omega \). The operator \( \mathcal{D}_\rho \) is compact by Lemma 7.11. Therefore \( q^{-\beta}F\tilde{\omega}(H_0)D_\rho \in \mathcal{S}_\infty \) because the operator \( q^{-\beta}F\tilde{\omega}(H_0) \) is bounded by Lemma 7.12. It remains to check that \( q^{-\beta}F\omega(H_0)D_\rho \in \mathcal{S}_\infty \). Since \( F\omega(H_0) = F\Delta \omega(H_0) \) and \( \omega(H_0) = \omega(H_0)F^*_\Delta \mathcal{F}_\Delta \), to that end we have to verify two inclusions:

\[ q^{-\beta}F\Delta \omega(H_0)D_\rho \omega(H_0)F^*_\Delta \in \mathcal{S}_\infty \quad \text{and} \quad q^{-\beta}F\Delta \omega(H_0)D_\rho \tilde{\omega}(H_0) \in \mathcal{S}_\infty. \quad (7.16) \]

According to Theorem 6.1, the first operator here equals

\[ \mathcal{Z}_\omega q^{-\beta}(P_-\mathcal{Y}_\rho P_+ + P_+\mathcal{Y}_\rho^* P_-) \mathcal{Z}_\omega^* \]

where the operators \( \mathcal{Z}_\omega \) and \( \mathcal{Y}_\rho \) are defined by formulas (6.5). Since the operator \( \mathcal{Z}_\omega : L^2(\mathbb{R};\mathcal{H}) \to L^2(\mathbb{R};\mathcal{N}) \) is bounded, it suffices to show that the operator \( q^{-\beta}(P_-\mathcal{Y}_\rho P_+ + P_+\mathcal{Y}_\rho^* P_-) \) is compact in the space \( L^2(\mathbb{R};\mathcal{H}) \). This fact follows from Corollary 4.4 applied to \( \Xi = Y\rho \) because the operator valued function \( Y(\lambda) \) takes compact values and is Hölder continuous for \( \lambda \in \text{supp} \rho \). The second operator in (7.16) is compact according to Lemma 7.13 where \( \varphi = \rho \) and \( v = \tilde{\omega} \). \( \square \)
7.5. Proof of Lemma 7.6. Let \( \rho \in C_0^\infty(\Omega) \) be such that \( \rho(\lambda) = 1 \) in a neighbourhood of \( \{\lambda_1, \ldots, \lambda_L\} \) and \( \rho(\lambda)\xi(\lambda) = 0 \). We set \( \tilde{\rho}(\lambda) = 1 - \rho(\lambda) \) so that \( \tilde{\rho}(\lambda)\xi(\lambda) = \xi(\lambda) \). Then we have

\[
\xi(H) = \rho(H_0)\xi(H_0) + \tilde{\rho}(H)\xi(H)\tilde{\rho}(H) + \tilde{\rho}(H)\xi(H_0)\rho(H_0),
\]

\[
\xi(H_0) = \rho(H)\xi(H_0) + \tilde{\rho}(H)\xi(H_0)\tilde{\rho}(H) + \tilde{\rho}(H)\xi(H_0)\rho(H)
\]

and hence

\[
D_\xi = -D_\rho \xi(H_0) + \tilde{\rho}(H)D_\xi \tilde{\rho}(H) - \tilde{\rho}(H)\xi(H_0)D_\rho.
\] (7.17)

Let us sandwich this expression by \( q^{-\beta}F \) and consider every term in the r.h.s. separately.

The first term in (7.17) yields

\[
-(q^{-\beta}F D_\rho) \left( \xi(H_0)F^*q^{-\beta} \right).
\]

The first factor here is a compact operator by Lemma 7.14 and the second factor is a bounded operator by Lemma 7.12.

The second term in (7.17) yields

\[
(q^{-\beta}F \tilde{\rho}(H))D_\xi (\tilde{\rho}(H)F^*q^{-\beta}).
\] (7.18)

We have

\[
q^{-\beta}F \tilde{\rho}(H) = -q^{-\beta}F D_\rho + q^{-\beta}F \tilde{\rho}(H_0) \in B.
\] (7.19)

Indeed, the first operator on the right is compact according to Lemma 7.14, and the second operator is bounded according to Lemma 7.12. Thus the first and third factors in (7.18) are bounded operators. It remains to use the fact that the operator \( D_\xi \) is compact by Lemma 7.11.

Finally, the third term in (7.17) yields

\[
-(q^{-\beta}F \tilde{\rho}(H)) \xi(H_0) (D_\rho F^*q^{-\beta}).
\]

The first factor here is bounded according to (7.19), and the last factor is compact according to Lemma 7.14.

7.6. Proof of Lemma 7.7. Set \( \tilde{\omega} = 1 - \omega \). We have to check two inclusions

\[
q^{-\beta}F \omega(H_0)D_\phi \tilde{\omega}(H_0)F^*q^{-\beta} \in \mathcal{S}_\infty \quad (7.20)
\]

and

\[
q^{-\beta}F \tilde{\omega}(H_0)D_\phi \omega(H_0)F^*q^{-\beta} \in \mathcal{S}_\infty. \quad (7.21)
\]

Let \( v \) be a \( C^\infty \) function satisfying condition (6.7) and such that \( v\tilde{\omega} = \tilde{\omega} \). We write operator (7.20) as a product of two factors

\[
(q^{-\beta}F \omega(H_0)D_\phi v(H_0)) (\tilde{\omega}(H_0)F^*q^{-\beta}).
\]

The first one is compact according to Lemma 7.13, and the second one is bounded according to Lemma 7.12.
Operator (7.21) can be factorized into a product of three factors
\[(q^{-\beta}Fv(H_0))(\bar{\omega}(H_0)D\varphi)(\bar{\omega}(H_0)F^*q^{-\beta}).\] (7.22)
The first and the third factors here are bounded operators by Lemma 7.12. Since \(\bar{\omega}(\lambda)\varphi(\lambda) = 0\), we can write the second factor as
\[\bar{\omega}(H_0)D\varphi = \bar{\omega}(H_0)\varphi(H) = -D\bar{\omega}\varphi(H) = D\omega\varphi(H).\] (7.23)
By Lemma 7.11 the operators \(D\omega\) and hence (7.23) are compact. This proves that operator (7.22) is also compact. \(\square\)

7.7. Proof of Lemma 7.8. According to representation (6.6) and the definition (1.5) of the SHO \(\Xi\), we have to check that the operator
\[q^{-\beta}Z_\omega P_-Y_\varphi P_+Z^*_\omega q^{-\beta} - q^{-\beta}P_-Z_\omega Y_\varphi Z^*_\omega P_+q^{-\beta}\]
\[= (q^{-\beta}(Z_\omega P_- - P_-Z_\omega)q^{-\beta})Y_\varphi(q^\beta P_+q^{-\beta})Z^*_\omega\]
\[+ (q^{-\beta}P_-q^{-\beta})Z_\omega Y_\varphi(q^{-\beta}(P_+Z^*_\omega - Z^*_\omega P_+)q^{-\beta})\]
is compact in \(L^2(\mathbb{R}; \mathcal{N})\). Here we have taken into account that the operators \(q^{-\beta}\) commute with \(Z_\omega\) and \(Y_\varphi\). Recall that, by Proposition 4.1, the operators \(q^{-\beta}P_\pm q^{-\beta}\) are bounded. Therefore it suffices to use the fact that, by Lemma 4.3 (see also Remark 4.5) applied to the operator valued function \(\Xi(\lambda) = Z(\lambda)\omega(\lambda)\), the operators
\[q^{-\beta}(Z_\omega P_- - P_-Z_\omega)q^{-\beta} : L^2(\mathbb{R}; \mathcal{H}) \rightarrow L^2(\mathbb{R}; \mathcal{N})\]
are compact. \(\square\)

7.8. Proof of Lemma 7.9. Putting together formulas (5.7), (7.12) and (7.13), we see that it suffices to check that the operator valued function
\[\Psi(\lambda) = Z(\lambda)Y(\lambda + i0)Z^*(\lambda)\varphi(\lambda)\omega^2(\lambda) - \sum_{\ell=1}^L \varkappa_\ell(S(\lambda\ell) - I)\zeta(\lambda - \lambda\ell)\] (7.24)
satisfies the assumptions of Lemma 4.3. Clearly, both terms here are continuous functions of \(\lambda\) away from the set \(\lambda_1, \ldots, \lambda_L\) and tend to zero as \(|\lambda| \rightarrow \infty\).

Observe that the functions \(\varphi\) and \(\theta\) have the same jump (1.2) at the point \(\lambda_\ell\). Therefore the jump of the first term in (7.24) at the point \(\lambda_\ell\) equals
\[\varkappa_\ell Z(\lambda_\ell)Y(\lambda_\ell + i0)Z^*(\lambda_\ell).\] (7.25)
Since \(\zeta(\pm0) = \pm1/2\), the jump at the point \(\lambda_\ell\) of the sum in (7.24) equals
\[\varkappa_\ell(S(\lambda_\ell) - I).\] (7.26)
It follows from the representation (2.16) of the scattering matrix \(S(\lambda)\) that expressions (7.25) and (7.26) coincide. Hence the operator valued function \(\Psi(\lambda)\) is continuous at each point \(\lambda_\ell\).
Finally, $\Psi(\lambda)$ satisfies condition (4.7) because $Z(\lambda)$ and $Y(\lambda + i0)$ are H"older continuous functions, $\theta(\lambda)$ satisfies condition (7.1) and $\zeta(\lambda)$ satisfies condition (3.3). □

Appendix A. Proof of Lemma 3.10

Proof. 1. First we recall some background information on the Legendre function. This function can be defined (see formulas (2.10.2) and (2.10.5) in [2]) in terms of the hypergeometric function

$$\begin{align*}
F(a, b, c; z) &= \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n} \frac{z^n}{n!}, \\
(a)_n &= \frac{\Gamma(a+n)}{\Gamma(a)}, \quad |z| < 1,
\end{align*}
$$

where $\Gamma$ is the gamma-function. Namely, for $x>1$, we have

$$P_{-\frac{1}{2}+i\tau}(x) = \text{Re} \left( m(\tau) F\left( \frac{1}{4} - i\frac{\tau}{2}, \frac{3}{4} - i\frac{\tau}{2}; 1 - i\tau; x^{-2}\right) x^{-\frac{1}{2}+i\tau} \right)$$

where

$$m(\tau) = \frac{\Gamma(i\tau)}{\sqrt{\pi\Gamma\left(\frac{1}{2} + i\tau\right)}} 2^{\frac{1}{2}+i\tau}. \quad (A.3)$$

Putting together (A.1) and (A.2), we see that

$$P_{-\frac{1}{2}+i\tau}(x) = \text{Re} \left( m(\tau) x^{-\frac{1}{2}+i\tau} \sum_{n=0}^{\infty} p_n(\tau)x^{-2n} \right), \quad (A.4)$$

where $p_0(\tau) = 1$ and

$$p_n(\tau) = \frac{\left( \frac{1}{4} - i\frac{\tau}{2} \right)_n \left( \frac{3}{4} - i\frac{\tau}{2} \right)_n}{(1 - i\tau)_n n!}.$$ 

According to the Stirling formula all coefficients $p_n(\tau)$ are uniformly (in $n$ and $\tau$) bounded for $\tau$ in compact intervals $\delta \subset \mathbb{R}_+$. Moreover,

$$|\partial p_n(\tau)/\partial \tau| \leq C \ln n$$

where $C$ does not depend on $n$ and $\tau$ in $\delta$. In particular, we see that $P_{-\frac{1}{2}+i\tau}(x)$ is a smooth function of $x>1$ and it has the asymptotics

$$P_{-\frac{1}{2}+i\tau}(x) = \text{Re} \left( m(\tau) x^{-\frac{1}{2}+i\tau} \right) + O(x^{-5/2}), \quad x \to \infty; \quad (A.5)$$

this asymptotics can be differentiated in $x$. The series (A.4) and hence the asymptotics (A.5) can also be differentiated in $\tau$.

Instead of (A.2), in a neighborhood of the point $x = 1$ we use another representation (see formula (3.2.2) in [2])

$$P_{-\frac{1}{2}+i\tau}(x) = F\left( \frac{1}{2} - i\tau, \frac{1}{2} + i\tau; 1; \frac{1-x}{2} \right).$$

It implies that

$$|P_{-\frac{1}{2}+i\tau}(x)| + |P'_{-\frac{1}{2}+i\tau}(x)| \leq C, \quad x \in [1, 2]. \quad (A.6)$$
2. Let us return to the function \( w_\tau(\nu) \). Let us write (3.16) as

\[
\sqrt{2\pi}e^{-i\lambda}w_\tau(\lambda) = \int_1^\infty P_{\frac{1}{2}+i\tau}(x)e^{-i\lambda x}dx.
\]

Integrating here by parts, we see that

\[
i\sqrt{2\pi}e^{-i\lambda}w_\tau(\lambda) = P_{-\frac{1}{2}+i\tau}(1)e^{-i\lambda} + \int_1^\infty e^{-i\lambda x}P'_{\frac{1}{2}+i\tau}(x)dx.
\]

According to (A.5) \( |P'_{-\frac{1}{2}+\tau}(x)| \leq Cx^{-3/2} \) if \( x \geq 2 \). According to (A.6) the function \( P_{\frac{1}{2}+i\tau}(x) \) is bounded if \( x \in [1, 2] \). Therefore the integral in the right-hand side is bounded uniformly in \( \lambda \) which yields the first estimate (3.22).

3. To obtain the first estimate (3.23), we observe that, again by (A.6), the function \( P_{-\frac{1}{2}+i\tau}(x) \) is bounded for \( x \in [1, 2] \). For \( x \geq 2 \) we use asymptotics (A.5). Note that the leading term

\[
\int_2^\infty e^{-i\lambda x}x^{-1/2+\tau}dx = |\lambda|^{-1/2-\tau} \int_2^\infty e^{\mp iy}y^{-1/2+\tau}dy, \quad \pm \lambda > 0, \quad (A.7)
\]

satisfies estimate (3.23). The contribution of the remainder \( O(x^{-5/2}) \) in (A.5) to the integral in (3.16) is uniformly bounded.

4. Estimates (3.22) and (3.23) on the derivative \( \partial w_\tau(\lambda)/\partial \tau \) can be obtained quite similarly because asymptotics (A.5) are differentiable in \( \tau \) and estimates (A.6) remain true for the derivative \( \partial P_{-\frac{1}{2}+i\tau}(x)/\partial \tau \). The only difference is that instead of (A.7) we now have the integral

\[
\int_2^\infty e^{-i\lambda x}x^{-1/2+\tau}\ln x\,dx
\]

which is bounded by \( |\lambda|^{-1/2} \ln|\lambda|| \). □

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